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Article

A Monge–Ampere Equation with an Unusual Boundary Condition

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Abstract: We consider a class of Monge–Ampere equations where the convex conjugate of the unknown function is prescribed on a boundary of its domain yet to be determined. We show the existence of a weak solution.

Keywords: Monge–Ampere; convex conjugate; optimal mass transport

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1. Introduction

Let $\Omega \subset \mathbf{R}^2$ be open, convex and bounded. We are interested in the following Monge–Ampere equations:

$$\begin{cases} \det D^2 u(x,y) = \frac{f(x,y)}{\mathbf{a}^2(\frac{\partial u}{\partial x})} & (x,y) \in \Omega \\ Du(\Omega) \subset \Lambda_h \\ u^*(h(z), z) = \mathbf{a}(h(z)) & \text{on } \{h > 0\} \end{cases} \tag{1}$$

where $\mathbf{a}$ and f are prescribed and $f > 0$; the unknowns are $u : \Omega \longrightarrow \mathbf{R}$ and $h : [0,1] \longrightarrow [0,1)$. For such h, we associate the set:

$$\Lambda_h = \{(s,z) \in \Lambda : 0 \le s \le h(z)\} \quad \text{with} \quad \Lambda := [0,1) \times [0,1].$$

The function $u^* : \Lambda \longrightarrow \mathbf{R}$ denotes the convex conjugate of u. Typically, the function $\mathbf{a} : [0,1) \longmapsto (0,\infty)$ is smooth and satisfies the following property:

$$\lim_{s \to 1} \mathbf{a}(s) = \infty \qquad \text{and} \qquad \lim_{s \to 1} (1-s)\mathbf{a}(s) = 1. \tag{2}$$

The Monge–Ampere equations are known to play an important role in the formulation of some problems in meteorology and fluid mechanics; semigeostrophic equations and their variants provide examples of such problems (see [1–3]). Recently, Cullen and this author have discovered that the so-called forced axisymmetric flows that arise in meteorology can be formulated as Monge–Ampere equations coupled with continuity equations. However, it is important to note that these Monge–Ampere equations come with a boundary condition that is unusual, as this condition is derived from the unique structure of forced axisymmetric flows. A treatment of forced axisymmetric flows can be found in [4]. We initiate a generalization of the problem by considering Equation (1). We note that the first boundary condition in Equation (1) is standard in the theory of optimal mass transport [5]. The second boundary condition in Equation (1) is unusual. More precisely, it requires the convex conjugate of the unknown in the Monge–Ampere equation to be prescribed on a boundary of its a priori undetermined domain. Our aim is to investigate a class of prescribed functions for which Equation (1) admits a solution. In this paper, we impose that $\mathbf{a}$ satisfies the following condition:

$$\mathbf{a}'(s) \geq L \tag{3}$$

for some $L > 0$ and all $s \in [0, 1)$. In addition to the above constraints, we assume that

$$f \in L^1(\Omega), \quad \Omega \subset [0, L_0] \times [\eta_0, L_0] \tag{4}$$

with

$$0 < \eta_0 < L_0 < L \tag{5}$$

and we require h to satisfy the balance of mass equation:

$$\int_\Omega f(x, y)dxdy = \int_{\Lambda_h} \mathbf{a}^2(s)dsdz \tag{6}$$

We propose a variational approach to Equation (1). Inspired by the Hamiltonian that comes along with the axisymmetric flows, we introduce the following functional:

$$J(u) = \int_\Omega -u(x, y)f(x, y)dxdy + \inf_{h\in\mathcal{H}} \int_0^1 dz \int_0^{h(z)} \Big(\mathbf{a}(s) - u^*(s, z)\Big)\mathbf{a}^2(s)ds \tag{7}$$

We show that the maximizer of J over the set:

$$\mathcal{U} = \big\{u \in C(\bar{\Omega}) : u = u^{**}\big\} \tag{8}$$

provides a solution for Equation (1). This paper is organized in the following way: In Section 2, we give some definitions and fix the notation. In Section 3, we provide some well-known results on the convex conjugate of functions. In Section 4, we consider the minimization problem involved in Equation (7) and establish some stability results. In Section 5, we prove our main result.

2. Notation and Definitions

In this section, we introduce some notation and recall some standard definitions.

- $\mathcal{H}$ denotes the set of all continuous functions $h : [0, 1] \longrightarrow [0, 1)$.
- Let $\mathbf{X} \subset \mathbf{R}^2$ be a convex set, $Y_1, Y_2 \in \mathbf{X}$ and $t \in [0, 1]$. A function $v : \mathbf{X} \longmapsto \mathbf{R}$ is convex if

$$v(tY_1 + (1-t)Y_2) \leq tv(Y_2) + (1-t)v(Y_2).$$

- Let $\mathbf{X} \subset \mathbf{R}^2$ be a convex set. If $v : \mathbf{X} \longmapsto \mathbf{R}$ is a convex function and $Y_0 \in \mathbf{X}$, the subdifferential of v at Y_0, denoted by $\partial_{\cdot} v(Y_0)$, is defined as:

$$\partial_{\cdot} v(Y_0) := \left\{ Z \in \mathbf{R}^2 : v(Y) \geq v(Y_0) + \langle Z, Y - Y_0 \rangle \ \forall Y \in \mathbf{X} \right\}.$$

- Given two Borel measures μ and ν of the same finite total mass on $\mathbf{R}^2$, we say that a Borel map T pushes forward μ onto ν, and we write $T\#\mu = \nu$ if

$$\mu\left(T^{-1}(A)\right) = \nu(A)$$

for all Borel sets $A \subset \mathbf{R}^2$.
- Given two Borel measures μ and ν of the same finite total mass on $\mathbf{R}^2$, $\Gamma(\mu, \nu)$ denotes the set of all transport plans γ, such that:

$$\Pi^1\#\gamma = \mu \quad \text{and} \quad \Pi^2\#\gamma = \nu.$$

Here, Π^1 and Π^2 denote, respectively, the first and second projection maps.

