

## Article

# Solving Multi-Objective Matrix Games with Fuzzy Payoffs through the Lower Limit of the Possibility Degree

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**Abstract:** In this article, we put forward the multi-objective matrix game model based on fuzzy payoffs. In order to solve the game model, we first discuss the relationship of two fuzzy numbers via the lower limit— $\frac{1}{2}$  of the possibility degree. Then, utilizing this relationship, we conclude that the equilibrium solution of this game model and the optimal solution of multicriteria linear optimization problems are of equal value. Finally, to illustrate the effectiveness and correctness of the obtained model, an example is provided.

**Keywords:** multi-objective game; possibility degree; equilibrium solution; multicriteria linear optimization

## 1. Introduction

The multicriteria zero sum game is a generalization of the standard zero sum game model. The multicriteria zero sum game is also known as the multi-objective matrix game as it can be represented by multiple payoffs. Along with the collision of distinct decision makers in the social and corporate circumstance, much research in recent years has focused on multiple objective matrix game problems.

The notions of maxmin and minmax values were first used to discuss the multi-objective game model in [1]. Zeleny [2] studied the matrix game based on multiple payoff through notions of compromise solutions and a decomposition of parametric spaces. Ghose et al. [3] proposed the concepts of Pareto-optimal, Pareto saddle points and security levels of the multicriteria matrix game and analyzed the existence of Pareto saddle points of this game problem. Afterwards, the same game model was discussed by Fernandez et al. [4] and they proved that efficient solutions of multicriteria linear optimization problems and Pareto-optimal security strategies (POSS) for each Player are of equal value. Meanwhile, they obtained the set of all POSS through alternative ways.

The fuzzy set theory was initially introduced in 1965 by Zadeh [5]. The fuzziness occurring in the game problems is categorized as the fuzzy game problems. Single objective fuzzy game problems and related problems have attracted a wide range of research [6–11]. Therefore, fuzzy games theory has been extensively studied in some fields, such as economics, engineering and management science [12–19]. In order to deal with the fuzzy matrix games problem, a method of robust ranking is formulated by Bhaumik [20]. In terms of fuzzy games problems, Tan et al. [21] presented a concept of the potential function. Furthermore, they also reached a conclusion that the solution of fuzzy games and the marginal value of potential functions are equivalent. In [19], in order to solve the game problem quickly, the gradient iterative algorithm was proposed. Cevik et al. [22] utilized the fuzzy relation method to find the solution of matrix games in terms of fuzzy goals and fuzzy payoffs. Chakeri et al. [23] used fuzzy logic to determine the priority of the payoff based on the linguistic

preference relation and proposed the notion of linguistic Nash equilibrium. Fuzzy preference relation has been widely used in fuzzy game theory [24–27]. At the same time, they [24] utilize the same method [23] to determine the priority of the payoff based on fuzzy preference relation. In order to deal with this game model, a new approach is put forward. Moreover, Sharifian et al. [28] also applied fuzzy linguistic preference relation to fuzzy game theory.

Although the research on single objective fuzzy matrix games has become increasingly widespread, there are still few conclusions in the multicriteria case. The major contributions in this aspect have been studied in [18,29–32]. Sakawa et al. [32] discussed the fuzzy multicriteria games model with fuzzy goals according to the theory of maxmin value. In order to solve multiple decision-making problems, a model of fuzzy multiple matrix games is presented by Peldschus et al. [18]. Subsequently, Chen [30] found that the equilibrium solution of multiple matrix games based on fuzzy payoffs is equivalent to the solution of the fuzzy multi-objective attribute decision-making problem. Inspired by [3,4], Aggarwal et al. [29] applied the notions of POSS and security levels of apiece players to research the multicriteria matrix game in terms of fuzzy goals and demonstrated that this game problem and fuzzy multiple objective linear optimization problems are of equal value. Taking elicitation from [29,33,34], we can take inspiration and put forward a new model of the multiple objective matrix game based on fuzzy payoffs according to the lower limit  $-\frac{1}{2}$  of the possibility degree.

The outline of this article is as follows: The background of this paper is introduced in Section 1. Section 2 introduces some basic definitions and recalls some results concerning crisp multi-objective matrix games and the fuzzy numbers. Furthermore, we discuss the relationship of two fuzzy numbers via the lower limit  $-\frac{1}{2}$  of the possibility degree. In Section 3, The multiple objective matrix game model based on fuzzy payoffs is considered. We conclude that the equilibrium solution of this game model and the optimal solution of multi-objective linear optimization problems are of equal value. In Section 4, a small numerical example is given.

## 2. Preliminaries

In this section, we begin to depict a crisp multiple objective matrix game in [29]. For this, we recall some definitions.

**Definition 1.** [3] *The set of mixed strategies for Player I is denoted by*

$$S^m = \{x = (x_1, x_2, \dots, x_m)^T \in \mathbb{R}^m \mid \sum_{i=1}^m x_i = 1, x_i \geq 0, i = 1, 2, \dots, m.\} \quad (1)$$

*Similarly, The set of mixed strategies for Player II is denoted by*

$$S^n = \{y = (y_1, y_2, \dots, y_n)^T \in \mathbb{R}^n \mid \sum_{j=1}^n y_j = 1, y_j \geq 0, j = 1, 2, \dots, n.\} \quad (2)$$

where  $x^T$  is the transposition of  $x$ ,  $\mathbb{R}^m$  and  $\mathbb{R}^n$  are  $m$ - and  $n$ -dimensional Euclidean spaces.

Multiple payoff matrixes of multicriteria matrix games are taken as follows [3]:

$$A^1 = \begin{pmatrix} a_{11}^1 & \cdots & a_{1n}^1 \\ \vdots & \ddots & \vdots \\ a_{m1}^1 & \cdots & a_{mn}^1 \end{pmatrix}, \dots, A^r = \begin{pmatrix} a_{11}^r & \cdots & a_{1n}^r \\ \vdots & \ddots & \vdots \\ a_{m1}^r & \cdots & a_{mn}^r \end{pmatrix}. \quad (3)$$

In order not to lose generality, we suppose that Player I and Player II are maximized players.

