

Article

Cubature Formulas of Multivariate Polynomials Arising from Symmetric Orbit Functions

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Abstract: The paper develops applications of symmetric orbit functions, known from irreducible representations of simple Lie groups, in numerical analysis. It is shown that these functions have remarkable properties which yield to cubature formulas, approximating a weighted integral of any function by a weighted finite sum of function values, in connection with any simple Lie group. The cubature formulas are specialized for simple Lie groups of rank two. An optimal approximation of any function by multivariate polynomials arising from symmetric orbit functions is discussed.

Keywords: cubature formulas; symmetric orbit functions; simple Lie groups; Weyl groups

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1. Introduction

The purpose of this paper is to extend the results of [1,2], where cubature formulas for numerical integration connected with three types of multivariate Chebyshev-like polynomials arising from Weyl group orbit functions are developed. The specific goal of this article is to derive the cubature rules and corresponding approximation methods for the family of the polynomials arising from symmetric exponential Weyl group orbit sums [3,4] and detail specializations of the general results for the two-variable polynomials.

The family of polynomials induced by the symmetric Weyl group orbit functions (*C*-functions) forms one of the most natural generalizations of the classical Chebyshev polynomials of one variable—indeed, the lowest symmetric orbit function arising from the Weyl group of A_1 coincides with the common cosine function of one variable and thus induces the family of Chebyshev polynomials of the first kind [5]. The continuous and discrete orthogonality of the sets of cosine functions $\cos(nx)$ generalize to the families of multivariate *C*-functions [4,6,7]. This generalization serves as an essential starting point for deriving the cubature formulas and approximation methods.

Cubature formulas for numerical integration constitute multivariate generalizations of classical quadrature formulas for functions of one variable. A weighted integral over some domain inside \mathbb{R}^n of any given function is estimated by a finite weighted sum of values of the same function on a specific set of points (nodes). A standard requirement is imposed: the cubature formula has to hold as an exact equality for polynomials up to a certain degree. Numerous types of cubature formulas with diverse shapes of the integration domains and various efficiencies exist [8]. The efficiency of a given cubature formula reflects how the achieved maximal degree of the polynomials relates to the number of the

necessary nodes. Optimal cubature formulas of the highest possible efficiency (Gaussian formulas) are for multivariate functions obtained for instance in [1,2,9,10].

The sequence of Gaussian cubature formulas derived in [9] arises from the antisymmetric orbit functions (S -functions) of the Weyl groups of type A_n , $n \in \mathbb{N}$. Generalization of these cubature formulas from [9] to polynomials of the S -functions of Weyl groups of any type and rank is achieved in [2]. A crucial concept, which allows the generalization of the A_n formulas, is a novel definition of a degree of the underlying polynomials. This generalized degree (m -degree) is based on invariants of the Weyl groups and their corresponding root systems. Besides the polynomials corresponding to the C - and S -functions, two additional families of multivariate polynomials arise from Weyl group orbit functions of mixed symmetries [1]. These hybrid orbit functions (S^s - and S^l -functions) exist only for root systems of Weyl groups with two different lengths of roots— B_n , C_n , F_4 and G_2 . The cubature formulas related to the polynomials of these S^s - and S^l -functions are developed in [1]. Deduction of the remaining cubature formulas, which correspond to the polynomials of the C -functions, completes in this paper the results of [1,2,9]. The integration domains and nodes of these cubature formulas are constructed in a similar way as one-dimensional Gauss-Chebyshev formulas and their Chebyshev nodes.

Instead of a classical one-dimensional interval, the multivariate C -functions are considered in the fundamental domain of the affine Weyl group—a simplex $F \subset \mathbb{R}^n$. The discrete orthogonality relations of C -functions are performed over a finite fragment of a grid $F_M \subset F$, with the parameter $M \in \mathbb{N}$ controlling the density of F_M inside F . The simplex F together with the set of points F_M have to be transformed via a transform which induces the corresponding family of polynomials (X -transform). This process results in the integration domain Ω of non-standard shape and the set of nodes Ω_M , with specifically distributed points inside Ω . The last ingredient, needed for successful practical implementation, is the explicit form of the weight polynomial K . For practical purposes, the explicit construction of all two-variable cases is presented.

Except for direct numerical integration, one of the most immediate applications of the developed cubature formulas is related multivariate polynomial approximation [11]. The Hilbert basis of the orthogonal multivariate polynomials induced by the C -functions guarantees that any function from the corresponding Hilbert space is expressed as a series involving these polynomials. A specific truncated sum of this expansion provides the best approximation of the function by the polynomials. Among other potential applications of the developed cubature formulas are calculations in fluid flows [12], laser optics [13], stochastic dynamics [14], magnetostatic modelling [15], micromagnetic simulations [16], electromagnetic wave propagation [17], liquid crystal colloids [18] and quantum dynamics [19].

The paper is organized as follows. In Section 2, notation and pertinent properties of Weyl groups, affine Weyl groups and C -functions are reviewed. In Section 3, the cubature formulas related to C -functions are deduced. In Section 4, the explicit cubature formulas of the rank two cases A_2 , C_2 , G_2 are constructed. In Section 5, polynomial approximation methods are developed.

2. Root Systems and Polynomials

2.1. Pertinent Properties of Root Systems and Weight Lattices

The notation, established in [7], is used. Recall that, to the Lie algebra of the compact, connected, simply connected simple Lie group G of rank n , corresponds the set of simple roots $\Delta = (\alpha_1, \dots, \alpha_n)$ [3,20–22]. The set Δ spans the Euclidean space \mathbb{R}^n , with the scalar product denoted by $\langle \cdot, \cdot \rangle$. The following standard objects related to the set of simple roots Δ are used.

- The marks m_1, \dots, m_n of the highest root $\xi \equiv -\alpha_0 = m_1\alpha_1 + \dots + m_n\alpha_n$.
- The Coxeter number $m = 1 + m_1 + \dots + m_n$ of G .
- The Cartan matrix C and its determinant.

$$c = \det C. \quad (1)$$

- The root lattice $Q = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_n$.

- The \mathbb{Z} -dual lattice to Q ,

$$P^\vee = \{\omega^\vee \in \mathbb{R}^n \mid \langle \omega^\vee, \alpha \rangle \in \mathbb{Z}, \forall \alpha \in \Delta\} = \mathbb{Z}\omega_1^\vee + \dots + \mathbb{Z}\omega_n^\vee$$

with the vectors ω_i^\vee given by

$$\langle \omega_i^\vee, \alpha_j \rangle = \delta_{ij}.$$

- The dual root lattice $Q^\vee = \mathbb{Z}\alpha_1^\vee + \dots + \mathbb{Z}\alpha_n^\vee$, where $\alpha_i^\vee = 2\alpha_i / \langle \alpha_i, \alpha_i \rangle$.
- The dual marks $m_1^\vee, \dots, m_n^\vee$ of the highest dual root $\eta \equiv -\alpha_0^\vee = m_1^\vee \alpha_1^\vee + \dots + m_n^\vee \alpha_n^\vee$. The marks and the dual marks are summarized in Table 1 in [7]. The highest dual root η satisfies for all $i = 1, \dots, n$

$$\langle \eta, \alpha_i \rangle \geq 0. \tag{2}$$

- The \mathbb{Z} -dual weight lattice to Q^\vee

$$P = \{\omega \in \mathbb{R}^n \mid \langle \omega, \alpha^\vee \rangle \in \mathbb{Z}, \forall \alpha^\vee \in Q^\vee\} = \mathbb{Z}\omega_1 + \dots + \mathbb{Z}\omega_n$$

with the vectors ω_i given by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. For $\lambda \in P$ the following notation is used,

$$\lambda = \lambda_1\omega_1 + \dots + \lambda_n\omega_n = (\lambda_1, \dots, \lambda_n). \tag{3}$$

- The partial ordering on P is given: for $\lambda, \nu \in P$ it holds that $\nu \leq \lambda$ if and only if $\lambda - \nu = k_1\alpha_1 + \dots + k_n\alpha_n$ with $k_i \in \mathbb{Z}^{\geq 0}$ for all $i \in \{1, \dots, n\}$.
- The half of the sum of the positive roots

$$\rho = \omega_1 + \dots + \omega_n.$$

- The cone of positive weights P^+ and the cone of strictly positive weights $P^{++} = \rho + P^+$

$$P^+ = \mathbb{Z}^{\geq 0}\omega_1 + \dots + \mathbb{Z}^{\geq 0}\omega_n, \quad P^{++} = \mathbb{N}\omega_1 + \dots + \mathbb{N}\omega_n.$$

- n reflections $r_\alpha, \alpha \in \Delta$ in $(n - 1)$ -dimensional “mirrors” orthogonal to simple roots intersecting at the origin denoted by

$$r_1 \equiv r_{\alpha_1}, \dots, r_n \equiv r_{\alpha_n}.$$

Following [2], we define so called m -degree of $\lambda \in P^+$ as the scalar product of λ with the highest dual root η , i.e., by the relation

$$|\lambda|_m = \langle \lambda, \eta \rangle = \lambda_1 m_1^\vee + \dots + \lambda_n m_n^\vee.$$

Let us denote a finite subset of the cone of the positive weights P^+ consisting of the weights of the m -degree not exceeding M by P_M^+ i.e.,

$$P_M^+ = \{\lambda \in P^+ \mid |\lambda|_m \leq M\}.$$

Recall also the separation lemma which asserts for $\lambda \in P^+, \lambda \neq 0$ and any $M \in \mathbb{N}$ that

$$|\lambda|_m < 2M \quad \Rightarrow \quad \lambda \notin MQ. \tag{4}$$

Note that this lemma is proved in [2] for $M > m$ only—the proof, however, can be repeated verbatim with any $M \in \mathbb{N}$.

