symmetry

MDPI

Article

Cosine Measures of Neutrosophic Cubic Sets for Multiple Attribute Decision-Making

Zhikang Lu and Jun Ye *

Department of Electrical and Information Engineering, Shaoxing University, 508 Huancheng West Road, Shaoxing 312000, China; luzhikang@usx.edu.cn
* Correspondence: yejun@usx.edu.cn

Received: 26 June 2017; Accepted: 11 July 2017; Published: 18 July 2017

Abstract: The neutrosophic cubic set can contain much more information to express its interval neutrosophic numbers and single-valued neutrosophic numbers simultaneously in indeterminate environments. Hence, it is a usual tool for expressing much more information in complex decision-making problems. Unfortunately, there has been no research on similarity measures of neutrosophic cubic sets so far. Since the similarity measure is an important mathematical tool in decision-making problems, this paper proposes three cosine measures between neutrosophic cubic sets based on the included angle cosine of two vectors, distance, and cosine functions, and investigates their properties. Then, we develop a cosine measures-based multiple attribute decision-making method under a neutrosophic cubic environment in which, from the cosine measure between each alternative (each evaluated neutrosophic cubic set) and the ideal alternative (the ideal neutrosophic cubic set), the ranking order of alternatives and the best option can be obtained, corresponding to the cosine measure values in the decision-making process. Finally, an illustrative example about the selection problem of investment alternatives is provided to illustrate the application and feasibility of the developed decision-making method.

Keywords: neutrosophic cubic set; decision-making; similarity measure; cosine measure; interval neutrosophic set; single-valued neutrosophic set

1. Introduction

The classic fuzzy set, as presented by Zadeh [1], is only described by the membership degree in the unit interval [0, 1]. In the real world, it is often difficult to express the value of a membership function by an exact value in a fuzzy set. In such cases, it may be easier to describe vagueness and uncertainty in the real world using both an interval value and an exact value, rather than unique interval/exact values. Thus, the hybrid form of an interval value and an exact value may be a very useful expression for a person to describe certainty and uncertainty due to his/her hesitant judgment in complex decision-making problems. For this purpose, Jun et al. [2] introduced the concept of (fuzzy) cubic sets, including internal cubic sets and external cubic sets, by the combination of both an interval-valued fuzzy number (IVFN) and a fuzzy value, and defined some logic operations of cubic sets, such as the P-union, P-intersection, R-union, and R-intersection of cubic sets. Also, Jun and Lee [3] and Jun et al. [4–6] applied the concept of cubic sets to BCK/BCI-algebras and introduced the concepts of cubic subalgebras/ideals, cubic o-subalgebras and closed cubic ideals in BCK/BCI-algebras.

However, the cubic set is described by two parts simultaneously, where one represents the membership degree range by the interval value and the other represents the membership degree by a fuzzy value. Hence, a cubic set is the hybrid set combined by both an IVFN and a fuzzy value. Obviously, the advantage of the cubic set is that it can contain much more information to express the IVFN and fuzzy value simultaneously.

As the generalization of fuzzy sets [1], interval-valued fuzzy sets (IVFSs) [7], intuitionistic fuzzy sets (IFSs) [8], and interval-valued intuitionistic fuzzy sets (IVIFSs) [9], Smarandache [10] initially introduced a concept of neutrosophic sets to express incomplete, indeterminate, and inconsistent information. As simplified forms of neutrosophic sets, Smarandache [10], Wang et al. [11,12] and Ye [13] introduced single-valued neutrosophic sets (SVNSs) and interval neutrosophic sets (INSs), and simplified neutrosophic sets (SNSs) as subclasses of neutrosophic sets for easy engineering applications. Since then, SVNSs, INSs, and SNSs have been widely applied to various areas, such as image processing [14–16], decision-making [17–32], clustering analyses [33,34], medical diagnoses [35,36], and fault diagnoses [37]. Recently, Ali et al. [38] and Jun et al. [39] have extended cubic sets to the neutrosophic sets and proposed the concepts of neutrosophic cubic sets (NCSs), including internal NCSs and external NCSs, subsequently introducing some logic operations of NCSs, such as the P-union, P-intersection, R-union, and R-intersection of NCSs. Furthermore, Ali et al. [38] introduced a distance measure between NCSs and applied it to pattern recognition. Subsequently, Banerjee et al. [40] further presented a multiple attribute decision-making (MADM) method with NCSs based on grey relational analysis, in which they introduced the Hamming distances of NCSs for weighted grey relational coefficients and standard (ideal) grey relational coefficients, and then gave the relative closeness coefficients in order to rank the alternatives.

From the above review, we can see that the existing literature mainly focus on the theoretical studies of cubic sets and NCSs, rather than the studies on their similarity measures and their applications. On the other hand, the NCS contains much more information than the general neutrosophic set (INS/SVNS) because the NCS is expressed by the combined information of both INS and SVNS. Hence, NCSs used for attribute evaluation in decision making may show its rationality and affectivity since general neutrosophic decision-making methods with INSs/SVNSs may lose some useful evaluation information (either INSs or SVNSs) of attributes, which may affect decision results, resulting in the distortion phenomenon. Moreover, the similarity measure is an important mathematical tool in decision-making problems. Currently, since there is no study on similarity measures of cubic sets and NCSs under a neutrosophic cubic environment, we need to develop new similarity measures for NCSs for MADM problems with neutrosophic cubic information, since the cubic set is a special case of the NCS. For these reasons, this paper aims to propose three cosine measures between NCSs based on the included angle cosine of two vectors, distance, and cosine function, and their MADM method in a neutrosophic cubic environment.

The remainder of the article is organized as follows. Section 2 briefly describes some concepts of cubic sets and NCSs. Section 3 presents three cosine measures of NCSs and discusses their properties. In Section 4, we develop an MADM approach based on the cosine measures of NCSs under a neutrosophic cubic environment. In Section 5, an illustrative example about the selection problem of investment alternatives is provided to illustrate the application and feasibility of the developed method. Section 6 contains conclusions and future research.

2. Some Basic Concepts of Cubic Sets and NCSs

By the combination of a fuzzy value and an IVFN, Jun et al. [2] defined a (fuzzy) cubic set.