A multiple objective matrix game (MOG) [29] is defined by

$$\text{MOG} = (S^m, S^n, A^k(1, 2, \dots, r)).$$

**Definition 2.** [3] When Player I chooses a mixed strategy  $x \in S^m$  and Player II chooses a mixed strategy  $y \in S^n$ , a vector

$$\begin{aligned} E(x, y, A) &= x^T A y = [E_1(x, y), E_2(x, y), \dots, E_r(x, y)] \\ &= [x^T A^1 y, x^T A^2 y, \dots, x^T A^r y] \end{aligned} \quad (4)$$

is called an expected payoff of Player I. As the multi-objective game (MOG) is zero-sum, the payoff for Player II is  $-x^T A y$ .

**Definition 3.**  $(x^*, y^*) \in S^m \times S^n$  is called a solution of the (MOG) model if

$$x^* A^k y \geq (V^k)^*, \quad \forall y \in S^n,$$

$$x A^k y^* \leq (V^k)^*, \quad \forall x \in S^m.$$

Here,  $x^*$  and  $y^*$  are called the equilibrium solution for Player I and Player II, respectively. Furthermore,  $(V^k)^* (k = 1, 2, \dots, r)$  are called the values of (MOG).

Given a multi-objective game (MOG), its solution can be obtained by solving the following pair of primal-dual multiple objective linear optimization problems (MOGLP) and (MOGLD).

$$\begin{aligned} (\text{MOGLP}) \quad & \max (V^1, V^2, \dots, V^r) \\ & \text{such that } \sum_{i=1}^m (a_{ij})^k x_i \geq V^k, (k = 1, 2, \dots, r, j = 1, 2, \dots, n), \\ & \quad \forall x \in S^m, \forall y \in S^n, \\ (\text{MOGLD}) \quad & \min (W^1, W^2, \dots, W^r) \\ & \text{such that } \sum_{j=1}^n (a_{ij})^k y_j \leq W^k, (k = 1, 2, \dots, r, i = 1, 2, \dots, m), \\ & \quad \forall x \in S^m, \forall y \in S^n. \end{aligned}$$

The following notations, definitions and results will be needed in the sequel.

We denote  $\mathcal{K}_C$  as the family of all bounded closed intervals in  $\mathbb{R}$  [35], that is,

$$\mathcal{K}_C = \{[a_L, a_R] | a_L, a_R \in \mathbb{R} \text{ and } a_L \leq a_R\}.$$

A fuzzy set  $\tilde{x}$  of  $\mathbb{R}$  is characterized by a membership function  $\mu_{\tilde{x}}: \mathbb{R} \rightarrow [0, 1]$  [5]. For each such fuzzy set  $\tilde{x}$ , we denote by  $[\tilde{x}]^\alpha = \{x \in \mathbb{R} : \mu_{\tilde{x}}(x) \geq \alpha\}$  for any  $\alpha \in (0, 1]$ , its  $\alpha$ -level set. We define the set  $[\tilde{x}]^0$  by  $[\tilde{x}]^0 = \overline{\bigcup_{\alpha \in (0, 1]} [\tilde{x}]^\alpha}$ , where  $\overline{A}$  denotes the closure of a crisp set  $A$ . A fuzzy number  $\tilde{x}$  is a fuzzy set with non-empty bounded closed level sets  $[\tilde{x}]^\alpha = [\tilde{x}_L(\alpha), \tilde{x}_R(\alpha)]$  for all  $\alpha \in [0, 1]$ , where  $[\tilde{x}_L(\alpha), \tilde{x}_R(\alpha)]$  denotes a closed interval with the left end point  $\tilde{x}_L(\alpha)$  and the right end point  $\tilde{x}_R(\alpha)$  [36]. We denote the class of fuzzy numbers by  $\mathcal{F}$ .

**Definition 4.** [37] Let  $\tilde{x}$  and  $\tilde{y}$  be fuzzy numbers. It is said that  $\tilde{x}$  precedes  $\tilde{y}$  ( $\tilde{x} \preceq \tilde{y}$ ), if  $\tilde{x}_R(\alpha) \leq \tilde{x}_R(\alpha)$  and  $\tilde{y}_L(\alpha) \leq \tilde{y}_L(\alpha)$ ,  $\alpha \in [0, 1]$ .

**Definition 5.** [5] Let  $\tilde{x}$  be fuzzy numbers, If the membership function  $u_{\tilde{x}}(x)$  of the fuzzy number  $\tilde{x}$  is denoted by

$$u_{\tilde{x}}(x) = \begin{cases} 0, & x < \hat{a}, x > \hat{a}, \\ \frac{x - \hat{a}}{a - \hat{a}}, & \hat{a} \leq x \leq a, \\ \frac{\hat{a} - x}{\hat{a} - a}, & a < x \leq \hat{a}. \end{cases}$$

Then,  $\tilde{x}$  is called a triangular fuzzy number. Furthermore, the triangular fuzzy number  $\tilde{x}$  is presented by  $\tilde{x} = (\acute{a}, a, \grave{a})$ .

Furthermore, the  $\alpha$ -level set of the triangular fuzzy number  $\tilde{x} = (\acute{a}, a, \grave{a})$  is the closed interval [5]

$$[\tilde{x}]^\alpha = [\tilde{x}_L(\alpha), \tilde{x}_R(\alpha)] = [(a - \acute{a})\alpha + \acute{a}, -(\grave{a} - a)\alpha + \grave{a}], \quad \alpha \in (0, 1]. \quad (5)$$

**Definition 6.** [38] Let  $\tilde{x}_i$  be fuzzy numbers and  $b_i \geq 0$  ( $i = 1, 2, \dots, n$ ) be real numbers. Then,  $\sum_{i=1}^n \tilde{x}_i b_i$  is a fuzzy number.

