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The Orthogonality between Complex Fuzzy Sets and Its Application to Signal Detection

Bo Hu [1], **Lvqing Bi** [2] and **Songsong Dai** [3,*]

[1] School of Mechanical and Electrical Engineering, Guizhou Normal University, Guiyang 550025, China; hb.gznu@gmail.com
[2] School of Electronics and Communication Engineering, Yulin Normal University, Yulin 537000, China; bilvqing108@163.com
[3] School of Information Science and Engineering, Xiamen University, Xiamen 361005, China
* Correspondence: ssdai@stu.xmu.edu.cn; Tel.: +86-592-258-0135

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Abstract: A complex fuzzy set is a set whose membership values are vectors in the unit circle in the complex plane. This paper establishes the orthogonality relation of complex fuzzy sets. Two complex fuzzy sets are said to be orthogonal if their membership vectors are perpendicular. We present the basic properties of orthogonality of complex fuzzy sets and various results on orthogonality with respect to complex fuzzy complement, complex fuzzy union, complex fuzzy intersection, and complex fuzzy inference methods. Finally, an example application of signal detection demonstrates the utility of the orthogonality of complex fuzzy sets.

Keywords: orthogonality; complex fuzzy sets; complex fuzzy operations; complex fuzzy inference

1. Introduction

Complex fuzzy sets [1] are an important extension of fuzzy set theory. In recent years, complex fuzzy sets have been successfully used in complex fuzzy inference systems for various applications, such as time series prediction [2–8], function approximation [9–11], and image restoration [12,13]. A complex fuzzy set A on a universe of discourse U is a mapping from U to the unit disc in the complex plane. Thus a complex fuzzy set is a set of vectors in the complex plane. This idea of membership vectors greatly inspired how definition of complex fuzzy sets operations [14,15] and complex fuzzy logic systems [16–21] and how to measures the difference between two complex fuzzy sets [22–24].

Orthogonality is an important concept in mathematics and computer science. For instance, orthogonality in geometry means that two vectors are perpendicular. Orthogonality in programming language design is the ability to use different language features in different combinations with consistent results [25]. Then, a question arises: why do we need to study the orthogonality between complex fuzzy sets?

The literature review shows two main reasons for studying the orthogonality between complex fuzzy sets. (1) Orthogonality is also an important tool in traditional fuzzy systems. Orthogonal transformation method, orthogonal rule, and orthogonal approximation concept are frequently applied to fuzzy rule-based models [26–28], fuzzy neural networks [29,30], and fuzzy control [31,32]. Complex fuzzy sets as an extension of fuzzy sets have been studied. When we apply these orthogonal methods to complex fuzzy systems, the concept of orthogonality should be considered. (2) Recently, complex fuzzy sets have been used in image restoration [12,13] and signal processing [1,23]. In these applications, various orthogonal data, such as orthogonal signals and orthogonal images, are often considered. So, the concept of orthogonality also should be considered when we apply complex fuzzy sets to these applications.

Our goal in this paper is to present the concept and properties of orthogonality for complex fuzzy sets. In our view, this is analogous to orthonormal vectors. As mentioned in [16], the membership function of a complex fuzzy set can be viewed as a vector in the complex plane. As is well known, a vector has length and direction. There are various properties concerning the direction of vectors. The orthogonality relation is an important concept in vector theory when a complex fuzzy set is viewed as a set of vectors and a complex fuzzy logic is viewed as a logic of vectors. We should consider the direction properties for complex fuzzy sets and complex fuzzy logic. Having this in mind, we propose the orthogonality for complex fuzzy sets. Then, their implications for the possible choices of complex fuzzy operations are examined. The representation of complex fuzzy set S is given by $r_A(x) \cdot e^{jw_A(x)}$, the amplitude term $r_A(x)$ and the phase term $w_A(x)$ are both real-valued, $r_A(x) \in [0,1]$, and $j = \sqrt{-1}$. We also can use the form $r_A(x)\cos(w_A(x)) + jr_A(x)\sin(w_A(x))$. This form is based on two orthonormal vectors, $(1,0)$ and $(0,1)$, corresponding to the complex numbers $1 + j0$ and $0 + j$, respectively. These two vectors are the most used orthonormal basis in complex plane. The concept of orthogonality has been used in complex fuzzy sets naturally, but it has not formed systemic cognition and there has been a lack of standardized study.

The main contribution of the study includes. (1) A concept of orthogonality for complex fuzzy sets; (2) An idea for the possible choices of complex fuzzy operators: a complex fuzzy operation should preserve the orthogonality. In the practical application of complex fuzzy sets, one of the key research issues is how to select a suitable complex fuzzy operation. We investigate this idea in depth, and then examine its implications for the possible choices of complex fuzzy inference; and (3) A signal detection method which involves the orthogonality complex fuzzy sets. Operations used in this method have the orthogonality preserving property.

The remainder of this paper is organized in the following way. In Section 2, the concept of orthogonality of complex fuzzy sets is introduced, and their properties investigated. In Section 3, we discuss whether orthogonality can be preserved for complex fuzzy complement, complex fuzzy union, complex fuzzy intersection and complex fuzzy inference system, and present an application example in signal detection. Discussion and conclusion are presented in Section 4.

2. Materials and Methods

In this section we introduce the concept of orthogonality of complex fuzzy sets, which should be contrasted with the concept of orthogonality in mathematics such as $x \perp y$ between two vectors x and y. By intuition, this is analogous to the orthogonality of vectors, orthogonality is the relation of two complex fuzzy sets where the angle between membership vectors of each element of the universal set are perpendicular. As shown in Figure 1 , two membership vectors A and B of an element of U are orthogonality (perpendicular).

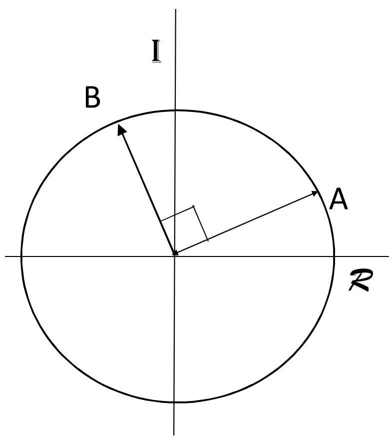

Figure 1. Orthogonality.