For two dominant weights $\lambda, \nu \in P^+$ for which $\nu \leq \lambda$ we have for their m -degrees

$$|\lambda|_m - |\nu|_m = \langle \lambda - \nu, \eta \rangle = \sum_{i=1}^n k_i \langle \alpha_i, \eta \rangle, \quad k_i \geq 0. \tag{5}$$

Taking into account Equation (2), we have the following proposition.

Proposition 1. For two dominant weights $\lambda, \nu \in P^+$ with $\nu \leq \lambda$ it holds that $|\nu|_m \leq |\lambda|_m$.

2.2. Affine Weyl Groups

The Weyl group W is generated by n reflections r_1, \dots, r_n and its order $|W|$ can be calculated using the formula

$$|W| = n! m_1 \dots m_n c. \tag{6}$$

The affine Weyl group W^{aff} is the semidirect product of the Abelian group of translations Q^\vee and of the Weyl group W ,

$$W^{\text{aff}} = Q^\vee \rtimes W. \tag{7}$$

The fundamental domain F of W^{aff} , which consists of precisely one point of each W^{aff} -orbit, is the convex hull of the points $\left\{0, \frac{\omega_1^\vee}{m_1}, \dots, \frac{\omega_n^\vee}{m_n}\right\}$. Considering $n + 1$ real parameters $y_0, \dots, y_n \geq 0$, we have

$$F = \left\{y_1 \omega_1^\vee + \dots + y_n \omega_n^\vee \mid y_0 + y_1 m_1 + \dots + y_n m_n = 1\right\}. \tag{8}$$

The volumes $\text{vol}(F) \equiv |F|$ of the simplices F are calculated in [7].

Considering the standard action of W on \mathbb{R}^n , we denote for $\lambda \in \mathbb{R}^n$ the isotropy group and its order by

$$\text{Stab}(\lambda) = \{w \in W \mid w\lambda = \lambda\}, \quad h_\lambda \equiv |\text{Stab}(\lambda)|,$$

and denote the orbit by

$$W\lambda = \{w\lambda \in \mathbb{R}^n \mid w \in W\}.$$

Then the orbit-stabilizer theorem gives for the orders

$$|W\lambda| = \frac{|W|}{h_\lambda}. \tag{9}$$

Considering the standard action of W on the torus \mathbb{R}^n / Q^\vee , we denote for $x \in \mathbb{R}^n / Q^\vee$ the order of its orbit by $\varepsilon(x)$, i.e.,

$$\varepsilon(x) = \left| \{wx \in \mathbb{R}^n / Q^\vee \mid w \in W\} \right|. \tag{10}$$

For an arbitrary $M \in \mathbb{N}$, the grid F_M is given as cosets from the W -invariant group $\frac{1}{M}P^\vee / Q^\vee$ with a representative element in the fundamental domain F

$$F_M \equiv \frac{1}{M}P^\vee / Q^\vee \cap F.$$

The representative points of F_M can be explicitly written as

$$F_M = \left\{ \frac{u_1}{M} \omega_1^\vee + \dots + \frac{u_n}{M} \omega_n^\vee \mid u_0, u_1, \dots, u_n \in \mathbb{Z}^{\geq 0}, u_0 + u_1 m_1 + \dots + u_n m_n = M \right\}. \tag{11}$$

The numbers of elements of F_M , denoted by $|F_M|$, are also calculated in [7] for all simple Lie algebras.

2.3. Orbit Functions

Symmetric orbit functions [4] are defined as complex functions $C_\lambda : \mathbb{R}^n \rightarrow \mathbb{C}$ with the labels $\lambda \in P^+$,

$$C_\lambda(x) = \sum_{v \in W\lambda} e^{2\pi i \langle v, x \rangle}, \quad x \in \mathbb{R}^n. \tag{12}$$

Note that in [7] the results for C-functions are formulated for the normalized C-functions Φ_λ which are related to the orbit sums (12) as

$$\Phi_\lambda = h_\lambda C_\lambda.$$

Due to the symmetries with respect to the Weyl group W as well as with respect to the shifts from Q^\vee

$$C_\lambda(wx) = C_\lambda(x), \quad C_\lambda(x + q^\vee) = C_\lambda(x), \quad w \in W, q^\vee \in Q^\vee, \quad (13)$$

it is sufficient to consider C-functions restricted to the fundamental domain of the affine Weyl group F . Moreover, the C-functions are continuously orthogonal on F ,

$$\int_F C_\lambda(x) \overline{C_{\lambda'}(x)} dx = \frac{|F||W|}{h_\lambda} \delta_{\lambda, \lambda'}. \quad (14)$$

and form a Hilbert basis of the space $\mathcal{L}^2(F)$ [4], i.e., any function $\tilde{f} \in \mathcal{L}^2(F)$ can be expanded into the series of C-functions

$$\tilde{f} = \sum_{\lambda \in P^+} c_\lambda C_\lambda, \quad c_\lambda = \frac{h_\lambda}{|F||W|} \int_F \tilde{f}(x) \overline{C_\lambda(x)} dx. \quad (15)$$

Special case of the orthogonality relations (14) is when one of the weights is equal to zero,

$$\int_F C_\lambda(x) dx = |F| \delta_{\lambda, 0}. \quad (16)$$

For any $M \in \mathbb{N}$, the C-functions from a certain subset of P^+ are also discretely orthogonal on F_M and form a basis of the space of discretized functions \mathbb{C}^{F_M} of dimension $|F_M|$ [7]; special case of these orthogonality relations is when one of the weights is equal to zero modulo the lattice MQ ,

$$\sum_{x \in F_M} \varepsilon(x) C_\lambda(x) = \begin{cases} cM^n & \lambda \in MQ, \\ 0 & \lambda \notin MQ. \end{cases} \quad (17)$$

The key point in developing the cubature formulas is comparison of Formulas (16) and (17) in the following proposition.

Proposition 2. For any $M \in \mathbb{N}$ and $\lambda \in P_{2M-1}^+$ it holds that

$$\frac{1}{|F|} \int_F C_\lambda(x) dx = \frac{1}{cM^n} \sum_{x \in F_M} \varepsilon(x) C_\lambda(x). \quad (18)$$

Proof. Suppose first that $\lambda = 0$. Then from (16) and (17) we obtain

$$\frac{1}{|F|} \int_F C_0(x) dx = 1 = \frac{1}{cM^n} \sum_{x \in F_M} \varepsilon(x) C_0(x).$$

Secondly let $\lambda \neq 0$ and $|\lambda|_m < 2M$. Then from the separation lemma (4) we have that $\lambda \notin MQ$ and thus

$$\frac{1}{|F|} \int_F C_\lambda(x) dx = 0 = \frac{1}{cM^n} \sum_{x \in F_M} \varepsilon(x) C_\lambda(x).$$

□

Let us denote for convenience the C-functions corresponding to the basic dominant weights ω_j by Z_j , i.e.,

$$Z_j \equiv C_{\omega_j}.$$

Recall from [3], Ch. VI, §4 that any W -invariant sum of the exponential functions $e^{2\pi i\langle v, a \rangle}$ can be expressed as a linear combination of some functions C_λ with $\lambda \in P^+$. Also for any $\lambda \in P^+$ a function of the monomial type $Z_1^{\lambda_1} Z_2^{\lambda_2} \dots Z_n^{\lambda_n}$ can be expressed as the sum of C -functions labeled by less or equal dominant weights than λ , i.e.,

$$Z_1^{\lambda_1} Z_2^{\lambda_2} \dots Z_n^{\lambda_n} = \sum_{v \leq \lambda, v \in P^+} c_v C_v, \quad c_v \in \mathbb{C}, \quad c_\lambda = 1. \tag{19}$$

Conversely, any function C_λ , $\lambda \in P^+$ can be expressed as a polynomial in variables Z_1, \dots, Z_n , i.e., there exist multivariate polynomials $\tilde{p}_\lambda \in \mathbb{C}[y_1, \dots, y_n]$ such that

$$C_\lambda = \tilde{p}_\lambda(Z_1, \dots, Z_n) = \sum_{v \leq \lambda, v \in P^+} d_v Z_1^{v_1} Z_2^{v_2} \dots Z_n^{v_n}, \quad d_v \in \mathbb{C}, \quad d_\lambda = 1. \tag{20}$$

Antisymmetric orbit functions [23] are defined as complex functions $S_\lambda : \mathbb{R}^n \rightarrow \mathbb{C}$ with the labels $\lambda \in P^{++}$,

$$S_\lambda(x) = \sum_{w \in W} \det(w) e^{2\pi i \langle w\lambda, x \rangle}, \quad x \in \mathbb{R}^n. \tag{21}$$

The antisymmetry with respect to the Weyl group W and the symmetry with respect to the shifts from Q^\vee holds

$$S_\lambda(wx) = (\det w) S_\lambda(x), \quad S_\lambda(x + q^\vee) = S_\lambda(x), \quad w \in W, q^\vee \in Q^\vee.$$

Recall that Proposition 9 in [23] states that for the lowest S -function S_ρ it holds that

$$S_\rho(x) = 0, \quad x \in F \setminus F^\circ, \tag{22}$$

$$S_\rho(x) \neq 0, \quad x \in F^\circ, \tag{23}$$

where F° denotes the interior of F . Since the square of the absolute value $|S_\rho|^2 = S_\rho \overline{S_\rho}$ is a W -invariant sum of exponentials it can be expressed as a linear combination of C -functions. Each C -function in this combination is moreover a polynomial of the form (20). Thus there exists a unique polynomial $\tilde{K} \in \mathbb{C}[y_1, \dots, y_n]$ such that

$$|S_\rho|^2 = \tilde{K}(Z_1, \dots, Z_n). \tag{24}$$

3. Cubature Formulas

3.1. The X -Transform

The key component in the development of the cubature formulas is the integration by substitution. The X -transform transforms the fundamental domain $F \subset \mathbb{R}^n$ to the domain $\Omega \subset \mathbb{R}^n$ on which are the cubature rules defined. In order to obtain a real valued transform we first need to examine the values of the C -functions.