A cubic set S in a universe of discourse X is constructed as follows [2]:

$$S = \{x, T(x), \mu(x) | x \in X\},$$

where $T(x) = [T^-(x), T^+(x)]$ is an IVFN for $x \in X$ and μ is a fuzzy value for $x \in X$. Then, we call

(i) $S = \{x, T(x), \mu(x) | x \in X\}$ an internal cubic set if $T^-(x) \leq \mu(x) \leq T^+(x)$ for $x \in X$;

(ii) $S = \{x, T(x), \mu(x) | x \in X\}$ an external cubic set if $\mu(x) \notin (T^-(x), T^+(x))$ for $x \in X$.

Then, Ali et al. [38] and Jun et al. [39] proposed a NCS based on the combination of an interval neutrosophic number (INN) and a single-valued neutrosophic number (SVNN) as the extension of the (fuzzy) cubic set.

A NCS S in X is constructed as the following form [38,39]:

$$P = \{x, < T(x), U(x), F(x) >, < t(x), u(x), f(x) > | x \in X\},$$

where $<T(x), U(x), F(x)>$ is an INN, and $T(x) = [T^-(x), T^+(x)] \subseteq [0, 1]$, $U(x) = [U^-(x), U^+(x)] \subseteq [0, 1]$, and $F(x) = [F^-(x), F^+(x)] \subseteq [0, 1]$ for $x \in X$ are the truth-interval, indeterminacy-interval, and falsity-interval, respectively; then $<t(x), u(x), f(x)>$ is a SVNN, and $t(x), u(x), f(x) \in [0, 1]$ for $x \in X$ are the truth, indeterminacy, and falsity degrees, respectively.

An NCS $P = \{x, < T(x), U(x), F(x) >, < t(x), u(x), f(x) > | x \in X\}$ is said to be [38,39]:

(i) An internal NCS $P = \{x, < T(x), U(x), F(x) >, < t(x), u(x), f(x) > | x \in X\}$ if $T^-(x) \le t(x) \le T^+(x)$, $U^-(x) \le u(x) \le U^+(x)$, and $F^-(x) \le f(x) \le F^+(x)$ for $x \in X$;

(ii) An external NCS $P = \{x, < T(x), U(x), F(x) >, < t(x), u(x), f(x) > | x \in X\}$ if $t(x) \notin (T^-(x), T^+(x))$, $u(x) \notin (U^-(x), U^+(x))$, and $f(x) \notin (F^-(x), F^+(x))$ for $x \in X$.

For convenience, a basic element $(x, < T(x), U(x), F(x) >, < t(x), u(x), f(x) >)$ in an NCS P is simply denoted by $p = (<T, U, F>, <t, u, f>)$, which is called a neutrosophic cubic number (NCN), where $T, U, F \subseteq [0, 1]$ and $t, u, f \in [0, 1]$, satisfying $0 \le T^+(x) + U^+(x) + F^+(x) \le 3$ and $0 \le t + u + f \le 3$.

Let $p_1 = (<T_1, U_1, F_1>, <t_1, u_1, f_1>)$ and $p_2 = (<T_2, U_2, F_2>, <t_2, u_2, f_2>)$ be two NCNs. Then, there are the following relations [38,39]:

(1) $p_1^c = \left(\left\langle [F_1^-, F_1^+], [1 - U_1^+, 1 - U_1^-], [T_1^-, T_1^+] \right\rangle, \langle f_1, 1 - u_1, t_1 \rangle\right)$ (complement of p_1);
(2) $p_1 \subseteq p_2$ if and only if $T_1 \subseteq T_2$, $U_1 \supseteq U_2$, $F_1 \supseteq F_2$,$t_1 \le t_2$, $u_1 \ge u_2$, and $f_1 \ge f_2$ (P-order);
(3) $p_1 = p_2$ if and only if $p_2 \subseteq p_1$ and $p_1 \subseteq p_2$, i.e., $<T_1, U_1, F_1> = <T_2, U_2, F_2>$ and $<t_1, u_1, f_1> = <t_2, u_2, f_2>$.

3. Cosine Measures of NCSs

In this section, we propose three cosine measures between NCSs.

Definition 1. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and two NCSs be $P = \{p_1, p_2, \ldots, p_n\}$ and $Q = \{q_1, q_2, \ldots, q_n\}$, where $p_j = (<T_{pj}, U_{pj}, F_{pj}>, <t_{pj}, u_{pj}, f_{pj}>)$ and $q_j = (<T_{qj}, U_{qj}, F_{qj}>, <t_{qj}, u_{qj}, f_{qj}>)$ for $j = 1, 2, \ldots, n$ are two collections of NCNs. Then, three cosine measures of P and Q are proposed based on the included angle cosine of two vectors, distance, and cosine function, respectively, as follows:*

(1) Cosine measure based on the included angle cosine of two vectors

$$S_1(P,Q) = \frac{1}{2n}\left\{\sum_{j=1}^{n}\frac{T_{pj}^-T_{qj}^- + T_{pj}^+T_{qj}^+ + U_{pj}^-U_{qj}^- + U_{pj}^+U_{qj}^+ + F_{pj}^-F_{qj}^- + F_{pj}^+F_{qj}^+}{\left\{\begin{array}{c}\sqrt{(T_{pj}^-)^2 + (T_{pj}^+)^2 + (U_{pj}^-)^2 + (U_{pj}^+)^2 + (F_{pj}^-)^2 + (F_{pj}^+)^2} \\ \times\sqrt{(T_{qj}^-)^2 + (T_{qj}^+)^2 + (U_{qj}^-)^2 + (U_{qj}^+)^2 + (F_{qj}^-)^2 + (F_{qj}^+)^2}\end{array}\right\}} + \sum_{j=1}^{n}\frac{t_{pj}t_{qj} + u_{pj}u_{qj} + f_{pj}f_{qj}}{\left\{\sqrt{t_{pj}^2 + u_{pj}^2 + f_{pj}^2} \times \sqrt{t_{qj}^2 + u_{qj}^2 + f_{qj}^2}\right\}}\right\} \tag{1}$$