A complex fuzzy set A, defined on a universe of discourse U, is characterized by a membership function $\mu_A(x)$ that assigns any element $x \in U$ a complex-valued grade of membership in A.

Let $F^C(U)$ be the set of all complex fuzzy sets on U. The complex fuzzy set A may be represented as the set of ordered pairs:

$$A = \{(x, \mu_A(x)) | x \in U\} \tag{1}$$

where the membership function $\mu_A(x)$ is of the form $r_A(x) \cdot e^{jw_A(x)}$, $j = \sqrt{-1}$, the amplitude term $r_A(x)$ and the phase term $w_A(x)$ are both real-valued, and $r_A(x) \in [0, 1]$.

The inner product $< \mu_A(x), \mu_B(x) >$ of membership vectors of $x \in U$ is defined as:

$$< \mu_A(x), \mu_B(x) >= r_A(x) \cdot r_B(x) \cdot \cos(|w_A(x) - w_B(x)|) \tag{2}$$

Definition 1. *Let A and B be two complex fuzzy sets on U, and $\mu_A(x) = r_A(x) \cdot e^{jw_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{jw_B(x)}$ their membership functions, respectively. A and B are said to be orthogonal if $< \mu_A(x), \mu_B(x) >= 0$ for each $x \in U$ and it is denoted by $A \perp B$. Similarly, for the subsets $\mathscr{C}, \mathscr{D} \subseteq F^C(U)$, we write $A \perp \mathscr{C}$ if $A \perp \mathscr{C}$ for all $C \in \mathscr{C}$ and $\mathscr{C} \perp \mathscr{D}$ if $C \perp D$ for all $C \in \mathscr{C}$ and all $D \in \mathscr{D}$.*

Theorem 1. *The orthogonality of complex fuzzy sets has the following properties:*

(i) $\mathbf{0} \perp A$ *for all* $A \in F^C(U)$,
(ii) *if* $A \perp B$ *then* $B \perp A$,
(iii) *if* $A \perp A$ *then* $A = \mathbf{0}$.

Proof. Trivial. □

The proof is simple and we omit the details. In this paper, we omit proofs of some results which can be easily obtained.

Remark 1. *$\mathbf{0}$ denote the complex fuzzy set where the membership vectors of each element of the universal set is the zero vector.*

In practice, orthogonality may be represented as independent or uncorrelated objects. The orthogonality between complex fuzzy sets may be seen as a theory or method for handling the relativity and dependence among complex data. The following problems are interesting.

Problem 1. *Suppose $A \perp B$ and $A \perp C$, then whether or not $A \perp (B \circ C)$, where $\circ$ is some operation of complex fuzzy sets.*

Problem 2. *Suppose $A \perp B$, then whether or not $f(A) \perp f(B)$, where f is some complex fuzzy inference method.*

3. Results

In this section, we discuss whether orthogonality can be preserved for complex fuzzy complement, complex fuzzy union, and complex fuzzy intersection, i.e., Problems 1 and 2.

The concept of orthogonality preserving is formally expressed as follows.

Definition 2. *Let $f : F^C(U) \to F^C(U)$ be a function, f is orthogonality preserving if:*

$$A \perp B \Rightarrow f(A) \perp f(B) \tag{3}$$

for any $A, B \in F^C(U)$.

Obviously, set rotation and reflection operations, described in [1], are orthogonality preserving.

Let A be a complex fuzzy set on U and $\mu_A(x) = r_A(x) \cdot e^{jw_A(x)}$ its membership functions. The rotation of A by θ radians, denoted $Rot_\theta(A)$ is defined as:

$$Rot_\theta(\mu_A(x)) = r_A(x) \cdot e^{j(w_A(x)+\theta)} \tag{4}$$

The reflection of A , denoted $Ref(A)$ is defined as:

$$Ref(\mu_A(x)) = r_A(x) \cdot e^{-jw_A(x)} \tag{5}$$

Theorem 2. *Let A and B be two complex fuzzy sets on U.*

(i) If $A \perp B$ then $Ref(A) \perp Ref(B)$.
(ii) If $A \perp B$ then $Rot_\theta(A) \perp Rot_\theta(B)$ for any θ radians.

Proof. Trivial. □

3.1. Complex Fuzzy Complement

The complement of a complex fuzzy set is an extension of the definition of the traditional fuzzy complement. Ramot et al. [1] introduced the following three complex fuzzy complements.

Let A be a complex fuzzy set on U and $\mu_A(x) = r_A(x) \cdot e^{jw_A(x)}$ its membership functions. The complex fuzzy complement of A, denoted $\neg A$ is specified by a function:

$$\mu_{\neg_1 A}(x) = c(r_A(x)) \cdot e^{j(-w_A(x))} \tag{6}$$

$$\mu_{\neg_2 A}(x) = c(r_A(x)) \cdot e^{j(w_A(x))} \tag{7}$$

$$\mu_{\neg_3 A}(x) = c(r_A(x)) \cdot e^{j(w_A(x)+\pi)} \tag{8}$$

where $c(r_A(x)) = 1 - r_A(x)$.