The C -functions of the algebras

$$A_1, B_n (n \geq 3), C_n (n \geq 2), D_{2k} (k \geq 2), E_7, E_8, F_4, G_2 \tag{25}$$

are real-valued [4]. Using the notation (3), for the remaining cases it holds that

$$\begin{aligned} A_n (n \geq 2) : \quad C_{(\lambda_1, \lambda_2, \dots, \lambda_n)}(x) &= \overline{C_{(\lambda_n, \lambda_{n-1}, \dots, \lambda_1)}(x)}, \\ D_{2k+1} (k \geq 2) : \quad C_{(\lambda_1, \lambda_2, \dots, \lambda_{2k-1}, \lambda_{2k}, \lambda_{2k+1})}(x) &= \overline{C_{(\lambda_1, \lambda_2, \dots, \lambda_{2k-1}, \lambda_{2k+1}, \lambda_{2k})}(x)}, \\ E_6 : \quad C_{(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)}(x) &= \overline{C_{(\lambda_5, \lambda_4, \lambda_3, \lambda_2, \lambda_1, \lambda_6)}(x)}. \end{aligned} \tag{26}$$

Specializing the relations (26) for the C-functions corresponding to the basic dominant weights Z_j , we obtain that the functions Z_j are real valued, except for the following cases for which it holds that

$$\begin{aligned}
 A_{2k}(k \geq 1) : Z_j &= \overline{Z_{2k-j+1}}, j = 1, \dots, k, \\
 A_{2k+1}(k \geq 1) : Z_j &= \overline{Z_{2k-j+2}}, j = 1, \dots, k, \\
 D_{2k+1}(k \geq 2) : Z_{2k} &= \overline{Z_{2k+1}}, \\
 E_6 : Z_2 &= \overline{Z_4}, Z_1 = \overline{Z_5}.
 \end{aligned}
 \tag{27}$$

Taking into account (27), we introduce the real-valued functions $X_j, j \in \{1, \dots, n\}$ as follows. For the cases (25), we set

$$X_j \equiv Z_j, \tag{28}$$

and for the remaining cases (27), we define

$$\begin{aligned}
 A_{2k} : X_j &= \frac{Z_j + Z_{2k-j+1}}{2}, X_{2k-j+1} = \frac{Z_j - Z_{2k-j+1}}{2i}, j = 1, \dots, k, \\
 A_{2k+1} : X_j &= \frac{Z_j + Z_{2k-j+2}}{2}, X_{k+1} = Z_{k+1}, X_{2k-j+2} = \frac{Z_j - Z_{2k-j+2}}{2i}, j = 1, \dots, k, \\
 D_{2k+1} : X_j &= Z_j, X_{2k} = \frac{Z_{2k} + Z_{2k+1}}{2}, X_{2k+1} = \frac{Z_{2k} - Z_{2k+1}}{2i}, j = 1, \dots, 2k - 1, \\
 E_6 : X_1 &= \frac{Z_1 + Z_5}{2}, X_2 = \frac{Z_2 + Z_4}{2}, X_3 = Z_3, X_4 = \frac{Z_2 - Z_4}{2i}, X_5 = \frac{Z_1 - Z_5}{2i}, X_6 = Z_6.
 \end{aligned}
 \tag{29}$$

Thus, we obtain a crucial mapping $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by

$$X(x) \equiv (X_1(x), \dots, X_n(x)). \tag{30}$$

The image $\Omega \subset \mathbb{R}^n$ of the fundamental domain F under the mapping X forms the integration domain on which the cubature rules will be formulated, i.e.,

$$\Omega \equiv X(F). \tag{31}$$

In order to use the mapping X for an integration by substitution we need to know that it is one-to-one except for possibly some set of zero measure. Since the image $\Omega_M \subset \mathbb{R}^n$ of the set of points F_M under the mapping X forms the set of nodes for the cubature rules, i.e.,

$$\Omega_M \equiv X(F_M), \tag{32}$$

a discretized version of the one-to-one correspondence of the restricted mapping X_M of X to F_M , i.e.,

$$X_M \equiv X \upharpoonright_{F_M} \tag{33}$$

is also essential. Note that due to the periodicity of C-functions (13), the restriction (33) is well-defined for the cosets from F_M .

Proposition 3. *The mapping $X : F \rightarrow \Omega$, given by (30), is a one-to-one correspondence except for some set of zero measure. For any $M \in \mathbb{N}$ is the restriction mapping $X_M : F_M \rightarrow \Omega_M$, given by (33), a one-to-one correspondence and thus it holds that*

$$|\Omega_M| = |F_M|. \tag{34}$$

Proof. Let us assume that there exists a set $F' \subset F$ of non-zero measure such that $X(x) = X(y)$ with $x, y \in F'$. Since the transforms (28) and (29) are as regular linear mappings one-to-one correspondences, this fact implies that $Z_1(x) = Z_1(y), \dots, Z_n(x) = Z_n(y)$ with $x, y \in F'$. Then from the polynomial

expression (20) we obtain for all $\lambda \in P^+$ that it holds that $C_\lambda(x) = C_\lambda(y)$. Since the C-functions $C_\lambda, \lambda \in P^+$ form a Hilbert basis of the space $\mathcal{L}^2(F)$ we conclude that for any $f \in \mathcal{L}^2(F)$ is valid that $f(x) = f(y), x, y \in F'$ which is a contradiction.

Retracing the steps of the continuous case above, let us assume that there exist two distinct points $x, y \in F_M, x \neq y$ such that $X(x) = X(y)$. Since the transforms (28) and (29) are as regular linear mappings one-to-one correspondences, this fact again implies that $Z_1(x) = Z_1(y), \dots, Z_n(x) = Z_n(y)$. Then from the polynomial expression (20) we obtain for all $\lambda \in P^+$ that it holds that $C_\lambda(x) = C_\lambda(y)$. The same equality has to hold for those C-functions C_λ which form a basis of the space \mathbb{C}^{F_M} . We conclude that for any $f \in \mathbb{C}^{F_M}$ is valid that $f(x) = f(y), x \neq y$ which is a contradiction. \square

The absolute value of the determinant of the Jacobian matrix of the X-transform (30) is essential for construction of the cubature formulas—its value is determined in the following proposition.

Proposition 4. *The absolute value of the Jacobian determinant $|J_x(X)|$ of the X-transform (30) is given by*

$$|J_x(X)| = \frac{\kappa(2\pi)^n}{|F||W|} |S_\rho(x)|, \tag{35}$$

where κ is defined as

$$\kappa = \begin{cases} 2^{-\lfloor \frac{n}{2} \rfloor} & \text{for } A_n \\ \frac{1}{2} & \text{for } D_{2k+1} \\ \frac{1}{4} & \text{for } E_6 \\ 1 & \text{otherwise.} \end{cases} \tag{36}$$

Proof. Note that the X-transform can be composed of the following two transforms: the transform $\zeta : x \mapsto (Z_1(x), \dots, Z_n(x))$ and the transform $R : (Z_1, \dots, Z_n) \mapsto (X_1, \dots, X_n)$ via relations (28) and (29). To calculate the Jacobian of the transform ζ , let us denote by α^\vee the matrix of the coordinates (in columns) of the vectors $\alpha_1^\vee, \dots, \alpha_n^\vee$ in the standard orthonormal basis of \mathbb{R}^n and by a_1, \dots, a_n the coordinates of a point $x \in \mathbb{R}^n$ in α^\vee -basis, i.e., $x = a_1\alpha_1^\vee + \dots + a_n\alpha_n^\vee$. If a denotes the coordinates a_1, \dots, a_n arranged in a column vector then it holds that $x = \alpha^\vee a$. The absolute value of the Jacobian of the mapping $a \mapsto (Z_1(\alpha^\vee a), \dots, Z_n(\alpha^\vee a))$ is according to Equation (32) in [2] given by $(2\pi)^n |S_\rho(\alpha^\vee a)|$. Using the chain rule, this implies for the absolute value of the Jacobian $|J_x(\zeta)|$ of the map ζ that

$$|J_x(\zeta)| = |\det \alpha^\vee|^{-1} (2\pi)^n |S_\rho(x)|.$$

It can be seen directly from Formula (6) and Proposition 2.1 in [7] that

$$|\det \alpha^\vee| = |W||F|.$$

The calculation of the absolute value of the Jacobian determinant $\kappa = |J_x(R)|$ is straightforward from definitions (28) and (29). \square

3.2. The Cubature Formula

We attach to any $\lambda \in P^+$ a monomial $y^\lambda \equiv y_1^{\lambda_1} \dots y_n^{\lambda_n} \in \mathbb{C}[y_1, \dots, y_n]$ and assign to this monomial the m -degree $|\lambda|_m$ of λ . The m -degree of a polynomial $p \in \mathbb{C}[y_1, \dots, y_n]$, denoted by $\deg_m p$, is defined as the largest m -degree of a polynomial occurring in $p(y) \equiv p(y_1, \dots, y_n)$. For instance we observe from Proposition 1 and (20) that the m -degree of each C-polynomial \tilde{p}_λ coincides with the m -degree of λ ,

$$\deg_m \tilde{p}_\lambda = |\lambda|_m. \tag{37}$$

The subspace $\Pi_M \subset \mathbb{C}[y_1, \dots, y_n]$ is formed by the polynomials of m -degree at most M , i.e.,

$$\Pi_M \equiv \left\{ p \in \mathbb{C}[y_1, \dots, y_n] \mid \deg_m p \leq M \right\}. \tag{38}$$

In order to investigate how the m -degree of a polynomial changes under the substitution of the type (29) we formulate the following proposition.

