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Article

On the Continuity of the Hutchinson Operator

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Abstract: We investigate when the Hutchinson operator associated with an iterated function system is continuous. The continuity with respect to both the Hausdorff metric and Vietoris topology is carefully considered. An example showing that the Hutchinson operator on the hyperspace of nonempty closed bounded sets need not be Hausdorff continuous is given. Infinite systems are also discussed. The work clarifies and generalizes several partial results scattered across the literature.

Keywords: iterated function system; Hutchinson operator; multifunction; boundedly uniformly continuous map; upper semicontinuity; strict attractor; Vietoris continuity; compact-open topology

MSC classifications: 54B20, 37B99

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1. Introduction

The question of when the Hutchinson operator is continuous has not received sufficient attention. Only recently has this question been shown to be of practical interest, e.g., [1–3]. It is well known that the Hutchinson operator inherits essentially all of the continuity properties of the functions of the underlying iterated function system (IFS), *cf.* [4–6]. However, some issues remain obscure and unexplored.

For the non-specialist, we mention here some reasons why we are interested in the continuity of the Hutchinson operator. Recently, in increasingly abstract settings, conditions have been established under which the Hutchinson operator has attractive fixed points, see for example [7]; these fixed points are points in hyperspaces and are called attractors of the IFS. In applications, in diverse areas of science and engineering, these attractors may be models for complicated physical objects. While these objects may be geometrically intricate and difficult to describe directly, the IFS or equivalently the Hutchinson operator, may be relatively simple to describe. (For example, it is well known that a two-dimensional fractal fern is described efficiently with four two-dimensional affine maps.) Moreover, the continuity of the Hutchinson operator underlies the feasibility of using random iteration algorithms for computing attractors, in general settings, see [1,3,8]. Thus there is a practical payoff from increased abstraction, as we illustrate in the next paragraph. The key requirement is that, whatever sophisticated extension of the basic contractive theory is made, the resulting Hutchinson operator must act continuously on the hyperspace in question.

By way of illustration of the benefits of such abstraction we mention the notion of a "super" Hutchinson operator. The latter may be constructed by defining IFS of Hutchinson operators acting continuously on a hyperspace, see [9]. In this case, attractors are collections of fractals with partial self-similarity, and comprise points in a "hyper-hyperspace". Such collections may be sampled by means of a chaos game algorithm whereby each iteration yields a member of the collection, say of related objects that all look like ferns. For such constructions to work, continuity of the Hutchinson operator needs to be assured at several levels.

In the present work, we show how to deduce the Hausdorff continuity of the Hutchinson operator F from its uniform continuity, when F is regarded as acting on the hyperspace of compact sets in a metric space. This is no longer true when F is regarded as acting on the hyperspace of closed bounded sets, as we demonstrate by means of an example. We also provide a criterion for when it is true. Additionally, we discuss how the invariance of a strict attractor is related to the continuity of the iterated function system.

The situation is markedly different when one asks for Vietoris continuity of the Hutchinson operator F on the hyperspace of closed subsets of a topological space; in this case, F is continuous.

We close the paper with a discussion of the Hutchinson operator for infinite iterated function systems.

2. Hyperspaces, Multifunctions, Iterated Function Systems

Let (X, d) be a metric space with metric d. This is the environment where we work for the first half of the paper. Thereafter we switch to topological spaces.

The closure of $B \subset X$ will be denoted by $\overline{B}$. The ε-neighbourhood of B is

$$N_\varepsilon B := \{x \in X : d(x, b) < \varepsilon \text{ for some } b \in B\}$$

The *Hausdorff distance* between $B \subset X$ and $C \subset X$ is given by

$$d_H(B, C) := \inf\{\varepsilon > 0 : B \subset N_\varepsilon C \text{ and } C \subset N_\varepsilon B\}$$

We distinguish three collections of subsets of X: $\mathcal{P}(X)$—all nonempty sets, $\mathcal{F}_b(X)$—nonempty bounded closed sets, and $\mathcal{K}(X)$—nonempty compact sets. The Hausdorff distance is a metric on both $\mathcal{K}(X)$ and $\mathcal{F}_b(X)$, while it is only an extended-valued semimetric on $\mathcal{P}(X)$.

We call any map $\varphi : X \to \mathcal{P}(X)$ a *multifunction*. As usual, a single-valued map $f : X \to X$ is identified with the multifunction $\varphi : X \to \mathcal{K}(X)$, $\varphi(x) := \{f(x)\}$ for $x \in X$. The *image* of a nonempty $B \subset X$ under φ is $\varphi(B) := \bigcup_{b\in B} \varphi(b)$.

Once $\mathcal{P}(X)$ is endowed with d_H we can speak of continuity and uniform continuity. We need for multifunctions yet another type of continuity—upper semicontinuity—ubiquitous in topological dynamics, *cf.* [10]. A multifunction $\varphi : X \to \mathcal{P}(X)$ is *upper semicontinuous* at $x_0 \in X$, if

$$\forall \varepsilon > 0,\ \exists \delta > 0 :\ \varphi(N_\delta\{x_0\}) \subset N_\varepsilon \varphi(x_0)$$

More information on hyperspaces and multifunctions can be found in [11].

The system $(X, f_i : i \in I)$ consisting of a family of maps $f_i : X \to X$ is called an *iterated function system* (IFS). When I is finite we speak about a finite IFS. A *multivalued IFS* is given by a multifunction $\varphi : X \to \mathcal{P}(X)$. An ordinary IFS becomes multivalued if we define φ by $\varphi(x) := \{f_i(x) : i \in I\}$, $x \in X$. IFSs with condensation (inhomogeneous fractals) and Markov–Feller theory of IFSs fall gently within multivalued framework, *cf.* [12]. On the other hand, we loose an important tool in fractal geometry—the coding map, *cf.* [5].