Theorem 3. *Let A and B be two complex fuzzy sets on U and $\mu_A(x) = r_A(x) \cdot e^{jw_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{jw_B(x)}$ their membership functions, respectively. Support $\mu_A(x) \neq 0$ and $\mu_B(x) \neq 0$ for all $x \in U$. If $A \perp B$, then $\neg_i A \perp \neg_i B$, $i \in \{1, 2, 3\}$.*

Proof. We only consider the case of $i = 1$. For any $x \in U$, by $A \perp B$, we have:

$$< \mu_A(x, \mu_B(x)) >= r_A(x) \cdot r_B(x) \cdot \cos(|w_A(x) - w_B(x)|) = 0$$

Because $\mu_A(x) \neq 0$ and $\mu_B(x) \neq 0$, thus:

$$\cos(|w_A(x) - w_B(x)|) = 0$$

By the definition of the complex fuzzy complement:

$$\mu_{\neg A}(x) = (1 - r_A(x)) \cdot e^{j(-w_A(x))},$$

$$\mu_{\neg B}(x) = (1 - r_B(x)) \cdot e^{j(-w_B(x))},$$

We obtain:

$$\begin{aligned} < \mu_{\neg_1 A}(x), \mu_{\neg_1 B}(x) > &= c(r_A(x)) \cdot c(r_B(x)) \cdot \cos(|-w_A(x) + w_B(x)|) \\ &= c(r_A(x)) \cdot c(r_B(x)) \cdot \cos(|w_A(x) - w_B(x)|) \\ &= 0. \end{aligned}$$

Thus $\neg_1 A \perp \neg_1 B$ □

Example 1. *Assume two complex fuzzy sets A and B are given by:*

$$\begin{aligned} A =& 0.1e^{j1.5\pi}/x_1 + 0.2e^{j0.3\pi}/x_2 \\ B =& 0.2e^{j\pi}/x_1 + 0.4e^{j0.8\pi}/x_2 \end{aligned}$$

We see $A \perp B$. Then we have:

$$\begin{aligned} \neg_1 A =& 0.9e^{j0.5\pi}/x_1 + 0.8e^{j1.7\pi}/x_2 \\ \neg_1 B =& 0.8e^{j\pi}/x_1 + 0.6e^{j1.2\pi}/x_2 \\ \neg_2 A =& 0.9e^{j1.5\pi}/x_1 + 0.8e^{j0.3\pi}/x_2 \\ \neg_2 B =& 0.8e^{j\pi}/x_1 + 0.6e^{j0.8\pi}/x_2 \\ \neg_3 A =& 0.9e^{j0.5\pi}/x_1 + 0.8e^{j1.3\pi}/x_2 \\ \neg_3 B =& 0.8e^{j2\pi}/x_1 + 0.6e^{j1.8\pi}/x_2. \end{aligned}$$

It is easy to verify $\neg_i A \perp \neg_i B$ for any $i \in \{1, 2, 3\}$

3.2. Complex Fuzzy Union

The complex fuzzy union of complex fuzzy sets are reviewed as follows (see [1]).

Let A and B be two complex fuzzy sets on U and $\mu_A(x) = r_A(x) \cdot e^{jw_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{jw_B(x)}$ their membership functions, respectively. The complex fuzzy union of A and B, denoted $A \cup B$, is specified by a function:

$$\mu_{A\cup B}(x) = (r_A(x) \oplus r_B(x)) \cdot e^{jw_{A\otimes B}(x)} \tag{9}$$

where $\oplus$ represents a t-conorm function.

The functions given below are possibilities for calculating $w_{A\otimes B}$:

Sum:	$w_{A\otimes B} = w_A + w_B$	(10)
Max:	$\mu_{A\cup B} = \max(w_A, w_B)$	(11)
Min:	$\mu_{A\cup B} = \min(w_A, w_B)$	(12)
Winner Take All:	$w_{A\otimes B} = \begin{cases} w_A, & r_A > r_B \\ w_B, & r_A < r_B \end{cases}$	(13)
Weighted Average:	$\mu_{A\cup B} = \dfrac{r_A w_A + r_B w_B}{r_A + r_B}$	(14)
Average:	$\mu_{A\cup B} = (w_A + w_B)/2$	(15)
Product:	$\mu_{A\cup B} = w_A \cdot w_B$	(16)
Difference:	$\mu_{A\cup B} = w_A - w_B$	(17)

Theorem 4. *Let A, B and C be three complex fuzzy sets on U. If $\mu_A(x) \neq 0$ and $\mu_B(x) \neq 0$ for all $x \in U$. Support* Sum *is used for determining the phase term of the complex fuzzy union. If $A \perp B$, then $A \cup C \perp B \cup C$,*

Proof. We consider two cases:

Case 1: $\mu_C(x) = 0$. Then $< \mu_A(x) \cup \mu_C(x), \mu_B(x) \cup \mu_C(x) >=< \mu_A(x), \mu_B(x) >= 0$.

Case 2: $\mu_C(x) \neq 0$. If $r_{A\oplus C}(x) = 0$ or $r_{B\oplus C}(x) = 0$ Then $< \mu_A(x) \cup \mu_C(x), \mu_B(x) \cup \mu_C(x) >= 0$.

If $r_{A\oplus C}(x) \neq 0$ and $r_{B\oplus C}(x) \neq 0$. Because:

$$\begin{aligned} < \mu_A(x) \cup \mu_C(x), \mu_B(x) \cup \mu_C(x) > &= r_{A\oplus C}(x) \cdot r_{B\oplus C}(x) \cdot \cos(|w_{A\otimes C}(x) - w_{B\otimes C}(x)|) \\ &= r_{A\oplus C}(x) \cdot r_{B\oplus C}(x) \cdot \cos(|w_A(x) + w_C(x) - w_B(x) - w_C(x)|) \\ &= r_{A\oplus C}(x) \cdot r_{B\oplus C}(x) \cdot \cos(|w_A(x) - w_B(x)|). \end{aligned}$$

Then we have $\cos(|w_A(x) - w_B(x)|) = 0$ by $A \perp B$. Thus $< \mu_A(x) \cup \mu_C(x), \mu_B(x) \cup \mu_C(x) >= 0$. □

Example 2. *Assume three complex fuzzy sets A, B, and C are given by:*

$$\begin{aligned} A =& \ 0.4e^{j0.1\pi}/x_1 + 0.2e^{j0.2\pi}/x_2 + 0.3e^{j1.2\pi}/x_3 \\ B =& \ 0.1e^{j0.6\pi}/x_1 + 0.7e^{j0.7\pi}/x_2 + 0.6e^{j0.7\pi}/x_3 \\ C =& \ 0.7e^{j1.2\pi}/x_1 + 0/x_2 + 0.9e^{j0.3\pi}/x_3 \end{aligned}$$

We see $A \perp B$.

