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The Interval Cognitive Network Process for Multi-Attribute Decision-Making

Xiuli Qi, Chengxiang Yin *, Kai Cheng and Xianglin Liao

Department of Simulation and Data Engineering in the College of Command Information System, PLA Army Engineering University, Nanjing 210007, China; qixiuli@189.cn (X.Q.); chengkai911@gmail.com (K.C.); liaoxianglin@189.cn (X.L.)

* Correspondence: 17721551707@189.cn; Tel.: +86-1595-1769-048

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Abstract: Aiming at combining the good characteristics of a differential scale in representing human cognition and the favorable properties of interval judgments in expressing decision-makers' uncertainty, this paper proposes the interval cognitive network process (I-CNP) to extend the primitive cognition network process (P-CNP) to handle interval judgments. The key points of I-CNP include a consistency definition for an interval pairwise opposite matrix (IPOM) and a method to derive interval utilities from an IPOM. This paper defines a feasible region-based consistency definition and a transitivity based consistency definition for an IPOM. Both of the two definitions are equivalent to the consistency definition for a crisp pairwise opposite matrix (POM) when an IPOM is reduced to a POM. Two methods that are able to derive interval utilities from both consistent and inconsistent IPOMs are developed based on the two definitions. Four numerical examples are used to illustrate the proposed methods and to compare I-CNP to interval analytic hierarchy process (IAHP). The results show that I-CNP reflects the decision-makers' cognition better, and that the suggestions provided by I-CNP are more convincing. I-CNP provides new useful alternative tools for multi-attribute decision-making problems.

Keywords: differential scale; ratio scale; interval cognitive network process; primitive cognitive network process; interval utilities; interval weights; multi-attribute decision making

1. Introduction

Multi-attribute decision-making (MADM) has grown as a part of operations research concerned with designing computational and mathematical tools for supporting the subjective evaluation of performance by decision-makers (DMs) [1]. In recent years, various methods, such as utility additive method (UTA) [2], robust ordinal regression (ROR) [3], analytic hierarchy process (AHP) [4], analytic network process (ANP) [5], primitive cognitive network process (P-CNP) [6], step-wise weight assessment ratio analysis (SWARA) [7,8], weighted aggregated sum product assessment (WASPAS) [9], factor relationship (FARE) method [10], the visekriterijumska optimizacija i kompromisno resenje (VIKOR) method [11], the characteristic objects method (COMET) [12], and COMET with hesitant fuzzy sets (HFS COMET) [13,14] have been proposed to handle MADM problems. Among these methods, AHP has been increasingly studied by plenty of researchers and applied in various applications due to its simplicity and practicability [15–18].

AHP derives priority weights from a pairwise reciprocal matrix, which is also called a multiplicative preference relation (MPR). With the increase of the complexity of decision-making problems, classical AHP has been extended in many aspects (scales used to measure the results of pairwise comparisons [6,19,20], the styles in which the pairwise comparisons are carried out [20–23], uncertainty concerns [24–26], etc.). Of all the extensions mentioned above, those extensions that can handle uncertainty and vagueness draw the most research attention.

Saaty and Vargas [24] proposed to use interval numbers to capture the vagueness and uncertainty in MADM problems, and adopted the Monte Carlo method to derive interval weights from the interval multiplicative preference relation (IMPR). Since then, interval judgments in pairwise comparisons have been natural and important tools to model uncertainty, and plenty of researchers have devoted effort to study consistency issues and weight-driving methods for interval preference relations. The commonly used interval fuzzy preference relations include IMPR and interval fuzzy preference relations (IFPR) [27].

The earlier research mainly focused on IMPR. Arbel [28] interpreted interval judgments as constraints on a weights space, and proposed to derive interval weights by a linear programming model. Kress [29] pointed out that the feasible region of Arbel's method was empty sometimes. Islam et al. [30] proposed a lexicographical goal-programming method to obtain crisp weights from the IMPR. Wang [31] showed that the weights and rankings obtained from upper and lower triangular judgments using Islam's method are different. Wang et al. [32] defined the consistency of an IMPR based on a feasible region restricted by interval judgments. They developed an eigenvector method-based nonlinear programming model to derive interval weights from inconsistent IMPRs. Sugihara et al. [33] put forward an interval regression method containing lower and upper approximate models. However, Guo et al. [34] showed that the lower approximate model is infeasible when the IMPR is inconsistent. Guo et al. [34] brought out a method to revise inconsistent IMPRs to obtain dual interval weights. Wang et al. [35] proposed a two-stage logarithmic goal-programming model, which firstly minimizes inconsistency then generates weights under the condition of minimal inconsistency for IMPRs. Wang et al. [36] designed a goal-programming model which was able to derive all of the interval weights by solving only one programming model. Liu [37] introduced consistency and acceptable consistency of IMPRs based on two converted crisp MPRs. Li et al. [38] showed that Liu's method is not robust to the permutations of decision makes'(DMs') judgments. Lan et al. [39] proposed to use an information mining method, which firstly generates a series of crisp weight vectors and then derives the final result by linear combination. The weight used in the linear combination is determined based on a deviation degree. Conde et al. [40] introduced a linear optimization problem to define a consistency index and to derive an interval weight vector for inconsistent IMPRs. Wang [41] defined geometric consistency for an IMPR and developed a two-stage goal-programming model to estimate the missing value for incomplete IMPRs. Wang [42] proposed an uncertainty index-based consistency measurement and a weight generation method with interval probabilities. Dong et al. [43] defined the consistency index of an IMPR based on logarithmic Manhattan distance, and developed a linear programming model to compute the consistency index. They also designed models to improve consistency and to derive weights. Zhang [44] developed a logarithmic least square method based on a parameterized transformation formula which converts a normalized interval weight vector into a consistent IMPR to derive interval weights. Meng et al. [45,46] proposed a new formulation of consistency for an IMPR that is based on the definition of a quasi-IMPR, and they analyzed the relationship between their definition, Liu's definition [37], and Wang et al.'s definition [32]. Based on the analysis of existing definitions of consistency, Krejčí [47] proposed a new definition which was invariant to the permutation of objects for an IMPR.

Tanino [48] defined additive consistency and multiplicative consistency for a fuzzy preference relation (FPR). Accordingly, the studies on IFPRs mainly focus on additive consistency and multiplicative consistency and the corresponding weight-deriving methods. Xu et al. [49] proposed additive consistency and multiplicative consistency for IFPR based on a feasible region restricted by interval judgments in an IFPR. An IFPR is additively or multiplicatively consistent if the corresponding feasible region is not empty. Xu et al. [50] defined an additive transitivity based consistency, which is generalized from a characterization of additive consistency of an FPR for an IFPR, but Wang [51] pointed out that this definition was highly dependent on alternative labels and not robust to the permutations of the DMs' judgments. Wang et al. [52] defined another additive transitivity based consistency and a multiplicative transitivity based consistency for IFPRs, and proposed goal-programming models to obtain interval priority weights. Wang et al. [53] introduced geometric transitivity based consistency for an IFPR and proved that geometric consistency is equivalent to the definition of multiplicative consistency in [52]. They also proposed a goal-programming model which minimizes the deviation between the logarithm of the ratio of the original judgments and the logarithm of the ratio of the consistent ones converted from a parametric transformation formula to derive interval weights. Liu et al. [54] transformed an IFPR into an IMPR and used the method in [37] to check for consistency. However, Li et al. [38] pointed out that the definition in [37] was technically deficient and yielded contradictory results for the same judgment matrix after the alternatives are re-labeled. Wang et al. [55] put forward a new method with a parameter to obtain priority weights from an FPR, and defined a new definition for additive consistency in an IFPR. Based on the new definition, linear programming models for deriving interval priority weights from both a consistent and an inconsistent IFPR were proposed. Dong et al. [56] defined an average-case consistency index as the average consistency degree of all FPRs associated with an IFPR. Zhang et al. [57] developed a goal-programming model according to the multiplicative consistency property to derive interval weights from an IFPR. Krejčí [58] reviewed and analyzed the definitions of additive consistency for IFPRs, and proposed a new additive consistency definition and additive weak consistency for IFPRs. Wan et al. [59] defined a geometric consistency index for an IFPR based on the max-consistency index and the min-consistency index, and proposed a goal-programming model to obtain an acceptable geometrically consistent IFPR with a fuzzy logarithmic programming model to drive interval weights.