(2) Cosine measure based on distance

$$S_2(P,Q) = \frac{1}{2n}\sum_{j=1}^{n}\left\{\begin{array}{c}\cos\left(\frac{\left|T_{pj}^- - T_{qj}^-\right| + \left|T_{pj}^+ - T_{qj}^+\right| + \left|U_{pj}^- - U_{qj}^-\right| + \left|U_{pj}^+ - U_{qj}^+\right| + \left|F_{pj}^- - F_{qj}^-\right| + \left|F_{pj}^+ - F_{qj}^+\right|}{12}\pi\right) \\ + \cos\left(\frac{\left|t_{pj} - t_{qj}\right| + \left|u_{pj} - u_{qj}\right| + \left|f_{pj} - f_{qj}\right|}{6}\pi\right)\end{array}\right\} \tag{2}$$

(3) *Cosine measure based on cosine function*

$$S_3(P,Q) = \frac{1}{2n}\left\{\frac{1}{3(\sqrt{2}-1)}\sum_{j=1}^{n}\left\{\begin{array}{l}\left[\sqrt{2}\cos\left(\frac{T_{pj}^{-}+T_{pj}^{+}-T_{qj}^{-}-T_{qj}^{+}}{8}\pi\right)-1\right]\\+\left[\sqrt{2}\cos\left(\frac{U_{pj}^{-}+U_{pj}^{+}-U_{qj}^{-}-U_{qj}^{+}}{8}\pi\right)-1\right]\\+\left[\sqrt{2}\cos\left(\frac{F_{pj}^{-}+F_{pj}^{+}-F_{qj}^{-}-F_{qj}^{+}}{8}\pi\right)-1\right]\end{array}\right\}+\frac{1}{3(\sqrt{2}-1)}\sum_{j=1}^{n}\left\{\begin{array}{l}\left[\sqrt{2}\cos\left(\frac{t_{pj}-t_{qj}}{4}\pi\right)-1\right]\\+\left[\sqrt{2}\cos\left(\frac{u_{pj}-u_{qj}}{4}\pi\right)-1\right]\\+\left[\sqrt{2}\cos\left(\frac{f_{pj}-f_{qj}}{4}\pi\right)-1\right]\end{array}\right\}\right\} \quad (3)$$

Obviously, the three cosine measures $S_k(P, Q)$ (k = 1, 2, 3) satisfy the following properties (S_1)–(S_3):

(S_1) $0 \leq S_k(P, Q) \leq 1$;
(S_2) $S_k(P, Q) = S_k(Q, P)$;
(S_3) *$S_k(P, Q) = 1$ if $P = Q$, i.e., $<T_{pj}, U_{pj}, F_{pj}>, = <T_{qj}, U_{qj}, F_{qj}>$ and $<t_{pj}, u_{pj}, f_{pj}> = <t_{qj}, u_{qj}, f_{qj}>$.*

Proof.

Firstly, we prove the properties (S_1)–(S_3) of $S_1(P, Q)$.

(S_1) The inequality $S_1(P, Q) \geq 0$ is obvious. Then, we only prove $S_1(P, Q) \leq 1$.

Based on the Cauchy–Schwarz inequality:

$$(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2 \leq \left(x_1^2 + x_2^2 + \cdots x_n^2\right) \times \left(y_1^2 + y_2^2 + \cdots y_n^2\right),$$

where $(x_1, x_2, \ldots, x_n) \in R^n$ and $(y_1, y_2, \ldots, y_n) \in R^n$, we can give the following inequality:

$$(x_1y_1 + x_2y_2 + \cdots + x_ny_n) \leq \sqrt{\left(x_1^2 + x_2^2 + \cdots x_n^2\right)} \times \sqrt{\left(y_1^2 + y_2^2 + \cdots y_n^2\right)}.$$

According to the above inequality, we have the following inequality:

$$T_{pj}^{-}T_{qj}^{-} + T_{pj}^{+}T_{qj}^{+} + U_{pj}^{-}U_{qj}^{-} + U_{pj}^{+}U_{qj}^{+} + F_{pj}^{-}F_{qj}^{-} + F_{pj}^{+}F_{qj}^{+} \leq$$
$$\sqrt{(T_{pj}^{-})^2 + (T_{pj}^{+})^2 + (U_{pj}^{-})^2 + (U_{pj}^{+})^2 + (F_{pj}^{-})^2 + (F_{pj}^{+})^2} \times \sqrt{(T_{qj}^{-})^2 + (T_{qj}^{+})^2 + (U_{qj}^{-})^2 + (U_{qj}^{+})^2 + (F_{qj}^{-})^2 + (F_{qj}^{+})^2},$$

$$t_{pj}t_{qj} + u_{pj}u_{qj} + f_{pj}f_{qj} \leq \sqrt{t_{pj}^2 + u_{pj}^2 + f_{pj}^2} \times \sqrt{t_{qj}^2 + u_{qj}^2 + f_{qj}^2}.$$

Hence, there is the following result:

$$\frac{1}{n}\sum_{j=1}^{n}\frac{T_{pj}^{-}T_{qj}^{-} + T_{pj}^{+}T_{qj}^{+} + U_{pj}^{-}U_{qj}^{-} + U_{pj}^{+}U_{qj}^{+} + F_{pj}^{-}F_{qj}^{-} + F_{pj}^{+}F_{qj}^{+}}{\left\{\begin{array}{l}\sqrt{(T_{pj}^{-})^2 + (T_{pj}^{+})^2 + (U_{pj}^{-})^2 + (U_{pj}^{+})^2 + (F_{pj}^{-})^2 + (F_{pj}^{+})^2}\\ \times\sqrt{(T_{qj}^{-})^2 + (T_{qj}^{+})^2 + (U_{qj}^{-})^2 + (U_{qj}^{+})^2 + (F_{qj}^{-})^2 + (F_{qj}^{+})^2}\end{array}\right\}} \leq 1,$$

$$\frac{1}{n}\sum_{j=1}^{n}\frac{t_{pj}t_{qj} + u_{pj}u_{qj} + f_{pj}f_{qj}}{\left\{\sqrt{t_{pj}^2 + u_{pj}^2 + f_{pj}^2} \times \sqrt{t_{qj}^2 + u_{qj}^2 + f_{qj}^2}\right\}} \leq 1.$$

Based on Equation (1), we have $S_1(P, Q) \leq 1$. Hence, $0 \leq S_1(P, Q) \leq 1$ holds.