When using the max *union function for calculating $r_{B\cup C}$ and the Sum method in (23) for determining $w_{B\cup C}$, the following results are obtained for $A \cup C$ and $B \cup C$:*

$$\begin{aligned} A \cup C =& \ 0.7e^{j1.3\pi}/x_1 + 0.2e^{j0.2\pi}/x_2 + 0.9e^{j0.5\pi}/x_3 \\ B \cup C =& \ 0.7e^{j1.8\pi}/x_1 + 0.7e^{j0.7\pi}/x_2 + 0.9e^{j\pi}/x_3 \end{aligned}$$

It is easy to verify $A \cup C \perp B \cup C$.

When using the max *union function for calculating $r_{B_1\cup C}$ and the Max method in (24) for determining $w_{B_1\cup C}$, the following results are obtained for $A \cup C$ and $B_1 \cup C$:*

$$\begin{aligned} A \cup C =& \ 0.7e^{j1.2\pi}/x_1 + 0.2e^{j0.2\pi}/x_2 + 0.9e^{j1.2\pi}/x_3 \\ B \cup C =& \ 0.7e^{j1.2\pi}/x_1 + 0.7e^{j0.7\pi}/x_2 + 0.9e^{j0.7\pi}/x_3 \end{aligned}$$

It is easy to verify $A \cup C \not\perp B \cup C$.

Now, we discuss the problem: whether we have:

$$A \perp B, A \perp C \Rightarrow A \perp f(B, C) \tag{18}$$

for all $A, B, C \in F^C(U)$.

Another property, denoted rotational invariance, is introduced in [16], i.e.,

$$Rot_\theta(A) \circ Rot_\theta(B) = Rot_\theta(A \circ B)$$

where $\circ$ is a union or intersection operator. Rotational invariance states that a union or intersection operator is invariant (in both amplitude and direction) under a simple rotation.

Theorem 5. *Let A and B be two complex fuzzy sets on U, and $\mu_A(x) = r_A(x) \cdot e^{jw_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{jw_B(x)}$ their membership functions, respectively. If the* Max, Min, *or* Winner Take All *method is used for determining $w_{A\otimes B}$, then $A \perp C \cup D$, if $A \perp B$, $A \perp C$ for any $A, B, C \in F^C(U)$.*

Proof. We only consider the case of $w_{A\otimes B}$ is *Max* method in (1).

Suppose *Max* method is used for determining $w_{A\otimes B}$, we consider four cases:

Case 1: $\mu_C(x) = 0$. Then $< \mu_C(x), \mu_A(x) \cup \mu_B(x) >= 0$.

Case 2: $\mu_C(x) \neq 0$ and $\mu_B(x) = 0$. Then $< \mu_C(x), \mu_A(x) \cup \mu_B(x) >=< \mu_C(x), \mu_A(x) >= 0$ by $C \perp A$.

Case 3: $\mu_C(x) \neq 0$ and $\mu_A(x) = 0$. Then $< \mu_C(x), \mu_A(x) \cup \mu_B(x) >=< \mu_C(x), \mu_B(x) >= 0$ by $C \perp B$.

Case 4: $\mu_C(x) \neq 0$, $\mu_A(x) \neq 0$ and $\mu_B(x) \neq 0$. If $r_{A \oplus B}(x) = 0$, then we have $< \mu_C(x), \mu_A(x) \cup \mu_B(x) >= 0$. If $r_{A \oplus B}(x) \neq 0$, then we have $\cos(|w_C(x) - \max(w_A(x), w_B(x))|) = 0$ from $\cos(|w_C(x) - w_A(x)|) = 0$ by $C \perp A$ and $\cos(|w_C(x) - w_B(x)|) = 0$ by $C \perp B$. Because:

$$\begin{aligned} < \mu_C(x), \mu_A(x) \cup \mu_B(x) > &= r_C(x) \cdot r_{A \oplus B}(x) \cdot \cos(|w_C(x) - w_{A \otimes B}(x)|) \\ &= r_C(x) \cdot r_{A \oplus B}(x) \cdot \cos(|w_C(x) - \max(w_A(x), w_B(x))|). \end{aligned}$$

Thus $< \mu_C(x), \mu_A(x) \cup \mu_B(x) >= 0$. □

From the above theorem, we see that the complex fuzzy union has the property of function (18) if the *Max*, *Min*, or *Winner Take All* method is used for determining the phase term. However, a complex fuzzy union using one of above three methods in the phase term does not preserve orthogonality. Moreover, a complex fuzzy union using *Sum* method in the phase term is orthogonality preserving (see Theorem 4) but does not have the property of function (18). Let us consider the following example.

Example 3. *Assume five complex fuzzy sets A ,B, and C are given by:*

$$\begin{aligned} A &= 0.1e^{j0.1\pi}/x_1 + 0.2e^{j0.2\pi}/x_2 + 0.3e^{j1.2\pi}/x_3 \\ B &= 0.4e^{j0.6\pi}/x_1 + 0.7e^{j0.7\pi}/x_2 + 0.6e^{j0.7\pi}/x_3 \\ C &= 0.7e^{j1.6\pi}/x_1 + 0.8e^{j1.7\pi}/x_2 + 0.9e^{j1.7\pi}/x_3 \end{aligned}$$

We see $A \perp B$, $A \perp C$.

Using the max *union function for calculating and the Sum method in* (11) *for determining the phase term, the following results are obtained for:*

$$B \cup C = 0.7e^{j0.2\pi}/x_1 + 0.8e^{j0.4\pi}/x_2 + 0.9e^{j0.4\pi}/x_3$$

It is easy to verify $A \not\perp (B \cup C)$.