Yuen [6,60–62] indicated that the basic ratio scale in an MPR inappropriately represents the human cognition of paired difference. Taking the comparison of two persons' height for example, if A is 1.79 m and B is 1.80 m, then we may say that B is slightly taller than A. However, if a ratio scale is applied, the comparison result will be 2, i.e., B is two times taller than A, which is obviously unreasonable. The exaggeration of difference may mislead the DM. Yuen [6,60–62] proposed the primitive cognitive network process (P-CNP) using a paired differential scale to replace the ratio scale. The numerical examples in [6,61] indicated that P-CNP performed better than AHP. Yuen [19] extended P-CNP to handle uncertainty and proposed the fuzzy cognitive network process (F-CNP). Zhang et al. [20] combined the ideas of P-CNP and the best worst method (BWM) [21,63] and proposed the cognitive best worst method (CBWM).

It is worth noting that the inappropriate definition of ratio scale for an MPR will also be inappropriate in an IMPR. Meanwhile, it will be difficult for DMs to provide crisp and consistent judgments for P-CNP due to a lack of information or the complexity of the MADM problem. Considering the good character of a differential scale and the wide use of interval judgments to capture uncertainty, it is natural to replace a ratio scale with a differential scale in an IMPR, or in other words to extend P-CNP to handle interval judgments. To the best of our knowledge, such a work has not been done. In this paper, we extend P-CNP to handle interval judgments and propose the interval cognitive network process (I-CNP). We define two definitions for an interval pairwise opposite matrix (IPOM) according to the existing consistency definition for an IMPR and an IFPR, and introduce some interesting properties related to the definitions. Based on the two definitions, a feasible region-based method and a transitivity based method are developed to derive interval utilities from an OIPM. We use four numerical examples to illustrate the proposed method and to show the difference between I-CNP and interval AHP.

The rest of the paper is organized as follows. The P-CNP method and some basic concepts are briefly reviewed in Section 2; the I-CNP method, including consistency issues and utility driving models, is investigated in Section 3; I-CNP is used for four numerical examples in Section 4; in Section 5, the differences between I-CNP and interval AHP are discussed; and the paper is concluded in Section 6.

2. Preliminaries

In this section, we first review the basic idea of P-CNP then introduce some basic concepts that will be needed to construct our method.

2.1. Primitive Cognitive Network Process

The P-CNP is a cognitive architecture which comprises the following cognitive decision processes: problem cognition process (PCP), cognitive assessment process (CAP), cognitive prioritization process (CPP), multiple information fusion process (MIP), and decision volition process (DVP).

In PCP, the DM determines the criteria $C = \{c_1, c_2, \cdots, c_n\}$ and the alternatives $X = \{x_1, x_2, \cdots, x_m\}$. Besides that, a measurement scale schema $(\aleph, \overline{X})$, which maps the verbal scales $\aleph$ to the corresponding numerical scales $\overline{X}$, should also be determined. As P-CNP adopts differential scales to measure pairwise differences, the numerical representations should be in the following form [6]:

$$\overline{X} = \{\alpha_i = i\kappa/\tau | \forall i \in \{-\tau, \cdots, 0, 1, \cdots, \tau\}, \kappa > 0\}. \tag{1}$$

κ is the normal utility, which is the mean of the individual utility of the objects involved in the pairwise comparisons. $2\tau + 1$ is the number of measurements in the scale schema. From the definition of $\overline{X}$, we can obtain $\max(\overline{X}) = \kappa$. For the convenience of computation, we set $\kappa = \tau$; then, each element in $\overline{X}$ will be an integer. Table 1 [6] gives an example of the ratio scale [4] and the interval scale [6] $(\tau = \kappa = 8)$.

Table 1. Scale schemas: ratio scales and differential scales.

i	Verbal Scales	Ratio Scales	Differential Scales
0	Equally	1	0
1	Weakly	2	1
2	Moderately	3	2
3	Moderately plus	4	3
4	Strongly	5	4
5	Strongly plus	6	5
6	Very Strongly	7	6
7	Very, very strongly	8	7
8	Extremely	9	8
$\{-i\}$	Reciprocals/opposites of above	(from 1/9 to 1)	(from -8 to 0)

In CAP, the pairwise comparisons are executed and a pairwise opposite matrix (POM) is constructed.

Definition 1. *A pairwise opposite matrix (POM) [6] on a set of n objects $x_1, x_2, \cdots, x_n$ is a complementary matrix $B = (b_{ij})_{n\times n}$ satisfying the following condition:*

$$-\kappa \le b_{ij} \le \kappa, b_{ii} = 0,\ b_{ij} + b_{ji} = 0,\ \text{for all } i, j = 1, 2, \cdots, n. \tag{2}$$

b_{ij} represents the preference intensity of object x_i over object x_j, and κ is the normal utility. In P-CNP, the value of b_{ij} is measured using differential scales. Let an ideal utility vector of the objects involved in the pairwise comparisons be $V = (v_1, v_2, \cdots, v_n)$, $v_i \ge 0$, and $\sum_{i=1}^{n} v_i = n\kappa$, then the ideal POM will be $B = (b_{ij})_{n\times n}$ where $b_{ij} = v_i - v_j$.

Definition 2. *A POM $B = (b_{ij})_{n\times n}$ is accordant if it satisfies the following condition [6]:*

$$b_{ik} + b_{kj} = b_{ij},\ \text{for all} i, j, k = 1, 2, \cdots, n. \tag{3}$$

The accordance index (AI) is used to measure the degree of accordance:

$$AI = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{\kappa}(b_{ik} + b_{kj} - b_{ij})\right)^2}. \tag{4}$$

If $AI = 0$, then B is perfectly accordant; if $0 < AI \leq 0.1$, then B is satisfactory; and if $AI > 0.1$, B is unsatisfactory.

The purpose of CPP is to derive a utility vector from the POM. Yuen [6] proposed the primitive least squares (PLS) method and least penalty squares (LPS) method.

The PLS method derives the utility vector through the following model:

$$\begin{aligned}
&\min \Delta = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (b_{ij} - v_i - v_j)^2 \\
&\text{s.t.} \\
&\sum_{i=1}^{n} v_i = n\kappa, \\
&v_i \geq 0, i = 1, 2, \cdots n.
\end{aligned} \tag{5}$$

Yuen [6] pointed out that the closed form solution of model (5) is:

$$v_i = \kappa + \frac{1}{n} \sum_{j=1}^{n} b_{ij}, \ \forall i = 1, 2, \cdots, n. \tag{6}$$

LPS is the weighted version of PLS, in which the weight is the penalty factor β. The utility vector can be derived by the following model:

$$\begin{aligned}
&\min \overset{\frown}{\Delta} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \beta_{ij} (b_{ij} - v_i - v_j)^2 \\
&\text{s.t.} \\
&\sum_{i=1}^{n} v_i = n\kappa, \\
&v_i \geq 0, i = 1, 2, \cdots n, \\
&\beta_{ij} = \begin{cases} \beta_1, (v_i > v_j \,\&\, b_{ij} > 0) or (v_i < v_j \,\&\, b_{ij} < 0), \\ \beta_2, (v_i = v_j \,\&\, b_{ij} \neq 0) or (v_i \neq v_j \,\&\, b_{ij} = 0), \\ \beta_3, \text{otherwise}, \end{cases} \\
&1 = \beta_1 \leq \beta_2 \leq \beta_3.
\end{aligned} \tag{7}$$

After obtaining the utility vector $V = (v_1, v_2, \cdots, v_n)$, the normalized weight vector $W = (w_1, w_2, \cdots, w_n)$ can be obtained by $w_i = \frac{v_i}{n\kappa}$, $i = 1, 2, \cdots, n$.

MIP is the same as the aggregation process traditionally mentioned in MADM, and DVP is the decision process based on the aggregated global utility of each alternative.

2.2. Basic Concepts

Definition 3. *A multiplicative preference relation (MPR) [27] on a set of n objects $x_1, x_2, \cdots, x_n$ is a reciprocal matrix $A = (a_{ij})_{n \times n}$ satisfying the following condition:*

$$a_{ij} > 0, a_{ii} = 1, \ a_{ij} a_{ji} = 1, \ \text{for all } i, j = 1, 2, \cdots, n. \tag{8}$$

a_{ij} represents the preference intensity of object x_i over object x_j. In AHP, the value of a_{ij} is measured by the '1–9' ratio scale.

