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E-Polytopes in Picard Groups of Smooth Rational Surfaces

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Academic Editor: Brigitte Servatius
Received: 30 September 2015; Accepted: 13 April 2016; Published: 20 April 2016

Abstract: In this article, we introduce special divisors (root, line, ruling, exceptional system and rational quartic) in smooth rational surfaces and study their correspondences to subpolytopes in Gosset polytopes k_{21}. We also show that the sets of rulings and exceptional systems correspond equivariantly to the vertices of 2_{k1} and 1_{k2} via E-type Weyl action.

Keywords: del Pezzo surface; Hirzebruch surface; Gosset polytope; E-polytope

MSC Classifications: 14J26; 14E05

1. Introduction

Rational surfaces are complex surfaces birational to $\mathbb{P}^2$. According to the classification of surfaces [1], the minimal surfaces of smooth rational surfaces are either a projective plane $\mathbb{P}^2$ or a Hirzebruch surface $\mathbf{F}$. In particular, the typical examples of del Pezzo surfaces, which are smooth surfaces S_r with the ample anticanonical divisor class $-K_{S_r}$, are obtained by blowing up r (<9) points of $\mathbb{P}^2$ in general position. Del Pezzo surfaces have drawn the attention of mathematicians and physicists because of their geometries and dualities involving mysterious symmetries. For example, the special divisor classes l (called lines) of a del Pezzo surface S_r satisfying $l^2 = l \cdot K_{S_r} = -1$ are bijectively related to the vertices of Gosset polytopes $(r-4)_{21}$, which are one type of semiregular polytopes given by the action of the E_r symmetry group (called E-polytopes). The well-known 27 lines in a cubic surface S_6 are bijectively related to the vertices of a Gosset 2_{21} obtained by an E_6-action, and it is well known that the configuration of the 27 lines can also be understood via the action of the Weyl group E_6 [2,3]. Coxeter [4] applied the bijection between the lines of S_6 and the vertices in 2_{21} to study the geometry of 2_{21}. The complete list [5] of bijections between lines in del Pezzo surfaces S_r and vertices in Gosset polytopes $(r-4)_{21}$ is well known, and the bijections play key roles in many different research fields[6–8]. In particular, the classical application appeared in the study by Du Val [9].

The first author constructed Gosset polytopes as convex hulls in the Picard group $\mathrm{Pic}(S_r)$ and extended the bijections between lines and vertices to correspondences between special divisors in the del Pezzo surface S_r and subpolytopes in $(r-4)_{21}$ via the Weyl E_r-action. correspondences were applied to study the geometry of del Pezzo surfaces and the geometry of Gosset polytopes [10–12].

We consider the special divisor classes D of del Pezzo surfaces S_r [10–12], which are *rational* with self intersection $D^2 = -2, -1, 0, 1, 2$ (called root, line, ruling, exceptional system, quartic rational divisor, respectively). Here, the rulings of del Pezzo surfaces S_r are $\mathbb{P}^1$-fibrations of S_r, and the exceptional systems produce rational maps from S_r to $\mathbb{P}^2$. In fact, one can define these special divisor classes in a smooth projective variety S. We define the following *special divisor (class)* in $\mathrm{Pic}(S)$:

$$\begin{aligned}
\mathbf{R}(S) &:= \left\{ d \in \mathrm{Pic}(S) \;\middle|\; d^2 = -2, d \cdot K_S = 0 \right\} \;(\textit{roots}) \\
\mathbf{L}(S) &:= \left\{ l \in \mathrm{Pic}(S) \;\middle|\; l^2 = l \cdot K_S = -1 \right\} \;(\textit{lines}) \\
\mathbf{M}(S) &:= \left\{ m \in \mathrm{Pic}(S) \;\middle|\; m^2 = 0, m \cdot K_S = -2 \right\} \;(\textit{rulings}) \\
\mathcal{E}(S) &:= \left\{ e \in \mathrm{Pic}(S) \;\middle|\; e^2 = 1, e \cdot K_S = -3 \right\} \;(\textit{exceptional systems}) \\
\mathcal{Q}(S) &:= \left\{ q \in \mathrm{Pic}(S) \;\middle|\; q^2 = 2, q \cdot K_S = -4 \right\} \;(\textit{quartic rational divisors})
\end{aligned}$$

For each root d in $\mathbf{R}(S)$, we consider a *reflection* σ_d on $(\mathbb{Z}K_S)^\perp$ in $\mathrm{Pic}(S)$ as defined by

$$\sigma_d(D) := D + (D \cdot d)\, d \;\text{ for } D \in (\mathbb{Z}K_S)^\perp$$

Since each reflection σ_d preserves the intersection of divisors and fixes the canonical divisor K_S, the action of the reflection σ_d can be extended naturally to the whole Picard group $\mathrm{Pic}(S)$. Therefore, the *Weyl group* $W(S)$ generated by the reflections on $(\mathbb{Z}K_S)^\perp$ acts on $\mathrm{Pic}(S)$, and it acts on each set of special divisors. In this article, we consider the surfaces S whose Weyl groups are of E_n-type whose extended list is given as follows.

n	3	4	5	6	7	8
E_n	$A_1 \times A_2$	A_4	D_5	E_6	E_7	E_8

For rational surfaces S, we add a condition $K_S^2 > 0$ so that the intersection is negative definite on each affine hypersurface given as

$$H_b(S) := \{ D \in \mathrm{Pic}(S) | -D \cdot K_S = b \}$$

In [10], the first author showed that the set of lines $\mathbf{L}(S_r) \subset H_1(S_r)$ of del Pezzo surfaces S_r turns out to be an orbit of an E_{r+4}-action, and its convex hull in $\mathrm{Pic}(S_r) \otimes \mathbb{Q}$ is the Gosset polytope $(r-4)_{21}$. Therefore, he obtained equivariant correspondences between the special divisors and the subpolytopes in Gosset polytopes and studied the geometry of del Pezzo surfaces, such as the configurations of lines [10–12].

One can ask to extend the class of surfaces from del Pezzo surfaces to rational surfaces (with a condition $K^2 > 0$ on the canonical divisor class K) and the class of E-polytopes from Gosset polytopes related to lines to other E-polytopes related to the special divisors. In this article, we give a summary of the first author's works [10–12] on the special divisors of del Pezzo surfaces and the subpolytopes in Gosset polytopes via Weyl actions and provide an extension of the studies on del Pezzo surfaces to the rational surfaces given as the blowing up of Hirzebruch surfaces. We also extend the family of E-polytopes from Gosset polytopes of lines to other E-polytopes of special divisors. As a matter of fact, the extension of [10–12] to the blowing up of Hirzebruch surfaces is so natural that most results are parallel to [10–12]. However, there are interesting characterizations different from the studies on del Pezzo surfaces appearing along the monoidal transformation of Hirzebruch surfaces. In this article, we announce the parallel results on fundamental issues without details. The studies related to the monoidal transformation of the blowing up of Hirzebruch surfaces will be explained in [13]. For the extension of families of E-polytopes, we consider families of polytopes 2_{k1} and 1_{k2} related to rulings and exceptional systems, respectively. Further studies on correspondences between subpolytopes in 2_{k1} and 1_{k2} and divisors in rational surfaces will be described in other articles.