Proposition 5. Let $j, k \in \{1, \dots, n\}$ be two distinct indices $j < k$ such that $m_j^\vee = m_k^\vee$ and $p, \tilde{p} \in \mathbb{C}[y_1, \dots, y_n]$ two polynomials such that

$$\tilde{p}(y_1, \dots, y_n) = p(y_1, \dots, y_{j-1}, \frac{y_j + y_k}{2}, \dots, y_{k-1}, \frac{y_j - y_k}{2i}, \dots, y_n)$$

holds. Then $\deg_m \tilde{p} = \deg_m p$.

Proof. Since any polynomial $p \in \Pi_M$ is a linear combination of monomials y^λ with $|\lambda|_m \leq M$, it is sufficient to prove $\tilde{p} \in \Pi_M$ for all monomials p of m -degree at most M . If p is a monomial y^λ with $|\lambda|_m \leq M$, then

$$\tilde{p}(y_1, \dots, y_n) = \left(\frac{y_j + y_k}{2}\right)^{\lambda_j} \left(\frac{y_j - y_k}{2i}\right)^{\lambda_k} \prod_{l \in \{1, \dots, n\} \setminus \{j, k\}} y_l^{\lambda_l}.$$

Using the binomial expansion, we obtain

$$\tilde{p}(y_1, \dots, y_n) = \frac{1}{i^{\lambda_k} 2^{\lambda_j + \lambda_k}} \sum_{r=0}^{\lambda_j} \sum_{s=0}^{\lambda_k} (-1)^{\lambda_k - s} \binom{\lambda_j}{r} \binom{\lambda_k}{s} y_j^{r+s} y_k^{\lambda_j + \lambda_k - (r+s)} \prod_{l \in \{1, \dots, n\} \setminus \{j, k\}} y_l^{\lambda_l}.$$

Therefore, the m -degree of the polynomial \tilde{p} is given by

$$\deg_m \tilde{p} = \max_{r,s} \{ (r+s)m_j^\vee + (\lambda_j + \lambda_k - (r+s))m_k^\vee + \sum_{l \in \{1, \dots, n\} \setminus \{j, k\}} \lambda_l m_l^\vee \}.$$

Since we assume that $m_j^\vee = m_k^\vee$, we conclude that $\deg_m \tilde{p} = \sum_{l=1}^n \lambda_l m_l^\vee = \deg_m p$. \square

Having the X_M -transform (33), it is possible to transfer uniquely the values (10) of $\varepsilon(x)$, $x \in F_M$ to the points of Ω_M , i.e., by the relation

$$\tilde{\varepsilon}(y) \equiv \varepsilon(X_M^{-1}y), \quad y \in \Omega_M. \tag{39}$$

Taking the inverse transforms of (28) and (29) and substituting them into the polynomials (24) and (20) we obtain the polynomials $K, p_\lambda \in \mathbb{C}[y_1, \dots, y_n]$ such that

$$|S_\rho|^2 = K(X_1, \dots, X_n). \tag{40}$$

Theorem 6 (Cubature formula). For any $M \in \mathbb{N}$ and any $p \in \Pi_{2M-1}$ it holds that

$$\int_\Omega p(y) K^{-\frac{1}{2}}(y) dy = \frac{\kappa}{c|W|} \left(\frac{2\pi}{M}\right)^n \sum_{y \in \Omega_M} \tilde{\varepsilon}(y) p(y). \tag{41}$$

Proof. Proposition 3 guarantees that the X -transform is one-to-one except for some set of measure zero and Proposition 4 together with (23) gives that the Jacobian determinant is non-zero except for the boundary of F . Thus using the integration by substitution $y = X(x)$ we obtain

$$\int_{\Omega} p(y)K^{-\frac{1}{2}}(y) dy = \frac{\kappa(2\pi)^n}{|F||W|} \int_F p(X(x)) dx.$$

The one-to-one correspondence for the points F_M and Ω_M from Proposition 3 enables us to rewrite the finite sum in (41) as

$$\frac{\kappa}{c|W|} \left(\frac{2\pi}{M}\right)^n \sum_{y \in \Omega_M} \tilde{\varepsilon}(y)p(y) = \frac{\kappa}{c|W|} \left(\frac{2\pi}{M}\right)^n \sum_{x \in F_M} \varepsilon(x)p(X(x)). \tag{42}$$

Successively applying Proposition 5 to perform the substitutions (29) in p we conclude that there exists a polynomial $\tilde{p} \in \Pi_{2M-1}$ such that $\tilde{p}(Z_1, \dots, Z_n) = p(X_1, \dots, X_n)$. Due to (19) we obtain for the polynomial \tilde{p} that

$$p(X_1, \dots, X_n) = \tilde{p}(Z_1, \dots, Z_n) = \sum_{\lambda \in P_{2M-1}^+} \tilde{c}_{\lambda} Z_1^{\lambda_1} Z_2^{\lambda_2} \dots Z_n^{\lambda_n} = \sum_{\lambda \in P_{2M-1}^+} \tilde{c}_{\lambda} \sum_{\nu \leq \lambda, \nu \in P^+} c_{\nu} C_{\nu}$$

and therefore it holds that

$$\frac{1}{|F|} \int_F p(X(x)) dx = \sum_{\lambda \in P_{2M-1}^+} \tilde{c}_{\lambda} \sum_{\nu \leq \lambda, \nu \in P^+} c_{\nu} \frac{1}{|F|} \int_F C_{\nu}(x) dx \tag{43}$$

and

$$\frac{1}{cM^n} \sum_{x \in F_M} \varepsilon(x)p(X(x)) = \sum_{\lambda \in P_{2M-1}^+} \tilde{c}_{\lambda} \sum_{\nu \leq \lambda, \nu \in P^+} c_{\nu} \frac{1}{cM^n} \sum_{x \in F_M} \varepsilon(x)C_{\nu}(x). \tag{44}$$

Since Proposition 1 states that for all $\nu \leq \lambda$ it holds that $|\nu|_m \leq |\lambda|_m \leq 2M - 1$, we connect Equations (43) and (44) by Proposition 2. \square

Note that for practical purposes it may be more convenient to use the cubature formula (41) in its less developed form resulting from (42),

$$\int_{\Omega} p(y)K^{-\frac{1}{2}}(y) dy = \frac{\kappa}{c|W|} \left(\frac{2\pi}{M}\right)^n \sum_{x \in F_M} \varepsilon(x)p(X(x)). \tag{45}$$

This form may be more practical since the explicit inverse transform to X_M is usually not available and, on the contrary, the calculation of the coefficients $\varepsilon(x)$ and the points $X(x)$, $x \in F_M$ is straightforward.

4. Cubature Formulas of Rank Two

In this section we specialize the cubature formula for the irreducible root systems of rank two. Let us firstly recall some basic facts about root systems of rank 2, i.e., A_2, C_2 and G_2 . They are characterized by two simple roots $\Delta = (\alpha_1, \alpha_2)$ which satisfy

$$\begin{aligned} A_2 : & \langle \alpha_1, \alpha_1 \rangle = 2, \quad \langle \alpha_2, \alpha_2 \rangle = 2, \quad \langle \alpha_1, \alpha_2 \rangle = -1, \\ C_2 : & \langle \alpha_1, \alpha_1 \rangle = 1, \quad \langle \alpha_2, \alpha_2 \rangle = 2, \quad \langle \alpha_1, \alpha_2 \rangle = -1, \\ G_2 : & \langle \alpha_1, \alpha_1 \rangle = 2, \quad \langle \alpha_2, \alpha_2 \rangle = \frac{2}{3}, \quad \langle \alpha_1, \alpha_2 \rangle = -1. \end{aligned} \tag{46}$$

The transformation rules among the root system and the remaining three bases is given as follows,

$$\begin{aligned}
 A_2: \quad & \alpha_1 = \alpha_1^\vee = 2\omega_1 - \omega_2, \quad \alpha_2 = \alpha_2^\vee = -\omega_1 + 2\omega_2, \quad \omega_1 = \omega_1^\vee, \quad \omega_2 = \omega_2^\vee, \\
 C_2: \quad & \alpha_1 = \frac{1}{2}\alpha_1^\vee = 2\omega_1 - \omega_2, \quad \alpha_2 = \alpha_2^\vee = -2\omega_1 + 2\omega_2, \quad \omega_1 = \frac{1}{2}\omega_1^\vee, \quad \omega_2 = \omega_2^\vee, \\
 G_2: \quad & \alpha_1 = \alpha_1^\vee = 2\omega_1 - 3\omega_2, \quad \alpha_2 = \frac{1}{3}\alpha_2^\vee = -\omega_1 + 2\omega_2, \quad \omega_1 = \omega_1^\vee, \quad \omega_2 = \frac{1}{3}\omega_2^\vee.
 \end{aligned}
 \tag{47}$$

Taking the weights in the standard form $\lambda = (\lambda_1, \lambda_2) = \lambda_1\omega_1 + \lambda_2\omega_2$, the corresponding Weyl group is generated by reflections r_1 and r_2 of the explicit form

$$\begin{aligned}
 A_2: \quad & r_1(\lambda_1, \lambda_2) = (-\lambda_1, \lambda_1 + \lambda_2), \quad r_2(\lambda_1, \lambda_2) = (\lambda_1 + \lambda_2, -\lambda_2), \\
 C_2: \quad & r_1(\lambda_1, \lambda_2) = (-\lambda_1, \lambda_1 + \lambda_2), \quad r_2(\lambda_1, \lambda_2) = (\lambda_1 + 2\lambda_2, -\lambda_2), \\
 G_2: \quad & r_1(\lambda_1, \lambda_2) = (-\lambda_1, 3\lambda_1 + \lambda_2), \quad r_2(\lambda_1, \lambda_2) = (\lambda_1 + \lambda_2, -\lambda_2).
 \end{aligned}
 \tag{48}$$