(S_2) It is straightforward.

(S_3) If $P = Q$, there are $<T_{pj}, U_{pj}, F_{pj}> = <T_{qj}, U_{qj}, F_{qj}>$ and $<t_{pj}, u_{pj}, f_{pj}> = <t_{qj}, u_{qj}, f_{qj}>$. Thus $T_{pj} = T_{qj}$, $U_{pj} = U_{qj}$, $F_{pj} = F_{qj}$, $t_{pj} = t_{qj}$, $u_{pj} = u_{qj}$, and $f_{pj} = f_{qj}$ for $j = 1, 2, \ldots, n$. Hence $S_1(P, Q) = 1$ holds.

Secondly, we prove the properties (S_1)–(S_3) of $S_2(P, Q)$.

(S_1) Let $x_1 = \left(\left|T_{pj}^{-} - T_{qj}^{-}\right| + \left|T_{pj}^{+} - T_{qj}^{+}\right| + \left|U_{pj}^{-} - U_{qj}^{-}\right| + \left|U_{pj}^{+} - U_{qj}^{+}\right| + \left|F_{pj}^{-} - F_{qj}^{-}\right| + \left|F_{pj}^{+} - F_{qj}^{+}\right|\right)/6$ and $x_2 = \left(\left|t_{pj} - t_{qj}\right| + \left|u_{pj} - u_{qj}\right| + \left|f_{pj} - f_{qj}\right|\right)/3$. It is obvious that there exist $0 \leq x_1 \leq 1$

and $0 \leq x_2 \leq 1$. Thus, there are $0 \leq \cos(x_1\pi/2) \leq 1$ and $0 \leq \cos(x_2\pi/2) \leq 1$. Hence, $0 \leq S_2(P, Q) \leq 1$ holds.

(S_2) It is straightforward.

(S_3) If $P = Q$, there are $<T_{pj}, U_{pj}, F_{pj}> = <T_{qj}, U_{qj}, F_{qj}>$ and $<t_{pj}, u_{pj}, f_{pj}> = <t_{qj}, u_{qj}, f_{qj}>$. Thus $T_{pj} = T_{qj}$, $U_{pj} = U_{qj}$, $F_{pj} = F_{qj}$, $t_{pj} = t_{qj}$, $u_{pj} = u_{qj}$, and $f_{pj} = f_{qj}$ for $j = 1, 2, \ldots, n$. Hence, $S_2(P, Q) = 1$ holds.

Thirdly, we prove the properties (S_1)–(S_3) of $S_3(P, Q)$.

(S_1) Let $y_1 = (T^-_{pj} + T^+_{pj} - T^-_{qj} - T^+_{qj})/2$, $y_2 = (U^-_{pj} + U^+_{pj} - U^-_{qj} - U^+_{qj})/2$, $y_3 = (F^-_{pj} + F^+_{pj} - F^-_{qj} - F^+_{qj})/2$, $y_4 = t_{pj} - t_{qj}$, $y_5 = u_{pj} - u_{qj}$, and $y_6 = f_{pj} - f_{qj}$. Obviously, there exists $-1 \leq y_k \leq +1$ for $k = 1, 2, \ldots, 6$. Thus, $\sqrt{2}/2 \leq \cos(y_k\pi/4) \leq 1$, and then there exists $0 \leq S_3(P, Q) \leq 1$.

(S_2) It is straightforward.

(S_3) If $P = Q$, there are $<T_{pj}, U_{pj}, F_{pj}> = <T_{qj}, U_{qj}, F_{qj}>$ and $<t_{pj}, u_{pj}, f_{pj}> = <t_{qj}, u_{qj}, f_{qj}>$. Thus $T_{pj} = T_{qj}$, $U_{pj} = U_{qj}$, $F_{pj} = F_{qj}$, $t_{pj} = t_{qj}$, $u_{pj} = u_{qj}$, and $f_{pj} = f_{qj}$ for $j = 1, 2, \ldots, n$. Hence, $S_3(P, Q) = 1$ holds. □

When the weight of the elements p_j and q_j ($j = 1, 2, \ldots, n$) is taken into account, $\boldsymbol{w} = \{w_1, w_2, \ldots, w_n\}$ is given as the weight vector of the elements p_j and q_j ($j = 1, 2, \ldots, n$) with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then, we have the following three weighted cosine measures between P and Q, respectively:

$$S_{w1}(P,Q) = \frac{1}{2}\left\{ \begin{array}{c} \sum\limits_{j=1}^{n} w_j \dfrac{T^-_{pj}T^-_{qj} + T^+_{pj}T^+_{qj} + U^-_{pj}U^-_{qj} + U^+_{pj}U^+_{qj} + F^-_{pj}F^-_{qj} + F^+_{pj}F^+_{qj}}{\left\{ \begin{array}{c} \sqrt{(T^-_{pj})^2 + (T^+_{pj})^2 + (U^-_{pj})^2 + (U^+_{pj})^2 + (F^-_{pj})^2 + (F^+_{pj})^2} \\ \times \sqrt{(T^-_{qj})^2 + (T^+_{qj})^2 + (U^-_{qj})^2 + (U^+_{qj})^2 + (F^-_{qj})^2 + (F^+_{qj})^2} \end{array} \right\}} \\ + \sum\limits_{j=1}^{n} w_j \dfrac{t_{pj}t_{qj} + u_{pj}u_{qj} + f_{pj}f_{qj}}{\left\{ \sqrt{t^2_{pj} + u^2_{pj} + f^2_{pj}} \times \sqrt{t^2_{qj} + u^2_{qj} + f^2_{qj}} \right\}} \end{array} \right\}, \quad (4)$$