We define the new relationship of two fuzzy numbers.

**Definition 7.** Let  $\tilde{x}$  and  $\tilde{y}$  be fuzzy numbers. The width of  $[\tilde{x}]^\alpha = [\tilde{x}_L(\alpha), \tilde{x}_R(\alpha)]$  and  $[\tilde{y}]^\alpha = [\tilde{y}_L(\alpha), \tilde{y}_R(\alpha)]$  respectively are given by

$$w(\tilde{x}) = \tilde{x}_R(\alpha) - \tilde{x}_L(\alpha), \quad w(\tilde{y}) = \tilde{y}_R(\alpha) - \tilde{y}_L(\alpha), \quad \alpha \in [0, 1].$$

We say that  $p(\tilde{x} \preceq \tilde{y})$  is possibility a degree of  $\tilde{x} \preceq \tilde{y}$ , where

$$p(\tilde{x} \preceq \tilde{y}) = 1 - \max\left\{\min\left\{1, \frac{\tilde{x}_R(\alpha) - \tilde{y}_L(\alpha)}{w(\tilde{x}) + w(\tilde{y})}\right\}, 0\right\}.$$

**Definition 8.** Let  $\tilde{x}$  and  $\tilde{y}$  be fuzzy numbers. If  $p(\tilde{x} \preceq \tilde{y}) \geq \frac{1}{2}$ , we say that  $\tilde{x}$  precedes  $\tilde{y}$  ( $\tilde{x} \preceq \tilde{y}$ ). Furthermore,  $\frac{1}{2}$  is the lower limit of the possibility degree of  $\tilde{x} \preceq \tilde{y}$ . That is,  $\tilde{x}$  precedes  $\tilde{y}$  with the possibility degree not less than  $\frac{1}{2}$ .

**Theorem 1.** Let  $\tilde{x}$  and  $\tilde{y}$  be fuzzy numbers. Then  $\tilde{x} \preceq \tilde{y}$  if and only if  $p(\tilde{x} \preceq \tilde{y}) \geq \frac{1}{2}$ .

**Proof.** Sufficiency: Since  $p(\tilde{x} \preceq \tilde{y}) \geq \frac{1}{2}$ . According to the above Definition 8, we get  $\tilde{x} \preceq \tilde{y}$ .  $\square$

Necessity: Since  $\tilde{x} \preceq \tilde{y}$ . Then, By Definition 4, we have

$$\tilde{x}_R(\alpha) \leq \tilde{y}_R(\alpha) \quad \text{and} \quad \tilde{x}_L(\alpha) \leq \tilde{y}_L(\alpha) \quad \alpha \in [0, 1].$$

Hence,

$$\frac{\tilde{x}_R(\alpha) + \tilde{x}_L(\alpha)}{2} \leq \frac{\tilde{y}_R(\alpha) + \tilde{y}_L(\alpha)}{2} \quad \alpha \in [0, 1].$$

$$\frac{\tilde{x}_R(\alpha) - \tilde{y}_L(\alpha)}{2} \leq \frac{\tilde{y}_R(\alpha) - \tilde{x}_L(\alpha)}{2} \quad \alpha \in [0, 1].$$

$$\tilde{x}_R(\alpha) - \tilde{y}_L(\alpha) \leq \frac{\tilde{y}_R(\alpha) - \tilde{y}_L(\alpha) + \tilde{x}_R(\alpha) - \tilde{x}_L(\alpha)}{2} \quad \alpha \in [0, 1].$$

$$\frac{\tilde{x}_R(\alpha) - \tilde{y}_L(\alpha)}{\tilde{y}_R(\alpha) - \tilde{y}_L(\alpha) + \tilde{x}_R(\alpha) - \tilde{x}_L(\alpha)} \leq \frac{1}{2} \quad \alpha \in [0, 1].$$

Thus,

$$\min\left\{1, \frac{\tilde{x}_R(\alpha) - \tilde{y}_L(\alpha)}{\tilde{y}_R(\alpha) - \tilde{y}_L(\alpha) + \tilde{x}_R(\alpha) - \tilde{x}_L(\alpha)}\right\} \leq \frac{1}{2} \quad \alpha \in [0, 1].$$

$$\max\left\{\min\left\{1, \frac{\tilde{x}_R(\alpha) - \tilde{y}_L(\alpha)}{\tilde{y}_R(\alpha) - \tilde{y}_L(\alpha) + \tilde{x}_R(\alpha) - \tilde{x}_L(\alpha)}\right\}, 0\right\} \leq \frac{1}{2} \quad \alpha \in [0, 1].$$

$$1 - \max\left\{\min\left\{1, \frac{\tilde{x}_R(\alpha) - \tilde{y}_L(\alpha)}{\tilde{y}_R(\alpha) - \tilde{y}_L(\alpha) + \tilde{x}_R(\alpha) - \tilde{x}_L(\alpha)}\right\}, 0\right\} \geq \frac{1}{2} \quad \alpha \in [0, 1].$$

Therefore, by Definition 7, we have that

$$1 - \max\{\min\{1, \frac{\tilde{x}_R(\alpha) - \tilde{y}_L(\alpha)}{w(\tilde{y}) + w(\tilde{x})}\}, 0\} \geq \frac{1}{2} \quad \alpha \in [0, 1].$$

That is to say,

$$p(\tilde{x} \preceq \tilde{y}) \geq \frac{1}{2}.$$

### 3. A Generalized Model for a Multi-Objective Fuzzy Matrix Game

$S^m$  and  $S^n$  are given in Section 2. Suppose that the elements of  $\tilde{A}^k$  ( $k = 1, 2, \dots, r$ ) are fuzzy numbers. Let  $\tilde{V}_0^k$  and  $\tilde{W}_0^k$  be the aspiration levels as fuzzy numbers of Player I and Player II, respectively. Therefore, the multiple objective matrix game based on fuzzy payoffs, denoted by *MOFP*, can be presented as

$$MOFP = (S^m, S^n, \tilde{A}^k, \tilde{V}_0^k, \tilde{W}_0^k, (k = 1, 2, \dots, r)).$$

Now, we have the following definition to define the solution of *MOFP*.