Any Weyl group orbit of a generic point $\lambda \in P$ consists of

$$\begin{aligned}
 A_2: \quad & \{(\lambda_1, \lambda_2), (-\lambda_1, \lambda_1 + \lambda_2), (\lambda_1 + \lambda_2, -\lambda_2), (-\lambda_2, -\lambda_1), (-\lambda_1 - \lambda_2, \lambda_1), (\lambda_2, -\lambda_1 - \lambda_2)\}, \\
 C_2: \quad & \{\pm(\lambda_1, \lambda_2), \pm(-\lambda_1, \lambda_1 + \lambda_2), \pm(\lambda_1 + 2\lambda_2, -\lambda_2), \pm(\lambda_1 + 2\lambda_2, -\lambda_1 - \lambda_2)\}, \\
 G_2: \quad & \{\pm(\lambda_1, \lambda_2), \pm(-\lambda_1, 3\lambda_1 + \lambda_2), \pm(\lambda_1 + \lambda_2, -\lambda_2), \pm(2\lambda_1 + \lambda_2, -3\lambda_1 - \lambda_2), \\
 & \pm(-\lambda_1 - \lambda_2, 3\lambda_1 + 2\lambda_2), \pm(-2\lambda_1 - \lambda_2, 3\lambda_1 + 2\lambda_2)\}.
 \end{aligned}
 \tag{49}$$

4.1. The Case A_2

If the points are considered in the α^\vee -basis, $x = a_1\alpha_1^\vee + a_2\alpha_2^\vee$, the symmetric C-functions (49) and antisymmetric S-functions of A_2 are explicitly given by

$$\begin{aligned}
 C_{(\lambda_1, \lambda_2)}(a_1, a_2) &= \frac{1}{h_\lambda} \left(e^{2\pi i(\lambda_1 a_1 + \lambda_2 a_2)} + e^{2\pi i(-\lambda_1 a_1 + (\lambda_1 + \lambda_2) a_2)} + e^{2\pi i((\lambda_1 + \lambda_2) a_1 - \lambda_2 a_2)} \right. \\
 &\quad \left. + e^{2\pi i(\lambda_2 a_1 - (\lambda_1 + \lambda_2) a_2)} + e^{2\pi i((-\lambda_1 - \lambda_2) a_1 + \lambda_1 a_2)} + e^{2\pi i(-\lambda_2 a_1 - \lambda_1 a_2)} \right), \\
 S_{(\lambda_1, \lambda_2)}(a_1, a_2) &= e^{2\pi i(\lambda_1 a_1 + \lambda_2 a_2)} - e^{2\pi i(-\lambda_1 a_1 + (\lambda_1 + \lambda_2) a_2)} - e^{2\pi i((\lambda_1 + \lambda_2) a_1 - \lambda_2 a_2)} \\
 &\quad + e^{2\pi i(\lambda_2 a_1 - (\lambda_1 + \lambda_2) a_2)} + e^{2\pi i((-\lambda_1 - \lambda_2) a_1 + \lambda_1 a_2)} - e^{2\pi i(-\lambda_2 a_1 - \lambda_1 a_2)},
 \end{aligned}$$

where the values h_λ can be found in Table 1.

Table 1. The values of h_λ are shown for A_2, C_2 and G_2 with \star denoting the corresponding coordinate different from 0.

$\lambda \in P^+$	h_λ		
	A_2	C_2	G_2
(0, 0)	6	8	12
(\star , 0)	2	2	2
(0, \star)	2	2	2
(\star , \star)	1	1	1

Performing the transform (29), the resulting real-valued functions X_1, X_2 are given by

$$\begin{aligned}
 X_1(a_1, a_2) &= \cos 2\pi a_1 + \cos 2\pi a_2 + \cos 2\pi(a_1 - a_2), \\
 X_2(a_1, a_2) &= \sin 2\pi a_1 - \sin 2\pi a_2 - \sin 2\pi(a_1 - a_2).
 \end{aligned}$$

The integration domain Ω can be described explicitly as

$$\Omega = \{(y_1, y_2) \in \mathbb{R}^2 \mid -(y_1^2 + y_2^2 + 9)^2 + 8(y_1^3 - 3y_1y_2^2) + 108 \geq 0\}.$$

The weight K -polynomial (40) is given explicitly as

$$K(y_1, y_2) = -(y_1^2 + y_2^2 + 9)^2 + 8(y_1^3 - 3y_1y_2^2) + 108.$$

The index set which will label the sets of points F_M and Ω_M is introduced via

$$I_M = \{[s_0, s_1, s_2] \in (\mathbb{Z}^{\geq 0})^3 \mid s_0 + s_1 + s_2 = M\}.$$

Thus the grid F_M consists of the points

$$F_M = \left\{ \frac{s_1}{M} \omega_1^\vee + \frac{s_2}{M} \omega_2^\vee \mid [s_0, s_1, s_2] \in I_M \right\}.$$

If for $j = [s_0, s_1, s_2] \in I_M$ we denote

$$(y_1^{(j)}, y_2^{(j)}) = \left(X_1 \left(\frac{2s_1 + s_2}{3M}, \frac{s_1 + 2s_2}{3M} \right), X_2 \left(\frac{2s_1 + s_2}{3M}, \frac{s_1 + 2s_2}{3M} \right) \right)$$

then the set of nodes Ω_M consists of the points

$$\Omega_M = \{(y_1^{(j)}, y_2^{(j)}) \in \mathbb{R}^2 \mid j \in I_M\}.$$

The integration domain Ω together with the set of nodes Ω_{15} is depicted in Figure 1.

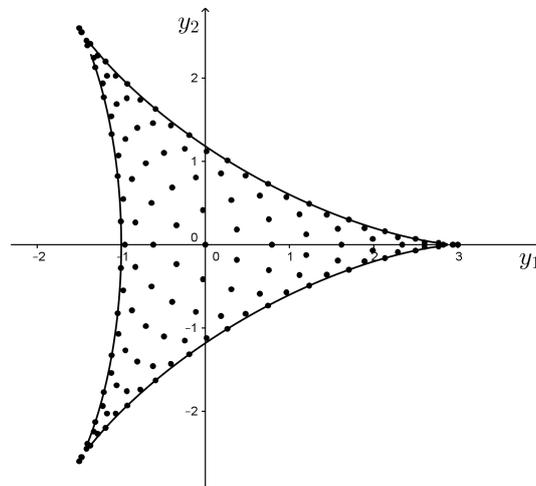


Figure 1. The region Ω of A_2 together with the points of Ω_{15} . The boundary of Ω is defined by the equation $K(y_1, y_2) = 0$.

Each point of F_M as well as of Ω_M is labeled by the index set I_M and it is convenient for the point $x \in F_M$ and its image in Ω_M labeled by $j \in I_M$ to denote

$$\varepsilon_j = \varepsilon(x) = \tilde{\varepsilon}(X_M(x)).$$

The values of ε_j can be found in Table 2. The cubature rule for any $p \in \Pi_{2M-1}$ is of the form

$$\int_{\Omega} p(y_1, y_2) K^{-\frac{1}{2}}(y_1, y_2) dy_1 dy_2 = \frac{\pi^2}{9M^2} \sum_{j \in I_M} \varepsilon_j p(y_1^{(j)}, y_2^{(j)}). \tag{50}$$

Table 2. The values of ε_j are shown for A_2, C_2 and G_2 with \star denoting the corresponding coordinate different from 0.

$j \in I_M$	ε_j		
	A_2	C_2	G_2
$[\star, 0, 0]$	1	1	1
$[0, \star, 0]$	1	2	3
$[0, 0, \star]$	1	1	2
$[\star, \star, 0]$	3	4	6
$[\star, 0, \star]$	3	4	6
$[0, \star, \star]$	3	4	6
$[\star, \star, \star]$	6	8	12

The cubature rule (50) is an analogue of the formula deduced in [24] using the generalized cosine functions TC_k and generalized sine functions TS_k defined by

$$TC_k(t) = \frac{1}{3} \left[e^{\frac{ik_2}{3}(k_2-k_3)(t_2-t_3)} \cos k_1 \pi t_1 + e^{\frac{ik_2}{3}(k_2-k_3)(t_3-t_1)} \cos k_1 \pi t_2 + e^{\frac{ik_2}{3}(k_2-k_3)(t_1-t_2)} \cos k_1 \pi t_3 \right],$$

$$TS_k(t) = \frac{1}{3} \left[e^{\frac{ik_2}{3}(k_2-k_3)(t_2-t_3)} \sin k_1 \pi t_1 + e^{\frac{ik_2}{3}(k_2-k_3)(t_3-t_1)} \sin k_1 \pi t_2 + e^{\frac{ik_2}{3}(k_2-k_3)(t_1-t_2)} \sin k_1 \pi t_3 \right],$$

where $t = (t_1, t_2, t_3) \in \mathbb{R}^3$ with $t_1 + t_2 + t_3 = 0$ and $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ with $k_1 + k_2 + k_3 = 0$. It can be shown by performing the following change of variables and parameters

$$t_1 = 2a_1 - a_2, \quad t_2 = -a_1 + 2a_2, \quad t_3 = -a_1 - a_2,$$

$$k_1 = \lambda_1, \quad k_2 = \lambda_2, \quad k_3 = -\lambda_1 - \lambda_2$$

that generalized cosine and sine functions actually coincide (up to scalar multiplication) with the symmetric C -functions and antisymmetric S -functions of A_2 . More precisely, we obtain

$$TC_k(t) = \frac{h_\lambda}{6} C_\lambda(x), \quad TS_k(t) = \frac{1}{6} S_\lambda(t).$$

Example 1. The cubature formula (50) is the exact equality of a weighted integral of any polynomial function of m -degree up to M with a weighted sum of finite number of polynomial values. It can be used in numerical integration to approximate a weighted integral of any function by finite summing.