2. Preliminary

2.1. Smooth Rational Surfaces with $K^2 > 0$

In this article, we consider smooth rational surfaces S with the condition $K_S^2 > 0$. By the classification of surfaces [1], each smooth rational surface has a minimal surface, a projective plane $\mathbb{P}^2$, or a Hirzebruch surface $\mathbf{F}$. In this article, we only consider the blow-ups of the projective plane $\mathbb{P}^2$ and the Hirzebruch surface $\mathbf{F}$ in general position. In fact, the studies of the blow-ups of the Hirzebruch surface $\mathbf{F}$ *in general position* have not done much so that there are no typical descriptions of the general positions of points in $\mathbf{F}$. Below, we give a description of the blow-ups of the Hirzebruch surface $\mathbf{F}$ in general position matching our purpose.

First, we consider the smooth rational surface S with $K_S^2 > 0$ whose minimal surface is $\mathbb{P}^2$. We have a birational morphism $\rho\colon S \longrightarrow \mathbb{P}^2$ decomposed by contractions $\rho_j\colon S_j \longrightarrow S_{j-1}$, $j = 1, 2, \ldots, r$ of (-1)-curves e'_j (*i.e.*, $\rho = \rho_1 \circ \cdots \circ \rho_{r-1} \circ \rho_r$), where $S_r = S$ and $S_0 = \mathbb{P}^2$. We remark that each e'_j, $j = 1, 2, \ldots, r$ can be obtained by a blow-up of a point p_j on S_{j-1}. The blow-ups at points p_j, $j = 1, 2, \ldots, r$ allow infinitely near ones on $\mathbb{P}^2$. When the points p_j, $j = 1, 2 \ldots, r$ on $\mathbb{P}^2$ are in general position, we call S a del Pezzo surface with a degree $K_S^2 = 9 - r > 0$ whose anticanonical divisor $-K_S$ is ample. We denote by e_j, $j = 1, 2, \ldots, r-1$ the total transforms of (-1)-curves e'_j by $\rho_{j+1} \circ \cdots \circ \rho_{r-1} \circ \rho_r$, and $e_r := e'_r$. Then, the Picard group of S is generated by h and e_j, $j = 1, 2, \ldots, r$ (*i.e.*, $\mathrm{Pic}(S) = \mathbb{Z}h \bigoplus \mathbb{Z}e_1 \bigoplus \cdots \bigoplus \mathbb{Z}e_r$), where h is a pull-back of a line in $\mathbb{P}^2$ by ρ. Note that the canonical divisor $K_S \equiv -3h + e_1 + e_2 + \cdots + e_r$, and $r \leq 8$ since $K_S^2 = 9 - r > 0$.

Also, we deal with a smooth rational surface X with $K_X^2 > 0$ whose minimal surface is a Hirzebruch surface $\mathbf{F}$. As above we denote a birational morphism $\rho\colon X \longrightarrow \mathbf{F}$ decomposed by contractions $\rho_j\colon \mathbf{F}_j \longrightarrow \mathbf{F}_{j-1}$, $j = 1, 2, \ldots, r$ of (-1)-curves e'_j (*i.e.*, $\rho = \rho_1 \circ \cdots \circ \rho_{r-1} \circ \rho_r$), where $\mathbf{F}_r = X$ and $\mathbf{F}_0 = \mathbf{F}$; moreover, we have a map $\varphi'\colon \mathbf{F} \longrightarrow \mathbb{P}^1$, which gives a fibration $\varphi := \varphi' \circ \rho\colon X \longrightarrow \mathbb{P}^1$. We denote by f and s a general fibre of φ and the special section of φ whose self-intersection number is a nonpositive integer $-p$, respectively. We use $\mathbf{F}_{p,r}$ instead of $\mathbf{F}_r$ unless there is no confusion. When we consider the contraction ρ_j of e'_j as a (possibly infinitely near) blow-up of a point p_j on $\mathbf{F}_{j-1}$, the point p_j is located in the special section s_{j-1} on $\mathbf{F}_{j-1}$ or not. If p_j is not a point in s_{j-1} then $s_j^2 = s_{j-1}^2$. On the other hand, if p_j is a point in s_{j-1} then $s_j^2 = s_{j-1}^2 - 1$. Thus, we define that distinct points $p_{j+1}, p_{j+2}, \ldots, p_r$ are located on $\mathbf{F}_{-s_j^2,j} := \mathbf{F}_j$ *in general position* if each point p_k, $k = j+1, \ldots, r$ is not in a special section s_j ($s_j^2 \neq 0$) and there are no two points p_{k_1}, p_{k_2} for $k_1 \neq k_2 \in \{j+1, \ldots, r\}$ which are in a common general fibre of $\varphi' \circ \rho_1 \circ \rho_2 \circ \cdots \circ \rho_j$. As above, we denote by e_j, $j = 1, 2, \ldots, r-1$ the total transforms of (-1)-curves e'_j by $\rho_{j+1} \circ \cdots \circ \rho_{r-1} \circ \rho_r$, and $e_r := e'_r$. Then the Picard group of X is generated by f, s and e_j, $j = 1, 2, \ldots, r$ (i.e. $\mathrm{Pic}(X) = \mathbb{Z}f \bigoplus \mathbb{Z}s \bigoplus \mathbb{Z}e_1 \bigoplus \cdots \bigoplus \mathbb{Z}e_r$). The canonical divisor K_X is written as $K_X \equiv (-2-p)f - 2s + e_1 + e_2 + \cdots + e_r$, and we have $r \leq 7$ since $K_X^2 = 8 - r > 0$.

Remark 1. *We can consider an isometric isomorphism $\psi\colon \mathrm{Pic}(\mathbf{F}_{p,r-1}) \longrightarrow \mathrm{Pic}(S_r)$ for $r = 3, 4, \ldots, 8$ defined by*

$$\begin{cases} \psi(e_i) = e_{i+1},\ 2 \leq i \leq r, \\ \psi(e_1) = h - e_1 - e_2, \\ \psi(f) = h - e_1, \\ \psi(s) = \frac{2-p}{2}h + \frac{p}{2}e_1 - e_2 \end{cases} \quad \text{if } p \text{ is even}$$

$$\begin{cases} \psi(e_i) = e_{i+1},\ 1 \leq i \leq r, \\ \psi(f) = h - e_1, \\ \psi(s) = \frac{1-p}{2}h + \frac{1+p}{2}e_1 \end{cases} \quad \text{if } p \text{ is odd}$$

Then ψ preserves intersection pairings, root systems and the canonical divisor classes between $\mathrm{Pic}(\mathbf{F}_{p,r-1})$ *and* $\mathrm{Pic}(S_r)$. *Note the isometric isomorphism in the proof of Lemma 3.2 in [14]. Thus, we can naturally extend the correspondence between the special divisors of a del Pezzo surface S_r and the subpolytopes of a Gosset polytope* *to one for* $\mathbf{F}_{p,r-1}$*. Because the correspondence was established for a del Pezzo surface in [10,12] at first, we use the argument of [10,12] in Sections 3 and 4 to present the original idea, although we have the isometric isomorphism* ψ*.*

2.2. *ADE-Polytopes*

The polytopes under consideration in this article are characterized by the highly nontrivial symmetries given by *Coxeter groups*. Moreover, the polytopes are determined by the corresponding Coxeter–Dynkin diagrams. In this subsection, we introduce the general theory of regular and semiregular polytopes according to their symmetry groups and the corresponding Coxeter–Dynkin diagrams. In particular, we consider a family of semiregular polytopes known as the Gosset polytopes, (k_{21} according to Coxeter), 2_{k1} and 1_{k2}. Here, we only present a brief introduction; for further detail, refer to [10,15].