An MPR $A = (a_{ij})_{n \times n}$ is consistent if multiplicative transitivity is satisfied:

$$a_{ij} = a_{ik} a_{kj}, \text{for all } i, j, k = 1, 2, \cdots, n. \tag{9}$$

Definition 4. *An interval multiplicative preference relation (IMPR) [27] on a set of n objects* $x_1, x_2, \cdots, x_n$ *is an interval matrix* $\widetilde{A} = (\widetilde{a}_{ij})_{n\times n}$, $\widetilde{a}_{ij} = [a_{ij}^-, a_{ij}^+]$ *satisfying the following condition:*

$$0 < a_{ij}^- \leq a_{ij}^+, a_{ii}^- = a_{ii}^+ = 1,\ a_{ij}^- a_{ji}^+ = a_{ij}^+ a_{ji}^- = 1,\ \text{for all } i, j = 1, 2, \cdots, n. \tag{10}$$

As discussed in Section 1, there has been plenty of research focusing on IMPRs. Here, we give the two mostly used consistency definitions for IMPRs.

Definition 5. *An IMPR* $\widetilde{A} = (\widetilde{a}_{ij})_{n\times n}$, $\widetilde{a}_{ij} = [a_{ij}^-, a_{ij}^+]$ *is consistent if the convex feasible region* S_W *is not empty [32].*

$$S_W = \left\{ W = (w_1, w_2, \cdots, w_n) \;\middle|\; \begin{array}{l} a_{ij}^- \leq \frac{w_i}{w_j} \leq a_{ij}^+,\ i, j = 1, 2, \cdots, n \\ w_i \geq 0,\ i = 1, 2, \cdots, n \\ \sum\limits_{i=1}^{n} w_i = 1 \end{array} \right\}. \tag{11}$$

Definition 6. *An IMPR* $\widetilde{A} = (\widetilde{a}_{ij})_{n\times n}$, $\widetilde{a}_{ij} = [a_{ij}^-, a_{ij}^+]$ *is (geometrically) consistent if the following transitivity condition is satisfied [41]:*

$$a_{ij}^- a_{ij}^+ = a_{ik}^- a_{ik}^+ a_{kj}^- a_{kj}^+,\ i, j, k = 1, 2, \cdots, n. \tag{12}$$

Definition 7. *A normalized interval weight vector [41] is defined as* $\widetilde{W} = (\widetilde{w}_1, \widetilde{w}_2, \cdots, \widetilde{w}_n)$, $\widetilde{w}_i = [w_i^-, w_i^+]$ *where for all* $i = 1, 2, \cdots, n$,

$$0 \leq w_i^- \leq w_i^+ \leq 1,\ \sum_{j=1, j\neq i}^{n} w_j^- + w_i^+ \leq 1,\ \sum_{j=1, j\neq i}^{n} w_j^U + w_i^L \geq 1. \tag{13}$$

3. Interval Cognitive Network Process

The basic steps of the interval cognitive network process (I-CNP) are the same as P-CNP. The main differences lie in the consistency definition and the method used to derive utility vectors. So, we mainly illustrate the consistency issues for an interval POM and design models to derive an interval utility vector from an interval POM.

Definition 8. *An interval pairwise opposite matrix (IPOM) on a set of n objects* $x_1, x_2, \cdots, x_n$ *is an interval matrix* $\widetilde{B} = (\widetilde{b}_{ij})_{n\times n}$, $\widetilde{b}_{ij} = [b_{ij}^-, b_{ij}^+]$ *satisfying the following condition:*

$$-\kappa \leq b_{ij}^- \leq b_{ji}^+ \leq \kappa, b_{ii}^- = b_{ii}^+ = 0,\ b_{ij}^- + b_{ji}^+ = b_{ij}^+ + b_{ji}^- = 0,\ \text{for all } i, j = 1, 2, \cdots, n. \tag{14}$$

Analogous to Definition 7, we give the definition of normalized interval utility vector.

Definition 9. *A normalized interval utility vector is defined as* $\widetilde{V} = (\widetilde{v}_1, \widetilde{v}_2, \cdots, \widetilde{v}_n)$, $\widetilde{v}_i = [v_i^-, v_i^+]$, *where for all* $i = 1, 2, \cdots, n$, *the following conditions are satisfied:*

$$0 \leq v_i^- \leq v_i^+, v_i^+ + \sum_{j=1, j\neq i}^{n} v_j^- \leq n\kappa,\ v_i^- + \sum_{j=1, j\neq i}^{n} v_j^+ \geq n\kappa. \tag{15}$$

3.1. Consistency Issues

Inspired by previous research on IMPRs and IFPRs, we introduce two kinds of definitions for an IPOM.

3.1.1. Feasible Region-Based Consistency

Definition 10. *An IPOM* $\widetilde{B} = (\widetilde{b}_{ij})_{n\times n}$, $\widetilde{b}_{ij} = [b_{ij}^-, b_{ij}^+]$ *is consistent if the convex feasible region* S_V *is not empty.*

$$S_V = \left\{ V = (v_1, v_2, \cdots, v_n) \middle| \begin{array}{l} b_{ij}^- \le v_i - v_j \le b_{ij}^+,\ i,j = 1,2,\cdots,n \\ \sum\limits_{i=1}^{n} v_i = n\kappa \\ v_i \ge 0,\ i = 1,2,\cdots,n \end{array} \right\} \tag{16}$$

Theorem 1. *An IPOM* $\widetilde{B} = (\widetilde{b}_{ij})_{n\times n}$, $\widetilde{b}_{ij} = [b_{ij}^-, b_{ij}^+]$ *is consistent if and only if it satisfies the following inequality constraints:*

$$\max_k (b_{ik}^- + b_{kj}^-) \le \min_k (b_{ik}^+ + b_{kj}^+),\ i,j,k = 1,2,\cdots,n. \tag{17}$$

Proof. If $\widetilde{B}$ is consistent, then S_V is not empty, i.e., there is no contradiction among the following inequality constraints:

$$b_{ik}^- \le v_i - v_k \le b_{ik}^+,\ i,k = 1,2,\cdots,n, \tag{18}$$

$$b_{kj}^- \le v_k - v_j \le b_{kj}^+,\ i,k = 1,2,\cdots,n. \tag{19}$$

Adding (18) and (19) leads to the following implied indirect inequalities:

$$b_{ik}^- + b_{kj}^- \le v_i - v_j \le b_{ik}^+ + b_{kj}^+,\ i,j,k = 1,2,\cdots,n. \tag{20}$$

Since (20) holds for any $k = 1,2,\cdots,n$, it follows that $\max\limits_k (b_{ik}^- + b_{kj}^-) \le \min\limits_k (b_{ik}^+ + b_{kj}^+)$ holds for all $i,j,k = 1,2,\cdots,n$. Conversely, if (17) holds, then $b_{ij}^- \le v_i - v_j \le b_{ij}^+$ holds for all $i,j = 1,2,\cdots,n$, i.e., S_V is not empty and $\widetilde{B}$ is consistent according to Definition 10. This completes the proof. □

Theorem 1 can be used to check the consistency of an IPOM without solving any programming model.

Theorem 2. *If* $\widetilde{B} = (\widetilde{b}_{ij})_{n\times n}$, $\widetilde{b}_{ij} = [b_{ij}^-, b_{ij}^+]$ *is reduced to a POM* $B = (b_{ij})_{n\times n}$, *Definition 10 implies Definition 2.*

Proof. When $\widetilde{B} = (\widetilde{b}_{ij})_{n\times n}$ is reduced to $B = (b_{ij})_{n\times n}$, the feasible region S_V becomes:

$$S_V' = \left\{ V = (v_1, v_2, \cdots, v_n) \middle| \begin{array}{l} v_i - v_j = b_{ij},\ i,j = 1,2,\cdots,n \\ \sum\limits_{i=1}^{n} v_i = n\kappa \\ v_i \ge 0,\ i = 1,2,\cdots,n \end{array} \right\}. \tag{21}$$

If S_V' is not empty, $b_{ik} = v_i - v_k$, $b_{kj} = v_k - v_j$, and $b_{ij} = v_i - v_j$ hold for all $i,j,k = 1,2,\cdots,n$, then $b_{ik} + b_{kj} = b_{ij}$ holds for all $i,j,k = 1,2,\cdots,n$. This completes the proof. □

It is worth noting that if the value of κ is determined properly, Definition 2 and Definition 10 will be equivalent.