Corollary 6. *Let $A, B_i \in F^C(U)$, $i = 1, 2, ..., n$. Suppose a complex fuzzy union $\cup$ has the property of function* (19)*, Let $A \perp B_i$ for each $i = 1, 2, ..., n$, then $A \perp \bigcup_{i=1}^{n} B_i$.*

Proof. Trivial. □

Moreover, we discuss this problem: whether we have:

$$A \perp B, C \perp D \Rightarrow f(A, B) \perp f(C, D) \tag{19}$$

for all $A, B, C \in F^C(U)$.

Theorem 7. *Let A, B, C, and D be four complex fuzzy sets on U. If $\mu_A(x) \neq 0$, $\mu_B(x) \neq 0$, $\mu_C(x) \neq 0$, and $\mu_D(x) \neq 0$ for all $x \in U$. If $A \perp B$ and $C \perp D$ then $A \cup C \parallel B \cup D$, when* Sum *is used for determining phase term of complex fuzzy union.*

Proof. If $r_{A \oplus C}(x) = 0$ or $r_{B \oplus D}(x) = 0$.

Then $< Rot_{0.5\pi}(\mu_A(x) \cup \mu_C(x)), \mu_B(x) \cup \mu_D(x) >= 0$. If $r_{A \oplus C}(x) \neq 0$ and $r_{B \oplus D}(x) \neq 0$. Because:

$$
\begin{aligned}
& < Rot_{0.5\pi}(\mu_A(x) \cup \mu_C(x)), \mu_B(x) \cup \mu_C(x) > \\
= \; & r_{A\oplus C}(x) \cdot r_{B\oplus D}(x) \cdot \cos(|0.5\pi + w_{A\otimes C}(x) - w_{B\otimes D}(x)|) \\
= \; & r_{A\oplus C}(x) \cdot r_{B\oplus D}(x) \cdot \cos(|0.5\pi + w_A(x) + w_C(x) - w_B(x) - w_D(x)|) \\
= \; & r_{A\oplus C}(x) \cdot r_{B\oplus D}(x) \cdot \cos(|0.5\pi + (w_A(x) - w_B(x)) + (w_C(x) - w_D(x))|)
\end{aligned}
$$

Because $\cos(|w_A(x) - w_B(x)|) = 0$ by $A \perp B$, we have $|w_A(x) - w_B(x)| = 0.5\pi + k\pi$, for some $k = 0, 1, 2, \ldots$. Similarly, we have $\cos(|w_C(x) - w_D(x)|) = 0$ by $C \perp D$, then $|w_C(x) - w_D(x)| = 0.5\pi + k\pi$, for some $k = 0, 1, 2, \ldots$.

Then we get $|0.5\pi + (w_A(x) - w_B(x)) + (w_C(x) - w_D(x))| = 0.5\pi + k\pi$, for some $k = 0, 1, 2, \ldots$. Thus $< Rot_{0.5\pi}(\mu_A(x) \cup \mu_C(x)), \mu_B(x) \cup \mu_D(x) >= 0$. □

Example 4. *Assume four complex fuzzy sets A, B, C, and D are given by:*

$$
\begin{aligned}
A =& 0.4e^{j0.1\pi}/x_1 + 0.2e^{j0.2\pi}/x_2 + 0.3e^{j1.2\pi}/x_3 \\
B =& 0.1e^{j0.6\pi}/x_1 + 0.7e^{j0.7\pi}/x_2 + 0.6e^{j0.7\pi}/x_3 \\
C =& 0.7e^{j1.2\pi}/x_1 + 0.6e^{j0.1\pi}/x_2 + 0.5e^{j1.8\pi}/x_3 \\
D =& 0.2e^{j0.7\pi}/x_1 + 0.7e^{j0.6\pi}/x_2 + 0.9e^{j0.3\pi}/x_3
\end{aligned}
$$

We see $A \perp B$ and $C \perp D$.

When using the max *union function for calculating the amplitude term and the Sum method in* (11) *for determining the phase term, the following results are obtained for $A \cup C$ and $B \cup D$:*

$$
\begin{aligned}
A \cup C =& 0.7e^{j1.3\pi}/x_1 + 0.6e^{j0.3\pi}/x_2 + 0.5e^{j\pi}/x_3 \\
B \cup D =& 0.2e^{j1.3\pi}/x_1 + 0.7e^{j1.3\pi}/x_2 + 0.9e^{j\pi}/x_3
\end{aligned}
$$

It is easy to verify $A \cup C \not\perp B \cup D$.

3.3. Complex Fuzzy Intersection

The complex fuzzy intersection of complex fuzzy sets are reviewed as follows (see [1]).

Let A and B be two complex fuzzy sets on U and $\mu_A(x) = r_A(x) \cdot e^{jw_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{jw_B(x)}$ their membership functions, respectively. The complex fuzzy intersection of A and B, denoted $A \cup B$, is specified by a function:

$$\mu_{A\cup B}(x) = (r_A(x) \star r_B(x)) \cdot e^{jw_{A\otimes B}(x)} \tag{20}$$

where $\star$ signifies any t-norm function.

Possible choices for calculating $w_{A\otimes B}$ are given in (10)–(17). Then, we have following results.

Theorem 8. *Let A, B, and C be three complex fuzzy sets on U. If $\mu_A(x) \neq 0$ and $\mu_B(x) \neq 0$ for all $x \in U$. Support* Sum *is used for determining the phase term of the complex fuzzy intersection. If $A \perp B$, then $A \cap C \perp B \cap C$.*

Proof. Similar to Theorem 4. □

Theorem 9. *Let A and B be two complex fuzzy sets on U and $\mu_A(x) = r_A(x) \cdot e^{jw_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{jw_B(x)}$ their membership functions, respectively. If the* Max, Min, *or* Winner Take All *method is used for determining $w_{A\otimes B}$, then $A \perp C \cap D$ if $A \perp B, A \perp C$ for any $A, B, C \in F^C(U)$.*

Proof. Similar to Theorem 5. □

Corollary 10. *Let $A, B_i \in F^C(U)$, $i = 1, 2, ..., n$. Suppose a complex fuzzy intersection $\cup$ has the property of function (19). Let $A \perp B_i$ for each $i = 1, 2, ..., n$, then $A \perp \bigcap_{i=1}^{n} B_i$.*

Proof. Trivial. □

Now we give a brief summary of orthogonality preserving of the complex fuzzy union and complex fuzzy intersection. The key is what method is used for determining the phase term of complex fuzzy operations. Our results are listed as follows:

(i) $A \perp B, \Rightarrow A \circ C \perp B \circ C$: *Sum* (See Theorems 4 and 7);
(ii) $A \perp B, A \perp C \Rightarrow A \perp B \circ C$: *Max*, *Min* and *Winner Take All* (See Theorems 5 and 8);

where $\circ$ is a complex fuzzy union or complex fuzzy intersection.