2.2.1. Regular and Semiregular Polytopes

Let P_n be a convex n-polytope in an n-dimensional Euclidean space. For each vertex O of P_n, when the set of midpoints of all the edges emanating from the vertex O in P_n is contained in an affine hyperplane, the set consists of the vertices of an $(n-1)$-polytope. This $(n-1)$-polytope is called the *vertex figure of* P_n at O. A *regular* polytope P_n $(n \geq 2)$ is a polytope whose facets and the vertex figure at each vertex are regular. Thus, the facets of a regular P_n are all congruent, and the vertex figures are the same. We also call a polytope P_n a *semiregular* one if its facets are regular and the symmetry group of P_n acts transitively on the vertices of P_n. Here we consider two kinds of regular polytopes, a regular simplex and a crosspolytope. A *regular simplex* α_n is an n-dimensional simplex with equilateral edges, and a *crosspolytope* β_n is an n-dimensional polytope whose $2n$-vertices are the intersections between an n-dimensional Cartesian coordinate frame and a sphere centered at the origin. Below we also consider three kinds of families of semiregular polytopes given by Weyl action of E_n.

2.2.2. Coxeter–Dynkin Diagrams

Reflection groups generated by the reflections with respect to hyperplanes (called mirrors) are called *Coxeter groups*, and the relationships among generating reflections are presented in the *Dynkin diagrams* of Coxeter groups.

The *Coxeter–Dynkin diagrams* of Coxeter groups are graphs with labels where their nodes present indexed mirrors and the labels on edges present the order n, which is the dihedral angle $\frac{\pi}{n}$ between two mirrors. If $n = 2$, namely, two mirrors are perpendicular, we denote it as no edge joining two nodes presenting the corresponding mirrors. This also implies that the corresponding mirrors commute. For $n = 3$, since the dihedral angle $\frac{\pi}{3}$ appears very often, we denote it with an edge between two nodes without labels. We only label the edges when the corresponding order is $n > 3$. Each Coxeter–Dynkin diagram contains at least one ringed node which represents an active mirror; *i.e.*, we choose a point off the mirrors corresponding to the ringed nodes, and on the mirrors corresponding to the nodes without rings. The construction of a polytope begins with reflecting the point by the active mirrors.

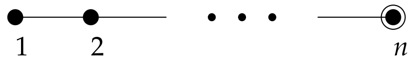
2.2.3. *ADE*-Type Coxeter Groups and Isotropy Groups

We consider Coxeter groups of *ADE*-type, and we call the polytopes given by Coxeter groups of *ADE*-type *ADE*-polytopes. In fact, the Coxeter–Dynkin diagrams of polytopes in this article have only one ringed node and no labeled edges. For these cases, the following simple procedure using the Coxeter–Dynkin diagrams describes possible subpolytopes and gives the total number of them. Each Coxeter–Dynkin diagram of subpolytopes $\tilde{P}$ is the connected subgraph Γ of the diagram containing the ringed node. The subgraph obtained by removing all the nodes joined with the subgraph Γ represents the isotropy group $G_{\tilde{P}}$ of $\tilde{P}$. Furthermore, the index between the symmetry group G of the ambient polytope and the isotropy group $G_{\tilde{P}}$ gives the total number of such subpolytopes. For example, by removing the ringed node, we obtain the subgraph corresponding to the isotropy group of a vertex where the isotropy group is the symmetry group of the vertex figure.

In fact, the complete list of ADE-polytopes consists of various polytopes given by ADE Weyl action. However, we consider only a few of them related to the studies in this article as below.

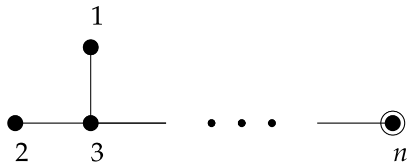
2.2.4. ADE-Polytopes

(1) (A-polytope) A **regular simplex** α_n is an n-dimensional simplex with equilateral edges. Inductively, α_n is constructed as a pyramid based on an $(n-1)$-dimensional simplex α_{n-1}. The facets of a regular simplex α_n are regular simplexes α_{n-1}, and the vertex figure of α_n is also α_{n-1}. For a regular simplex α_n, only regular simplexes α_k, $0 \leq k \leq n-1$ appear as subpolytopes.



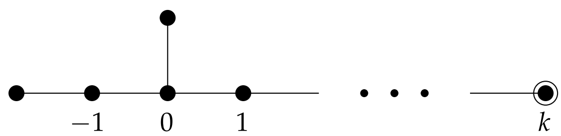
Coxeter–Dynkin diagram of α_n

(2) (D-polytope) A **crosspolytope** β_n is an n-dimensional polytope whose $2n$-vertices are given as the intersections between an n-dimensional Cartesian coordinate frame and a sphere centered at the origin. β_n is also inductively constructed as a bipyramid based on an $(n-1)$-dimensional crosspolytope β_{n-1}, and the n-vertices in β_n form a simplex α_{n-1} if any two vertices are not chosen from the same Cartesian coordinate line. The vertex figure of a crosspolytope β_n is also a crosspolytope β_{n-1}, and the facets of β_n are simplexes α_{n-1}. For a crosspolytope β_n, only regular simplexes α_k, $0 \leq k \leq n-1$ appear as subpolytopes.



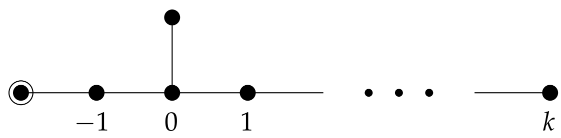
Coxeter–Dynkin diagram of β_n

(3) (E-polytope) **Gosset polytopes** k_{21} $(k = -1, 0, 1, 2, 3, 4)$ are $(k+4)$-dimensional semiregular polytopes of the Coxeter groups E_{k+4} discovered by Gosset. The vertex figure of k_{21} is $(k-1)_{21}$. For $k \neq -1$ the facets of k_{21}-polytopes are the regular simplexes α_{k+3} and the crosspolytopes β_{k+3}, but all the lower dimensional subpolytopes are regular simplexes. In fact, Coxeter called 4_{21}, 3_{21} and 2_{21} Gosset polytopes. We extend the list of Gosset polytopes along the extended list of E_n. Note that a Gosset polytope $(-1)_{21}$, a triangular prism, especially has an isosceles triangle as the vertex figure different from an equilateral triangle.



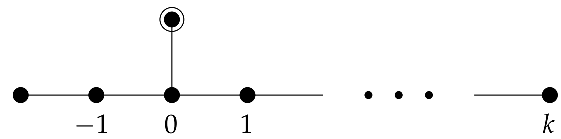
Coxeter–Dynkin diagram of k_{21} $k \neq -1$

(4) (E-polytope) 2_{k1} $(k = -1, 0, 1, 2, 3, 4)$ are $(k+4)$-dimensional semiregular polytopes of the Coxeter groups E_{k+4}. Here the vertex figure of 2_{k1} is a $(k+3)$-demicube. For $k \neq -1$ the facets of 2_{k1}-polytopes are regular simplexes α_{k+3} and semiregular polytopes $2_{(k-1)1}$.