Definition 2.1. Let $u : \bar{\Omega} \longrightarrow \mathbf{R}$. We say that v is the convex conjugate of u if

$$v(Y) = \sup_{X \in \bar{\Omega}} \{\langle X, Y \rangle - u(X)\} \qquad \text{for all } Y \in \Lambda \tag{9}$$

and we write $v = u^*$. Similarly, let $v : \bar{\Lambda} \longrightarrow \mathbf{R}$. We say that u is the convex conjugate of v if:

$$u(X) = \sup_{Y \in \bar{\Lambda}} \{\langle X, Y \rangle - v(Y)\} \qquad \text{for all } X \in \Omega \tag{10}$$

and we write $u = v^*$.

Remark 2.2. If u is convex and lower semicontinuous then:

$$Y \in \partial_{\cdot} u(X) \quad \text{if and only if} \quad u(X) + u^*(Y) = \langle X, Y \rangle.$$

We consider the Brenier solutions of the Monge–Ampere equation (see [6,7]).

Definition 2.3. (Solution in the sense of Brenier) We say that (u, h) is a weak solution for Equation (1) if u is Lipschitz continuous, h is continuous and,

$$\begin{cases} Du\#\chi_{\Omega} f = \chi_{\Lambda_h} \mathbf{a}^2 \\ u^*(h(z), z) = \mathbf{a}(h(z)) \quad \text{on } \{h > 0\}. \end{cases} \tag{11}$$

Remark 2.4. Note that for a solution u of Equation (1), in the sense of Brenier, with only Lipschitz regularity, "$Du(\Omega) \subset \Lambda_h$" is to be understood as $Du(x, y) \in \Lambda_h$ for $a.e\ (x, y) \in \Omega$.

3. Preliminaries

In this section, we collect some standard results on convex conjugate functions. We will give a sketchy proof and refer the reader to relevant references. Let us consider the Lipschitz continuous functions $v : \Lambda \longrightarrow \mathbf{R}$, such that:

$$0 \leq \partial_s v \leq L_0 \quad \text{and} \quad \eta_0 < \partial_z v \leq L_0 \tag{12}$$

Lemma 3.1. *Let $\lambda \in \mathbf{R}$ and $u \in C(\Omega)$. Then,*

(i) $(u+\lambda)^* = u^* - \lambda$.

(ii) $\partial_{.} u^* \subset \bar{\Omega}$. *As a consequence, u^* is Lipschitz continuous and satisfies Equation (12).*

(iii) If $u = u^{**}$, then $\partial_{.} u \subset \bar{\Lambda}$. In this case, if we assume in addition that $u^*(0,0) = 0$, then there exists a constant C_L only dependent on L, such that:

$$||u||_{L^\infty(\Omega)} + ||Du||_{L^\infty(\Omega)} \leq C_{L_0} \quad \textit{and} \quad ||u^*||_{L^\infty(\Lambda)} + ||Du^*||_{L^\infty(\Lambda)} \leq C_{L_0}.$$

Proof. (i) is trivial. To obtain (ii), we observe that u^* is the supremum of affine (and so, convex) functions. Therefore, u^* is convex and lower semicontinuous. In light of Remark 2.2, Equation (9) implies that $\partial_{.} u^* \subset \bar{\Omega}$. In view of the second equation of Equation (4), u^* is Lipschitz continuous and satisfies Equation (12). If $u = u^{**}$, then a similar argument as in (ii) yields $\partial_{.} u \subset \bar{\Lambda}$. Set

$$C_L := 2 + 8L_0.$$

Since $\partial_{.} u^* \subset \bar{\Omega}$, we have:

$$||Du^*||_{L^\infty(\Lambda)} \leq 2L_0 \tag{13}$$

And so,

$$|u^*(Y) = |u^*(Y) - u^*(0,0)| \leq 2L_0|Y| \leq 4L_0 \tag{14}$$

for all $X \in \Omega$. Thus,

$$||u^*||_{L^\infty(\Omega)} + ||Du^*||_{L^\infty(\Omega)} \leq C_L$$

Similarly, as $\partial_{.} u \subset \bar{\Lambda}$,

$$||Du||_{L^\infty(\Omega)} \leq 2 \tag{15}$$

We exploit Equations (10) and (14) to obtain that

$$|u(X)| \leq 8L_0$$

so that

$$||u||_{L^\infty(\Omega)} + ||Du||_{L^\infty(\Omega)} \leq C_L.$$

This proves (iii). $\square$

The proof of the following Lemma can be seen in [8,9].

Lemma 3.2. *Let $u \in C(\Omega)$, $\varphi \in C_c(\Omega)$ and $\varepsilon > 0$.*

Then,

$$\lim_{\varepsilon \to 0} \frac{(u + \varepsilon\varphi)^* - u^*}{\varepsilon} = -\varphi \circ Du^* \quad a.e.$$

and

$$\Big|\Big| \frac{(u + \varepsilon\varphi)^* - u^*}{\varepsilon} \Big|\Big|_\infty \leq \|\varphi\|_\infty.$$

4. A Minimization Problem and Some Stability Results

For any $v \in C(\Lambda)$, we define

$$S_v(s, z) := \int_0^s \Big(\mathbf{a}(t) - v(t, z)\Big)\mathbf{a}^2(t)dt, \qquad (s, z) \in \Lambda.$$

Lemma 4.1. *Let v and $\{v_n\}_{n=0}^\infty \subset C(\bar{\Lambda})$ satisfy Equation (12).*

(i) The sub-levels of $S_v(\cdot, z)$ are bounded, uniformly for all $z \in [0, 1]$: for $m \in \mathbf{R}$, there exists a constant $c_{v,m}$, such that $0 < c_{v,m} < 1$ and

$$\left\{ \inf_{0 \leq z \leq 1} S_v(\cdot, z) \leq m \right\} \subset [0, c_{v,m}] \tag{16}$$