3.1.2. Transitivity Based Consistency

Definition 11. *An IPOM* $\widetilde{B} = (\widetilde{b}_{ij})_{n\times n}$, $\widetilde{b}_{ij} = [b_{ij}^-, b_{ij}^+]$ *is consistent if the following additive transitivity is satisfied:*

$$\widetilde{b}_{ij} + \widetilde{b}_{jk} + \widetilde{b}_{ki} = \widetilde{b}_{kj} + \widetilde{b}_{ji} + \widetilde{b}_{ik},\ i, j, k = 1, 2, \cdots, n. \tag{22}$$

Theorem 3. *If* $\widetilde{B} = (\widetilde{b}_{ij})_{n\times n}$, $\widetilde{b}_{ij} = [b_{ij}^-, b_{ij}^+]$ *is reduced to a POM* $B = (b_{ij})_{n\times n}$, *Definition 11 is equivalent to Definition 2.*

Proof. When all of the interval elements in $\widetilde{B} = (\widetilde{b}_{ij})_{n\times n}$ are reduced to a crisp number, (22) will become $b_{ij} + b_{jk} + b_{ki} = b_{kj} + b_{ji} + b_{ik}$, $i, j, k = 1, 2, \cdots, n$. From the definition of a POM, we can get $b_{ij} = -b_{ji}$, $b_{jk} = -b_{kj}$, and $b_{ki} = -b_{ik}$; then, $b_{ij} + b_{jk} + b_{ki} = -(b_{ij} + b_{jk} + b_{ki})$, $i, j, k = 1, 2, \cdots, n$. So, it is obvious that $b_{ij} + b_{jk} + b_{ki} = 0$ and $b_{ij} + b_{jk} = b_{ik}$, $i, j, k = 1, 2, \cdots, n$. It is obvious that the deduction above is reversible. This completes the proof. □

Let $\widetilde{V} = (\widetilde{v}_1, \widetilde{v}_2, \cdots, \widetilde{v}_n)$, $\widetilde{v}_i = [v_i^-, v_i^+]$ be a normalized interval utility vector. We construct an interval matrix $\widetilde{P} = (\widetilde{p}_{ij})_{n\times n}$ as follows:

$$\widetilde{p}_{ij} = [p_{ij}^-, p_{ij}^+] = \begin{cases} [0,0], & i = j, \\ [v_i^- - v_j^+, v_i^+ - v_j^-], & i \neq j. \end{cases} \tag{23}$$

Theorem 4. *The interval matrix* $\widetilde{P} = (\widetilde{p}_{ij})_{n\times n}$ *is a consistent IPOM according to Definition 11.*

Proof. We first show that $\widetilde{P}$ is an IPOM. As $\widetilde{V}$ is a normalized interval utility vector, it is obvious that $v_i^- - v_j^+ \leq v_i^+ - v_j^-$. Then, for all $i, j = 1, 2, \cdots, n$, $p_{ij}^- + p_{ji}^+ = v_i^- - v_j^+ + v_j^+ - v_i^- = 0$, $p_{ij}^+ + p_{ji}^- = v_i^+ - v_j^- + v_j^- - v_i^+ = 0$. According to Definition 8, we can say that $\widetilde{P}$ is an IPOM. Now, we show that $\widetilde{P}$ is a consistent IPOM by the following deductions. For all $i, j, k = 1, 2, \cdots, n$,

$$\begin{aligned} &\widetilde{p}_{ij} + \widetilde{p}_{jk} + \widetilde{p}_{ki} \\ &= [(v_i^- - v_j^+) + (v_j^- - v_k^+) + (v_k^- - v_i^+), (v_i^+ - v_j^-) + (v_j^+ - v_k^-) + (v_k^+ - v_i^-)] \\ &= [(v_k^- - v_j^+) + (v_j^- - v_i^+) + (v_i^- - v_k^+), (v_k^+ - v_j^-) + (v_j^+ - v_i^-) + (v_i^+ - v_k^-)] \\ &= \widetilde{p}_{kj} + \widetilde{p}_{ji} + \widetilde{p}_{ik}. \end{aligned} \tag{24}$$

This completes the proof. □

Corollary 1. Let $\widetilde{B} = (\widetilde{b}_{ij})_{n\times n}$, $\widetilde{b}_{ij} = [b_{ij}^-, b_{ij}^+]$ be an IPOM. If there exists a normalized interval utility vector $\widetilde{V} = (\widetilde{v}_1, \widetilde{v}_2, \cdots, \widetilde{v}_n)$, $\widetilde{v}_i = [v_i^-, v_i^+]$ such that:

$$\widetilde{b}_{ij} = [b_{ij}^-, b_{ij}^+] = \begin{cases} [0,0], & i = j, \\ [v_i^- - v_j^+, v_i^+ - v_j^-], & i \neq j. \end{cases} \tag{25}$$

where $\widetilde{V}$ satisfies (15), then $\widetilde{B}$ is an consistent IPOM.

From Corollary 1, it is obvious that if an IPOM is consistent according to Definition 11, then it must be consistent according to Definition 10.

3.2. Deriving an Interval Utility Vector from an IPOM

We develop two utility driving methods based on the two consistency definitions in Section 3.1.

3.2.1. Deriving an Interval Utility Vector Based on a Feasible Region-Based Definition

If an IPOM $\widetilde{B} = (\widetilde{b}_{ij})_{n\times n}$, $\widetilde{b}_{ij} = [b_{ij}^{-}, b_{ij}^{+}]$ is consistent according to Definition 10, then the interval utility vector $\widetilde{V} = (\widetilde{v}_1, \widetilde{v}_2, \cdots, \widetilde{v}_n)$, $\widetilde{v}_i = [v_i^{-}, v_i^{+}]$ can be obtained by the following two programming models:

$$v_i^{-} = \min v_i \text{ s.t. } S_V,\ i = 1, 2, \cdots, n. \tag{26}$$

$$v_i^{+} = \max v_i \text{ s.t. } S_V,\ i = 1, 2, \cdots, n. \tag{27}$$

S_V is the feasible region defined in (16).

As for all $i, j = 1, 2, \cdots, n$, $v_j - v_i \geq v_{ji}^{-} \Leftrightarrow v_i - v_j \leq v_{ij}^{+}$, $v_j - v_i \leq v_{ji}^{+} \Leftrightarrow v_i - v_j \geq v_{ij}^{-}$, (26) and (27) can be simplified using the upper triangular elements in $\widetilde{B}$, as follows:

$$\begin{array}{l} v_i^{-} = \min v_i \\ \text{s.t.} \\ v_i - v_j \geq b_{ij}^{-},\ i = 1, \cdots, n-1,\ j = i+1, \cdots, n, \\ v_i - v_j \leq b_{ij}^{+},\ i = 1, \cdots, n-1,\ j = i+1, \cdots, n, \\ \sum\limits_{i=1}^{n} v_i = n\kappa, \\ v_i \geq 0,\ i = 1, 2, \cdots, n. \end{array} \cdot \tag{28}$$

$$\begin{array}{l} v_i^{+} = \max v_i \\ \text{s.t.} \\ v_i - v_j \geq b_{ij}^{-},\ i = 1, \cdots, n-1,\ j = i+1, \cdots, n, \\ v_i - v_j \leq b_{ij}^{+},\ i = 1, \cdots, n-1,\ j = i+1, \cdots, n, \\ \sum\limits_{i=1}^{n} v_i = n\kappa, \\ v_i \geq 0,\ i = 1, 2, \cdots, n. \end{array} \cdot \tag{29}$$

When $\widetilde{B}$ is inconsistent, i.e., the feasible region S_V is empty, we cannot obtain the utility vector using (28) and (29). To solve this issue, we relax the constraints in (28) and (29) by introducing the non-negative deviation variables p_{ij} and q_{ij}, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$, such that:

$$b_{ij}^{-} - p_{ij} \leq v_i - v_j \leq b_{ij}^{+} + q_{ij},\ i = 1, \cdots, n-1,\ j = i+1, \cdots, n. \tag{30}$$

It is obvious that the smaller the deviation variables p_{ij} and q_{ij}, the closer $\widetilde{B}$ is to a consistent IPOM. So, we can obtain the following linear programming model:

$$\begin{array}{l} \min J_1 = \sum\limits_{i=1}^{n-1} \sum\limits_{j=i+1}^{n} (p_{ij} + q_{ij}) \\ \text{s.t.} \\ v_i - v_j \geq b_{ij}^{-} - p_{ij},\ i = 1, \cdots, n-1,\ j = i+1, \cdots, n, \\ v_i - v_j \leq b_{ij}^{+} + q_{ij},\ i = 1, \cdots, n-1,\ j = i+1, \cdots, n, \\ p_{ij}, q_{ij} \geq 0,\ i = 1, \cdots, n-1,\ j = i+1, \cdots, n, \\ \sum\limits_{i=1}^{n} v_i = n\kappa, \\ v_i \geq 0,\ i = 1, 2, \cdots, n. \end{array} \cdot \tag{31}$$