Coxeter–Dynkin diagram of 2_{k1} $k \neq -1$

(5) (E-polytope) 1_{k2} ($k = -1, 0, 1, 2, 3, 4$) are semiregular polytopes which are $(k+4)$-dimensional polytopes whose symmetry groups are the Coxeter groups E_{k+4}. Here the vertex figure of 1_{k2} is a birectified $(k+4)$-simplex. For $k \neq -1$, the facets of 1_{k2}-polytopes are the semiregular polytopes $1_{(k-1)2}$ and $(k+3)$-demicubes, but all the lower dimensional subpolytopes are regular simplexes.



Coxeter–Dynkin diagram of 1_{k2} $k \neq -1$

2.2.5. Subpolytopes in Gosset Polytopes k_{21}

Below, it is useful to know the total numbers of subpolytopes in $(r-4)_{21}$ for each $r \in \{3, 4, \ldots, 8\}$. The numbers of simplexes α_i, $i = 1, 2, \ldots, r-1$ in a Gosset polytope $(r-4)_{21}$ are as follows in Tables 1 and 2.

Table 1. Total numbers of simplexes α_i in $(r-4)_{21}$ for $1 \leq$ i ≤ 7.

r	3	4	5	6	7	8
α_1	9	30	80	216	756	6720
α_2	2	30	160	720	4032	60,480
α_3	0	5	120	1080	10,080	241,920
α_4	0	0	16	648	12,096	483,840
α_5	0	0	0	72	6048	483,840
α_6	0	0	0	0	576	207,360
α_7	0	0	0	0	0	17,280

The numbers of crosspolytopes β_{r-1} in $(r-4)_{21}$ for each $r \in \{3, 4, \ldots, 8\}$ are as follows.

Table 2. Total numbers of crosspolytopes β_{r-1} in $(r-4)_{21}$.

r	3	4	5	6	7	8
β_{r-1}	3	5	10	27	126	2160

2.3. Del Pezzo Surfaces and Gosset Polytopes

A *del Pezzo surface* is a smooth rational surface whose anticanonical divisor is ample. All del Pezzo surfaces are S_r, a blow-up of $\mathbb{P}^2$ at r points in general position, for $r = 0, 1, \ldots, 8$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

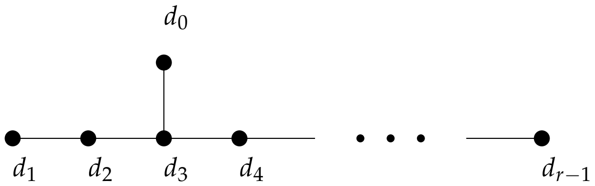
To define reflections on $(\mathbb{Z}K_{S_r})^{\perp}$ in $\mathrm{Pic}(S_r)$, we consider a root system:

$$\mathbf{R}(S_r) = \left\{ d \in \mathrm{Pic}(S_r) \mid d^2 = -2,\ d \cdot K_{S_r} = 0 \right\}$$

With a choice of simple roots

$$d_0 = h - e_1 - e_2 - e_3,\ d_i = e_i - e_{i+1},\ 1 \leq i \leq r-1$$

we have a Weyl group $W(S_r)$ of E_r-type with the Dynkin diagram:



Dynkin diagram of E_r $r \geq 3$

In [10] we obtained correspondences between special divisors on del Pezzo surfaces S_r and the faces of Gosset polytopes $(r-4)_{21}$. For this purpose we define *affine hyperplane sections* in $\mathrm{Pic}(S_r) \otimes \mathbb{Q}$ as

$$\tilde{H}_b(S_r) := \{D \in \mathrm{Pic}(S_r) \otimes \mathbb{Q} \mid -D \cdot K_{S_r} = b\}$$

where b is a real number.

After fixing a center $\frac{-b}{K_{S_r}^2} K_{S_r}$ in $\tilde{H}_b(S_r)$, for each divisor $D \in \tilde{H}_b(S_r)$ we obtain

$$\left(D + \frac{b}{K_{S_r}^2} K_{S_r}\right)^2 = D^2 - \frac{b^2}{K_{S_r}^2} \leq 0$$

and $D \equiv_{\mathbb{Q}} \frac{-b}{K_{S_r}^2} K_{S_r}$ if $\left(D + \frac{b}{K_{S_r}^2} K_{S_r}\right)^2 = 0$ by Hodge index theorem and $K_{S_r}^2 > 0$. Therefore, we get a negative definite norm in $\tilde{H}_b(S_r)$ induced by the intersection product when we fix a center $\frac{-b}{K_{S_r}^2} K_{S_r}$ in $\tilde{H}_b(S_r)$.

Then, by applying reflections defined by the roots for the lines of del Pezzo surfaces S_r, we obtained the following theorem (in [10]) about the vertices of Gosset polytopes $(r-4)_{21}$ in $\tilde{H}_1(S_r) \subset \mathrm{Pic}(S_r) \otimes \mathbb{Q}$:

Theorem 2 (Theorem 4.2 in [10])**.** *For each del Pezzo surface S_r, the set $\mathbf{L}(S_r)$ of lines on S_r is the set of vertices of a Gosset polytope $(r-4)_{21}$ in a hyperplane section $\tilde{H}_1(S_r)$.*

Remark 3. *In fact, the convex hull of the set $\mathbf{L}(S_r)$ of lines in $\tilde{H}_1(S_r)$ is the polytope $(r-4)_{21}$.*

We consider the set of *skew a-lines* defined by

$$\mathbf{L}^a(S_r) := \{D \in \mathrm{Pic}(S_r) \mid D \equiv l_1 + l_2 + \cdots + l_a \text{ for disjoint lines } l_i \text{ in } \mathbf{L}(S_r)\}.$$

The following theorem gives a correspondence among the exceptional systems of S_r, the $(r-1)$-simplexes of $(r-4)_{21}$, and the skew r-lines of S_r.

After we observe the bijections between exceptional systems in del Pezzo surfaces and the top degree simplexes in $(r-4)_{21}$ except for $r = 8$, we deduced the following theorem (in [10]).

Theorem 4 (Theorem 5.3 in [10])**.** *When $3 \leq r \leq 8$, each $(r-1)$-simplex in $(r-4)_{21}$ corresponds to an exceptional system in the del Pezzo surfaces S_r. Moreover, for $3 \leq r \leq 7$, the Weyl group $W(S_r)$ acts transitively on $\mathcal{E}(S_r)$;* i.e., *the set of exceptional systems in the del Pezzo surface S_r. Finally $\mathcal{E}(S_r)$ is bijective to $\mathbf{L}^r(S_r)$, the set of skew r-lines in* $\mathrm{Pic}(S_r)$.