Moreover, if $\{v_n\}_{n=1}^\infty$ is uniformly bounded, then, for $m \in \mathbf{R}$, there exists a constant $\mathcal{C}_m$, such that:

$$0 < \mathcal{C}_m < 1 \tag{17}$$

and

$$\bigcup_{n=0}^\infty \left\{ \inf_{0 \leq z \leq 1} S_{v_n}(\cdot, z) \leq m \right\} \subset [0, \mathcal{C}_m] \tag{18}$$

(ii) Fix $z \in [0, 1]$. There exists $\lambda \in [0, 1)$ such that

$$S_v(\lambda, z) \leq S_v(s, z) \quad \textit{for all } s \in [0, 1) \tag{19}$$

Furthermore, either $\lambda = 0$ with $\mathbf{a}(0) \geq v(0, z)$ or $\lambda \in (0, 1)$ and satisfies

$$\mathbf{a}(\lambda) = v(\lambda, z) \tag{20}$$

(iii) Let $\{z_n\}_{n=0}^\infty \subset [0, 1]$, such that $\{z_n\}_{n=1}^\infty$ converges to z_0. Let λ_n satisfy Equation (19) with z replaced by z_n and v replaced by v_n for $n \geq 0$. Assume that λ_0 satisfies Equation (19) uniquely with z replaced by z_0, v replaced by v_0 and that $\{v_n\}_{n=1}^\infty$ converges uniformly to v_0. Then, $\{\lambda_n\}_{n=1}^\infty$ converges to λ_0.

Proof. 1. Since $\lim_{t \to 1}(1 - t)\mathbf{a}(t) = 1$, we can choose $s_0 \in [0, 1)$, such that

$$1 \leq 2\mathbf{a}(t)(1 - t) \tag{21}$$

for all $s_0 \leq t < 1$. Setting $d = ||v||_\infty$ and invoking the fact that $\lim_{s \to 1} \mathbf{a}(s) = \infty$, we can further choose s_0, such that

$$\mathbf{a}(t) \geq 2d \tag{22}$$

for all $s_0 \leq t < 1$. We exploit Equations (21) and (22) to obtain

$$\frac{1}{16(1-t)^3} \leq \frac{\mathbf{a}^3(t)}{2} \leq \mathbf{a}^3(t) - d\mathbf{a}^2(t) \tag{23}$$

for all $s_0 \leq t < 1$. Note that:

$$\mathbf{a}^3(t) - d\mathbf{a}^2(t) \leq \Big(\mathbf{a}(t) - v(t,z)\Big)\mathbf{a}^2(t) \tag{24}$$

for all $s_0 \leq t < 1$. We combine Equations (23) and (24) to get

$$\int_{s_0}^{s} \frac{1}{2(1-t)^3} dt \leq \int_{s_0}^{s} \Big(\mathbf{a}(t) - v(t,z)\Big)\mathbf{a}^2(t)dt \tag{25}$$

for all $s_0 \leq s \leq 1$ and $0 \leq z \leq 1$. Therefore,

$$\int_0^{s_0} \Big(\mathbf{a}(t) - v(s,z)\Big)\mathbf{a}^2(t)dt + \int_{s_0}^{s} \frac{1}{2(1-t)^3} dt \leq S_v(s,z) \tag{26}$$

Note that the first term of Equation (26) is finite and that

$$\lim_{s \to 1^-} \int_{s_0}^{s} \frac{1}{2(1-t)^3} dt = \infty \tag{27}$$

Let $m \in \mathbf{R}$. In view of Equation (27), the Equation (26) implies that if

$$\inf_{0 \leq z \leq 1} S_v(s,z) \leq m,$$

then there exists a constant $c_{v,m}$, such that

$$0 \leq s \leq c_{v,m} < 1.$$

In other words, the sub-levels of $S_v(\cdot, z)$ are bounded, uniformly for all $z \in [0,1]$.

2. Consider $\{v_n\}_{n=1}^\infty \subset C(\Lambda)$. Following the reasoning above, we obtain

$$\int_0^{s_0} \Big(\mathbf{a}(t) - v_n(s,z)\Big)\mathbf{a}^2(t)dt + \int_{s_0}^{s} \frac{1}{2(1-t)^3} dt \leq S_{v_n}(s,z) \tag{28}$$

Assume that $\{v_n\}_{n=1}^\infty$ is uniformly bounded. Then, the first term in Equation (28) is bounded. In view of Equation (27), Equation (28) implies that if

$$\inf_{z \in (0,1)} S_{v_n}(s,z) \leq m$$

then there exists a constant $\mathcal{C}_m$, such that

$$0 \leq s \leq \mathcal{C}_m < 1.$$

3. Fix $z \in [0,1]$. The continuity of $S_v(\cdot, z)$ ensures that:

$$\{s : S_v(\cdot, z) \leq m\}$$

is closed and then compact in view of Equation (16). We use again the continuity of $S_v(\cdot, z)$ to obtain that $S_v(\cdot, z)$ has a minimizer in $[0, c_{v,m}]$. This ensures the existence of λ in Equation (19). If $\lambda > 0$, we use the differentiability of $S_v(\cdot, z)$ on $(0,1)$ to obtain that $\partial_s S_v(\lambda, z) = 0$, that is $\mathbf{a}(\lambda) = v(\lambda, z)$. If $\lambda = 0$, then $\mathbf{a}(0) - v(0, z) = \partial_s S_v(0, z) \geq 0$. This proves (ii).