$$S_{w2}(P,Q) = \frac{1}{2}\sum_{j=1}^{n} w_j \left\{ \begin{array}{c} \cos\left(\dfrac{\left|T^-_{pj} - T^-_{qj}\right| + \left|T^+_{pj} - T^+_{qj}\right| + \left|U^-_{pj} - U^-_{qj}\right| + \left|U^+_{pj} - U^+_{qj}\right| + \left|F^-_{pj} - F^-_{qj}\right| + \left|F^+_{pj} - F^+_{qj}\right|}{12}\pi \right) \\ + \cos\left(\dfrac{\left|t_{pj} - t_{qj}\right| + \left|u_{pj} - u_{qj}\right| + \left|f_{pj} - f_{qj}\right|}{6}\pi \right) \end{array} \right\}, \quad (5)$$

$$S_{w3}(P,Q) = \frac{1}{2}\left\{ \frac{1}{3(\sqrt{2}-1)} \sum_{j=1}^{n} w_j \left\{ \left(\begin{array}{c} \left[\sqrt{2}\cos\left(\frac{T^-_{pj} + T^+_{pj} - T^-_{qj} - T^+_{qj}}{8}\pi\right) - 1\right] \\ + \left[\sqrt{2}\cos\left(\frac{U^-_{pj} + U^+_{pj} - U^-_{qj} - U^+_{qj}}{8}\pi\right) - 1\right] \\ + \left[\sqrt{2}\cos\left(\frac{F^-_{pj} + F^+_{pj} - F^-_{qj} - F^+_{qj}}{8}\pi\right) - 1\right] \end{array} \right) + \left(\begin{array}{c} \left[\sqrt{2}\cos\left(\frac{t_{pj} - t_{qj}}{4}\pi\right) - 1\right] \\ + \left[\sqrt{2}\cos\left(\frac{u_{pj} - u_{qj}}{4}\pi\right) - 1\right] \\ + \left[\sqrt{2}\cos\left(\frac{f_{pj} - f_{qj}}{4}\pi\right) - 1\right] \end{array} \right) \right\} \right\}. \quad (6)$$

It is obvious that the three cosine measures $S_{wk}(P, Q)$ (k=1, 2, 3) also satisfy the following properties (S_1)–(S_3):

(S_1) $0 \leq S_{wk}(P, Q) \leq 1$;

(S_2) $S_{wk}(P, Q) = S_{wk}(Q, P)$;

(S_3) $S_{wk}(P, Q) = 1$ if $P = Q$, i.e., $<T_{pj}, U_{pj}, F_{pj}> = <T_{qj}, U_{qj}, F_{qj}>$ and $<t_{pj}, u_{pj}, f_{pj}> = <t_{qj}, u_{qj}, f_{qj}>$.

By similar proof ways, we can prove the properties (S_1)–(S_3) for $S_{wk}(P, Q)$ (k = 1, 2, 3). Their proofs are omitted here.

4. Decision-Making Method Using Cosine Measures

In this section, we propose an MADM method by using one of three cosine measures to solve decision-making problems with neutrosophic cubic information.

In an MADM problem, let $P = \{P_1, P_2, \ldots, P_m\}$ be a set of m alternatives and $R = \{R_1, R_2, \ldots, R_n\}$ be a set of n attributes. The evaluation value of an attribute R_j ($j = 1, 2, \ldots, n$) with respect to an alternative P_i (i = 1, 2, ... , m) is expressed by a NCN p_{ij} = (<T_{ij}, U_{ij}, F_{ij}>, <t_{ij}, u_j, f_{ij}>) (j = 1, 2, ... , n; i = 1, 2, ... , m), where $T_{ij}, U_{ij}, F_{ij} \subseteq [0,1]$ and $t_{ij}, u_{ij}, f_{ij} \in [0,1]$. Therefore, all the evaluation values expressed by NCNs can be constructed as the neutrosophic cubic decision matrix $P = (p_{ij})_{m\times n}$. Then, the weight vector of the attributes R_j (j = 1, 2, ... , n) is considered as $\boldsymbol{w}$ = (w_1, w_2, ... , w_n), satisfying $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. In this case, the proposed decision steps are described as follows:

Step 1: Establish an ideal solution (ideal alternative) $P^* = \{p_1^*, p_2^*, \ldots, p_n^*\}$ by the ideal NCN

$$p_j^* = \left(\left\langle \left[\max_i(T_{ij}^-), \max_i(T_{ij}^+)\right], \left[\min_i(U_{ij}^-), \min_i(U_{ij}^+)\right], \left[\min_i(F_{ij}^-), \min_i(F_{ij}^+)\right] \right\rangle, \left\langle \max_i(t_{ij}), \min_i(u_{ij}), \min_i(f_{ij}) \right\rangle\right)$$

corresponding to the benefit type of attributes and

$$p_j^* = \left(\left\langle \left[\min_i(T_{ij}^-), \min_i(T_{ij}^+)\right], \left[\max_i(U_{ij}^-), \max_i(U_{ij}^+)\right], \left[\max_i(F_{ij}^-), \max_i(F_{ij}^+)\right] \right\rangle, \left\langle \min_i(t_{ij}), \max_i(u_{ij}), \max_i(f_{ij}) \right\rangle\right)$$

corresponding to the cost type of attributes.

Step 2: Calculate the weighted cosine measure values between an alternative P_i (i = 1, 2, ... , m) and the ideal solution P^* by using Equation (4) or Equation (5) or Equation (6) and get the values of $S_{w1}(P_i, P^*)$ or $S_{w2}(P_i, P^*)$ or $S_{w3}(P_i, P^*)$ (i = 1, 2, ... , m).

Step 3: Rank the alternatives in descending order corresponding to the weighted cosine measure values and select the best one(s) according to the bigger value of $S_{w1}(P_i, P^*)$ or $S_{w2}(P_i, P^*)$ or $S_{w3}(P_i, P^*)$.

Step 4: End.

5. Illustrative Example and Comparison Analysis

In this section, an illustrative example of the selection problem of investment alternatives is provided in order to demonstrate the application of the proposed MADM method with neutrosophic cubic information.