Remark 5. *To explain the correspondence, we consider a transformation $\Phi(e) = K_{S_r} + 3e$ from $\mathcal{E}(S_r)$ to $\mathbf{L}^r(S_r)$ which satisfies*

$$\Phi(e) \cdot K_{S_r} = (K_{S_r} + 3e) \cdot K_{S_r} = -r, \ \Phi(e)^2 = -r$$

When $r = 8$, the set of exceptional systems consists of two orbits. One orbit, with 17280 elements, corresponds to the set of skew 8*-lines in S_8, and the other orbit, with 240 elements, corresponds to the set of E_8-roots, because* *each E_8-root d gives an exceptional system $-3K_{S_8} + 2d$. To distinguish elements of two orbits, we check if $e + 3K_{S_8}$ is in $2\mathrm{Pic}(S_8)$ or not. If so, there is a root satisfying $e + 3K_{S_8} = 2d$, which otherwise corresponds to a skew 8-line via $3e + K_{S_8}$.*

Also, we had correspondences between rulings of S_r and $(r-1)$-crosspolytopes of $(r-4)_{21}$ in [10]:

Theorem 6 (Theorem 5.4 in [10])**.** *For each ruling f in a del Pezzo surface S_r, $3 \leq r \leq 8$, there is a pair of lines l_1 and l_2 with $l_1 \cdot l_2 = 1$ such that f is equivalent to the sum $l_1 + l_2$. Furthermore, the set of rulings in S_r is bijective to the set of $(r-1)$-crosspolytopes in $(r-4)_{21}$ and is acted transitively upon by the Weyl group $W(S_r)$.*

One can observe that there are two types of $(r-2)$-simplexes in $(r-4)_{21}$. In [12], an $(r-2)$-simplex in $(r-4)_{21}$ is called *of A-type* if it is contained in an $(r-1)$-simplex in $(r-4)_{21}$, and of *B-type* otherwise. In fact, $(r-2)$-simplexes in $(r-4)_{21}$ form two Weyl orbits according to types and the total numbers of the simplexes of each orbit are as follows in Tables 3 and 4.

Table 3. Total numbers of $(r-2)$-simplexes in $(r-4)_{21}$.

$(r-4)_{21}$	-1_{21}	0_{21}	1_{21}	2_{21}	3_{21}	4_{21}
total #	9	30	120	648	6048	207,360
A, B	3, 6	10, 20	40, 80	216, 432	2016, 4032	69,120, 138,240

Table 4. Total numbers of quartic rational divisor classes.

r	3	4	5	6	7	8
total #	3	10	40	216	2072	82,560
I, II	3, 0	10, 0	40, 0	216, 0	2016, 56	69,120, 13,440

For $r = 3, 4, \ldots, 6$ the set $\mathcal{Q}(S_r)$ of quartic rational divisors consists of one orbit, $W(S_r) \cdot (2h - e_1 - e_2)$. However, for $r = 7, 8$ the set $\mathcal{Q}(S_r)$ consists of two orbits, $W(S_r) \cdot (2h - e_1 - e_2) \dot\cup W(S_r) \cdot (3h - \sum_{i=1}^{6} e_i + e_7)$. We say that a quartic rational divisor is *of type* I if it is in the orbit $W(S_r) \cdot (2h - e_1 - e_2)$ and *of type* II if it is in the orbit $W(S_r) \cdot (3h - \sum_{i=1}^{6} e_i + e_7)$. The total number of such divisor classes in S_r is finite and is given in the following table.

Then, we had correspondences between $(r-2)$-simplexes of A-type in $(r-4)_{21}$ and quartic rational divisors of type I in $\mathcal{Q}(S_r)$ as follows (in [12]):

Theorem 7 (Theorem 1 in [12])**.** *For disjoint lines l_i, $1 \leq i \leq r-1$ on a del Pezzo surface S_r, they produce a contraction to $\mathbb{P}^1 \times \mathbb{P}^1$ if there is a quartic rational divisor class q on S_r satisfying $2q + K_{S_r} \equiv l_1 + l_2 + \cdots + l_{r-1}$. Moreover, the quartic rational divisor classes of type I are bijectively related to $(r-2)$-simplexes of A-type in $(r-4)_{21}$.*

3. Special Divisors of Blown-up Hirzebruch Surfaces

In this section we consider a smooth rational surface X with $K_X^2 > 0$ whose minimal surface is a Hirzebruch surface $\mathbf{F}$. For a Hirzebruch surface $\mathbf{F}_p$, we allow a blowing up of points in $\mathbf{F}_p$ in general position (Note Section 2.1) up to 7 points so that the obtained surface $\mathbf{F}_{p,r}$ satisfies $K_{\mathbf{F}_{p,r}}^2 > 0$. The results in this section for $\mathbf{F}_{p,r}$ are naturally parallel to studies of a del Pezzo surface S_{r+1} (Note Remark 1). The particular and unique characterization on $\mathbf{F}_{p,r}$ comparing to S_{r+1} will be in [13].

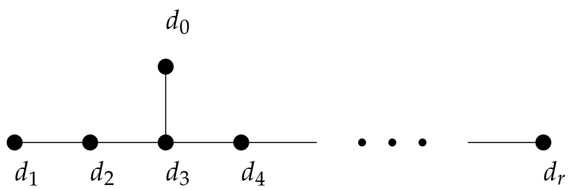
3.1. Special Divisors of $\mathbf{F}_{p,r}$

For $2 \le r \le 7$ we consider simple roots of a root system $\mathbf{R}(\mathbf{F}_{p,r})$ on $\left(\mathbb{Z}K_{\mathbf{F}_{p,r}}\right)^{\perp}$ in $\mathrm{Pic}(\mathbf{F}_{p,r})$:

$$\begin{cases} d_0 = e_1 - e_2, & \\ d_1 = \frac{p-2}{2}f + s, & \\ d_2 = f - e_1 - e_2, & \text{if } p \text{ is even} \\ d_i = e_{i-1} - e_i,\ 3 \le i \le r & \\ \text{(Note that we have only } d_0, d_1, d_2 \text{ when } r = 2) & \end{cases}$$

$$\begin{cases} d_0 = f - e_1 - e_2, & \\ d_1 = \frac{p-1}{2}f + s - e_1, & \text{if } p \text{ is odd} \\ d_i = e_{i-1} - e_i,\ 2 \le i \le r & \end{cases}$$

with a corresponding Coxeter–Dynkin diagram of the type E_{r+1}



Note that $E_3 = A_1 \times A_2$, $E_4 = A_4$ and $E_5 = D_5$. The Weyl group $W(\mathbf{F}_{p,r})$ is of the type E_{r+1} for $2 \le r \le 7$.

We get a number of special divisors on $\mathbf{F}_{p,r}$ by a simple calculation of the equations of numerical conditions or a coefficient of a theta series of a dual lattice of a root lattice E_{r+1} (see [10]). For $r = 2, 3, \ldots, 7$, by simple calculation we obtain the total numbers of special divisors in $\mathbf{F}_{p,r}$ which is the same as the one of a del Pezzo surface S_{r+1}.

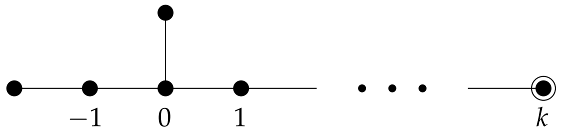
3.2. Blown-Up Hirzebruch Surfaces and Gosset Polytopes

We have correspondences between special divisors of a del Pezzo surface S_r and faces of a Gosset polytope $(r-4)_{21}$ in Section 2.3. For the special divisors of a blow-up $\mathbf{F}_{p,r}$ of a Hirzebruch surface in general position (Note Section 2.1) and the subpolytopes of a Gosset polytope $(r-3)_{21}$ we can also get parallel correspondences as in the case in Section 2.3 by applying similar proofs in [10]. We define an affine hyperplane $\tilde{H}(\mathbf{F}_{p,r})$ and the set $\mathbf{L}^a(\mathbf{F}_{p,r})$ of skew a-lines for $\mathbf{F}_{p,r}$ like the cases of a del Pezzo surface S_r (See Section 2.3).