4. Now, let us prove (iii). Note that $\{\lambda_n\}_{n=1}^{\infty} \subset [0, \mathcal{C}_0]$, and so, there exists a subsequence of $\{\lambda_n\}_{n=1}^{\infty}$ still denoted by $\{\lambda_n\}_{n=1}^{\infty}$ that converges to some λ^*. For $s \in [0,1)$, we have:

$$S_{v_n}(\lambda_n, z_n) \leq S_{v_0}(s, z_n) \tag{29}$$

Let M be a constant, such that $max(s, \mathcal{C}_m) \leq M < 1$. As $\{v_n\}_{n=1}^{\infty}$ converges uniformly to v_0, we have that $\{S_{v_n}\}_{n=1}^{\infty}$ converges uniformly to S_{v_0} on $[0, M] \times [0,1]$. This, along with the continuity of S_{v_n} and Equation (29), yields:

$$S_{v_0}(\lambda^*, z_0) = \lim_{n\to\infty} S_{v_n}(\lambda_n, z_n) \leq \lim_{n\to\infty} S_{v_n}(s, z_n) = S_{v_0}(s, z_0) \tag{30}$$

As s is arbitrary and λ_0 is the unique solution of Equation (19) with z replaced by z_0, we see Equation (30) to conclude that $\lambda_0 = \lambda^*$, and so, the whole sequence $\{\lambda_n\}_{n=1}^{\infty}$ converges to λ_0. $\square$

Lemma 4.2. *We assume that v satisfies Equation (12).*

(1) Let $z \in [0,1]$ and λ_i satisfy Equation (19), $i = 1, 2$. Then,

$$\lambda_1 = \lambda_2.$$

(2) Let $z_1, z_2 \in [0,1]$ and λ_1, λ_2 satisfy Equation (19), respectively, for z replaced, respectively, by z_1 and z_2. Then:

$$z_1 < z_2 \Longrightarrow \lambda_1 \leq \lambda_2 \tag{31}$$

Proof. 1. Fix $z \in [0,1]$, and note that $S_v(0, z) = 0$. If λ is as in Equation (19) and $c_{v,0}$ as in Equation (16), then

$$0 \leq \lambda \leq c_{v,0}.$$

By Lemma 4.1 (ii), either $\lambda = 0$ with $\mathbf{a}(0) - v(0, z) \geq 0$ or $\lambda \in (0,1)$ with $\mathbf{a}(\lambda) = v(\lambda, z)$. Assume $\lambda = 0$. In view of Equations (3), (5) and (12), we have that $\partial_s(\mathbf{a} - v(\cdot, s)) > 0$, so that:

$$0 \leq \mathbf{a}(0) - v(0, z) < \mathbf{a}(s) - v(s, z) \qquad \text{for } 0 < s < 1.$$

And so,

$$S_v(0, z) = 0 < \int_0^s (\mathbf{a}(t) - v(t, z))\, \mathbf{a}^2(t) ds = S_v(s, z) \qquad \text{for } 0 < s < 1.$$

It follows that if $\lambda = 0$, then Equation (19) holds uniquely for $\lambda = 0$.

Let $\lambda_1, \lambda_2 \in (0,1)$, such that

$$\mathbf{a}(\lambda_i) = v(\lambda_i, z), \qquad i = 1, 2.$$

Since $\partial_s v(s,z) \le L_0$, we have

$$\mathbf{a}(\lambda_2) - \mathbf{a}(\lambda_1) = v(\lambda_2, z) - v(\lambda_1, z) \le L_0(\lambda_2 - \lambda_1) \tag{32}$$

On the other hand, we use Equation (3) to obtain:

$$\mathbf{a}(\lambda_2) - \mathbf{a}(\lambda_1) \ge L(\lambda_2 - \lambda_1) \tag{33}$$

We combine Equations (32) and (33) to get that

$$\lambda_1 = \lambda_2.$$

2. Let $z_1, z_2 \in [0,1]$, such that $z_1 < z_2$, and let λ_1, λ_2 satisfy Equation (19) for z replaced respectively by z_1 and z_2. As $\partial_z v(s,\cdot) > 0$, we have

$$v(s, z_1) \le v(s, z_2) \quad \text{for} \quad 0 \le s < 1.$$

and so,

$$S_v(s, z_2) \le S_v(s, z_1) \quad \text{for} \quad 0 \le s < 1 \tag{34}$$

If $\lambda_1 = 0$, then Equation (31) trivially holds. Assume $\lambda_2 = 0$. Then, we use the fact that λ_2 satisfies Equation (19) for z replaced by z_2 and Equation (34) to get

$$0 = S_v(\lambda_2, z_2) \le S_v(\lambda_1, z_2) \le S_v(\lambda_1, z_1) \tag{35}$$

Again, as λ_1 satisfies Equation (19) for z replaced by z_1, we have

$$S_v(\lambda_1, z_1) \le S_v(0, z_1) = 0 \tag{36}$$

We combine Equations (35) and (36) to obtain that

$$S_v(\lambda_1, z_1) = 0 \tag{37}$$

By the uniqueness result in Part (1),

$$\lambda_1 = 0 \tag{38}$$

Thus, Equation (31) holds. Assume next that $\lambda_1, \lambda_2 \in (0,1)$. Then,

$$v(\lambda_i, z_i) = \mathbf{a}(\lambda_i), \quad i = 1, 2 \tag{39}$$

We use again the fact that $\partial_z v(s,\cdot) > 0$ to obtain:

$$v(\lambda_1, z_1) \le v(\lambda_1, z_2) \tag{40}$$

In light of Equation (39), the equation in Equation (40) becomes

$$\mathbf{a}(\lambda_1) \le v(\lambda_1, z_2) \tag{41}$$

and so,

$$\mathbf{a}(\lambda_1) - v(\lambda_1, z_2) \le 0 = \mathbf{a}(\lambda_2) - v(\lambda_1, z_2) \tag{42}$$

As $\partial_s(\mathbf{a} - v(\cdot, z_2)) > 0$, we have that $\mathbf{a} - v(\cdot, z_2)$ is monotone increasing. Thus, Equation (42) yields:

$$\lambda_1 \le \lambda_2$$

so that Equation (31) holds. $\square$

Proposition 4.3. *Let* $\{v_n\}_{n=0}^{\infty} \subset C(\bar{\Lambda})$ *satisfying Equation (12).*

(i) The functional

$$\mathcal{F}_{v_0} : h \longmapsto \int_0^1 S_{v_0}(h(z), z)dz \tag{43}$$

has a unique minimizer h_0 *over the set of all continuous functions* $h : [0,1] \longrightarrow [0,1)$*. Moreover,* h_0 *is monotone, and* $h_0(z)$ *satisfies Equation (19) for* v *replaced by* v_0*.*