If $J_1^* = \min J_1 = 0$, $\widetilde{B}$ is a consistent IPOM. Otherwise, the interval utility vector can be obtained by the following two models:

$$\begin{array}{l} v_i^- = \min v_i \\ \text{s.t.} \\ v_i - v_j \geq b_{ij}^- - p_{ij},\ i = 1, \cdots, n-1,\ j = i+1, \cdots, n, \\ v_i - v_j \leq b_{ij}^+ + q_{ij},\ i = 1, \cdots, n-1,\ j = i+1, \cdots, n, \\ p_{ij}, q_{ij} \geq 0,\ i = 1, \cdots, n-1,\ j = i+1, \cdots, n, \\ \sum\limits_{i=1}^{n} v_i = n\kappa, \\ v_i \geq 0,\ i = 1, 2, \cdots, n, \\ \sum\limits_{i=1}^{n-1} \sum\limits_{j=i+1}^{n} (p_{ij} + q_{ij}) \leq J_1^*. \end{array} . \tag{32}$$

$$\begin{array}{l} v_i^+ = \max v_i \\ \text{s.t.} \\ v_i - v_j \geq b_{ij}^- - p_{ij},\ i = 1, \cdots, n-1,\ j = i+1, \cdots, n, \\ v_i - v_j \leq b_{ij}^+ + q_{ij},\ i = 1, \cdots, n-1,\ j = i+1, \cdots, n, \\ p_{ij}, q_{ij} \geq 0,\ i = 1, \cdots, n-1,\ j = i+1, \cdots, n, \\ \sum\limits_{i=1}^{n} v_i = n\kappa, \\ v_i \geq 0,\ i = 1, 2, \cdots, n, \\ \sum\limits_{i=1}^{n-1} \sum\limits_{j=i+1}^{n} (p_{ij} + q_{ij}) \leq J_1^*. \end{array} . \tag{33}$$

The value of J_1^* can be used to measure the consistency degree of an IPOM.

3.2.2. Deriving an Interval Utility Vector Based on a Transitivity Based Definition

In many situations, DMs are not able to provide completely consistent judgments, i.e., (25) is not always satisfied. In this case, we turn to seek an interval utility vector $\widetilde{V} = (\widetilde{v}_1, \widetilde{v}_2, \cdots, \widetilde{v}_n)$ such that $[v_i^- - v_j^+, v_i^+ - v_j^-]$ is as close to $[b_{ij}^-, b_{ij}^+]$ as possible. Consequently, we can obtain the following goal-programming model:

$$\begin{array}{l} \min J_2 = \sum\limits_{i=1}^{n} \sum\limits_{j=1, j\neq i}^{n} \left(\left| v_i^- - v_j^+ - b_{ij}^- \right| + \left| v_i^+ - v_j^- - b_{ij}^+ \right| \right) \\ \text{s.t.} \\ 0 \leq v_i^- \leq v_i^+ \leq n\kappa,\ i = 1, \cdots, n, \\ v_i^+ + \sum\limits_{j=1, j\neq i}^{n} v_j^- \leq n\kappa,\ i = 1, \cdots, n, \\ v_i^- + \sum\limits_{j=1, j\neq i}^{n} v_j^+ \geq n\kappa,\ i = 1, \cdots, n. \end{array} . \tag{34}$$

Since for all $i, j = 1, 2, \cdots, n$, $v_j^- - v_i^+ - b_{ji}^- = -(v_i^+ - v_j^- - b_{ij}^+)$, $v_j^+ - v_i^- - b_{ji}^+ = -(v_i^- - v_j^+ - b_{ij}^-)$, we can simplify (34) to (35).

$$\begin{array}{l} \min J_2 = \sum\limits_{i=1}^{n-1} \sum\limits_{j=i+1}^{n} \left(\left| v_i^- - v_j^+ - b_{ij}^- \right| + \left| v_i^+ - v_j^- - b_{ij}^+ \right| \right) \\ \text{s.t.} \\ 0 \leq v_i^- \leq v_i^+ \leq n\kappa,\ i = 1, \cdots, n, \\ v_i^+ + \sum\limits_{j=1, j\neq i}^{n} v_j^- \leq n\kappa,\ i = 1, \cdots, n, \\ v_i^- + \sum\limits_{j=1, j\neq i}^{n} v_j^+ \geq n\kappa,\ i = 1, \cdots, n. \end{array} . \tag{35}$$

It is worth noting that if $\widetilde{B} = (\widetilde{b}_{ij})_{n\times n}$ is reduced to a POM $B = (b_{ij})_{n\times n}$ and the interval utility vector is reduced to a crisp utility vector, then (35) will become:

$$\begin{array}{l} \min J_2' = 2\sum\limits_{i=1}^{n-1}\sum\limits_{j=i+1}^{n} \left|v_i - v_j - b_{ij}\right| \\ \text{s.t.} \\ v_i \geq 0,\ i = 1,\cdots,n, \\ \sum\limits_{i=1}^{n} v_i = n\kappa. \end{array} . \tag{36}$$

It is obvious that the ideas of (5) and (36) are the same, so, (35) can also be used to derive crisp utility vectors from POMs.

For all $i = 1,2,\cdots,n-1$, $j = i+1,\cdots,n$, let $\varepsilon_{ij} = v_i^- - v_j^+ - b_{ij}^-$, $\delta_{ij} = v_i^+ - v_j^- - b_{ij}^+$, $\varepsilon_{ij}^+ = \frac{|\varepsilon_{ij}|+\varepsilon_{ij}}{2}$, $\varepsilon_{ij}^- = \frac{|\varepsilon_{ij}|-\varepsilon_{ij}}{2}$, $\delta_{ij}^+ = \frac{|\delta_{ij}|+\delta_{ij}}{2}$, and $\delta_{ij}^- = \frac{|\delta_{ij}|-\delta_{ij}}{2}$. Then, $\left|\varepsilon_{ij}\right| = \varepsilon_{ij}^+ + \varepsilon_{ij}^-$, $\left|\delta_{ij}\right| = \delta_{ij}^+ + \delta_{ij}^-$, $\varepsilon_{ij} = \varepsilon_{ij}^+ - \varepsilon_{ij}^-$, $\delta_{ij} = \delta_{ij}^+ - \delta_{ij}^-$, $\varepsilon_{ij}^+\varepsilon_{ij}^- = 0$, $\delta_{ij}^+\delta_{ij}^- = 0$. Substituting ε_{ij}^+, ε_{ij}^-, δ_{ij}^+, and δ_{ij}^- into (35), we can obtain the following linear programming model:

$$\begin{array}{l} \min J_2 = \sum\limits_{i=1}^{n-1}\sum\limits_{j=i+1}^{n} (\varepsilon_{ij}^+ + \varepsilon_{ij}^- + \delta_{ij}^+ + \delta_{ij}^-) \\ \text{s.t.} \\ 0 \leq v_i^- \leq v_i^+ \leq n\kappa,\ i = 1,\cdots,n, \\ v_i^+ + \sum\limits_{j=1,j\neq i}^{n} v_j^- \leq n\kappa,\ i = 1,\cdots,n, \\ v_i^- + \sum\limits_{j=1,j\neq i}^{n} v_j^+ \geq n\kappa,\ i = 1,\cdots,n, \\ \varepsilon_{ij}^+ - \varepsilon_{ij}^- = v_i^- - v_j^+ - b_{ij}^-,\ i = 1,\cdots,n-1,\ j = i+1,\cdots,n, \\ \delta_{ij}^+ - \delta_{ij}^- = v_i^+ - v_j^- - b_{ij}^+,\ i = 1,\cdots,n-1,\ j = i+1,\cdots,n. \end{array} . \tag{37}$$

Solving (37), we can obtain $J_2^* = \min J_2$ and the corresponding interval utility vector. If $J_2^* = 0$, then $\widetilde{B}$ is a consistent IPOM as per Definition 11. The value of J_2^* can also be used to measure the consistency degree of an IPOM.

After obtaining the interval utility vector $\widetilde{V} = (\widetilde{v}_1,\widetilde{v}_2,\cdots,\widetilde{v}_n)$, $\widetilde{v}_i = [v_i^-, v_i^+]$, the interval weight vector $\widetilde{W} = (\widetilde{w}_1,\widetilde{w}_2,\cdots,\widetilde{w}_n)$, $\widetilde{w}_i = [w_i^-, w_i^+]$ can be obtained by:

$$w_i^- = \frac{v_i^-}{n\kappa},\ w_i^+ = \frac{v_i^+}{n\kappa},\ i = 1,2,\cdots,n. \tag{38}$$

3.3. Interval Cognitive Network Process

We summarize the steps of the interval cognitive network process here as shown in Table 2.

Table 2. The basic steps of interval cognitive network process.