For correspondences between the lines of a blow-up of a Hirzebruch surface and the vertices of a Gosset polytope, we have the following theorem which extends Theorem 2 to Hirzebruch surfaces:

Theorem 8. *Let $\mathbf{F}_r$ be a blown-up of a Hirzebruch surface in general position for $r \in \{2, 3, \ldots, 7\}$. Then the convex hull of the set $\mathbf{L}(\mathbf{F}_r)$ of lines in an affine hyperplane $\tilde{H}_1(\mathbf{F}_r)$ is a Gosset polytope $(r-3)_{21}$.*

Proof. The lines in $\mathrm{Pic}(\mathbf{F}_r)$ are on the sphere of a radius $-1 - \frac{1}{K_{\mathbf{F}_r}^2}$ with a center $-\frac{1}{K_{\mathbf{F}_r}^2}K_{\mathbf{F}_r}$ in $\tilde{H}_1(\mathbf{F}_r)$. Thus a convex hull of $\mathbf{L}(\mathbf{F}_r)$ is a convex polytope in $\tilde{H}_1(\mathbf{F}_r)$. We consider the line e_r in $\mathbf{L}(\mathbf{F}_r)$. For each simple root d_i, $i = 0, 1, \ldots, r$ in $\mathbf{R}(\mathbf{F}_r)$ explicitly described in the previous subsection, reflections σ_{d_i}, $i = 0, \ldots, r-1$ fix the line e_r and the reflection σ_{d_r} moves the line e_r in $\mathbf{L}(\mathbf{F}_r)$. Thus σ_{d_r} is the only active reflection for the line e_r. Therefore, the Weyl E_{r+1}-orbit of e_r in $\mathrm{Pic}(\mathbf{F}_r)$ is the set of the vertices of an $(r-3)_{21}$-polytope obtained by the Coxeter–Dynkin diagram.



Coxeter–Dynkin diagram of k_{21} $k \neq -1$

Since $|\mathbf{L}(\mathbf{F_r})|$ = the number of the vertices of an $(r-3)_{21}$-polytope (Table 5), we show that $\mathbf{L}(\mathbf{F}_r)$ is a Weyl E_{r+1}-orbit of e_r and acted transitively by the Coxeter group of E_{r+1}-type. Thus it is the set of the vertices of a $(r-3)_{21}$-polytope. Therefore, we conclude that the convex hull of $\mathbf{L}(\mathbf{F}_r)$ is a Gosset polytope $(r-3)_{21}$ for each $r = 2, 4, \ldots, 7$. □

Table 5. Total numbers of special divisors.

r	1	2	3	4	5	6	7
$\mathbf{R}(\mathbf{F}_{p,r})$	2	8	20	40	72	126	240
$\mathbf{L}(\mathbf{F}_{p,r})$	3	6	10	16	27	56	240
$\mathbf{M}(\mathbf{F}_{p,r})$	2	3	5	10	27	126	2160
$\mathcal{E}(\mathbf{F}_{p,r})$	1	2	5	16	72	576	17,520
$\mathcal{Q}(\mathbf{F}_{p,r})$	1	3	10	40	216	2072	82,560

The following theorem, which extends Theorem 4 to a Hirzebruch surface, gives correspondences among the set $\mathcal{E}(\mathbf{F}_{p,r})$ of exceptional systems, the set $\mathbf{L}^{r+1}(\mathbf{F}_{p,r})$ of the skew $(r+1)$-lines and the set of the r-simplexes α_r of a Gosset polytope $(r-3)_{21}$.

Theorem 9. *Let $\mathbf{F}_r$ be a blow-up of a Hirzebruch surface in general position for $r \in \{2, 3, \ldots, 7\}$. Then there is a bijection compatible with the action of Weyl group $W(\mathbf{F}_r)$ between the set $\mathcal{E}(\mathbf{F}_r)$ of exceptional systems in $\mathrm{Pic}(\mathbf{F}_r)$ and the set of the simplexes α_r in a Gosset polytope $(r-3)_{21}$. Moreover, for $2 \leq r \leq 6$, the Weyl group $W(\mathbf{F}_r)$ transitively acts on $\mathcal{E}(\mathbf{F}_r)$, and $\mathcal{E}(\mathbf{F}_r)$ is bijective to $\mathbf{L}^{r+1}(\mathbf{F}_r)$, the set of the skew $(r+1)$-lines in $\mathrm{Pic}(\mathbf{F}_r)$.*

Remark 10. *Similarly to two Weyl orbits in the set $\mathcal{E}(S_8)$ of the del Pezzo surface S_8, the set $\mathcal{E}(\mathbf{F}_7)$ of exceptional systems of a blown up Hirzebruch surface $\mathbf{F}_7$ has two orbits. The set $\mathbf{L}^8(\mathbf{F}_7)$ of skew 8-lines in $\mathbf{F}_7$ consists of one orbit with 17280 elements. The set of E_8-roots consists of the other orbit with 240 elements. To distinguish elements of two orbits, we check if $e + 3K_{\mathbf{F}_7}$ is in $2\mathrm{Pic}(\mathbf{F}_7)$ or not. If so, there is a root satisfying $e + 3K_{\mathbf{F}_7} = 2d$, which otherwise corresponds to a skew 8-line via $3e + K_{\mathbf{F}_7}$.*

We can also obtain correspondences between the set $\mathbf{M}(\mathbf{F}_r)$ of rulings and the set of the crosspolytopes β_r of a Gosset polytope $(r-3)_{21}$. In the following, we consider correspondences between

$$\mathbb{L}(\mathbf{F}_r) := \{(l_1, l_2) \in \mathbf{L}(\mathbf{F}_r) \times \mathbf{L}(\mathbf{F}_r) \mid l_1 \cdot l_2 = 1\} /_\sim$$

and the crosspolytopes of a Gosset polytope $(r-3)_{21}$, where the relation $(l_1, l_2) \sim (\bar{l}_1, \bar{l}_2)$ is induced by $l_1 + l_2 \equiv \bar{l}_1 + \bar{l}_2$ for any (l_1, l_2) and $(\bar{l}_1, \bar{l}_2)$ in $\mathbf{L}(\mathbf{F}_r) \times \mathbf{L}(\mathbf{F}_r)$. Therefrom we obtain the following theorem which extends Theorem 6 to Hirzebruch surfaces.

Theorem 11. *Let $\mathbf{F}_r$ be a blow-up of a Hirzebruch surface in general position for $r \in \{2, 3, \ldots, 7\}$. Then there is a bijection compatible with the action of Weyl group $W(\mathbf{F}_r)$ between the set $\mathbf{M}(\mathbf{F}_r)$ of rulings in $\mathrm{Pic}(\mathbf{F}_r)$ and the set of the crosspolytopes β_r in a Gosset polytope $(r-3)_{21}$. Moreover, the Weyl group $W(\mathbf{F}_r)$ transitively acts on $\mathbf{M}(\mathbf{F}_r)$, and each ruling f in $\mathbf{M}(\mathbf{F}_r)$ is linearly equivalent to $l_1 + l_2$ for two lines l_1, l_2 in $\mathbf{L}(\mathbf{F}_r)$ with $l_1 \cdot l_2 = 1$.*