If we choose the function $f(y_1, y_2) = K^{\frac{1}{2}}(y_1, y_2)$ as our test function, then we can estimate the integral of 1 over Ω

$$\int_{\Omega} f(y_1, y_2) K^{-\frac{1}{2}}(y_1, y_2) dy_1 dy_2 = \int_{\Omega} 1 dy_1 dy_2$$

by finite weighted sums with different M 's and compare the obtained results with the exact value of the integral of 1 which is $2\pi \doteq 6.2832$. Table 3 shows the values of the finite weighted sums for $M = 10, 20, 30, 50, 100$.

Table 3. The table shows the estimations of the integrals of 1 over the regions Ω of A_2, C_2 and G_2 resp. by finite weighted sums of the right-hand side of (50)–(52) together with the number of points in Ω_M resp. for $M = 10, 20, 30, 50, 100$.

M	10	20	30	50	100	Exact Value
A_2	6.0751	6.2314	6.2602	6.2749	6.2811	$2\pi \doteq 6.2832$
$ \Omega_M $	66	231	496	1326	5151	-
C_2	10.056	10.5133	10.5985	10.6421	10.6605	$10.666\bar{6}$
$ \Omega_M $	36	121	256	676	2601	-
G_2	7.4789	8.2561	8.4092	8.4885	8.5221	$8.533\bar{3}$
$ \Omega_M $	14	44	91	234	884	-

4.2. The Case C_2

If the points are considered in the α^\vee -basis, $x = a_1\alpha_1^\vee + a_2\alpha_2^\vee$, the symmetric C-functions (49) and antisymmetric S-functions of C_2 are explicitly given by

$$C_{(\lambda_1, \lambda_2)}(a_1, a_2) = \frac{2}{h_\lambda} (\cos 2\pi(\lambda_1 a_1 + \lambda_2 a_2) + \cos 2\pi(-\lambda_1 a_1 + (\lambda_1 + \lambda_2)a_2) \\ + \cos 2\pi((\lambda_1 + 2\lambda_2)a_1 - \lambda_2 a_2) + \cos 2\pi((\lambda_1 + 2\lambda_2)a_1 - (\lambda_1 + \lambda_2)a_2)) , \\ S_{(\lambda_1, \lambda_2)}(a_1, a_2) = 2 (\cos 2\pi(\lambda_1 a_1 + \lambda_2 a_2) - \cos 2\pi(-\lambda_1 a_1 + (\lambda_1 + \lambda_2)a_2) \\ - \cos 2\pi((\lambda_1 + 2\lambda_2)a_1 - \lambda_2 a_2) + \cos 2\pi((\lambda_1 + 2\lambda_2)a_1 - (\lambda_1 + \lambda_2)a_2)) .$$

Performing the transform (28), the resulting real-valued functions X_1, X_2 are given by

$$X_1 = 2(\cos 2\pi a_1 + \cos 2\pi(a_1 - a_2)), \quad X_2 = 2(\cos 2\pi a_2 + \cos 2\pi(2a_1 - a_2)).$$

The integration domain Ω can be described explicitly as

$$\Omega = \left\{ (y_1, y_2) \in \mathbb{R}^2 \mid -2y_1 - 4 \leq y_2, 2y_1 - 4 \leq y_2, \frac{1}{4}y_1^2 \geq y_2 \right\}.$$

The weight K -polynomial (40) is given explicitly as

$$K(y_1, y_2) = (y_1^2 - 4y_2)((y_2 + 4)^2 - 4y_1^2).$$

The index set which will label the sets of points F_M and Ω_M is introduced via

$$I_M = \left\{ [s_0, s_1, s_2] \in (\mathbb{Z}^{\geq 0})^3 \mid s_0 + 2s_1 + s_2 = M \right\}.$$

Thus the grid F_M consists of the points

$$F_M = \left\{ \frac{s_1}{M}\omega_1^\vee + \frac{s_2}{M}\omega_2^\vee \mid [s_0, s_1, s_2] \in I_M \right\}.$$

If for $j = [s_0, s_1, s_2] \in I_M$ we denote

$$(y_1^{(j)}, y_2^{(j)}) = \left(X_1 \left(\frac{2s_1 + s_2}{2M}, \frac{s_1 + s_2}{M} \right), X_2 \left(\frac{2s_1 + s_2}{2M}, \frac{s_1 + s_2}{M} \right) \right)$$

then the set of nodes Ω_M consists of the points

$$\Omega_M = \left\{ (y_1^{(j)}, y_2^{(j)}) \in \mathbb{R}^2 \mid j \in I_M \right\}.$$

The integration domain Ω together with the set of nodes Ω_{15} is depicted in Figure 2.

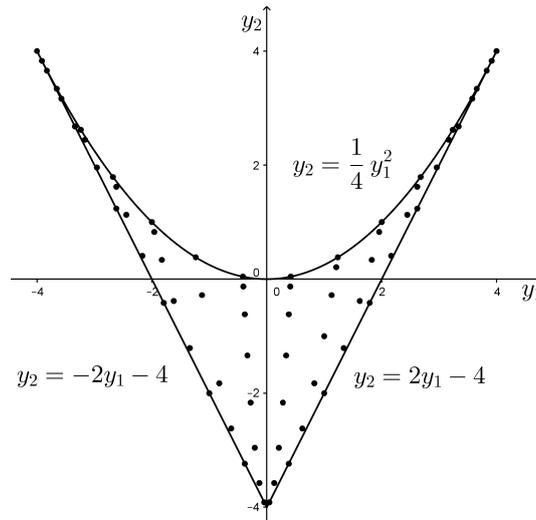


Figure 2. The region Ω of C_2 together with the points Ω_{15} . The boundary is described by three equations $y_2 = -2y_1 - 4$, $y_2 = 2y_1 - 4$ and $y_2 = \frac{1}{4}y_1^2$.

Similarly to the case A_2 , each point of F_M as well as of Ω_M is labeled by the index set I_M and it is convenient for the point $x \in F_M$ and its image in Ω_M labeled by $j \in I_M$ to denote

$$\varepsilon_j = \varepsilon(x) = \varepsilon(X_M(x)).$$

The values of ε_j can be found in Table 2.

The cubature rule for any $p \in \Pi_{2M-1}$ takes the form

$$\int_{\Omega} p(y_1, y_2) K^{-\frac{1}{2}}(y_1, y_2) dy_1 dy_2 = \frac{\pi^2}{4M^2} \sum_{j \in I_M} \varepsilon_j p(y_1^{(j)}, y_2^{(j)}). \tag{51}$$

Example 2. The cubature formula (51) is the exact equality of a weighted integral of any polynomial function of m -degree up to M with a weighted sum of finite number of polynomial values. It can be used in numerical integration to approximate a weighted integral of any function by finite summing.

Similarly to Example 1, if we choose the function $f(y_1, y_2) = K^{\frac{1}{2}}(y_1, y_2)$ as our test function, then we can estimate the integral of 1 over Ω

$$\int_{\Omega} f(y_1, y_2) K^{-\frac{1}{2}}(y_1, y_2) dy_1 dy_2 = \int_{\Omega} 1 dy_1 dy_2$$

by finite weighted sums with different M 's and compare the obtained results with the exact value of the integral of 1 which is $\frac{32}{3} = 10.666\bar{6}$. Table 3 shows the values of the finite weighted sums for $M = 10, 20, 30, 50, 100$.

4.3. The Case G_2

If the points are considered in the α^\vee -basis, $x = a_1\alpha_1^\vee + a_2\alpha_2^\vee$, the symmetric C-functions (49) and antisymmetric S-functions of G_2 are explicitly given by

$$\begin{aligned} C_{(\lambda_1, \lambda_2)}(a_1, a_2) &= \frac{2}{h_\lambda} (\cos 2\pi(\lambda_1 a_1 + \lambda_2 a_2) + \cos 2\pi(-\lambda_1 a_1 + (3\lambda_1 + \lambda_2)a_2) \\ &\quad + \cos 2\pi((\lambda_1 + \lambda_2)a_1 - \lambda_2 a_2) + \cos 2\pi((2\lambda_1 + \lambda_2)a_1 - (3\lambda_1 + \lambda_2)a_2) \\ &\quad + \cos 2\pi((-\lambda_1 - \lambda_2)a_1 + (3\lambda_1 + 2\lambda_2)a_2) + \cos 2\pi((-2\lambda_1 - \lambda_2)a_1 + (3\lambda_1 + 2\lambda_2)a_2)) , \\ S_{(\lambda_1, \lambda_2)}(a_1, a_2) &= 2 (\cos 2\pi(\lambda_1 a_1 + \lambda_2 a_2) - \cos 2\pi(-\lambda_1 a_1 + (3\lambda_1 + \lambda_2)a_2) \\ &\quad - \cos 2\pi((\lambda_1 + \lambda_2)a_1 - \lambda_2 a_2) + \cos 2\pi((2\lambda_1 + \lambda_2)a_1 - (3\lambda_1 + \lambda_2)a_2) \\ &\quad + \cos 2\pi((-\lambda_1 - \lambda_2)a_1 + (3\lambda_1 + 2\lambda_2)a_2) - \cos 2\pi((-2\lambda_1 - \lambda_2)a_1 + (3\lambda_1 + 2\lambda_2)a_2)) . \end{aligned}$$