Interval Cognitive Network Process (I-CNP)	
Step 1.	Problem Cognition Process. Determining the criteria $C = \{c_1, c_2, \cdots, c_n\}$, the alternatives $X = \{x_1, x_2, \cdots, x_m\}$, and the measurement scale schema $(\aleph, \overline{X})$. The value of κ and τ in Equation (1) should be determined in this step.

Table 2. *Cont.*

Interval Cognitive Network Process (I-CNP)	
Step 2.	Cognitive Assessment Process. Executing the pairwise comparisons and constructing the interval pairwise opposite matrices (IPOMs) whose entries are represented by interval differential scales. The consistencies of the IPOMs are checked according to Definition 10 and Definition 11 in this step.
Step 3.	Cognitive Prioritization Process. Deriving the interval utility vectors and interval weight vectors from the IPOMs using models (31)–(33) and model (37).
Step 4.	Multiple Information fusion Process. Aggregating the weights of criteria and the weights of alternatives with respect to each criterion to get an integrated score which is also represented by an interval number for each alternative.
Step 5.	Decision Volition Process. Making the final decision according to the integrated score of each alternative.

4. Numerical Examples

This section illustrates the methods in Section 3 by four examples. For the sake of brevity, we denote the feasible region-based method as I-CNP_FR and the transitivity based method as I-CNP_T. To compare two interval numbers, we adopt the method proposed by Xu et al. [64]. Let $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$ be two interval numbers. The degree of possibility that $\widetilde{a} \geq \widetilde{b}$ is defined as [64]:

$$p(\widetilde{a} \geq \widetilde{b}) = \frac{\max(0, a^+ - b^-) - \max(0, a^- - b^+)}{(a^+ - a^-) + (b^+ - b^-)}. \tag{39}$$

To rank a set of interval numbers $\{\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n\}$, we need to construct the preference degree matrix $P = (p_{ij})_{n\times n}$ where $p_{ij} = p(\widetilde{a}_i \geq \widetilde{a}_j)$. The optimal degree $\theta_i = \sum\limits_{j=1}^{n} p_{ij}$ of each interval number can be used to rank the interval numbers.

We compare our methods to the goal-programming model designed for an IMPR in Wang et al. [36]. We denote the model in [36] as IAHP_GP and convert IPOMs to IMPRs according to Table 1 to use IAHP_GP. We adopt a fitted error, which is the average difference between the original matrix and the matrix constructed from the interval utilities (weights), to measure the quality of interval utilities (weights) derived by different models.

Supposing the interval utility vector derived from an IPOM $\widetilde{S} = (\widetilde{s}_{ij})_{n\times n}$, $\widetilde{s}_{ij} = [s_{ij}^-, s_{ij}^+]$ is $\widetilde{V} = (\widetilde{v}_1, \widetilde{v}_2, \cdots, \widetilde{v}_n)$, $\widetilde{v}_i = [v_i^-, v_i^+]$, then the constructed IPOM will be $\widetilde{T} = (\widetilde{t}_{ij})$, $\widetilde{t}_{ij} = [t_{ij}^-, t_{ij}^+] = [v_i^- - v_j^+, v_i^+ - v_j^-]$, and we define the fitted error of $\widetilde{V}$ as:

$$F_D(\widetilde{V}) = \frac{1}{n^2 - n} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\left| s_{ij}^- - t_{ij}^- \right| + \left| s_{ij}^+ - t_{ij}^+ \right| \right). \tag{40}$$

It is easy to verify that $F_D(\widetilde{V}) = \frac{J^*}{n^2-n}$ where J^* is the optimal value of model (37). As a result, we can obtain the conclusion that if an IPOM is consistent according to Definition 11, then the fitted error of the derived utility vector will be 0. For this reason, the fitted error of I-CNP_FR cannot be smaller than I-CNP_T.

It is worth noting that we cannot calculate the difference between two ratio scale judgments by subtraction. For example, the difference between 1/2 and 1/3 should be 1 rather than 1/6, and the difference between 1/2 and 2 should be 2 rather than 1.5. So, we introduce a mapping function $f(x) = \begin{cases} x - 1, & x \geq 1, \\ 1 - \frac{1}{x}, & x < 1. \end{cases}$, by which we can obtain the difference between two ratio scale judgments x and y as $|f(x) - f(y)|$.

Supposing the interval weight vector derived from an IMPR $\widetilde{P} = (\widetilde{p}_{ij})_{n\times n}$, $\widetilde{p}_{ij} = [p_{ij}^-, p_{ij}^+]$ is $\widetilde{W} = (\widetilde{w}_1, \widetilde{w}_2, \cdots, \widetilde{w}_n)$, $\widetilde{w}_i = [w_i^-, w_i^+]$, then the constructed IMPR will be $\widetilde{Q} = (\widetilde{q}_{ij})$, $\widetilde{q}_{ij} = [q_{ij}^-, q_{ij}^+] = [\frac{w_i^-}{w_j^+}, \frac{w_i^+}{w_j^-}]$, and we define the fitted error of $\widetilde{W}$ as:

$$F_R(\widetilde{W}) = \frac{1}{n^2 - n} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\left| f(p_{ij}^-) - f(q_{ij}^-) \right| + \left| f(p_{ij}^+) - f(q_{ij}^+) \right| \right). \tag{41}$$

Both F_D and F_R represent the deviation of the derived judgments from the original judgments in the sense of a verbal scale; therefore, they are comparable for the same MADM problem. It is obvious that the smaller the fitted error is, the better the derived utilities (weights) represent a DM's cognition.

4.1. Example 1

Consider an IPOM $\widetilde{B}_1$ whose corresponding IMPR is $\widetilde{A}_1$ obtained by pairwise comparisons of four objects x_1, x_2, x_3, x_4 with $\kappa = 8$.

$$\widetilde{B}_1 = \begin{pmatrix} [0,0] & [1,3] & [2,4] & [3,5] \\ [-3,-1] & [0,0] & [0,2] & [1,3] \\ [-4,-2] & [-2,0] & [0,0] & [0,2] \\ [-5,-3] & [-3,-1] & [-2,0] & [0,0] \end{pmatrix}, \tag{42}$$

$$\widetilde{A}_1 = \begin{pmatrix} [1,1] & [2,4] & [3,5] & [4,6] \\ [1/4,1/2] & [1,1] & [1,3] & [2,4] \\ [1/5,1/3] & [1/3,1] & [1,1] & [1,3] \\ [1/6,1/4] & [1/4,1/2] & [1/3,1] & [1,1] \end{pmatrix}. \tag{43}$$

We use model (31) and model (37) to check the consistency of $\widetilde{B}_1$, then adopt the corresponding models to derive the interval utility vector; meanwhile, we execute IAHP_GP on $\widetilde{A}_1$. Table 3 lists the optimal values J^*, the weight vectors, the fitted errors, and the rankings derived by the three methods. In IAHP_GP, the value of J^* is the optimal value of the objective function and represents the consistency of the IMPR. The number above '$\succ$' in the rankings represents the probability of $p(\widetilde{w}_i \geq \widetilde{w}_j)$.

Table 3. Results of Example 1.

Methods	J^*	Interval Weights	Fitted Error	Rankings
I-CNP_FR	0	$\widetilde{w}_1 = [0.2926, 0.3437]$, $\widetilde{w}_2 = [0.2344, 0.2812]$, $\widetilde{w}_3 = [0.2031, 0.2500]$, $\widetilde{w}_4 = [0.1719, 0.2187]$	0.500	$\widetilde{w}_1 \overset{1}{\succ} \widetilde{w}_2 \overset{0.83}{\succ} \widetilde{w}_3 \overset{0.83}{\succ} \widetilde{w}_4$
I-CNP_T	0	$\widetilde{w}_1 = [0.3043, 0.3355]$, $\widetilde{w}_2 = [0.2418, 0.2730]$, $\widetilde{w}_3 = [0.2105, 0.2418]$, $\widetilde{w}_4 = [0.1793, 0.2105]$	0	$\widetilde{w}_1 \overset{1}{\succ} \widetilde{w}_2 \overset{1}{\succ} \widetilde{w}_3 \overset{1}{\succ} \widetilde{w}_4$
IAHP_GP	0.018	$\widetilde{w}_1 = [0.5056, 0.5374]$, $\widetilde{w}_2 = [0.1755, 0.2752]$, $\widetilde{w}_3 = [0.1015, 0.1818]$, $\widetilde{w}_4 = [0.0671, 0.1052]$	0.459	$\widetilde{w}_1 \overset{1}{\succ} \widetilde{w}_2 \overset{0.965}{\succ} \widetilde{w}_3 \overset{0.968}{\succ} \widetilde{w}_4$

From Table 3, $\widetilde{B}_1$ is consistent according to Definition 10 and Definition 11, but $\widetilde{A}_1$ is inconsistent according to IAHP_GP. The interval weights obtained by I-CNP_FR and I-CNP_T are slightly different, but they are very different from the interval weights obtained by IAHP_GP. The rankings obtained by the three methods are the same. The fitted error of I-CNP_T is zero, i.e., the interval weights obtained by I-CNP_T perfectly reflect a DM's cognition. The fitted error of IAHP_GP is slightly better than that of I-CNP_FR.