3.4. Complex Fuzzy Inference

Next, we discuss the Problem 2, which also can be known as: whether or not the orthogonality can be preserved for complex fuzzy reasoning, i.e., will the orthogonality of input cause the orthogonality of the output of complex fuzzy reasoning?

First, we recall complex fuzzy inference in [17] for convenience. Let U and V be two universes of discourse. The form of generalized modus ponens (GMP) of complex fuzzy inference may be written as follows:

Premise:	X is A^*;
Rule:	IF X is A, THEN Y is B;
Consequence:	Y is B^* (denote $CFI(A, B; A^*)$).

The sets $A, A^* \in F^C(U)$ and $B, B^* \in F^C(V)$ are all complex fuzzy sets. In this paper, we only consider the case of GMP form of complex fuzzy inference.

The amplitude term and the phase term of complex fuzzy implication relation $\mu_{A\to B}(x, y)$ in [17] are characterized by:

$$r_{A\to B}(x, y) = r_A(x) \cdot r_B(y) \tag{21}$$
$$w_{A\to B}(x, y) = w_A(x) + w_B(y) \tag{22}$$

Then amplitude term and the phase term of B^* is given by:

$$r_{B^*}(y) = \sup_{x\in U}[r_{A^*}(x) \star (r_A(x) \cdot r_B(y))] \tag{23}$$
$$w_{B^*}(y) = f[g(w_{A^*}(x), (w_A(x) + w_B(y)))] \tag{24}$$

Possible forms of g are the functions (10)–(17). Furthermore, possible forms of f are:

$$w_{B^*}(y) = \sup_{x\in U}[g(w_{A^*}(x), (w_A(x) + w_B(y)))] \tag{25}$$
$$w_{B^*}(y) = \inf_{x\in U}[g(w_{A^*}(x), (w_A(x) + w_B(y)))] \tag{26}$$
$$w_{B^*}(y) = g(w_{A^*}(x'), (w_A(x') + w_B(y))) \tag{27}$$
$$w_{B^*}(y) = \sum_{x\in U}[g(w_{A^*}(x), (w_A(x) + w_B(y)))] \tag{28}$$

where x' in (27) is the value of x for which the supremum, $\sup_{x\in U}[r_{A^*}(x) \star (r_A(x) \cdot r_B(y))]$ is obtained. Functions (25)–(27) are introduced in [17] and Function (28) is a new adding method.

Similar to Definition 2, the concept of orthogonality preserving complex fuzzy inference is formally expressed as follows.

Definition 3. *A complex fuzzy inference method is an* $F^C(U) \to F^C(U)$ *mapping* f*, i.e., each input* $A^* \in F^C(U)$ *corresponds to the output* $B^* \in F^C(V)$*. A method* f *is an orthogonality preserving complex fuzzy inference if for any* $A^*, A^{*'} \in F^C(U)$*:*

$$A^* \perp A^{*'} \Rightarrow f(A^*) \perp f(A^{*'}). \tag{29}$$

Theorem 11. *Suppose that* B^* *and* $B^{*'}$ *are the results of GFI*$(A, B; A^*)$ *and GFI*$(A, B, A^{*'})$ *respectively, if forms of* f *in* (24) *is* (27) *or* (28)*, and forms of* g *in* (24) *is* Sum *in* (10)*, then:*

$$A^* \perp A^{*'} \Rightarrow B^* \perp B^{*'} \tag{30}$$

when $|U|$ *is an odd number.*

Proof. Trivial from Theorem 8 and corollary 10. □

From the above theorem, we see that the results are not only related to the choices of f and g but also the size of U. It is interesting to note that the results are not related to the size of V. Let us consider the following example of the case $|U|$ is an odd number.

Example 5. *Let:*

$$\begin{aligned} A =& 0.5e^{j0.1\pi}/x_1 + 0.6e^{j0.2\pi}/x_2 + 1e^{j1.2\pi}/x_3 \\ B =& 0.4e^{j0.6\pi}/y_1 + 0.8e^{j0.7\pi}/y_2 \end{aligned}$$

we get following results from (21) *and* (22)*:*

$$A \to B = \begin{bmatrix} 0.2e^{j0.7\pi} & 0.4e^{j0.8\pi} \\ 0.24e^{j0.8\pi} & 0.48e^{j0.9\pi} \\ 0.4e^{j1.8\pi} & 0.8e^{j1.9\pi} \end{bmatrix}$$

Suppose:

$$\begin{aligned} A^* =& 0.4e^{j0.1\pi}/x_1 + 0.2e^{j0.2\pi}/x_2 + 0.3e^{j1.2\pi}/x_3 \\ A^{*'} =& 0.1e^{j0.6\pi}/x_1 + 0.7e^{j0.7\pi}/x_2 + 0.6e^{j0.7\pi}/x_3 \end{aligned}$$

We see $A^* \perp A^{*'}$*.*
If g *is* Sum *in (10),* $\star$ *is Min t-norm, form of* f *is function in* (28)*. Then:*

$$\begin{aligned} B^* =& 0.3e^{j0.8\pi}/y_1 + 0.6e^{j1.1\pi}/y_2 \\ B^{*'} =& 0.4e^{j1.3\pi}/y_1 + 0.6e^{j1.6\pi}/y_2 \end{aligned}$$

Thus $B^* \perp B^{*'}$*.*
If form of f *is function in* (25)*:*

$$\begin{aligned} B^* =& 0.3e^{j\pi}/y_1 + 0.6e^{j1.1\pi}/y_2 \\ B^{*'} =& 0.4e^{j1.5\pi}/y_1 + 0.6e^{j1.6\pi}/y_2 \end{aligned}$$

$B^* \not\perp B^{*'}$.