**Definition 9.** Let  $(x, y) \in S^m \times S^n$ . If  $x \in S^m$  and  $y \in S^n$  satisfy the following the conditions:

$$(\tilde{V}_0^k)^* \preceq x^{*T} \tilde{A}^k y, \quad \forall y \in S^n,$$

$$x \tilde{A}^k y^* \preceq (\tilde{W}_0^k)^*, \quad \forall x \in S^m.$$

Then,  $(x^*, y^*) \in S^m \times S^n$  is called the equilibrium solution of (*MOFP*).

In order to obtain the equilibrium solution of (*MOFP*), we conclude the following the theorem.

**Theorem 2.** Let  $\alpha \in (0, 1]$  be fixed.  $(x, ((\tilde{V}_0^1)_L(\alpha) + (\tilde{V}_0^1)_R(\alpha), \dots, (\tilde{V}_0^r)_L(\alpha) + (\tilde{V}_0^r)_R(\alpha)))$  and  $(y, ((\tilde{W}_0^1)_L(\alpha) + (\tilde{W}_0^1)_R(\alpha), \dots, (\tilde{W}_0^r)_L(\alpha) + (\tilde{W}_0^r)_R(\alpha)))$  are optimal solutions of multi-objective linear optimization problems (*MOCLP1*) and (*MOCLD1*) if and only if  $(x, y)$  is the equilibrium solution of (*MOFP*).

$$\begin{aligned} (\text{MOCLP1}) \quad & \max [(\tilde{V}_0^1)_L(\alpha) + (\tilde{V}_0^1)_R(\alpha), \dots, (\tilde{V}_0^r)_L(\alpha) + (\tilde{V}_0^r)_R(\alpha)] \\ & \text{such that } (\tilde{V}_0^k)_L(\alpha) + (\tilde{V}_0^k)_R(\alpha) \leq \sum_{i=1}^m [(\tilde{a}_{ij}^k)_L(\alpha) + (\tilde{a}_{ij}^k)_R(\alpha)] x_i, \\ & 0 \leq \alpha \leq 1, \\ & \forall x \in S^m, \forall (k = 1, 2, \dots, r; j = 1, 2, \dots, n). \\ (\text{MOCLD1}) \quad & \min [(\tilde{W}_0^1)_L(\alpha) + (\tilde{W}_0^1)_R(\alpha), \dots, (\tilde{W}_0^r)_L(\alpha) + (\tilde{W}_0^r)_R(\alpha)] \\ & \text{such that } \sum_{j=1}^n [(\tilde{a}_{ij}^k)_L(\alpha) + (\tilde{a}_{ij}^k)_R(\alpha)] y_j \leq (\tilde{W}_0^k)_L(\alpha) + (\tilde{W}_0^k)_R(\alpha), \\ & 0 \leq \alpha \leq 1, \\ & \forall y \in S^n, (k = 1, 2, \dots, r; i = 1, 2, \dots, m). \end{aligned}$$

**Proof.** By utilizing Definition 9, we obtain that  $(x, y)$  is the equilibrium solution of (*MOFP*) if and only if  $(x, y)$  is the optimal solution of multiple objective fuzzy optimization problems (*MOFP1*) and (*MOFD1*).

□

$$\begin{aligned} (\text{MOFP1}) \quad & \text{Find } x \in S^m \text{ such that} \\ & \tilde{V}_0^k \preceq (x^T \tilde{A}^k y), \quad \forall y \in S^n, \\ (\text{MOFD1}) \quad & \text{Find } y \in S^n \text{ such that} \\ & (x^T \tilde{A}^k y) \preceq \tilde{W}_0^k, \quad \forall x \in S^m, (k = 1, 2, \dots, r). \end{aligned}$$

Furthermore, by Theorem 1, the problems (MOFP1) and (MOFD1) respectively are equivalent to

$$\begin{aligned}
 (\text{MOP1}) \quad & \text{Find } x \in S^m \text{ such that} \\
 & p(\tilde{V}_0^k \preceq (x^T \tilde{A}^k y)) \geq \frac{1}{2}, \quad \forall y \in S^n, \\
 (\text{MOD1}) \quad & \text{Find } y \in S^n \text{ such that} \\
 & p((x^T \tilde{A}^k y) \preceq \tilde{W}_0^k) \geq \frac{1}{2}, \quad \forall x \in S^m, (k = 1, 2, \dots, r).
 \end{aligned}$$

By Definition 7, the problems (MOP1) and (MOD1) can be rewritten as (MOP2) and (MOD2), respectively.

$$\begin{aligned}
 (\text{MOP2}) \quad & \max [((\tilde{V}_0^1)_L(\alpha), \dots, (\tilde{V}_0^r)_L(\alpha)), ((\tilde{V}_0^1)_R(\alpha), \dots, (\tilde{V}_0^r)_R(\alpha))] \\
 & \text{such that } 1 - \max\{\min\{1, \frac{(\tilde{V}_0^k)_R(\alpha) - (x^T \tilde{A}_L^k(\alpha)y)}{(\tilde{V}_0^k)_R(\alpha) - (\tilde{V}_0^k)_L(\alpha) + x^T(\tilde{A}_R^k(\alpha) - \tilde{A}_L^k(\alpha))y}\}, 0\} \geq \frac{1}{2}, \\
 & 0 \leq \alpha \leq 1, \\
 & \forall x \in S^m, \forall y \in S^n, (k = 1, 2, \dots, r). \\
 (\text{MOD2}) \quad & \min [((\tilde{W}_0^1)_L(\alpha), \dots, (\tilde{W}_0^r)_L(\alpha)), ((\tilde{W}_0^1)_R(\alpha), \dots, (\tilde{W}_0^r)_R(\alpha))] \\
 & \text{such that } 1 - \max\{\min\{1, \frac{(x^T \tilde{A}_R^k(\alpha)y) - (\tilde{W}_0^k)_L(\alpha)}{x^T(\tilde{A}_R^k(\alpha) - \tilde{A}_L^k(\alpha))y + (\tilde{W}_0^k)_R(\alpha) - (\tilde{W}_0^k)_L(\alpha)}\}, 0\} \geq \frac{1}{2}, \\
 & 0 \leq \alpha \leq 1, \\
 & \forall x \in S^m, \forall y \in S^n, (k = 1, 2, \dots, r).
 \end{aligned}$$