4.2. Example 2

Consider an IPOM $\widetilde{B}_2$ whose corresponding IMPR is $\widetilde{A}_2$ also obtained by pairwise comparisons of four objects x_1, x_2, x_3, x_4 with $\kappa = 8$.

$$\widetilde{B}_2 = \begin{pmatrix} [0,0] & [1,3] & [4,6] & [7,8] \\ [-3,-1] & [0,0] & [3,5] & [6,7] \\ [-6,-4] & [-5,-3] & [0,0] & [2,4] \\ [-8,-7] & [-7,-6] & [-4,-2] & [0,0] \end{pmatrix}, \tag{44}$$

$$\widetilde{A}_2 = \begin{pmatrix} [1,1] & [2,4] & [5,7] & [8,9] \\ [1/4,1/2] & [1,1] & [4,6] & [7,8] \\ [1/7,1/5] & [1/6,1/4] & [1,1] & [3,5] \\ [1/9,1/8] & [1/8,1/7] & [1/5,1/3] & [1,1] \end{pmatrix}. \tag{45}$$

Table 4 lists the optimal values, the weight vectors, the fitted errors, and the rankings derived by the three methods.

Table 4. Results of Example 2.

Methods	*J**	Interval Weights	Fitted Error	Rankings
I-CNP_FR	0	$\widetilde{w}_1 = [0.3438, 0.3750]$, $\widetilde{w}_2 = [0.3047, 0.3359]$, $\widetilde{w}_3 = [0.1797, 0.2266]$, $\widetilde{w}_4 = [0.1016, 0.1328]$	0.417	$\widetilde{w}_1 \overset{1}{\succ} \widetilde{w}_2 \overset{1}{\succ} \widetilde{w}_3 \overset{1}{\succ} \widetilde{w}_4$
I-CNP_T	3	$\widetilde{w}_1 = [0.3487, 0.3720]$, $\widetilde{w}_2 = [0.3125, 0.3272]$, $\widetilde{w}_3 = [0.1845, 0.2237]$, $\widetilde{w}_4 = [0.1085, 0.1250]$	0.250	$\widetilde{w}_1 \overset{1}{\succ} \widetilde{w}_2 \overset{1}{\succ} \widetilde{w}_3 \overset{1}{\succ} \widetilde{w}_4$
IAHP_GP	0.155	$\widetilde{w}_1 = [0.5088, 0.6320]$, $\widetilde{w}_2 = [0.2497, 0.3444]$, $\widetilde{w}_3 = [0.0838, 0.1123]$, $\widetilde{w}_4 = [0.0345, 0.0345]$	2.270	$\widetilde{w}_1 \overset{1}{\succ} \widetilde{w}_2 \overset{1}{\succ} \widetilde{w}_3 \overset{1}{\succ} \widetilde{w}_4$

We can find that $\widetilde{B}_2$ is consistent according to Definition 10, but inconsistent according to Definition 11 from Table 4. The reason of this phenomenon is that Definition 11 is stricter than Definition 10. $\widetilde{A}_2$ is inconsistent according to IAHP_GP. The interval weights obtained by the three methods are different, but the rankings are the same. The fitted error of IAHP_GP is larger than that of both I-CNP_FR and I-CNP_T, which indicates that our two methods reflect a DM's cognition better than IAHP_GP.

4.3. Example 3

Consider an IPOM $\widetilde{B}_3$ whose corresponding IMPR is $\widetilde{A}_3$ also obtained by pairwise comparisons of four objects x_1, x_2, x_3, x_4 with $\kappa = 8$.

$$\widetilde{B}_3 = \begin{pmatrix} [0,0] & [1,4] & [3,6] & [5,7] \\ [-4,-1] & [0,0] & [1,4] & [2,5] \\ [-6,-3] & [-4,-1] & [0,0] & [2,3] \\ [-7,-5] & [-5,-2] & [-3,-2] & [0,0] \end{pmatrix}, \tag{46}$$

$$\widetilde{A}_3 = \begin{pmatrix} [1,1] & [2,5] & [4,7] & [6,8] \\ [1/5,1/2] & [1,1] & [2,5] & [3,6] \\ [1/7,1/4] & [1/5,1/2] & [1,1] & [3,4] \\ [1/8,1/6] & [1/6,1/3] & [1/4,1/3] & [1,1] \end{pmatrix}. \quad (47)$$

We do the same work as in the previous two examples here. Table 5 list the results obtained by the three methods.

Table 5. Results of Example 3.

Methods	J^*	Interval Weights	Fitted Error	Rankings
I-CNP_FR	7	$\widetilde{w}_1 = [0.3203, 0.3750]$, $\widetilde{w}_2 = [0.2500, 0.3047]$, $\widetilde{w}_3 = [0.2031, 0.2422]$, $\widetilde{w}_4 = [0.1328, 0.1719]$	1.500	$\widetilde{w}_1 \overset{1}{\succ} \widetilde{w}_2 \overset{1}{\succ} \widetilde{w}_3 \overset{1}{\succ} \widetilde{w}_4$
I-CNP_T	11	$\widetilde{w}_1 = [0.3203, 0.3828]$, $\widetilde{w}_2 = [0.2578, 0.2891]$, $\widetilde{w}_3 = [0.1953, 0.2266]$, $\widetilde{w}_4 = [0.1641, 0.1641]$	0.917	$\widetilde{w}_1 \overset{1}{\succ} \widetilde{w}_2 \overset{1}{\succ} \widetilde{w}_3 \overset{1}{\succ} \widetilde{w}_4$
IAHP_GP	1.636	$\widetilde{w}_1 = [0.4607, 0.6581]$, $\widetilde{w}_2 = [0.2284, 0.4259]$, $\widetilde{w}_3 = [0.0753, 0.0753]$, $\widetilde{w}_4 = [0.0382, 0.0382]$	3.255	$\widetilde{w}_1 \overset{1}{\succ} \widetilde{w}_2 \overset{1}{\succ} \widetilde{w}_3 \overset{1}{\succ} \widetilde{w}_4$

$\widetilde{B}_3$ is inconsistent according to both Definition 10 and Definition 11. $\widetilde{A}_3$ is inconsistent according to IAHP_GP. It is worth noting that the values of J^* for the three models in Example 3 are larger than those in Example 2. So, we can say that $\widetilde{B}_2$ is more consistent than $\widetilde{B}_3$ and $\widetilde{A}_2$ is more consistent than $\widetilde{A}_3$. Similar to Example 2, our two methods capture a DM's cognition more accurately than IAHP_GP as the fitted errors of our two methods are smaller.

4.4. Example 4

In this example, we consider the portfolio selection problem that has been investigated by Islam et al. [30] and Wang et al. [36]. A person wants to select a portfolio to invest his money. The alternatives are bank deposit (BD), debentures (DB), government bonds (GB), and shares (SH). There are four criteria which will affect his decision: return (Re), risk (Ri), tax benefits (Tb), and liquidity (Li). In [30,36], the pairwise comparison results for the four criteria as well as the four alternatives are represented by IMPRs. Here, we convert these IMPRs into IPOMs according to Table 1 with $\kappa = 8$. The five IPOMs are listed below. $\widetilde{B}_C$ is the pairwise comparison result of the four criteria, and $\widetilde{B}_{Re}$, $\widetilde{B}_{Ri}$, $\widetilde{B}_{Tb}$, and $\widetilde{B}_{Li}$ are the pairwise comparison results, represented by interval differential scales, of the four alternatives on Re, Ri, Tb, and Li, respectively.