(ii) Assume that $\{v_n\}_{n=1}^{\infty}$ *is uniformly convergent to* v_0 *and* h_n *is the minimizer of* $\mathcal{F}_{v_n}$*. Then,*

$$\mathcal{F}_{v_n}(h_n) \longrightarrow \mathcal{F}_{v_0}(h_0) \tag{44}$$

Proof. Define h_n, $n \geq 0$, in the following way:

$$h_n(z) \text{ is the minimizer of } S_{v_n}(\cdot, z) \text{ over } [0,1).$$

Lemma 4.2 (2) shows that h_n is monotone increasing; Lemma 4.1 (iii) ensures that each h_n is continuous. In order to prove (i), we claim that h_0 is the unique solution for the following minimization problem:

$$\inf_h \int_0^1 S_v(h(z), z)dz \tag{45}$$

The fact that h_0 is a solution for Equation (45) is straightforward as a result of Equation (19). Assume that $\tilde{h}$ is another minimizer of $\mathcal{F}_{v_0}$ as above. Then,

$$\int_0^1 S_v(h_0(z), z)dz = \int_0^1 S_v(\tilde{h}(z), z)dz \tag{46}$$

and

$$S_v(h_0(z), z) \leq S_v(\tilde{h}(z), z) \tag{47}$$

We use Equations (46) and (47) to obtain that

$$S_v(h_0(z), z) = S_v(\tilde{h}(z), z) \qquad a.e. \tag{48}$$

By the uniqueness of the minimizer in Lemma 4.2 (1), we use Equation (48) to conclude that $h_0 = \tilde{h}$ $a.e.$, and the continuity of h_0 and $\tilde{h}$ yields $h_0 = \tilde{h}$.

3. Since $S_{v_0}(0, z) = 0$ for all $z \in [0,1]$, we have that $0 \leq h_n, h_0 \leq \mathcal{C}_0$, where $\mathcal{C}_0$ is provided by Equation (17). As h_n is monotone, the Helly theorem implies that there exists a subsequence $\{h_{n_k}\}_{k=1}^{\infty}$ of $\{h_n\}_{n=1}^{\infty}$, such that $\{h_{n_k}\}_{k=1}^{\infty}$ converges pointwise to some function g. In view of Lemma 4.1 (iii), we have that $g = h_0$.

Observe that

$$\begin{aligned} |S_{v_{n_k}}(h_{n_k}(z), z) - S_v(h_0(z), z)| \leq & \left| \int_0^{h_0(z)} |v(s,z) - v_{n_k}(s,z)| \mathbf{a}^2(s)ds \right| \\ & + \left| \int_{h_0(z)}^{h_{n_k}(z)} (\mathbf{a}(s) - v_{n_k}(s,z))\, \mathbf{a}^2(s)ds \right| \end{aligned} \tag{49}$$

Note that

$$\begin{aligned}\limsup_{k\to\infty}\left|\int_0^{h_0(z)}\left(v(s,z)-v_{n_k}(s,z)\right)\mathbf{a}^2(s)ds\right| &\leq \limsup_{n\to\infty}\int_0^{\mathcal{C}_0}|v(s,z)-v_{n_k}(s,z)|\mathbf{a}^2(s)ds \\ &\leq \limsup_{k\to\infty}\Big(\int_0^{\mathcal{C}_0}\mathbf{a}^2(s)ds\Big)||v-v_{n_k}||_\infty \\ &= 0\end{aligned} \tag{50}$$

As $\{v_n\}_{n=1}^\infty$ converges uniformly, it is bounded in the uniform norm by a constant, say C_0. Note that

$$M_0 := \Big(||\mathbf{a}||_{L^\infty([0,\mathcal{C}_0])}+C_0\Big)||\mathbf{a}||^2_{L^\infty([0,\mathcal{C}_0])}<\infty.$$

We use the fact that $\{v_n\}_{n=1}^\infty$ is bounded along with the pointwise convergence of $\{h_{n_k}\}_{k=1}^\infty$ to obtain

$$\begin{aligned}\limsup_{n\to\infty}\left|\int_{h_0(z)}^{h_{n_k}(z)}\left(\mathbf{a}(s)-v_{n_k}(s,z)\right)\mathbf{a}^2(s)ds\right| &\leq M_0\limsup_{n\to\infty}\left|\int_{h_0(z)}^{h_{n_k}(z)}ds\right| \\ &\leq M_0\limsup_{n\to\infty}|h_0(z)-h_{n_k}(z)| \\ &= 0\end{aligned} \tag{51}$$

We combine Equations (49)–(51) to obtain that

$$\lim_{k\to\infty}S_{v_{n_k}}(h_{n_k}(z),z)=S_v(h_0(z),z) \tag{52}$$

Note that

$$\begin{aligned}\Big|S_{v_{n_k}}(h_n(z),z)\Big| &\leq \int_0^{\mathcal{C}_0}\Big(||\mathbf{a}||_{L^\infty([0,\mathcal{C}_0])}+||v_n||_\infty\Big)||\mathbf{a}||^2_{L^\infty([0,\mathcal{C}_0])}ds \\ &\leq \mathcal{C}_0M_0||\mathbf{a}||^2_{L^\infty([0,\mathcal{C}_0])}\end{aligned} \tag{53}$$

Invoking the Lebesgue-dominated convergence theorem, we use Equations (53) and (52) to prove Equation (44).