When $|U|$ is an even number, let us consider the following example.

Example 6. *Let:*

$$A = 0.5e^{j0.1\pi}/x_1 + 0.6e^{j0.2\pi}/x_2$$
$$B = 0.4e^{j0.6\pi}/y_1 + 0.8e^{j0.7\pi}/y_2 + 1e^{j1.2\pi}/y_3$$

We get following results from (24) and (25):

$$A \to B = \begin{bmatrix} 0.2e^{j0.7\pi} & 0.4e^{j0.8\pi} & 0.5e^{j1.3\pi} \\ 0.24e^{j0.8\pi} & 0.48e^{j0.9\pi} & 0.6e^{j1.4\pi} \end{bmatrix}$$

Suppose:

$$A^* = 0.4e^{j0.1\pi}/x_1 + 0.2e^{j0.2\pi}/x_2$$
$$A^{*'} = 0.1e^{j0.6\pi}/x_1 + 0.7e^{j0.7\pi}/x_2$$

We see $A^* \perp A^{*'}$.
If g is Sum *in (11), $\star$ is Min t-norm, form of f is function in Equation (30). Then:*

$$B^* = 0.2e^{j1.8\pi}/y_1 + 0.4e^{j2\pi}/y_2 + 0.4e^{j\pi}/y_3$$
$$B^{*'} = 0.24e^{j0.8\pi}/y_1 + 0.48e^{j\pi}/y_2 + 0.6e^{j2\pi}/y_3$$

It is easy to verify $B^* \not\perp B^{*'}$.

3.5. Example Application

We consider a signal detection example below which involves the orthogonality complex fuzzy sets. For the rationale of using complex fuzzy sets we refer to Ramot et al. [1,17].

The problem is an example that demonstrates the use of complex fuzzy set theory in an application that identifies a particular signal of interest out of several different signals received by a digital receiver.

Ramot et al. [1,17] first used the complex fuzzy set theory in the signal detection application that identifies a given signal of interest out of several different signals received by a digital receiver. Zhang et al. [23] considered this application by using a concept of approximate equality theory of complex fuzzy sets.

When we identify a particular signal out of several different signals, one of the most important step is measuring the difference between two complex fuzzy sets. Methods in [1,23] are combing the differences in the amplitude terms and the phase terms. For example, let $A = 0.5/x$, $B = 1/x$, and $C = e^{j0.5\pi}/x$, as shown in Figure 2 . Then we have $d(A,B) = d(B,C)$ by the method in [23]. However, in our point of view, the signal A is easier to be detected than the signal C for a given signal B, since A and B are in the same direction and B and C are orthogonal. To avoid this, our method to calculate the distance of two complex fuzzy sets is combing the differences in the in-phase terms and quadrature terms.

Let $S_l(t)$ $(1 \le l \le L)$ be L different signals. R is a given signal. Every $S_l(t)$ and $R(t)$ are sampled N times i.e., $1 \le t \le N$.

Each signal $S_l(t)$ can be represented as:

$$S_l(t) = \frac{1}{N}\sum_{n=1}^{N} C_{l,n} \cdot e^{j\frac{2\pi(n-1)(t-1)}{N} + a_{l,n}}$$

where $C_{l,n}$ are the complex Fourier coefficients S_l.

Then $S_l(t)$ can be rewritten in the form:

$$S_l(t) = \frac{1}{N}\sum_{n=1}^{N} A_{l,n} \cdot e^{j\frac{2\pi(n-1)(t-1)}{N} + a_{l,n}}$$

where $C_{l,n} = A_{l,n} \cdot e^{ja_{l,n}}$, with $A_{l,n}$, $a_{l,n}$ real-valued and $A_{l,n} \geq 0$ for all n ($1 \leq n \leq N$).

Similarly, let $C_{R,n}$ be the Fourier coefficients of R, the signal $R(t)$ can be represented as:

$$R(t) = \frac{1}{N}\sum_{n=1}^{N} A_{R,n} \cdot e^{j\frac{2\pi(n-1)(t-1)}{N} + a_{R,n}}$$

where $C_{R,n} = A_{R,n} \cdot e^{ja_{R,n}}$, with $A_{R,n}$, $a_{R,n}$ real-valued and $A_{R,n} \geq 0$ for all n ($1 \leq n \leq N$).

Apply the following method to compare the different signals.

Step 1 Normalize the amplitudes of all Fourier coefficients. Let $\mathbf{A}_l = (A_{l,1}, A_{l,2}, ..., A_{l,N})$ be the vector of amplitudes of S_l's Fourier coefficients, ($1 \leq l \leq L$). Let $\mathbf{A}_R = (A_{R,1}, A_{R,2}, ..., A_{R,N})$ be the vector of amplitudes of S_R's Fourier coefficients, ($1 \leq l \leq L$). Let $\mathbf{B}_l$ be the normalized vector $1/norm(\mathbf{A}_l) \cdot \mathbf{A}_l$, where $norm(\mathbf{A}_l) = \sqrt{\Sigma_{n=1}^{N}(A_{l,n})^2}$. Let $\mathbf{B}_R$ be the normalized vector $1/norm(\mathbf{A}_R) \cdot \mathbf{A}_R$. Then $\mathbf{B}_l = (B_{l,1}, B_{l,2}, ..., B_{l,N})$ is the vector of normalized amplitudes of S_l's Fourier coefficients. $\mathbf{B}_R = (A_{R,1}, A_{R,2}, ..., A_{R,N})$ is the vector of normalized amplitudes of R's Fourier coefficients.