$$\widetilde{B}_C = \begin{pmatrix} [0,0] & [2,3] & [4,5] & [5,6] \\ [-3,-2] & [0,0] & [3,4] & [4,5] \\ [-5,-4] & [-4,-3] & [0,0] & [2,3] \\ [-6,-5] & [-5,-4] & [-3,-2] & [0,0] \end{pmatrix}, \quad (48)$$

$$\widetilde{B}_{Re} = \begin{pmatrix} [0,0] & [-3,-2] & [2,3] & [-5,-4] \\ [2,3] & [0,0] & [5,6] & [-4,-3] \\ [-3,-2] & [-6,-5] & [0,0] & [-6,-5] \\ [4,5] & [3,4] & [5,6] & [0,0] \end{pmatrix}, \quad (49)$$

$$\widetilde{B}_{Ri} = \begin{pmatrix} [0,0] & [2,3] & [3,4] & [5,6] \\ [-3,-2] & [0,0] & [2,3] & [4,5] \\ [-4,-3] & [-3,-2] & [0,0] & [3,4] \\ [-6,-5] & [-5,-4] & [-4,-3] & [0,0] \end{pmatrix}, \tag{50}$$

$$\widetilde{B}_{Tb} = \begin{pmatrix} [0,0] & [0,0] & [-5,-4] & [-3,-2] \\ [0,0] & [0,0] & [-5,-4] & [-3,-2] \\ [4,5] & [4,5] & [0,0] & [3,4] \\ [2,3] & [2,3] & [-4,-3] & [0,0] \end{pmatrix}, \tag{51}$$

$$\widetilde{B}_{Li} = \begin{pmatrix} [0,0] & [2,3] & [5,5] & [5,6] \\ [-3,-2] & [0,0] & [2,3] & [2,3] \\ [-5,-5] & [-3,-2] & [0,0] & [2,3] \\ [-6,-5] & [-3,-2] & [-3,-2] & [0,0] \end{pmatrix}. \tag{52}$$

We use I-CNP_FR and I-CNP_T to analyze this problem. Among the five IPOMs above, only $\widetilde{B}_{Tb}$ is consistent according to Definition 10. We convert the interval utilities into interval weights using (38) and adopt the method proposed by Bryson et al. [65] to compute the composite weight.

For a multi-criteria decision-making problem with m criteria $(c_1, c_2, \cdots, c_m)$ and n alternatives $(x_1, x_2, \cdots, x_n)$, we denote $\widetilde{w}_i = [w_i^-, w_i^+]$ as the weight of c_i, and $\widetilde{w}_{ij} = [w_{ij}^-, w_{ij}^+]$ as the weight of alternative x_i with respect to c_j. Then, the composite weight $\widetilde{w}_{x_i} = [w_{x_i}^-, w_{x_i}^+]$ of alternative x_i can be obtained by the following two models [65].

$$\begin{aligned} & w_{x_i}^- = \min \sum_{j=1}^{m} w_{ij}^- w_j \\ & \text{s.t.} \\ & w_j^- \le w_j \le w_j^+,\ j = 1, 2, \cdots, n, \\ & \sum_{j=1}^{m} w_j = 1. \end{aligned} \tag{53}$$

$$\begin{aligned} & w_{x_i}^+ = \max \sum_{j=1}^{m} w_{ij}^- w_j \\ & \text{s.t.} \\ & w_j^- \le w_j \le w_j^+,\ j = 1, 2, \cdots, n, \\ & \sum_{j=1}^{m} w_j = 1. \end{aligned} \tag{54}$$

Tables 6 and 7 show the results obtained by the two proposed methods. The ranking obtained by I-CNP_FR is $BD \overset{0.60}{\succ} DB \overset{0.91}{\succ} SH \overset{1}{\succ} GB$, and the ranking obtained by I-CNP_T is $BD \overset{0.52}{\succ} DB \overset{0.86}{\succ} SH \overset{1}{\succ} GB$. Although the interval weights are different slightly, the final rankings obtained by the two methods are the same.

Table 6. Results obtained by I-CNP_FR for Example 4.

Portfolios	Re	Ri	Tb	Li	Composite Weight
	[0.336, 0.359]	**[0.281, 0.305]**	**[0.195, 0.219]**	**[0.141, 0.164]**	
BD	[0.203, 0.227]	[0.328, 0.352]	[0.195, 0.195]	[0.359, 0.359]	[0.259, 0.280]
DB	[0.273, 0.305]	[0.273, 0.297]	[0.195, 0.195]	[0.266, 0.266]	[0.255, 0.275]
GB	[0.125, 0.156]	[0.219, 0.242]	[0.352, 0.352]	[0.203, 0.203]	[0.208, 0.232]
SH	[0.336, 0.375]	[0.133, 0.156]	[0.258, 0.258]	[0.172, 0.172]	[0.232, 0.259]

BD: bank deposit; DB: debentures; GB: government bonds; SH: shares; Re: return; Ri: risk; Tb: tax benefits; Li: liquidity.

Table 7. Results obtained by I-CNP_T for Example 4.

Portfolios	Re	Ri	Tb	Li	Composite Weight
	[0.342, 0.356]	**[0.286, 0.300]**	**[0.200, 0.217]**	**[0.144, 0.161]**	
BD	[0.212, 0.219]	[0.333, 0.347]	[0.195, 0.195]	[0.352, 0.352]	[0.263, 0.274]
DB	[0.281, 0.306]	[0.279, 0.292]	[0.195, 0.195]	[0.258, 0.289]	[0.258, 0.277]
GB	[0.125, 0.149]	[0.222, 0.24]	[0.320, 0.352]	[0.195, 0.195]	[0.203, 0.227]
SH	[0.344, 0.368]	[0.136, 0.154]	[0.258, 0.289]	[0.164, 0.195]	[0.236, 0.265]

5. Discussion

Examples 1–3 show that both I-CNP_FR and I-CNP_T are able to handle consistent and inconsistent IPOMs. They can be new useful tools for uncertain MADM problems. The comparative analysis against IAHP_GP demonstrates that our proposed methods reflect a DM's cognition better, especially when the pairwise comparison matrix is inconsistent.

Wang et al. [36] also investigated Example 4 using IAHP_GP, and the ranking obtained is $SH \overset{1}{\succ} BD \overset{0.71}{\succ} DB \overset{1}{\succ} GB$. It is obvious that the results obtained by IAHP_GP are very different from our methods. The fundamental cause of such difference is the scales used to measure the paired difference. There are substantial differences between the ratio scale and the differential scale when representing the same verbal scale. We will show this difference by the following example.

$$A_1 = \begin{pmatrix} 1 & 9 \\ 1/9 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 8 \\ -8 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 2 \\ 1/2 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (55)$$

A_1 and B_1 represent the maximal difference between two objects using the ratio scale and the differential scale, respectively; in the same way, A_2 and B_2 represent the minimal difference. The weights obtained by A_1 and B_1 are [0.9, 0.1] and [0.75, 0.25], and the weights obtained by A_2 and B_2 are [0.67, 0.33] and [0.52, 0.47]. From the above example, we can find that the scope of difference expressed by the ratio scale is larger than that of the differential scale; however, the differential scale is able to express a more delicate difference than the ratio scale. In order to determine which scale captures a DM's cognition better, we calculate the fitted errors, shown in Table 8 of the three methods for the five matrices. It is clear that the fitted errors of our methods are smaller than IAHP_GP for all five matrices. As our methods reflect a DM's cognition of the problem better, the results obtained by our methods are more convincing, i.e., the differential scale performs better than the ratio scale.

Table 8. Fitted errors of different methods for the five matrices.

Methods	Criteria	Re	Ri	Tb	Li
I-CNP_FR	0.583	0.792	0.583	0.417	0.583
I-CNP_T	0.500	0.583	0.500	0.250	0.417
IAHP_GP	2.395	2.607	2.402	1.381	1.882

6. Conclusions

In this paper, we combine the ideas of P-CNP and interval AHP and propose the interval cognitive network process (I-CNP). Two kinds of consistency definitions and some corresponding properties are introduced. Based on the two definitions, we develop two methods to derive interval utilities. To illustrate the proposed methods, we provide four numerical examples. The comparative analysis against IAHP_GP demonstrates that I-CNP reflects a DM's cognition better and the suggestions provided by I-CNP are more reliable. I-CNP extends P-CNP to handle interval judgments and provides new alternative tools for MADM problems.

The good features of I-CNP are summarized as follows:

(1) I-CNP extends P-CNP by introducing interval judgments and makes it possible to handle uncertainty in MADM problems using differential scales.
(2) The two definitions of consistency for an IPOM are all consistent with the original consistency definition for a POM when an IPOM is reduced to a POM.
(3) Both I-CNP_FR and I-CNP_T can be used to check the consistency of an IPOM, and I-CNP_T can also be used to derive utility vectors from a POM.
(4) As differential scales represent a DM's cognition better than ratio scales, the fitted errors of our proposed methods are smaller than IAHP_GP and the results obtained by I-CNP are more convincing.

Future research will focus on designing new models to handle incomplete IPOMs and to revise inconsistent IPOMs. In addition, applying I-CNP to group decision problems will also be an interesting research direction.

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