That is equivalent to

$$\begin{aligned}
 (\text{MOP3}) \quad & \max [((\tilde{V}_0^1)_L(\alpha), \dots, (\tilde{V}_0^r)_L(\alpha)), ((\tilde{V}_0^1)_R(\alpha), \dots, (\tilde{V}_0^r)_R(\alpha))] \\
 & \text{such that } \max\{\min\{1, \frac{(\tilde{V}_0^k)_R(\alpha) - (x^T \tilde{A}_L^k(\alpha)y)}{(\tilde{V}_0^k)_R(\alpha) - (\tilde{V}_0^k)_L(\alpha) + x^T(\tilde{A}_R^k(\alpha) - \tilde{A}_L^k(\alpha))y}\}, 0\} \leq \frac{1}{2}, \\
 & 0 \leq \alpha \leq 1, \\
 & \forall x \in S^m, \forall y \in S^n, (k = 1, 2, \dots, r). \\
 (\text{MOD3}) \quad & \min [((\tilde{W}_0^1)_L(\alpha), \dots, (\tilde{W}_0^r)_L(\alpha)), ((\tilde{W}_0^1)_R(\alpha), \dots, (\tilde{W}_0^r)_R(\alpha))] \\
 & \text{such that } \max\{\min\{1, \frac{(x^T \tilde{A}_R^k(\alpha)y) - (\tilde{W}_0^k)_L(\alpha)}{x^T(\tilde{A}_R^k(\alpha) - \tilde{A}_L^k(\alpha))y + (\tilde{W}_0^k)_R(\alpha) - (\tilde{W}_0^k)_L(\alpha)}\}, 0\} \leq \frac{1}{2}, \\
 & 0 \leq \alpha \leq 1, \\
 & \forall x \in S^m, \forall y \in S^n, (k = 1, 2, \dots, r).
 \end{aligned}$$

That is to say,

$$\begin{aligned}
 (\text{MOP4}) \quad & \max [((\tilde{V}_0^1)_L(\alpha), \dots, (\tilde{V}_0^r)_L(\alpha)), ((\tilde{V}_0^1)_R(\alpha), \dots, (\tilde{V}_0^r)_R(\alpha))] \\
 & \text{such that } \frac{(\tilde{V}_0^k)_R(\alpha) - (x^T \tilde{A}_L^k(\alpha)y)}{(\tilde{V}_0^k)_R(\alpha) - (\tilde{V}_0^k)_L(\alpha) + (x^T \tilde{A}_R^k(\alpha)y) - (x^T \tilde{A}_L^k(\alpha)y)} \leq \frac{1}{2}, \\
 & 0 \leq \alpha \leq 1, \\
 & \forall x \in S^m, \forall y \in S^n, (k = 1, 2, \dots, r). \\
 (\text{MOD4}) \quad & \min [((\tilde{W}_0^1)_L(\alpha), \dots, (\tilde{W}_0^r)_L(\alpha)), ((\tilde{W}_0^1)_R(\alpha), \dots, (\tilde{W}_0^r)_R(\alpha))] \\
 & \text{such that } \frac{(x^T \tilde{A}_R^k(\alpha)y) - (\tilde{W}_0^k)_L(\alpha)}{x^T(\tilde{A}_R^k(\alpha) - \tilde{A}_L^k(\alpha))y + (\tilde{W}_0^k)_R(\alpha) - (\tilde{W}_0^k)_L(\alpha)} \leq \frac{1}{2}, \\
 & 0 \leq \alpha \leq 1, \\
 & \forall x \in S^m, \forall y \in S^n, (k = 1, 2, \dots, r).
 \end{aligned}$$

By arranging the models (MOP4) and (MOD4), we have

$$\begin{aligned}
(MOP5) \quad & \max [((\tilde{V}_0^1)_L(\alpha), \dots, (\tilde{V}_0^r)_L(\alpha)), ((\tilde{V}_0^1)_R(\alpha), \dots, (\tilde{V}_0^r)_R(\alpha))] \\
& \text{such that } (\tilde{V}_0^k)_L(\alpha) + (\tilde{V}_0^k)_R(\alpha) \leq x^T(\tilde{A}_R^k(\alpha) + \tilde{A}_L^k(\alpha))y, \\
& 0 \leq \alpha \leq 1, \\
& \forall x \in S^m, \forall y \in S^n, (k = 1, 2, \dots, r). \\
(MOD5) \quad & \min [((\tilde{W}_0^1)_L(\alpha), \dots, (\tilde{W}_0^r)_L(\alpha)), ((\tilde{W}_0^1)_R(\alpha), \dots, (\tilde{W}_0^r)_R(\alpha))] \\
& \text{such that } x^T(\tilde{A}_R^k(\alpha) + \tilde{A}_L^k(\alpha))y \leq (\tilde{W}_0^k)_L(\alpha) + (\tilde{W}_0^k)_R(\alpha), \\
& 0 \leq \alpha \leq 1, \\
& \forall x \in S^m, \forall y \in S^n, (k = 1, 2, \dots, r).
\end{aligned}$$