□

5. A Maximization Problem and Main Result

We recall that

$$J(u)=\int_\Omega -u(x,y)f(x,y)dxdy+\inf_{h\in\mathcal{H}}\int_0^1 dz\int_0^{h(z)}\Big(\mathbf{a}(s)-u^*(s,z)\Big)\mathbf{a}^2(s)ds$$

and:

$$\mathcal{U}=\left\{u\in C(\bar{\Omega}):u=u^{**}\right\}.$$

Lemma 5.1. *The functional J is bounded above on $C(\Omega)$.*

Proof. Let $h_0\in\mathcal{H}$ be a constant function, such that Equation (6) holds, and let $\gamma_0\in\Gamma\Big(\chi_\Omega f,\chi_{\Lambda_{h_0}}\mathbf{a}^2\Big)$. Observe that:

$$|\langle X,Y\rangle|\leq|X||Y|\leq 4L_0 \qquad \text{for all } X\in\Omega,\ Y\in\Lambda,$$

and

$$\gamma_0(\Omega \times \Lambda_{h_0}) = ||f||_{L^1(\Omega)}.$$

Note that

$$M_0 := 4L_0||f||_{L^1(\Omega)} + \int_0^{h_0} \mathbf{a}^3(s)ds < \infty \tag{54}$$

Let $u \in C(\Omega)$. Then, using Equation (9), we have

$$-u(X) - u^*(Y) \leq -\langle X, Y\rangle \qquad \text{with} \quad X := (x,y) \in \Omega,\ Y := (s,z) \in \Lambda.$$

and so,

$$\begin{aligned}\int_\Omega -u(x,y)f(x,y)dxdy + \int_0^1 dz \int_0^{h_0} -u^*(s,z)\mathbf{a}^2 ds &= \int_{\Omega\times\Lambda_{h_0}} -u(X) - u^*(Y)\\ &= \int_{\Omega\times\Lambda_{h_0}} -\langle X, Y\rangle d\gamma_0.\end{aligned}$$

Therefore,

$$\int_\Omega -u(x,y)f(x,y)dxdy + \int_0^1 dz \int_0^{h_0} \Big(\mathbf{a}(s) - u^*(s,z)\Big)\mathbf{a}^2(s)ds \leq \int_{\Omega\times\Lambda_{h_0}} -\langle X, Y\rangle d\gamma_0 + \int_0^{h_0} \mathbf{a}^3(s)ds.$$

In light of Equation (54), we have

$$\int_\Omega -u(x,y)f(x,y)dxdy + \int_0^1 dz \int_0^{h_0} \Big(\mathbf{a}(s) - u^*(s,z)\Big)\mathbf{a}^2(s)ds \leq M_0.$$

Thus, in view of the infimum term in $J(u)$, we have

$$J(u) \leq M_0.$$

which proves the Lemma. □

Proposition 5.2. *The functional $u \longmapsto J(u)$ admits a maximizer on $\mathcal{U}$.*

Proof. Let $u \in \mathcal{U}$. In light of Lemma 5.1, set

$$j = \sup_{u\in\mathcal{U}} J(u) \tag{55}$$

Let $\{u_n\}_n \subset C(\bar{\Omega})$ be a maximizing sequence for the maximization problem in Equation (55). In what follows, we show in Step 1 that $\{u_n\}_n \in C(\bar{\Lambda})$ converges up to a subsequence and in Step 2; we show that its limit is a maximizer in Equation (55).

Step 1.

Let $m \in [0,1)$, and as $S_{u_n^*}(0,z) = 0$, note that

$$\inf_{h\in\mathcal{H}} \int_0^1 dz \int_0^{h(z)} \Big(\mathbf{a}(s) - u_n^*(s,z)\Big)\mathbf{a}^2(s)ds \leq \int_0^1 dz \int_0^m \Big(\mathbf{a}(s) - u_n^*(s,z)\Big)\mathbf{a}^2(s)ds \tag{56}$$

Therefore,

$$J(u_n) \leq \int_\Omega -u_n(x,y)f(x,y)dxdy + \int_0^1 dz \int_0^m \Big(\mathbf{a}(s) - u_n^*(s,z)\Big)\mathbf{a}^2(s)ds \tag{57}$$

Next, we set

$$\bar{u}_n := u_n + u_n^*(0,0).$$

By Lemma 3.1 (i),

$$\bar{u}_n^* := u_n^* - u_n^*(0,0).$$

These, combined with Equation (57), yield that

$$\begin{aligned} J(u_n) \leq & \int_\Omega -\bar{u}_n(x,y)f(x,y)dxdy + u_n^*(0,0)||f||_{L^1(\Omega)} \\ & + \int_0^1 dz \int_0^m \Big(\mathbf{a}(s) - \bar{u}_n^*(s,z) - u_n^*(0,0)\Big)\mathbf{a}^2(s)ds \end{aligned} \tag{58}$$

As $\bar{u}_n^*(0,0) = 0$ and $\bar{u}_n^{**} = \bar{u}_n$, we use Lemma 3.1 (iii) to obtain

$$||\bar{u}_n||_{L^\infty(\Omega)} + ||D\bar{u}_n||_{L^\infty(\Omega)} \leq C_{L_0} \quad \text{and} \quad ||\bar{u}_n^*||_{L^\infty(\Omega)} + ||D\bar{u}_n^*||_{L^\infty(\Omega)} \leq C_{L_0} \tag{59}$$

Therefore, Equations (58) and (59) imply that

$$J(u_n) \leq C_L||f||_{L^1(\Omega)} + u_n^*(0,0)||f||_{L^1(\Omega)} + \int_0^m \Big(\mathbf{a}(s) + C_L - u_n^*(0,0)\Big)\mathbf{a}^2(s)ds \tag{60}$$

That is,

$$J(u_n) - C_L||f||_{L^1(\Omega)} - \int_0^m \Big(\mathbf{a}(s) + C_L\Big)\mathbf{a}^2(s)ds \leq u_n^*(0,0)\left(||f||_{L^1(\Omega)} - \int_0^m \mathbf{a}^2(s)ds\right) \tag{61}$$