Proof. Let β_r be a crosspolytope in a Gosset polytope $(r-3)_{21}$. By Theorem 8, we have $2r$ lines in $\mathrm{Pic}(\mathbf{F}_r)$ corresponding to the vertices of β_r. Choose distinct two pairs (l_1, l_2), $(\bar{l}_1, \bar{l}_2)$ of distinct two lines corresponding two vertices of β_r which are not in an edge of β_r. For the corresponding center of β_r, we get $l_1 + l_2 \equiv \bar{l}_1 + \bar{l}_2$ in $\mathrm{Pic}(\mathbf{F}_r)$. Moreover, we have $l_1 \cdot \bar{l}_2 = 0$ (*resp*. $l_2 \cdot \bar{l}_2 = 0$) because vertices corresponding to $l_1, \bar{l}_2$ (*resp*. $l_2, \bar{l}_2$) are in an edge of β_r in $(r-3)_{21}$. Thus we obtain $\bar{l}_1 \cdot \bar{l}_2 = 1$ and similarly get $l_1 \cdot l_2 = 1$. We define an injective map as

$$cr\colon \{\text{crosspolytopes } \beta_r \text{ in a Gosset polytope } (r-3)_{21}\} \longrightarrow \mathbb{L}(\mathbf{F}_r)$$

by $cr(\beta_r) := (l_1, l_2)$. The Weyl group $W(\mathbf{F}_r)$ is the Coxeter group of the type E_{r+1}. By applying Table 5, we get the theorem. □

Note: For the proof of Theorem 11, we apply a fact that for two distinct lines l_1 and l_2 in $\mathbf{F}_r$, $l_1 \cdot l_2 = 0$ if and only if the corresponding vertices in $(r-3)_{21}$ are joined by an edge. This fact can be obtained by simple calculations as in the case of del Pezzo surfaces ([10]).

We define a subset $\mathcal{Q}_I(\mathbf{F}_{p,r})$ of the set $\mathcal{Q}(\mathbf{F}_{p,r})$ of quartic rational divisors in $\mathrm{Pic}\big(\mathbf{F}_{p,r}\big)$ as follows:

$$\mathcal{Q}_I(\mathbf{F}_{p,r}) := \{q \in \mathcal{Q}(\mathbf{F}_{p,r}) \mid 2q + K_{\mathbf{F}_{p,r}} \equiv l_1 + \cdots + l_r \text{ for mutually disjoint lines } l_i\}.$$

For $r = 2, 3, \ldots, 7$ the set $\mathcal{Q}_I(\mathbf{F}_{p,r})$ consists of one orbit, $W(\mathbf{F}_{p,r}) \cdot q_0$, where $q_0 \equiv \left(\frac{p+2}{2}\right) f + s$ (*resp*. $\left(\frac{p+3}{2}\right) f + s - e_1$) for a nonnegative even integer (*resp*. a nonnegative odd integer) p. Then we obtain correspondence between the set $\mathcal{Q}_I(\mathbf{F}_{p,r})$ and the subset of simplexes α_{r-1} in a Gosset polytope $(r-3)_{21}$ as the following theorem which extends Theorem 7 for Hirzebruch surfaces:

Theorem 12. *Let $\mathbf{F}_r$ be a blow-up of a Hirzebruch surface in general position for $r \in \{2, 3, \ldots, 7\}$. Then there is a bijection compatible with an action of Weyl group $W(\mathbf{F}_r)$ between the set $\mathcal{Q}_I(\mathbf{F}_r)$ and the set of simplexes α_{r-1} of type A which is the set of simplexes α_{r-1} not contained in any simplexes α_r in a Gosset polytope $(r-3)_{21}$.*

Remark 13. *We obtain $\mathcal{Q}(\mathbf{F}_{p,r}) = \mathcal{Q}_I(\mathbf{F}_{p,r})$ for $2 \leq r \leq 5$. On the other hand, we have $\big|\mathcal{Q}(\mathbf{F}_{p,r}) - \mathcal{Q}_I(\mathbf{F}_{p,r})\big| = 56$ for $r = 7$, and $\big|\mathcal{Q}(\mathbf{F}_{p,r}) - \mathcal{Q}_I(\mathbf{F}_{p,r})\big| = 13440$ for $r = 8$. Indeed, for each $r \in \{7, 8\}$, the set $\mathcal{Q}(\mathbf{F}_{p,r}) - \mathcal{Q}_I(\mathbf{F}_{p,r}) = W\left(\mathbf{F}_{p,r}\right) \cdot q_1$ which consists of one orbit in $\mathcal{Q}(\mathbf{F}_{p,r})$ via an action of Weyl group $W\left(\mathbf{F}_{p,r}\right)$, where $q_1 \equiv (p+2) f + 2s - \sum_{i=1}^{5} e_i + e_6$. Thus*

$$\mathcal{Q}(\mathbf{F}_{p,r}) = W\left(\mathbf{F}_{p,r}\right) \cdot q_0 \,\dot{\cup}\, W\left(\mathbf{F}_{p,r}\right) \cdot q_1.$$

4. E-Polytopes in Picard Groups of Smooth Rational Surfaces

In the above Theorems 2 and 8, we know that the convex hulls of both $\mathbf{L}(S_r)$, the set of the lines of a del Pezzo surface S_r, and $\mathbf{L}(\mathbf{F}_{r-1})$, the set of the lines of a blow-up Hirzebruch surface $\mathbf{F}_{r-1}$, are Gosset polytopes $(r-4)_{21}$.

Below, we show that the sets of the special divisors are also identified as the vertices of E-polytopes by verifying the corresponding convex hulls are E-polytopes.

4.1. 2_{k1} and Rulings

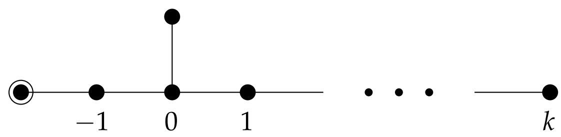
Above, we recall that the set $\mathbf{M}(S_r)$ of rulings in a del Pezzo surface S_r and the set of the crosspolytopes in $(r-4)_{21}$ are equivariantly corresponded via E_r-type Weyl action (Theorem 6), and we state the similar conclusion for the set $\mathbf{M}(\mathbf{F}_r)$ of rulings in blown-up Hirzebruch surfaces (Theorem 11). In the following Table 6, we observe that the bijections between rulings and the crosspolytopes in $(r-4)_{21}$ can be extended to the vertices of $2_{(r-4)1}$. Moreover, we show that the convex hulls of rulings are E-polytopes $2_{(r-4)1}$.

Table 6. Vertices of $2_{(r-4)1}$, crosspolytopes of $(r-4)_{21}$, and rulings of S_r $(\mathbf{F}_{r-1})$.