Since  $S^m$  and  $S^n$  are convex polytopes. Furthermore, the problems (MOP5) and (MOD5) are crisp multiple objective linear optimization problems; it is sufficient to consider only the extreme points of  $S^m$  and  $S^n$ . Thus, the problems (MOP5) and (MOD5) can be converted into

$$\begin{aligned}
(MOCLP) \quad & \max [((\tilde{V}_0^1)_L(\alpha), \dots, (\tilde{V}_0^r)_L(\alpha)), ((\tilde{V}_0^1)_R(\alpha), \dots, (\tilde{V}_0^r)_R(\alpha))] \\
& \text{such that } (\tilde{V}_0^k)_L(\alpha) + (\tilde{V}_0^k)_R(\alpha) \leq \sum_{i=1}^m [(\tilde{a}_{ij}^k)_L(\alpha) + (\tilde{a}_{ij}^k)_R(\alpha)]x_i, \\
& 0 \leq \alpha \leq 1, \\
& \forall x \in S^m, (k = 1, 2, \dots, r; j = 1, 2, \dots, n). \\
(MOCLD) \quad & \min [((\tilde{W}_0^1)_L(\alpha), \dots, (\tilde{W}_0^r)_L(\alpha)), ((\tilde{W}_0^1)_R(\alpha), \dots, (\tilde{W}_0^r)_R(\alpha))] \\
& \text{such that } \sum_{j=1}^n [(\tilde{a}_{ij}^k)_L(\alpha) + (\tilde{a}_{ij}^k)_R(\alpha)]y_j \leq (\tilde{W}_0^k)_L(\alpha) + (\tilde{W}_0^k)_R(\alpha), \\
& 0 \leq \alpha \leq 1, \\
& \forall y \in S^n, (k = 1, 2, \dots, r; i = 1, 2, \dots, m).
\end{aligned}$$

That is equal to

$$\begin{aligned}
(MOCLP1) \quad & \max [(\tilde{V}_0^1)_L(\alpha) + (\tilde{V}_0^1)_R(\alpha), \dots, (\tilde{V}_0^r)_L(\alpha) + (\tilde{V}_0^r)_R(\alpha)] \\
& \text{such that } (\tilde{V}_0^k)_L(\alpha) + (\tilde{V}_0^k)_R(\alpha) \leq \sum_{i=1}^m [(\tilde{a}_{ij}^k)_L(\alpha) + (\tilde{a}_{ij}^k)_R(\alpha)]x_i, \\
& 0 \leq \alpha \leq 1, \\
& \forall x \in S^m, (k = 1, 2, \dots, r; j = 1, 2, \dots, n). \\
(MOCLD1) \quad & \min [(\tilde{W}_0^1)_L(\alpha) + (\tilde{W}_0^1)_R(\alpha), \dots, (\tilde{W}_0^r)_L(\alpha) + (\tilde{W}_0^r)_R(\alpha)] \\
& \text{such that } \sum_{j=1}^n [(\tilde{a}_{ij}^k)_L(\alpha) + (\tilde{a}_{ij}^k)_R(\alpha)]y_j \leq (\tilde{W}_0^k)_L(\alpha) + (\tilde{W}_0^k)_R(\alpha), \\
& 0 \leq \alpha \leq 1, \\
& \forall y \in S^n, (k = 1, 2, \dots, r; i = 1, 2, \dots, m).
\end{aligned}$$

**Remark 1.** When the elements of  $\tilde{A}^k$ ,  $\tilde{V}_0^k$  and  $\tilde{W}_0^k$  ( $k = 1, 2, \dots, r$ ) are crisp numbers, the MOFP model reduces the MOG model, and the optimization problems MOCLP1 and MOCLD1 become the optimization problems MOGLP and MOGLD.

**Theorem 3.** Let  $\alpha \in (0, 1]$  be fixed. If  $(x, ((\tilde{V}_0^1)_L(\alpha) + (\tilde{V}_0^1)_R(\alpha), \dots, (\tilde{V}_0^r)_L(\alpha) + (\tilde{V}_0^r)_R(\alpha)))$  and  $(y, ((\tilde{W}_0^1)_L(\alpha) + (\tilde{W}_0^1)_R(\alpha), \dots, (\tilde{W}_0^r)_L(\alpha) + (\tilde{W}_0^r)_R(\alpha)))$  are the optimal solutions of (MOCLP1) and (MOCLD1), then,

$$\sum_{k=1}^r [(\tilde{V}_0^k)_L(\alpha) + (\tilde{V}_0^k)_R(\alpha)] \leq \sum_{k=1}^r [(\tilde{W}_0^k)_L(\alpha) + (\tilde{W}_0^k)_R(\alpha)].$$

**Proof.** By Theorem 2, we obtain

$$(\tilde{V}_0^k)_L(\alpha) + (\tilde{V}_0^k)_R(\alpha) \leq \sum_{i=1}^m [(\tilde{a}_{ij}^k)_L(\alpha) + (\tilde{a}_{ij}^k)_R(\alpha)]x_i \quad (6)$$

and

$$\sum_{j=1}^n [(\tilde{a}_{ij}^k)_L(\alpha) + (\tilde{a}_{ij}^k)_R(\alpha)] y_j \leq (\tilde{W}_0^k)_L(\alpha) + (\tilde{W}_0^k)_R(\alpha) \quad (7)$$

Since  $\forall x \in S^m, \forall y \in S^n$ , we have

$$\begin{aligned} (\tilde{V}_0^k)_L(\alpha) + (\tilde{V}_0^k)_R(\alpha) &= \sum_{j=1}^n [(\tilde{V}_0^k)_L(\alpha) + (\tilde{V}_0^k)_R(\alpha)] y_j \\ &\leq \sum_{j=1}^n \sum_{i=1}^m [(\tilde{a}_{ij}^k)_L(\alpha) + (\tilde{a}_{ij}^k)_R(\alpha)] x_i y_j \end{aligned} \quad (8)$$