Step 2 Composition the t samples $S_l(t)$, $1 \leq t \leq N$ for each signal S_l ($1 \leq l \leq L$). Define new complex fuzzy sets as:

$$B_l e^{a_l} = B_{l,1} \oplus B_{l,2} \oplus \cdots \oplus B_{l,n} e^{a_{l,1} \otimes a_{l,2} \otimes \cdots \otimes a_{l,N}} \tag{31}$$

Similarly, define a new complex set as:

$$B_R e^{a_R} = B_{R,1} \oplus B_{R,2} \oplus \cdots \oplus B_{R,n} e^{a_{R,1} \otimes a_{R,2} \otimes \cdots \otimes a_{R,N}} \tag{32}$$

Step 3 For each B_l ($1 \leq l \leq L$), define its in-phase and quadrature terms, respectively, as:

$$I_l = B_l \cos(a_l) \tag{33}$$

and

$$Q_l = B_l \sin(a_l). \tag{34}$$

Similarly, define R's in-phase and quadrature terms, respectively, as:

$$I_R = B_R \cos(a_R) \tag{35}$$

and

$$Q_R = B_R \sin(a_R). \tag{36}$$

Step 4 Calculate the distance between S_l ($1 \leq l \leq L$) and R:

$$d(S_l, R) = \frac{1}{2}\max(|I_R - I_l|, |Q_R - Q_l|) \tag{37}$$

Step 5 In order to conclude if S_l may be identified as R, compare $1 - d(S_l, R)$ to a threshold δ. If $1 - d(S_l, R)$ exceeds the threshold, identify S_l as R.

Above *Step 1* is the same as that in [1]. Other steps are different. Some complex fuzzy operations are used in Step 2. The *Sum*, *Max*, *Min*, and *Winner Take All* methods may be better than others for $\otimes$ in Step 2 , since these operations have the orthogonality preserving property (see Theorems 4, 5, 7 and 8). Step 3 is the orthogonal decomposition method, which is known as orthogonal demodulation in digital signal processing.

Example 7. *Two discrete time signals are represented by:*

$$S_1(t) = \frac{1}{5}\sum_{n=1}^{5} A_{R,n} \cdot e^{j\frac{2\pi(n-1)(t-1)}{5}+\frac{\pi}{6}}, \quad t = 1,2,...,5,$$
$$S_2(t) = \frac{1}{5}\sum_{n=1}^{5} A_{R,n} \cdot e^{j\frac{2\pi(n-1)(t-1)}{5}+\frac{2\pi}{3}}, \quad t = 1,2,...,5.$$

respectively. The given signal R is:

$$R(t) = \frac{1}{5}\sum_{n=1}^{5} A_{R,n} \cdot e^{j\frac{2\pi(n-1)(t-1)}{5}}, \quad t = 1,2,...,5.$$

Assume that $\oplus$ is Max and $\otimes$ is Sum. By applying Eqs.(31)-(32), we have $1/5e^{\frac{5\pi}{6}}$, $1/5e^{\frac{4\pi}{3}}$ and $1/5$. According to Equations (33)–(36), we obtain:

$$d(S_1, R) = 0.05, \quad d(S_2, R) = 0.15$$

If $\delta = 0.9$, identify S_1 as R.

In order to illustrate the superiority of the proposed method, a comparison between the proposed method and the existing methods [1,23] is conducted based on the numerical cases. By the method in Reference [1], we have same distance between $S_l(l = 1,2)$ and R in the above Example 7. It is impossible to know which signal is more likely to be identified, which cannot satisfy the application of signal detection. Method in Reference [23] has same problem (see the example of Figure 2). Therefore, we can say that our proposal is a satisfactory method.

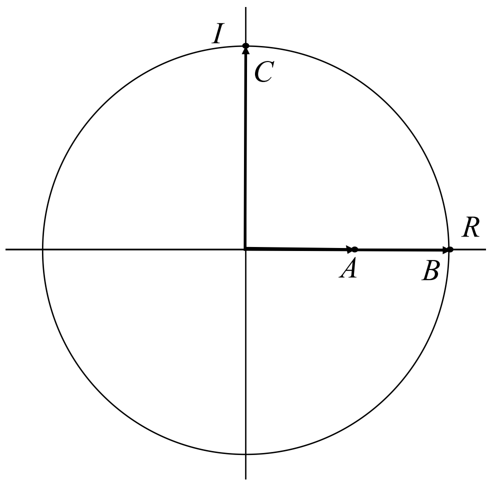

Figure 2. Membership vectors A, B, and C.

4. Discussion and Conclusions

In this paper, we introduced the orthogonality of complex fuzzy sets. We presented several results on orthogonality with respect to complex fuzzy operations and complex fuzzy inference method. The *Sum*, *Max*, *Min*, and *Winner Take All* methods in determining the phase term of the complex fuzzy union (intersection) have the orthogonality preserving property (see Theorems 4, 5, 7 and 8). A signal detection method was presented. In some special cases where some of the existing methods cannot provide reasonable results, the proposed method shows great capacity for discriminating complex fuzzy sets. Moreover, this method uses the choice of operations that have the orthogonality preserving property. However, our proposed method is not an absolutely perfect one. It is stuck with a lack of theoretical support. Furthermore, we have not show that Equation (37) in our method is a distance measure. Further efforts include continuing to look for a more excellent method (or distance measure) for much better exploration and exploitation on complex fuzzy sets.

Complex fuzzy sets, described in [1], are viewed as vectors in the complex plane, rather than scalar quantities. We should measure the difference of complex fuzzy sets from the viewpoint of vector theory. This idea of membership vectors induces new properties of complex fuzzy sets which are quite different from the properties of fuzzy set. Dick [16] analyzed the idea of rotational invariance for complex fuzzy logic. They also use figure of vector in complex plane to illustrate this point visibility. It is interesting to study complex fuzzy sets and complex fuzzy logic from the viewpoint of vector theory.

As we know, various orthogonal data are often applied in image processing and signal processing. The concept of orthogonality of complex fuzzy sets may be useful in these applications. Therefore, it will be meaningful to further discuss.

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