5.1. Illustrative Example

An investment company wants to invest a sum of money for one of four potential alternatives: (a) P_1 is a textile company; (b) P_2 is an automobile company; (c) P_3 is a computer company; (d) P_4 is a software company. The evaluation requirements of the four alternatives are on the basis of three attributes: (a) R_1 is the risk; (b) R_2 is the growth; (c) R_3 is the environmental impact; where the attributes R_1 and R_2 are benefit types, and the attribute R_3 is a cost type. The weight vector of the three attributes is $\boldsymbol{w}$ = (0.32, 0.38, 0.3). When the expert or decision maker is requested to evaluate the four potential alternatives on the basis of the above three attributes using the form of NCNs. Thus, we can construct the following neutrosophic cubic decision matrix:

$$P = \begin{bmatrix} (\langle[0.5,0.6],[0.1,0.3],[0.2,0.4]\rangle,\langle 0.6,0.2,0.3\rangle) & (\langle[0.5,0.6],[0.1,0.3],[0.2,0.4]\rangle,\langle 0.6,0.2,0.3\rangle) & (\langle[0.6,0.8],[0.2,0.3],[0.1,0.2]\rangle,\langle 0.7,0.2,0.1\rangle) \\ (\langle[0.6,0.8],[0.1,0.2],[0.2,0.3]\rangle,\langle 0.7,0.1,0.2\rangle) & (\langle[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle,\langle 0.6,0.1,0.2\rangle) & (\langle[0.6,0.7],[0.3,0.4],[0.1,0.2]\rangle,\langle 0.7,0.4,0.1\rangle) \\ (\langle[0.4,0.6],[0.2,0.3],[0.1,0.3]\rangle,\langle 0.6,0.2,0.2\rangle) & (\langle[0.5,0.6],[0.2,0.3],[0.3,0.4]\rangle,\langle 0.6,0.3,0.4\rangle) & (\langle[0.5,0.7],[0.2,0.3],[0.3,0.4]\rangle,\langle 0.6,0.2,0.3\rangle) \\ (\langle[0.7,0.8],[0.1,0.2],[0.1,0.2]\rangle,\langle 0.8,0.1,0.2\rangle) & (\langle[0.6,0.7],[0.1,0.2],[0.1,0.3]\rangle,\langle 0.7,0.1,0.2\rangle) & (\langle[0.6,0.7],[0.3,0.4],[0.2,0.3]\rangle,\langle 0.7,0.3,0.2\rangle) \end{bmatrix}.$$

Hence, the proposed MADM method can be applied to this decision-making problem with NCSs by the following steps:

Firstly, corresponding to the benefit attributes R_1, R_2, and the cost attribute R_3, we establish an ideal solution (ideal alternative):

$$P^* = \{p_1^*, p_2^*, \ldots, p_n^*\} = \left\{ \begin{array}{c} (\langle[0.7,0.8],[0.1,0.2],[0.1,0.2]\rangle,\langle 0.8,0.1,0.2\rangle), \\ (\langle[0.6,0.7],[0.1,0.2],[0.1,0.3]\rangle,\langle 0.7,0.1,0.2\rangle), \\ (\langle[0.5,0.7],[0.3,0.4],[0.3,0.4]\rangle,\langle 0.6,0.4,0.3\rangle) \end{array} \right\}.$$

Then, we calculate the weighted cosine measure values between an alternative P_i (i = 1, 2, 3, 4) and the ideal solution P^* by using Equation (4) or Equation (5) or Equation (6), get the values of $S_{w1}(P_i, P^*)$ or $S_{w2}(P_i, P^*)$ or $S_{w3}(P_i, P^*)$ (i = 1, 2, 3, 4), and rank the four alternatives, which are shown in Table 1.

Table 1. All the cosine measure values between P_i and P^* and ranking orders of the four alternatives.

$S_{wk}(P_i, P^*)$	Cosine Measure Value	Ranking Order	The Best Alternative
$S_{w1}(P_i, P^*)$	0.9564, 0.9855, 0.9596, 0.9945	$P_4 > P_2 > P_3 > P_1$	P_4
$S_{w2}(P_i, P^*)$	0.9769, 0.9944, 0.9795, 0.9972	$P_4 > P_2 > P_3 > P_1$	P_4
$S_{w3}(P_{\mathrm{i}}, P^{*'})$	0.9892, 0.9959, 0.9897, 0.9989	$P_4 > P_2 > P_3 > P_1$	P_4

From the results of Table 1, we can see that all the ranking orders of the four alternatives and best choice return the same results corresponding to the three cosine measures in the decision-making problem with neutrosophic cubic information. It is obvious that P_4 is the best one.

5.2. Related Comparison

For relative comparison, we compare our decision-making method with the only existing related decision-making method based on the grey relational analysis under neutrosophic cubic environment [40]. Because the decision-making problem/method with CNS weights in [40] is different from ours, which has exact/crisp weights, we cannot compare them under different decision-making conditions. However, we only gave the comparison of decision-making complexity to show our simple method.

The proposed decision-making method based on the cosine measures of NCSs directly uses the cosine measures between an alternative P_i ($i = 1, 2, \ldots, m$) and the ideal alternative (ideal solution) P^* to rank all the alternatives; while the existing decision-making method with NCSs introduced in [40] firstly determines the Hamming distances of NCSs for weighted grey relational coefficients and standard (ideal) grey relational coefficients, and then derives the relative closeness coefficients in order to rank the alternatives. It is obvious that our decision-making method is simpler and easier than the existing decision-making method with NCSs introduced in [40]. But, our decision-making method can only deal with decision-making problems with exact/crisp weights, rather than NCS weights [40].

Compared with existing related decision-making methods with general neutrosophic sets (INSs or SVNSs) [17–39], the proposed decision-making method with NCSs contains much more evaluation information of attributes, which consists of both INSs and SVNSs; while the existing decision-making methods [17–39] contain either INS or SVNS information, which may lose some useful evaluation information of attributes in the decision-making process and affect the decision results, resulting in the distortion phenomenon. Furthermore, the existing decision-making methods [17–39] cannot deal with the decision-making problem with NCSs.