Performing the transform (28), the resulting real-valued functions X_1, X_2 are given by

$$\begin{aligned} X_1 &= 2(\cos 2\pi a_1 + \cos 2\pi(a_1 - 3a_2) + \cos 2\pi(2a_1 - 3a_2)) , \\ X_2 &= 2(\cos 2\pi a_2 + \cos 2\pi(a_1 - a_2) + \cos 2\pi(a_1 - 2a_2)) . \end{aligned}$$

The integration domain Ω can be described explicitly as

$$\Omega = \left\{ (y_1, y_2) \in \mathbb{R}^2 \mid -2((y_2 + 3)^{\frac{3}{2}} + 3y_2 + 6) \leq y_1 \leq 2((y_2 + 3)^{\frac{3}{2}} - 3y_2 - 6), y_1 \geq \frac{1}{4}y_2^2 - 3 \right\} .$$

The weight K -polynomial (40) is given explicitly as

$$K(y_1, y_2) = (y_2^2 - 4y_1 - 12)(y_1^2 - 4y_2^3 + 12y_1y_2 + 24y_1 + 36y_2 + 36) .$$

The index set which will label the sets of points F_M and Ω_M is introduced via

$$I_M = \{[s_0, s_1, s_2] \in (\mathbb{Z}^{\geq 0})^3 \mid s_0 + 2s_1 + 3s_2 = M\} .$$

Thus the grid F_M consists of the points

$$F_M = \left\{ \frac{s_1}{M}\omega_1^\vee + \frac{s_2}{M}\omega_2^\vee \mid [s_0, s_1, s_2] \in I_M \right\} .$$

If for $j = [s_0, s_1, s_2] \in I_M$ we denote

$$(y_1^{(j)}, y_2^{(j)}) = \left(X_1 \left(\frac{2s_1 + 3s_2}{M}, \frac{s_1 + 2s_2}{M} \right), X_2 \left(\frac{2s_1 + 3s_2}{M}, \frac{s_1 + 2s_2}{M} \right) \right)$$

then the set of nodes Ω_M consists of the points

$$\Omega_M = \left\{ (y_1^{(j)}, y_2^{(j)}) \in \mathbb{R}^2 \mid j \in I_M \right\} .$$

The integration domain Ω together with the set of nodes Ω_{15} is depicted in Figure 3.

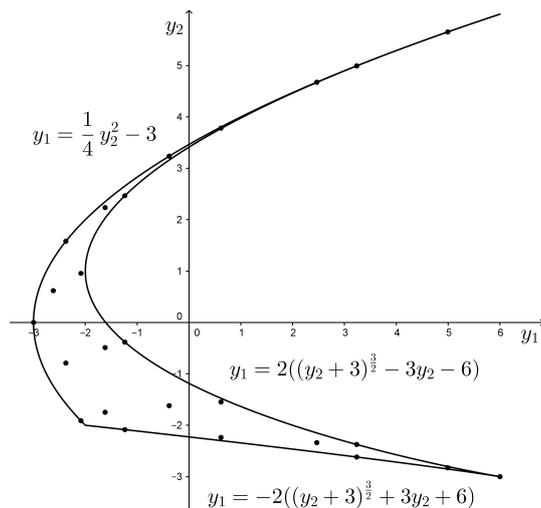


Figure 3. The region Ω of G_2 together with the points of Ω_{15} . The boundary of Ω is described by three equations $y_1 = \frac{1}{4}y_2^2 - 3$, $y_1 = 2((y_2 + 3)^{\frac{3}{2}} - 3y_2 - 6)$ and $y_1 = -2((y_2 + 3)^{\frac{3}{2}} + 3y_2 + 6)$.

Similarly to the case A_2 , each point of F_M as well as of Ω_M is labeled by the index set I_M and it is convenient for the point $x \in F_M$ and its image in Ω_M labeled by $j \in I_M$ to denote

$$\varepsilon_j = \varepsilon(x) = \tilde{\varepsilon}(X_M(x)).$$

The values of ε_j can be found in Table 2. The cubature rule for any $p \in \Pi_{2M-1}$ is of the form

$$\int_{\Omega} p(y_1, y_2) K^{-\frac{1}{2}}(y_1, y_2) dy_1 dy_2 = \frac{\pi^2}{3M^2} \sum_{j \in I_M} \varepsilon_j p(y_1^{(j)}, y_2^{(j)}). \tag{52}$$

As in the case A_2 , the cubature rule (52) is similar to the Gauss-Lobatto cubature formula derived in [25], where Xu et al study four types of functions $CC_k(t)$, $SC_k(t)$, $CS_k(t)$ and $SS_k(t)$ closely related to the orbit functions over Weyl groups. Concretely, the functions $CC_k(t)$ and $SS_k(t)$ are given by

$$\begin{aligned}
 CC_k(t) &= \frac{1}{3} \left[\cos \frac{\pi(k_1 - k_3)(t_1 - t_3)}{3} \cos \pi k_2 t_2 + \cos \frac{\pi(k_1 - k_3)(t_2 - t_1)}{3} \cos \pi k_2 t_3 \right. \\
 &\quad \left. + \cos \frac{\pi(k_1 - k_3)(t_3 - t_2)}{3} \cos \pi k_2 t_1 \right], \\
 SS_k(t) &= \frac{1}{3} \left[\sin \frac{\pi(k_1 - k_3)(t_1 - t_3)}{3} \sin \pi k_2 t_2 + \sin \frac{\pi(k_1 - k_3)(t_2 - t_1)}{3} \sin \pi k_2 t_3 \right. \\
 &\quad \left. + \sin \frac{\pi(k_1 - k_3)(t_3 - t_2)}{3} \sin \pi k_2 t_1 \right],
 \end{aligned}$$

where the variable t is given by homogeneous coordinates, i.e.,

$$t = (t_1, t_2, t_3) \in \mathbb{R}_H^3 = \{t \in \mathbb{R}^3 \mid t_1 + t_2 + t_3 = 0\}$$

and parameter $k = (k_1, k_2, k_3) \in \mathbb{Z}^3 \cap \mathbb{R}_H^3$. It can be verified that the linear transformations of variable and parameter

$$\begin{aligned}
 t_1 &= -a_1 + 3a_2, & t_2 &= 2a_1 - 3a_2, & t_3 &= -a_1, \\
 k_1 &= \lambda_1 + \lambda_2, & k_2 &= \lambda_1, & k_3 &= -2\lambda_1 - \lambda_2
 \end{aligned}$$

give the connection with C-functions and S-functions of G_2 :

$$C_\lambda(x) = \frac{12}{h_\lambda} CC_k(t), \quad S_\lambda(x) = -12SS_k(t).$$

Example 3. The cubature formula (52) is the exact equality of a weighted integral of any polynomial function of m -degree up to M with a weighted sum of finite number of polynomial values. It can be used in numerical integration to approximate a weighted integral of any function by finite summing.

Similarly to Example 1, if we choose the function $f(y_1, y_2) = K^{\frac{1}{2}}(y_1, y_2)$ as our test function, then we can estimate the integral of 1 over Ω

$$\int_{\Omega} f(y_1, y_2) K^{-\frac{1}{2}}(y_1, y_2) dy_1 dy_2 = \int_{\Omega} 1 dy_1 dy_2$$

by finite weighted sums with different M 's and compare the obtained results with the exact value of the integral of 1 which is $\frac{128}{15} = 8.533\bar{3}$. Table 3 shows the values of the finite weighted sums for $M = 10, 20, 30, 50, 100$.

5. Polynomial Approximations

5.1. The Optimal Polynomial Approximation

Since the polynomial function (40) is continuous and strictly positive in Ω° , its square root $K^{-\frac{1}{2}}$ can serve as a weight function for the weighted Hilbert space $\mathcal{L}_K^2(\Omega)$, i.e., a space of complex-valued cosets of measurable functions f such that $\int_{\Omega} |f|^2 K^{-\frac{1}{2}} < \infty$ with an inner product defined by

$$(f, g)_K = \frac{1}{\kappa(2\pi)^n} \int_{\Omega} f(y) \overline{g(y)} K^{-\frac{1}{2}}(y) dy. \tag{53}$$

Our aim is to construct a suitable Hilbert basis of $\mathcal{L}_K^2(\Omega)$. Taking the inverse transforms of (28) and (29) and substituting them into the polynomials (20) we obtain the polynomials $p_\lambda \in \mathbb{C}[y_1, \dots, y_n]$ such that

$$C_\lambda = p_\lambda(X_1, \dots, X_n). \tag{54}$$

Moreover, successively applying Proposition 5 to perform the substitutions (29) in p_λ and taking into account (37) we obtain that

$$\deg_m p_\lambda = |\lambda|_m. \tag{55}$$

Calculating the scalar product (53) for the p_λ polynomials (54), we obtain that the continuous orthogonality of the C-functions (14) is inherited, i.e.,

$$(p_\lambda, p_{\lambda'})_K = h_\lambda^{-1} \delta_{\lambda, \lambda'}, \quad \lambda, \lambda' \in P^+. \tag{56}$$

Assigning to any function $f \in \mathcal{L}_K^2(\Omega)$ a function $\tilde{f} \in \mathcal{L}^2(F)$ by the relation $\tilde{f}(x) = f(X(x))$ and taking into account the expansion (15) we obtain for its expansion coefficients c_λ that

$$c_\lambda = \frac{h_\lambda}{|W||F|} \int_F \tilde{f}(x) \overline{C_\lambda(x)} dx = \frac{h_\lambda}{\kappa(2\pi)^n} \int_{\Omega} f(y) \overline{p_\lambda(y)} K^{-\frac{1}{2}}(y) dy. \tag{57}$$

Therefore any $f \in \mathcal{L}_K^2(\Omega)$ can be expanded in terms of p_λ ,

$$f = \sum_{\lambda \in P^+} a_\lambda p_\lambda, \quad a_\lambda = h_\lambda (f, p_\lambda)_K \tag{58}$$

and the set of C-polynomials $p_\lambda, \lambda \in P^+$ is a Hilbert basis of $\mathcal{L}_K^2(\Omega)$.