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^n [(\tilde{a}_{ij}^k)_L(\alpha) + (\tilde{a}_{ij}^k)_R(\alpha)] x_i y_j &\leq \sum_{i=1}^m [(\tilde{W}_0^k)_L(\alpha) + (\tilde{W}_0^k)_R(\alpha)] x_i \\ &= (\tilde{W}_0^k)_L(\alpha) + (\tilde{W}_0^k)_R(\alpha) \end{aligned} \quad (9)$$

Therefore, observe that

$$(\tilde{V}_0^k)_L(\alpha) + (\tilde{V}_0^k)_R(\alpha) \leq (\tilde{W}_0^k)_L(\alpha) + (\tilde{W}_0^k)_R(\alpha)$$

Then,

$$\sum_{k=1}^r [(\tilde{V}_0^k)_L(\alpha) + (\tilde{V}_0^k)_R(\alpha)] \leq \sum_{k=1}^r [(\tilde{W}_0^k)_L(\alpha) + (\tilde{W}_0^k)_R(\alpha)].$$

□

**Theorem 4.** Let  $\alpha \in (0, 1]$  be fixed. If the elements of  $\tilde{A}^k$  ( $k = 1, 2, \dots, r$ ) are triangular fuzzy numbers, then  $(x, ((\tilde{V}_0^1)_L(\alpha) + (\tilde{V}_0^1)_R(\alpha), \dots, (\tilde{V}_0^r)_L(\alpha) + (\tilde{V}_0^r)_R(\alpha)))$  and  $(y, ((\tilde{W}_0^1)_L(\alpha) + (\tilde{W}_0^1)_R(\alpha), \dots, (\tilde{W}_0^r)_L(\alpha) + (\tilde{W}_0^r)_R(\alpha)))$  are the optimal solutions of (MOCLP2) and (MOCLD2) if and only if  $(x, y)$  is the equilibrium solution of (MOFP).

$$\begin{aligned} \text{(MOCLP2)} \quad & \max [(\tilde{V}_0^1)_L(\alpha) + (\tilde{V}_0^1)_R(\alpha), \dots, (\tilde{V}_0^r)_L(\alpha) + (\tilde{V}_0^r)_R(\alpha)] \\ & \text{such that } (\tilde{V}_0^k)_L(\alpha) + (\tilde{V}_0^k)_R(\alpha) \leq \sum_{i=1}^m [(\tilde{a}_{ij}^k)_L(\alpha) + (\tilde{a}_{ij}^k)_R(\alpha)] (1 - \alpha) + 2\alpha \tilde{a}_{ij}^k x_i \\ & 0 \leq \alpha \leq 1, \\ & \forall x \in S^m, (k = 1, 2, \dots, r; j = 1, 2, \dots, n). \\ \text{(MOCLD2)} \quad & \min [(\tilde{W}_0^1)_L(\alpha) + (\tilde{W}_0^1)_R(\alpha), \dots, (\tilde{W}_0^r)_L(\alpha) + (\tilde{W}_0^r)_R(\alpha)] \\ & \text{such that } \sum_{j=1}^n [(\tilde{a}_{ij}^k)_L(\alpha) + (\tilde{a}_{ij}^k)_R(\alpha)] (1 - \alpha) + 2\alpha \tilde{a}_{ij}^k y_j \leq (\tilde{W}_0^k)_L(\alpha) + (\tilde{W}_0^k)_R(\alpha), \\ & 0 \leq \alpha \leq 1, \\ & \forall y \in S^n, (k = 1, 2, \dots, r; i = 1, 2, \dots, m). \end{aligned}$$

**Proof.** Since the elements of  $\tilde{A}^k$  ( $k = 1, 2, \dots, r$ ) are triangular fuzzy numbers, using (5), we have

$$[\tilde{a}_{ij}^k]^\alpha = [(\tilde{a}_{ij}^k)_L(\alpha), (\tilde{a}_{ij}^k)_R(\alpha)] = [(a_{ij}^k - \tilde{a}_{ij}^k)\alpha + \tilde{a}_{ij}^k, -(\tilde{a}_{ij}^k - a_{ij}^k)\alpha + \tilde{a}_{ij}^k], \alpha \in (0, 1].$$

Hence,

$$(\tilde{a}_{ij}^k)_L(\alpha) + (\tilde{a}_{ij}^k)_R(\alpha) = (\tilde{a}_{ij}^k + \tilde{a}_{ij}^k)(1 - \alpha) + 2\alpha \tilde{a}_{ij}^k$$



By utilizing Theorem 2, we have that  $(x, ((\tilde{V}_0^1)_L(\alpha) + (\tilde{V}_0^1)_R(\alpha), \dots, (\tilde{V}_0^r)_L(\alpha) + (\tilde{V}_0^r)_R(\alpha)))$  and  $(y, ((\tilde{W}_0^1)_L(\alpha) + (\tilde{W}_0^1)_R(\alpha), \dots, (\tilde{W}_0^r)_L(\alpha) + (\tilde{W}_0^r)_R(\alpha)))$  are optimal solutions of (MOCLP2) and (MOCLD2) if and only if  $(x, y)$  is the equilibrium solution of (MOFP).