5.3. Sensitive Analysis

To show the sensitivities of these cosine measures on the decision results, we can only change the internal NCS of the alternative P_4 into the external NCS and reconstruct the following neutrosophic cubic decision matrix:

$$P' = \begin{bmatrix} (\langle[0.5,0.6],[0.1,0.3],[0.2,0.4]\rangle,\langle 0.6,0.2,0.3\rangle) & (\langle[0.5,0.6],[0.1,0.3],[0.2,0.4]\rangle,\langle 0.6,0.2,0.3\rangle) & (\langle[0.6,0.8],[0.2,0.3],[0.1,0.2]\rangle,\langle 0.7,0.2,0.1\rangle) \\ (\langle[0.6,0.8],[0.1,0.2],[0.2,0.3]\rangle,\langle 0.7,0.1,0.2\rangle) & (\langle[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle,\langle 0.6,0.1,0.2\rangle) & (\langle[0.6,0.7],[0.3,0.4],[0.1,0.2]\rangle,\langle 0.7,0.4,0.1\rangle) \\ (\langle[0.4,0.6],[0.2,0.3],[0.1,0.3]\rangle,\langle 0.6,0.2,0.2\rangle) & (\langle[0.5,0.6],[0.2,0.3],[0.3,0.4]\rangle,\langle 0.6,0.3,0.4\rangle) & (\langle[0.5,0.7],[0.2,0.3],[0.3,0.4]\rangle,\langle 0.6,0.2,0.3\rangle) \\ (\langle[0.7,0.8],[0.1,0.2],[0.1,0.2]\rangle,\langle 0.9,0.3,0.3\rangle) & (\langle[0.6,0.7],[0.1,0.2],[0.1,0.3]\rangle,\langle 0.8,0.3,0.4\rangle) & (\langle[0.6,0.7],[0.3,0.4],[0.2,0.3]\rangle,\langle 0.8,0.5,0.4\rangle) \end{bmatrix}.$$

Then, the corresponding ideal solution (ideal alternative) is changed into the following form:

$$P^* = \{p_1^*, p_2^*, \ldots, p_n^*\} = \left\{ \begin{array}{c} (\langle [0.7, 0.8], [0.1, 0.2], [0.1, 0.2] \rangle, \langle 0.9, 0.1, 0.2 \rangle), \\ (\langle [0.6, 0.7], [0.1, 0.2], [0.1, 0.3] \rangle, \langle 0.8, 0.1, 0.2 \rangle), \\ (\langle [0.5, 0.7], [0.3, 0.4], [0.3, 0.4] \rangle, \langle 0.6, 0.5, 0.4 \rangle) \end{array} \right\}.$$

According to the results of Table 2, both the cosine measure based on the included angle cosine of two vectors S_{w1} and the cosine measure based on cosine function S_{w3} still hold the same ranking orders; while the cosine measure based on distance S_{w2} shows another ranking form. In this case, S_{w2} is sensitive to the change of the evaluation values, since its ranking order changes with the change of the evaluation values for the alternative P_4.

Table 2. All the cosine measure values between $P_i{}'$ and $P^{*\prime}$ and ranking orders of the four alternatives.

$S_{wk}(P_i{}', P^{*\prime})$	Cosine Measure Value	Ranking Order	The Best Alternative
$S_{w1}(P_i{}', P^{*\prime})$	0.9451, 0.9794, 0.9524, 0.9846	$P_4 > P_2 > P_3 > P_1$	P_4
$S_{w2}(P_i{}', P^{*\prime})$	0.9700, 0.9906, 0.9732, 0.9877	$P_2 > P_4 > P_3 > P_1$	P_2
$S_{w3}(P_i{}', P^{*\prime})$	0.9867, 0.9942, 0.9877, 0.9968	$P_4 > P_2 > P_3 > P_1$	P_4

Nevertheless, this study provides a new and effective method for decision makers, due to the limited study on similarity measures and decision-making methods with NCSs in the existing literature. In this study, decision makers can select one of three cosine measures of NCSs to apply to MADM problems, according to their preferences and actual requirements.

6. Conclusions

This paper proposed three cosine measures of NCSs based on the included angle cosine of two vectors, distance, and cosine function, and discussed their properties. Then, we developed an MADM method with neutrosophic cubic information by using one of three cosine measures of NCSs. An illustrative example about the selection problem of investment alternatives was provided to demonstrate the applications of the proposed MADM method with neutrosophic cubic information.

The cosine measures-based MADM method developed in this paper is simpler and easier than the existing decision-making method with neutrosophic cubic information based on the grey related analysis, and shows the main advantage of its simple and easy decision-making process. However, this study can only deal with decision-making problems with exact/crisp weights, rather than NCS weights [40], which is its chief limitation. Therefore, the three cosine measures of NCSs that were developed, and their decision-making method are the main contributions of this paper. The developed MADM method provides a new and effective method for decision makers under neutrosophic cubic environments. In future work, we will further propose some new similarity measures of NCSs and their applications in other fields, such as image processing, medical diagnosis, and fault diagnosis.

Acknowledgments: This paper was supported by the National Natural Science Foundation of China (No. 71471172).

Author Contributions: Jun Ye proposed three cosine measures of NCSs and their decision-making method; Zhikang Lu provided the illustrative example and related comparison analysis; we wrote the paper together.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
2. Jun, Y.B.; Kim, C.S.; Yang, K.O. Cubic sets. *Ann. Fuzzy Math. Inf.* **2012**, *4*, 83–98.
3. Jun, Y.B.; Lee, K.J. Closed cubic ideals and cubic o-subalgebras in BCK/BCI-algebras. *Appl. Math. Sci.* **2010**, *4*, 3395–3402.