To construct a basis of the space of multivariate polynomials Π_M suffices to note that Equation (19) guarantees that each monomial $Z_1^{\lambda_1} Z_2^{\lambda_2} \dots Z_n^{\lambda_n}$ can be expanded in terms of C_λ with $\lambda \in P_M^+$; the same can be said about the transformed monomials $X_1^{\lambda_1} X_2^{\lambda_2} \dots X_n^{\lambda_n}$. Thus by the same argument as above we obtain for any $p \in \Pi_M$ the expansion

$$p = \sum_{\lambda \in P_M^+} b_\lambda p_\lambda, \quad b_\lambda = h_\lambda(p, p_\lambda)_K. \tag{59}$$

Truncating the series (58) to the finite set P_M^+ we obtain a polynomial approximation $u_M[f] \in \Pi_M$ of the functions $f \in \mathcal{L}_K^2(\Omega)$,

$$u_M[f] = \sum_{\lambda \in P_M^+} a_\lambda p_\lambda, \quad a_\lambda = h_\lambda(f, p_\lambda)_K. \tag{60}$$

Relative to the $\mathcal{L}_K^2(\Omega)$ norm is this approximation indeed optimal among all polynomials from Π_M as states the following proposition.

Proposition 7. For any $f \in \mathcal{L}_K^2(\Omega)$ is the $u_M[f]$ polynomial (60) the best approximation of f , relative to the $\mathcal{L}_K^2(\Omega)$ -norm, by any polynomial from Π_M .

Proof. Consider any $p \in \Pi_M$ a polynomial of the form (59), any $f \in \mathcal{L}_K^2(\Omega)$ expanded by (58) and $u_M[f]$ an approximation polynomial (60). Then we calculate that

$$\begin{aligned} (f - p, f - p)_K &= (f, f)_K - (f, p)_K - (p, f)_K + (p, p)_K \\ &= (f, f)_K - \sum_{\lambda \in P_M^+} h_\lambda^{-1} a_\lambda \bar{b}_\lambda - \sum_{\lambda \in P_M^+} h_\lambda^{-1} b_\lambda \bar{a}_\lambda + \sum_{\lambda \in P_M^+} h_\lambda^{-1} |b_\lambda|^2 \\ &= (f - u_M[f], f - u_M[f])_K + \sum_{\lambda \in P_M^+} h_\lambda^{-1} |b_\lambda - a_\lambda|^2 \geq (f - u_M[f], f - u_M[f])_K. \end{aligned}$$

□

5.2. The Cubature Polynomial Approximation

Rather than the optimal polynomial approximation (60) one may consider for practical applications its weakened version. Such a weaker version is obtained by using the cubature formula for an approximate calculation of $(f, p_\lambda)_K$, i.e., we set

$$v_M[f] = \sum_{\lambda \in P_M^+} a_\lambda p_\lambda, \quad a_\lambda = \frac{h_\lambda}{c|W|M^n} \sum_{y \in \Omega_M} \tilde{\varepsilon}(y) f(y) \overline{p_\lambda(y)}. \tag{61}$$

Since for $f \in \Pi_{M-1}$ and $p_\lambda, \lambda \in P_M^+$ it holds that $\overline{f p_\lambda} \in \Pi_{2M-1}$ and the cubature formula is thus valid, we obtain that the optimal approximation coincides with $v_M[f]$,

$$v_M[f] = u_M[f], \quad f \in \Pi_{M-1}. \tag{62}$$

Example 4. As a specific example of a continuous model function in the case C_2 , we consider

$$f(y_1, y_2) = e^{-(y_1^2 + (y_2 + 1.8)^2) / (2 \times 0.35^2)}$$

defined on Ω . The graph of f together with its approximations $v_M[f]$ for $M = 10, 20, 30$ is shown in Figure 4.

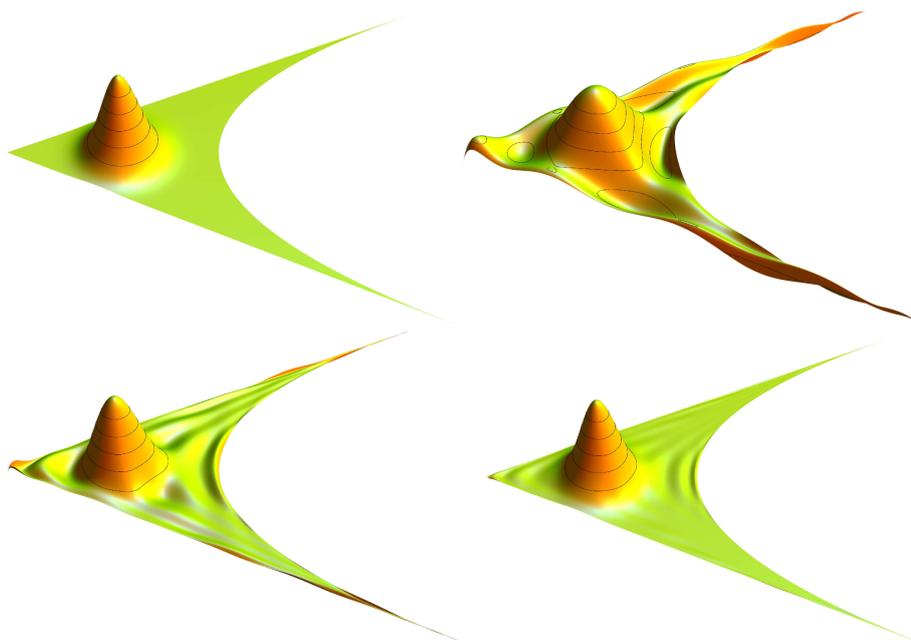


Figure 4. The figure shows the model function f and its approximations $v_M[f]$ for $M = 10, 20, 30$ on Ω of C_2 .

Integral error estimates of the polynomial approximations $v_M[f]$

$$\int_{\Omega} |f(y_1, y_2) - v_M[f](y_1, y_2)|^2 K^{-\frac{1}{2}}(y_1, y_2) dy_1 dy_2$$

can be found in Table 4.

Table 4. The table shows the values of integral error estimates of the polynomial approximations $v_M[f]$ for $M = 10, 20, 30$.

M	10	20	30
$\int_{\Omega} f - v_M[f] ^2 K^{-\frac{1}{2}} dy_1 dy_2$	0.0636842	0.0035217	0.0000636

6. Conclusions

- Due to the generality of the present construction of the cubature formulas, some of the cases presented in this paper appeared already in the literature. The case A_2 is closely related to two-variable analogues of Jacobi polynomials on Steiner’s hypocycloid [26]; the case C_2 is related to two-variable analogues of Jacobi polynomials on a domain bounded by two lines and a parabola [26–28] and the corresponding Gaussian cubature formulas induced by these polynomials are studied for example in [10]. The case G_2 and its cubature formulas are detailed in [25].
- The Chebyshev polynomials of the first kind induce the cubature formula of the maximal efficiency—Gauss-Chebyshev quadrature [5,29]. The nodes of this formula are M roots of the Chebyshev polynomials of the first kind of degree M and it exactly evaluates a weighted integral for any polynomial of degree at most $2M - 1$. The set of nodes Ω_M of A_1 does not correspond to the set of roots of the Gauss-Chebyshev quadrature— Ω_M of A_1 consists of $M + 1$ points and includes two boundary points of the interval. Thus, the number of points Ω_M exceeds by one the minimum number of nodes and the resulting cubature formula is not optimal. This phenomenon, already observed for the A_n sequence in [9], generalizes to all cases of Weyl groups. Even though

the developed formulas are slightly less efficient than the optimal ones, they contain points on the entire boundary of Ω_M —this might be useful for some applications.

- The technique for construction of the cubature formulas in this article is based on discrete orthogonality of C -functions over the set F_M . For the case A_1 the cosine transform corresponding to the discrete orthogonality of cosines is standardly known as discrete cosine transform of the type DCT-I [30]. The M roots of the Chebyshev polynomials of the first kind, which lead to the optimal cubature formula, enter the discrete orthogonality relations of the type DCT-II. This indicates that the discrete orthogonality relations of C -functions, which generalize the transforms of the type DCT-II, DCT-III and DCT-IV for admissible cases of Weyl groups [31], might lead to cubature formulas of higher efficiency.
- The weaker version of the polynomial approximation (60) assigns to any complex function $f : \Omega \rightarrow \mathbb{C}$ a polynomial functional series $\{u_M[f]\}_{M=1}^{\infty}$. Existence of conditions for convergence of these functional series together with an estimate of the approximation error $\int |f - u_M[f]|^2 K^{-\frac{1}{2}}$ poses an open problem.
- The Clenshaw-Curtis method for deriving cubature formulas is based on expressing a given function into a series of the corresponding set of orthogonal polynomials and then integrating the series term by term [32,33]. The coefficients in the series are calculated using discrete orthogonality properties. The comparison of the resulting cubature formulas by the Clenshaw-Curtis technique and the method used in the present paper and in [1,2] deserves further study.
- The polynomials of the orbit functions used in the present paper and in [1,2] to derive the cubature formulas are directly related to the Jacobi polynomials associated to root systems as well as to the Macdonald polynomials. The discrete orthogonality of the Macdonald polynomials with unitary parameters, achieved in [34], opens a possibility of an extension of the current cubature formulas to the corresponding subset of the Macdonald polynomials.

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