$$\begin{aligned}
 (\text{MOCLP2}) \quad & \max [(\tilde{V}_0^1)_L(\alpha) + (\tilde{V}_0^1)_R(\alpha), \dots, (\tilde{V}_0^r)_L(\alpha) + (\tilde{V}_0^r)_R(\alpha)] \\
 & \text{such that } (\tilde{V}_0^k)_L(\alpha) + (\tilde{V}_0^k)_R(\alpha) \leq \sum_{i=1}^m [(\hat{a}_{ij}^k + \hat{a}_{ij}^k)(1 - \alpha) + 2\alpha \hat{a}_{ij}^k] x_i \\
 & 0 \leq \alpha \leq 1, \\
 & \forall x \in S^m, (k = 1, 2, \dots, r; j = 1, 2, \dots, n). \\
 (\text{MOCLD2}) \quad & \min [(\tilde{W}_0^1)_L(\alpha) + (\tilde{W}_0^1)_R(\alpha), \dots, (\tilde{W}_0^r)_L(\alpha) + (\tilde{W}_0^r)_R(\alpha)] \\
 & \text{such that } \sum_{j=1}^n [(\hat{a}_{ij}^k + \hat{a}_{ij}^k)(1 - \alpha) + 2\alpha \hat{a}_{ij}^k] y_j \leq (\tilde{W}_0^k)_L(\alpha) + (\tilde{W}_0^k)_R(\alpha), \\
 & 0 \leq \alpha \leq 1, \\
 & \forall y \in S^n, (k = 1, 2, \dots, r; i = 1, 2, \dots, m).
 \end{aligned}$$

□

#### 4. Example

In order to illustrate the effectiveness and correctness of the obtained model, we consider multiple payoffs of the multiple objective matrix game based on fuzzy payoffs (MOFP) that are taken as

$$\tilde{A}^1 = \begin{bmatrix} (2.90, 3.00, 3.10) & (7.75, 8.00, 8.20) \\ (3.50, 4.00, 4.20) & (0.90, 1.00, 1.20) \end{bmatrix}$$

and

$$\tilde{A}^2 = \begin{bmatrix} (4.80, 5.00, 5.50) & (1.80, 2.00, 2.75) \\ (5.75, 6.00, 6.40) & (0.75, 1.00, 1.20) \end{bmatrix}.$$

In order to solve the game for a given  $\alpha$ , by Theorem 4, we have to solve the following problems (MOGCLP3) and (MOGCLD3).

$$\begin{aligned}
 (\text{MOCLP3}) \quad & \max [(\tilde{V}_0^1)_L(\alpha) + (\tilde{V}_0^1)_R(\alpha), (\tilde{V}_0^2)_L(\alpha) + (\tilde{V}_0^2)_R(\alpha)] \\
 & \text{such that } (\tilde{V}_0^1)_L(\alpha) + (\tilde{V}_0^1)_R(\alpha) \leq 6x_1 + (7.7 + 0.3\alpha)x_2 \\
 & (\tilde{V}_0^1)_L(\alpha) + (\tilde{V}_0^1)_R(\alpha) \leq (15.95 + 0.05\alpha)x_1 + (2.1 - 0.1\alpha)x_2 \\
 & (\tilde{V}_0^2)_L(\alpha) + (\tilde{V}_0^2)_R(\alpha) \leq (10.3 - 0.3\alpha)x_1 + (12.15 - 0.15\alpha)x_2 \\
 & (\tilde{V}_0^2)_L(\alpha) + (\tilde{V}_0^2)_R(\alpha) \leq (4.55 - 0.55\alpha)x_1 + (1.95 + 0.05\alpha)x_2 \\
 & 0 \leq \alpha \leq 1, \\
 & x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0, (k = 1, 2, \dots, r). \\
 (\text{MOCLD3}) \quad & \min [(\tilde{W}_0^1)_L(\alpha) + (\tilde{W}_0^1)_R(\alpha), (\tilde{W}_0^2)_L(\alpha) + (\tilde{W}_0^2)_R(\alpha)] \\
 & \text{such that } 6y_1 + (15.95 + 0.05\alpha)y_2 \leq (\tilde{W}_0^1)_L(\alpha) + (\tilde{W}_0^1)_R(\alpha), \\
 & (7.7 + 0.3\alpha)y_1 + (2.1 - 0.1\alpha)y_2 \leq (\tilde{W}_0^1)_L(\alpha) + (\tilde{W}_0^1)_R(\alpha), \\
 & (10.3 - 0.3\alpha)y_1 + (4.55 - 0.55\alpha)y_2 \leq (\tilde{W}_0^2)_L(\alpha) + (\tilde{W}_0^2)_R(\alpha), \\
 & (12.15 - 0.15\alpha)y_1 + (1.95 + 0.05\alpha)y_2 \leq (\tilde{W}_0^2)_L(\alpha) + (\tilde{W}_0^2)_R(\alpha), \\
 & 0 \leq \alpha \leq 1, \\
 & y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0, (k = 1, 2, \dots, r).
 \end{aligned}$$

By solving the above problems (MOGCLP3) and (MOGCLD3), particularly, let  $\alpha^* = 1$ , then we can obtain that  $((x_1^* = 0.20, x_2^* = 0.80), ((\tilde{V}_0^1)_L(\alpha^*) + (\tilde{V}_0^1)_R(\alpha^*) = 4.80, (\tilde{V}_0^2)_L(\alpha^*) + (\tilde{V}_0^2)_R(\alpha^*) = 2.40))$  is the optimal solution of (MOGCLP3) and  $((y_1^* = 0.80, y_2^* = 0.20), ((\tilde{W}_0^1)_L(\alpha^*) + (\tilde{W}_0^1)_R(\alpha^*) = 8.00, (\tilde{W}_0^2)_L(\alpha^*) + (\tilde{W}_0^2)_R(\alpha^*) = 10.00))$  is optimal solution of (MOGCLD3). By Theorem 4, we have that  $((x_1^* = 0.20, x_2^* = 0.80), (y_1^* = 0.80, y_2^* = 0.20))$  is the equilibrium solution of (MOFP).

## 5. Conclusions

In this paper, we proposed the multicriteria matrix game model based on fuzzy payoffs. In order to solve the game model, we first discussed the relationship of two fuzzy numbers via the lower limit— $\frac{1}{2}$  of the possibility degree. Then, utilizing this relationship, we conclude that the equilibrium solution of this game model and optimal solutions of a pair of multiple objective linear optimization problems are of equal value. We will use other more effective methods to study the matrix game in the future.

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