4. Jun, Y.B.; Kim, C.S.; Kang, M.S. Cubic subalgebras and ideals of BCK/BCI-algebras. *Far East J. Math. Sci.* **2010**, *44*, 239–250.
5. Jun, Y.B.; Kim, C.S.; Kang, J.G. Cubic q-ideals of BCI-algebras. *Ann. Fuzzy Math. Inf.* **2011**, *1*, 25–34.
6. Jun, Y.B.; Lee, K.J.; Kang, M.S. Cubic structures applied to ideals of BCI-algebras. *Comput. Math. Appl.* **2011**, *62*, 3334–3342. [CrossRef]
7. Zadeh, L.A. The concept of a linguistic variable and its application to approximate reasoning. Part 1. *Inf. Sci.* **1975**, *8*, 199–249. [CrossRef]
8. Atanassov, K. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
9. Atanassov, K.; Gargov, G. Interval valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *31*, 343–349. [CrossRef]
10. Smarandache, F. *Neutrosophy: Neutrosophic Probability, Set, and Logic*; American Research Press: Rehoboth, DE, USA, 1998.
11. Wang, H.; Smarandache, F.; Zhang, Y.Q.; Sunderraman, R. *Interval Neutrosophic Sets and Logic: Theory and Applications in Computing*; Hexis: Phoenix, AZ, USA, 2005.
12. Wang, H.; Smarandache, F.; Zhang, Y.Q.; Sunderraman, R. Single valued neutrosophic sets. *Multispace Multistruct.* **2010**, *4*, 410–413.
13. Ye, J. A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *J. Intell. Fuzzy Syst.* **2014**, *26*, 2459–2466.
14. Cheng, H.D.; Guo, Y. A new neutrosophic approach to image thresholding. *New Math. Nat. Comput.* **2008**, *4*, 291–308. [CrossRef]
15. Guo, Y.; Cheng, H.D. New neutrosophic approach to image segmentation. *Pattern Recognit.* **2009**, *42*, 587–595. [CrossRef]
16. Guo, Y.; Sengur, A.; Ye, J. A novel image thresholding algorithm based on neutrosophic similarity score. *Measurement* **2014**, *58*, 175–186. [CrossRef]
17. Ye, J. Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. *Int. J. Gen. Syst.* **2013**, *42*, 386–394. [CrossRef]
18. Ye, J. Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making. *J. Intell. Fuzzy Syst.* **2014**, *26*, 165–172.
19. Liu, P.D.; Chu, Y.C.; Li, Y.W.; Chen, Y.B. Some generalized neutrosophic number Hamacher aggregation operators and their application to group decision making. *J. Intell. Fuzzy Syst.* **2014**, *16*, 242–255.
20. Liu, P.D.; Wang, Y.M. Multiple attribute decision-making method based on single valued neutrosophic normalized weighted Bonferroni mean. *Neural Comput. Appl.* **2014**, *25*, 2001–2010. [CrossRef]
21. Şahin, R.; Küçük, A. Subsethood measure for single valued neutrosophic sets. *J. Intell. Fuzzy Syst.* **2015**, *29*, 525–530. [CrossRef]
22. Şahin, R. Cross-entropy measure on interval neutrosophic sets and its applications in multicriteria decision making. *Neural Comput. Appl.* **2017**, *28*, 1177–1187. [CrossRef]
23. Liu, P.D.; Tang, G.L. Some power generalized aggregation operators based on the interval neutrosophic numbers and their application to decision making. *J. Intell. Fuzzy Syst.* **2016**, *30*, 2517–2528. [CrossRef]
24. Liu, P.D.; Wang, Y.M. Interval neutrosophic prioritized OWA operator and its application to multiple attribute decision making. *J. Sci. Complex.* **2016**, *29*, 681–697. [CrossRef]
25. Liu, P.D. The aggregation operators based on Archimedean t-conorm and t-norm for the single valued neutrosophic numbers and their application to decision making. *Int. J. Fuzzy Syst.* **2016**, *18*, 849–863. [CrossRef]
26. Şahin, R.; Liu, P.D. Maximizing deviation method for neutrosophic multiple attribute decision making with incomplete weight information. *Neural Comput. Appl.* **2016**, *27*, 2017–2029. [CrossRef]
27. Şahin, R.; Liu, P.D. Possibility-induced simplified neutrosophic aggregation operators and their application to multicriteria group decision making. *J. Exp. Theor. Artif. Intell.* **2016**. [CrossRef]
28. Zavadskas, E.K.; Bausys, R.; Lazauskas, M. Sustainable assessment of alternative sites for the construction of a waste incineration plant by applying WASPAS method with single-valued neutrosophic set. *Sustainability* **2015**, *7*, 15923–15936. [CrossRef]
29. Stanujkic, D.; Zavadskas, E.K.; Smarandache, F.; Brauers, W.K.M.; Karabasevic, D. A neutrosophic extension of the MULTIMOORA method. *Informatica* **2017**, *28*, 181–192.

30. Pouresmaeil, H.; Shivanian, E.; Khorram, E.; Fathabadi, H.S. An extended method using TOPSIS and VIKOR for multiple attribute decision making with multiple decision makers and single valued neutrosophic numbers. *Adv. Appl. Stat.* **2017**, *50*, 261–292. [CrossRef]
31. Chen, J.Q.; Ye, J. Some single-valued neutrosophic Dombi weighted aggregation operators for multiple attribute decision-making. *Symmetry* **2017**, *9*, 82. [CrossRef]
32. Ye, J. Multiple attribute decision-making method using correlation coefficients of normal neutrosophic sets. *Symmetry* **2017**, *9*, 80. [CrossRef]
33. Ye, J. Single valued neutrosophic minimum spanning tree and its clustering method. *J. Intell. Syst.* **2014**, *23*, 311–324. [CrossRef]
34. Ye, J. Clustering methods using distance-based similarity measures of single-valued neutrosophic sets. *J. Intell. Syst.* **2014**, *23*, 379–389. [CrossRef]
35. Ye, J. Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses. *Artif. Intell. Med.* **2015**, *63*, 171–179. [CrossRef] [PubMed]
36. Ye, J.; Fu, J. Multi-period medical diagnosis method using a single valued neutrosophic similarity measure based on tangent function. *Comput. Methods Progr. Biomed.* **2016**, *123*, 142–149. [CrossRef] [PubMed]
37. Ye, J. Single valued neutrosophic similarity measures based on cotangent function and their application in the fault diagnosis of steam turbine. *Soft Comput.* **2017**, *21*, 817–825. [CrossRef]
38. Ali, M.; Deli, I.; Smarandache, F. The theory of neutrosophic cubic sets and their applications in pattern recognition. *J. Intell. Fuzzy Syst.* **2016**, *30*, 1957–1963. [CrossRef]
39. Jun, Y.B.; Smarandache, F.; Kim, C.S. Neutrosophic cubic sets. *New Math. Nat. Comput.* **2017**, *13*, 41–45. [CrossRef]
40. Banerjee, D.; Giri, B.C.; Pramanik, S.; Smarandache, F. GRA for multi attribute decision making in neutrosophic cubic set environment. *Neutrosophic Sets Syst.* **2017**, *15*, 64–73.