Del Pezzo Surface S_r $\left(\begin{array}{c}\text{Blown-up}\\ \text{Hirzebruch } \mathbf{F}_{r-1}\end{array}\right)$	S_3 $(\mathbf{F}_2)$	S_4 $(\mathbf{F}_3)$	S_5 $(\mathbf{F}_4)$	S_6 $(\mathbf{F}_5)$	S_7 $(\mathbf{F}_6)$	S_8 $(\mathbf{F}_7)$
rulings	3	5	10	27	126	2160
$2_{(r-4)1}$	2_{-11}	2_{01}	2_{11}	2_{21}	2_{31}	2_{41}
vertices of $2_{(r-4)1}$	3	5	10	27	126	2160
$(r-4)_{21}$	-1_{21}	0_{21}	1_{21}	2_{21}	3_{21}	4_{21}
midrule $(r-1)$-crosspolytopes	3	5	10	27	126	2160

Theorem 14. *For $r \in \{3,4,5,6,7,8\}$, the convex hulls of $\mathbf{M}(S_r)$ (resp. $\mathbf{M}(\mathbf{F}_{r-1})$) in $\tilde{H}_2(S_r) \subset$ Pic $(S_r) \otimes \mathbb{Q}$ (resp. $\tilde{H}_2(\mathbf{F}_{r-1}) \subset \mathrm{Pic}(\mathbf{F}_{r-1}) \otimes \mathbb{Q}$) are $2_{(r-4)1}$-polytopes.*

Proof. For a del Pezzo surface S_r, we consider the ruling $h - e_1 \in \mathbf{M}(S_r) \subset \tilde{H}_2(S_r)$ and its Weyl E_r-orbit. From the Dynkin diagram of E_r, the reflection $\sigma_{e_2-e_1}$ given by $d_1 = e_2 - e_1$ is the only active reflection moving $h - e_1$ among the reflections. Thus, the Weyl E_r-orbit of $h - e_1$ in $\mathrm{Pic}(S_r)$ is the set of the vertices of a $2_{(r-4)1}$-polytope obtained by the following Coxeter–Dynkin diagram.



Coxeter–Dynkin diagram of $2_{(r-4)1}$ $r \neq 3$

Since $|\mathbf{M}(S_r)| =$ the number of the vertices of a $2_{(r-4)1}$-polytope (Table 6), we show that $\mathbf{M}(S_r)$ is a Weyl E_r-orbit of $h - e_1$ and transitively acted by the Coxeter group of E_r-type. Therefore, it is the set of the vertices of a $2_{(r-4)1}$-polytope. Similarly, for a blown-up of Hirzebruch surface $\mathbf{F}_{r-1}$, we choose the ruling f of $\mathbf{F}_{r-1}$ to get the conclusion. This proves the theorem. □

4.2. 1_{k2} and Exceptional Systems

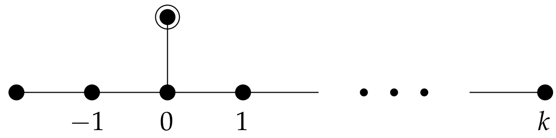
As above, we observe that the total numbers of exceptional systems in del Pezzo surfaces S_r and the total numbers of the top degree $(r-1)$ simplexes in $(r-4)_{21}$ are the same except for $r = 8$. Again in the following Table 7, we observe the bijections between exceptional systems and the $(r-1)$ simplexes in $(r-4)_{21}$ can be extended to the vertices of $1_{(r-4)2}$. Moreover, we show that the convex hulls of exceptional systems are $1_{(r-4)2}$. When $r = 8$, the set of exceptional systems has two orbits. We show that the convex hull of one of the orbits is 1_{42}.

Table 7. Vertices of $1_{(}r-4)2$, $(r-1)$-simplexes in $(r-4)_{21}$ and exceptional systems of S_r $(\mathbf{F}_{r-1})$.

Del Pezzo Surface S_r $\left(\begin{array}{c}\text{Blown-up}\\ \text{Hirzebruch } \mathbf{F}_{r-1}\end{array}\right)$	S_3 $(\mathbf{F}_2)$	S_4 $(\mathbf{F}_3)$	S_5 $(\mathbf{F}_4)$	S_6 $(\mathbf{F}_5)$	S_7 $(\mathbf{F}_6)$	S_8 $(\mathbf{F}_7)$
exceptional systems	2	5	16	72	576	17,520
$1_{(r-4)2}$	1_{-11}	1_{01}	1_{11}	1_{21}	1_{31}	1_{41}
vertices of $1_{(r-4)2}$	2	5	16	72	576	17,280
$(r-4)_{21}$	-1_{21}	0_{21}	1_{21}	2_{21}	3_{21}	4_{21}
$(r-1)$-simplexes	2	5	16	72	576	17,280

Theorem 15. *For $r \in \{3,4,5,6,7\}$, the convex hulls of $\mathcal{E}(S_r)$ (resp. $\mathcal{E}(\mathbf{F}_{r-1})$) in $\tilde{H}_3(S_r) \subset \mathrm{Pic}(S_r) \otimes \mathbb{Q}$ (resp. $\tilde{H}_3(\mathbf{F}_{r-1}) \subset \mathrm{Pic}(\mathbf{F}_{r-1}) \otimes \mathbb{Q}$) are $1_{(r-4)2}$-polytopes.*

Proof. For del Pezzo surfaces S_r , we consider the exceptional system $h \in \mathcal{E}(S_r) \subset \tilde{H}_3(S_r)$ and its Weyl E_r-orbit. From the Dynkin diagram of E_r, the reflection $\sigma_{h-e_1-e_2-e_3}$ given by $d_0 = h - e_1 - e_2 - e_3$ is the only active reflection moving h among the reflections in the Dynkin diagram. Thus the Weyl E_r-orbit of h in $\mathrm{Pic}\,(S_r)$ is a set of the vertices of an $1_{(r-4)2}$-polytope obtained by the following Coxeter–Dynkin diagram.



Coxeter–Dynkin diagram of 1_{k2} $k \neq -1$

Since $|\mathcal{E}(S_r)|$ = the number of vertices of an $1_{(r-4)2}$-polytope, we show that $\mathcal{E}(S_r)$ is the Weyl E_r-orbit of h and acted transitively by the Coxeter group of E_r-type. Thus it is the set of the vertices of an $1_{(r-4)2}$-polytope. Similarly, for a blown-up Hirzebruch surface $\mathbf{F}_{r-1}$, we choose the exceptional system $s + \frac{p+1}{2} f$ for odd p ($s + \frac{p+2}{2} f - e_1$ for even p) of $\mathbf{F}_{r-1}$. Then the only active reflection from the Dynkin diagram is given as σ_{d_0} where $d_0 = f - e_1 - e_2$ for odd p ($d_0 = e_1 - e_2$ for even p), and we get the conclusion. This proves the theorem. □

For $r = 8$, there are two orbits of Weyl action in $\mathcal{E}(S_8)$ and the Weyl orbit of h in $\mathcal{E}(S_8)$ is bijectively related to the vertices of 1_{42}. Also, for $\mathcal{E}(\mathbf{F}_7)$ the Weyl orbit of $s + \frac{p+1}{2} f$ for odd p ($s + \frac{p+2}{2} f - e_1$ for even p) is bijectively related to the vertices of 1_{42}. Therefore, the proof of the above Theorem 15 gives the following Corollary.

Corollary 16. *For $r = 8$, the convex hull of the Weyl orbit of exceptional system h (resp. $s + \frac{p+1}{2} f$ for odd p, $s + \frac{p+2}{2} f - e_1$ for even p) in $\tilde{H}_3(S_8) \subset \mathrm{Pic}(S_8) \otimes \mathbb{Q}$ (resp. $\tilde{H}_3(\mathbf{F}_7) \subset \mathrm{Pic}(\mathbf{F}_7) \otimes \mathbb{Q}$) is an E_8-polytope 1_{42}.*

Acknowledgments: The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology(No.2013R1A1A2012783) and the second author was supported by Shanghai Center for Mathematical Sciences. The authors are grateful to the reviewer for her/his comments, which contributed to improve the quality of the article.

Author Contributions: The article was written by both authors.

Conflicts of Interest: The authors declare no conflict of interest.

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