In view of Equation (2), we use Equation (21) to get

$$\frac{1}{4(1-m)} + \frac{1}{4(1-s_0)} = \int_{s_0}^m \frac{1}{4(1-s)^2}ds \leq \int_{s_0}^m \mathbf{a}^2(s)ds \tag{62}$$

for some $s_0 \in [0,1)$. We use Equation (62) to obtain:

$$\lim_{m\to 1}\int_0^m \mathbf{a}^2(s)ds = +\infty.$$

Therefore, we can choose $m_0 \in [0,1)$, such that:

$$||f||_{L^1(\Omega)} < \int_0^{m_0} \mathbf{a}^2(s)ds.$$

Setting first $m = 0$ and then $m = m_0$ in Equation (61), we obtain:

$$\frac{J(u_n) - C_{L_0}||f||_{L^1(\Omega)}}{||f||_{L^1(\Omega)}} \leq u_n^*(0,0) \leq \frac{J(u_n) - C_L||f||_{L^1(\Omega)} - \int_0^{m_0}\Big(\mathbf{a}(s) + C_{L_0}\Big)\mathbf{a}^2(s)ds}{\Big(||f||_{L^1(\Omega)} - \int_0^{m_0}\mathbf{a}^2(s)ds\Big)} \tag{63}$$

We next assume, without loss of generality, that

$$j - 1 \leq J(u_n) \qquad \text{for all } n \geq 1.$$

Then,

$$\frac{j-1-C_L||f||_{L^1(\Omega)}}{||f||_{L^1(\Omega)}} \leq u_n^*(0,0) \leq \frac{j-C_L||f||_{L^1(\Omega)}-\int_0^{m_0}\Big(\mathbf{a}(s)+C_L\Big)\mathbf{a}^2(s)ds}{\Big(||f||_{L^1(\Omega)}-\int_0^{m_0}\mathbf{a}^2(s)ds\Big)} \tag{64}$$

In view of Equation (64), $\{u_n^*(0,0)\}_{n=1}^{\infty}$ is bounded, and so, there exists a subsequence still denoted $\{u_n^*(0,0)\}_{n=1}^{\infty}$ that converges to some $C^* \in \mathbf{R}$. In light of Equation (59), we use Ascoli–Azerla to conclude that there exists a subsequence of $\{u_{n_k}\}_{k=1}^{\infty}$ still denoted by $\{u_{n_k}\}_{k=1}^{\infty}$, such that

$$\{\bar{u}_{n_k}\}_{k=1}^{\infty} \text{ converges uniformly to } \bar{u}_0$$

and

$$\{\bar{u}_{n_k}^*\}_{k=1}^{\infty} \text{ converges uniformly to } \bar{u}_0^*.$$

We use the last two displayed convergence results and the convergence of $\{u_{n_k}(0,0)\}_{k=1}^{\infty}$ to obtain that:

$$\{u_{n_k}\}_{k=1}^{\infty} \text{ converges uniformly to } u_0=\bar{u}_0+C^*$$

and

$$\{u_{n_k}^*\}_{k=1}^{\infty} \text{ converges uniformly to } u_0^*=\bar{u}_0^*+C^*.$$

Step 2.

To show the existence of a maximizer, it will be enough to study the continuity in the second term in the expression of J. Let $\tilde{h}_{n_k}$ and $\tilde{h}$ denote respectively the minimizers in the second term of $J(u_{n_k})$ and $J(u_0)$. As $\{u_{n_k}^*\}_n \subset C(\bar{\Lambda})$ satisfies Equation (12) and converges uniformly to u_0^*, we use Equation (44) to get:

$$\mathcal{F}_{u_{n_k}^*}(\tilde{h}_{n_k}) \longrightarrow \mathcal{F}_{u^*}(\tilde{h}) \tag{65}$$

It follows that

$$J(u_{n_k}) \longrightarrow J(u_0) \tag{66}$$

As $\{u_n\}_n$ is a maximizing sequence, we have

$$J(u_0)=\sup_{u\in\mathcal{U}} J(u),$$

which concludes the proof. □

Theorem 5.3. *Let $u_0 \in \mathcal{U}$, such that:*

$$J(u_0)=\sup_{u\in\mathcal{U}} J(u),$$

and h_0 the minimizer in Equation (43) for v_0 replaced by u_0^*. Then, (u_0,h_0) provides a weak solution for Equation (1).

Proof. Let $\varepsilon>0$ and $\varphi \in C_c(\Omega)$. We define the functions h_0 and h_ε from $[0,1]$ to $[0,1)$, such that for each $z\in[0,1]$ fixed, $h_0(z)$ and $h_\varepsilon(z)$ satisfy respectively Equation (19) for v replaced by u_0^* and Equation (19) for v replaced by $(u_0+\varepsilon\varphi)^*$. Then, in light of Proposition 4.3 (i),

$$J(u_0)=\int_\Omega -u_0 f(x,y)dxdy+\int_0^1 dz\int_0^{h_0(z)}\Big(\mathbf{a}(s)-u_0^*(s,z)\Big)\mathbf{a}^2(s)ds \tag{67}$$

and

$$J(u_0+\varepsilon\varphi)=\int_\Omega -(u_0+\varepsilon\varphi)f dxdy+\int_0^1 dz\int_0^{h_\varepsilon(z)}\Big(\mathbf{a}-(u_0+\varepsilon\varphi)^*\Big)\mathbf{a}^2(s)ds \tag{68}$$

We use the definition of h_0 and h_ε to establish that

$$S_{u_0}(h_0(z),z)-S_{u_0+\varepsilon\varphi}(h_\epsilon(z),z)\leq S_{u_0}(h_\epsilon(z),z)-S_{u_0+\varepsilon\varphi}(h_\epsilon(z),z)=\int_0^{h_\varepsilon(z)}\Big(u_0^*-(u_0+\varepsilon\varphi)^*\Big)\mathbf{a}^2(s)ds.$$

Likewise,

$$S_{u_0+\varepsilon\varphi}(h_\epsilon(z),z)-S_u(h_0(z),z)\leq S_{u+\varepsilon\varphi}(h_0(z),z)-S_u(h_0(z),z)=\int_0^{h_0(z)}\Big((u_0+\varepsilon\varphi)^*-u^*\Big)\mathbf{a}^2(s)ds.$$

Combining the last two displayed equations, we obtain:

$$\int_0^{h_\epsilon(z)}\Big((u_0+\varepsilon\varphi)^*-u_0^*\Big)\mathbf{a}^2(s)ds\leq S_{u_0+\varepsilon\varphi}(h_\epsilon(z),z)-S_{u_0}(h_0(z),z)\leq\int_0^{h_0(z)}\Big((u_0+\varepsilon\varphi)^*-u_0^*\Big)\mathbf{a}^2(s)ds \tag{69}$$

Note that

$$\begin{aligned}&\limsup_{\varepsilon\to0}\frac{1}{\varepsilon}\Big|\int_0^{h_\epsilon(z)}\Big((u_0+\varepsilon\varphi)^*-u_0^*\Big)\mathbf{a}^2(s)ds-\int_0^{h_0(z)}\Big((u_0+\varepsilon\varphi)^*-u_0^*\Big)\mathbf{a}^2(s)ds\Big|\\&\quad\leq\Big|\int_{h_0(z)}^{h_\epsilon(z)}\left(\frac{(u_0+\varepsilon\varphi)^*-u_0^*}{\varepsilon}\right)\mathbf{a}^2(s)ds\Big|\leq||\varphi||_\infty\left(\int_0^1\mathbf{a}^2(s)ds\right)\limsup_{\varepsilon\to0}|h_\epsilon(z)-h_0(z)|\end{aligned} \tag{70}$$

In view of Lemma 4.1 (iii), we use the fact that $(u_0+\varepsilon\varphi)^*$ uniformly converges to u_0^* as obtained Lemma 3.2 and standard arguments on sequences to show that

$$\lim_{\epsilon\to0}h_\epsilon(z)=h_0(z) \tag{71}$$

for all $z\in[0.1]$. Combining Equations (70) and (71), it follows that

$$\limsup_{\varepsilon\to0}\frac{1}{\varepsilon}\Big|\int_0^{h_\epsilon(z)}\Big((u+\varepsilon\varphi)^*-u_0^*\Big)\mathbf{a}^2(s)ds-\int_0^{h_0(z)}\Big((u+\varepsilon\varphi)^*-u^*\Big)\mathbf{a}^2(s)ds\Big|=0 \tag{72}$$

We use Equations (69) and (72) and Lemma 3.2 to obtain:

$$\begin{aligned}\lim_{\varepsilon\to0}\frac{S_{u_0+\varepsilon\varphi}(h_\epsilon(z),z)-S_{u_0}(h_0(z),z)}{\varepsilon}&=\lim_{\varepsilon\to0}\int_0^{h_0(z)}\frac{(u_0+\varepsilon\varphi)^*-u_0^*}{\varepsilon}\mathbf{a}^2(s)ds\\&=-\int_0^{h_0(z)}\varphi\circ Du_0^*\,\mathbf{a}^2(s)ds\end{aligned} \tag{73}$$

Using Lemma 3.2, we note that

$$\Big|\int_0^{h_0(z)}\frac{(u_0+\varepsilon\varphi)^*-u_0^*}{\varepsilon}\mathbf{a}^2(s)ds\Big|\leq||\varphi||_\infty\left(\int_0^{\mathcal{C}_0}\mathbf{a}^2(s)ds\right) \tag{74}$$

where $\mathcal{C}_0$ is provided as in Lemma 4.1. Thus, in light of Equations (73) and (74), we use the Lebesgue-dominated convergence theorem,

$$\lim_{\varepsilon\to0}\int_0^1\frac{S_{u+\varepsilon\varphi}(h_\epsilon(z),z)-S_{u_0}(h_0(z),z)}{\varepsilon}dz=-\int_0^1\int_0^{h_0(z)}\varphi\circ Du_0^*\,\mathbf{a}^2(s)dsdz.$$

Note that

$$\frac{J(u_0+\varepsilon\varphi)-J(u_0)}{\varepsilon}=\int_\Omega \varphi f dxdy+\int_0^1 \frac{S_{u_0+\varepsilon\varphi}(h_\epsilon(z),z)-S_{u_0}(h(z),z)}{\varepsilon}dz \tag{75}$$

As u_0 is a maximizer for J, we have

$$0=\lim_{\varepsilon\to 0}\frac{J(u_0+\varepsilon\varphi)-J(u_0)}{\varepsilon}=\int_\Omega \varphi f(x,y)dxdy-\int_0^1\int_0^{h_0(z)}\varphi\circ Du_0^*\,\mathbf{a}^2(s)dsdz \tag{76}$$

Since φ is arbitrary, we obtain from Equation (76) that

$$Du_0^*\#\chi_{\Lambda_{h_0}}\mathbf{a}^2=\chi_\Omega f \tag{77}$$

As $u_0=u_0^{**}$ and Du_0, Du_0^* exist almost everywhere with respect to the Lebesgue measure, we have

$$u_0(X)+u_0^*(Du_0(X))=\langle X, Du_0(X)\rangle \quad \chi_\Omega f\ a.e.,$$

and

$$u_0(Du_0^*(Y))+u_0^*(Y)=\langle Du_0^*(Y),Y\rangle \quad \chi_\Lambda\mathbf{a}^2\ a.e.$$

It follows that

$$Du_0^*\circ Du_0(X)=X \quad \chi_\Omega f\ a.e. \quad \text{and} \qquad Du_0\circ Du_0^*(Y)=Y \quad \chi_\Lambda\mathbf{a}^2\ a.e.$$

As a consequence, Equation (77) implies that:

$$Du\#\chi_\Omega f=\chi_{\Lambda_{h_0}}\mathbf{a}^2 \tag{78}$$

Note that as $h_0(z)$ satisfies Equation (19), the Equation (20) holds, as well, whenever $h_0>0$, that is,

$$u_0^*(h_0(z),z)=\mathbf{a}(h_0(z)) \quad \{h_0>0\} \tag{79}$$

We combine Equations (78) and (79) to obtain a weak solution of Equation (1). $\square$

Conflicts of Interest

The author declares no conflict of interest.

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