The *Hutchinson operator* $F : \mathcal{P}(X) \to \mathcal{P}(X)$ associated with a system given by the multifunction $\varphi : X \to \mathcal{P}(X)$ is defined as

$$F(B) := \overline{\bigcup_{b\in B} \varphi(b)}$$

for $B \in \mathcal{P}(X)$. In the case of IFS $(X, f_i : i \in I)$ this means that $F(B) = \overline{\bigcup_{i\in I} f_i(B)}$. Throughout the paper the letter F will be reserved for the Hutchinson operator. Note that for our purposes we can assume that φ assumes closed values. Indeed, let $\psi : X \to \mathcal{P}(X)$, $\psi(x) := \overline{\varphi(x)}$ for $x \in X$. Then $F(B) = \overline{\varphi(B)} = \overline{\psi(B)}$ for $B \in \mathcal{P}(X)$.

The most important instance of the Hutchinson operator is its restriction to the hyperspace of compacta $F : \mathcal{K}(X) \to \mathcal{K}(X)$, since the hyperspace $\mathcal{K}(X)$ is often perceived as a habitat for fractals. To be more precise one has to assume that F sends compacta onto compacta. Indeed, this is fulfilled when the system $\{f_i\}_{i\in I}$ consists of continuous maps and I is finite. Still more general condition can be provided for multivalued IFSs.

Proposition 1. *Let $\varphi : X \to \mathcal{K}(X)$ be an upper semicontinuous multifunction with compact values. Then the induced Hutchinson operator $F : \mathcal{P}(X) \to \mathcal{P}(X)$ transforms compacta into compacta. In particular the restriction $F : \mathcal{K}(X) \to \mathcal{K}(X)$ is well-defined and*

$$F(B) = \varphi(B)$$

for $B \in \mathcal{K}(X)$.

Proof. It is well known that under our assumptions the image of a compact set is again compact ([11], (Proposition 6.2.11, p. 196)). $\square$

3. Continuity on $\mathcal{K}(X)$

In this section we establish a positive result concerning the continuity of the Hutchinson operator F induced by a multifunction φ.

Proposition 2. *Let $K \subset X$. If $\varphi : K \to \mathcal{K}(X)$ is uniformly continuous, then $F : \mathcal{K}(K) \to \mathcal{K}(X)$ is uniformly continuous too.*

Proof. Fix $\varepsilon > 0$. Find $\delta > 0$ such that for every pair $x_1, x_2 \in X$, $d(x_1, x_2) < \delta$ implies $d_H(\varphi(x_1), \varphi(x_2)) < \varepsilon$.

Now let $S_1, S_2 \in \mathcal{K}(K)$ be such that $d_H(S_1, S_2) < \delta$. Then $S_2 \subset N_\delta S_1$. Therefore

$$\varphi(S_2) \subset \varphi(N_\delta S_1) \subset N_\varepsilon \varphi(S_1)$$

By symmetry we get $d_H(\varphi(S_1), \varphi(S_2)) < \varepsilon$ as desired. $\square$

Theorem 1. *Let $\varphi : X \to \mathcal{K}(X)$ be a continuous multifunction with compact values. Then the Hutchinson operator $F : \mathcal{K}(X) \to \mathcal{K}(X)$ induced by φ is continuous.*

Proof. Let $S_n, S \in \mathcal{K}(X)$, $S_n \to S$ with respect to d_H. Put $K := \bigcup_{n=1}^{\infty} S_n \cup S = \overline{\bigcup_{n=1}^{\infty} S_n}$. Of course $K \in \mathcal{K}(X)$. Since φ is continuous, it is uniformly continuous on K. Hence $F : \mathcal{K}(K) \to \mathcal{K}(X)$ is uniformly continuous by Proposition 2. This yields $F(S_n) \to F(S)$. $\square$

In particular the Hutchinson operator associated with the finite IFS of continuous maps is continuous. However, simple examples show that an upper semicontinuous multifunction on a compact space need not induce a continuous Hutchinson operator, e.g., [4], (Counter-Example 1) and [5], (Proposition 1.5.3).

4. Lack of Continuity on $\mathcal{F}_b(X)$

This section is devoted to two major issues with the Hutchinson operator: why it is not continuous in general and what is the key ingredient in its continuity. Our presentation is heavily influenced by a work of A. Izzo [13].

Let an IFS comprising one single-valued map $f : X \to X$ be given. Let $F : \mathcal{P}(X) \to \mathcal{P}(X)$ be the Hutchinson operator induced by f. Assume that f maps bounded sets, $\mathcal{F}_b(X)$, onto bounded sets. Then the restriction of F from $\mathcal{P}(X)$ to $\mathcal{F}_b(X)$, $F : \mathcal{F}_b(X) \to \mathcal{F}_b(X)$, makes sense.

We say that $f : X \to X$ is *boundedly uniformly continuous* provided the restrictions $f|B$ of f to $B \subset X$ are uniformly continuous for all bounded sets B. Recall that $f : B \to X$ is uniformly continuous when for each pair of sequences x_n, y_n, $d(x_n, y_n) \to 0$ implies $d(f(x_n), f(y_n)) \to 0$.

Theorem 2. *(Criterion of continuity of F). Let $f : X \to X$ map bounded sets onto bounded sets. Let $F : \mathcal{F}_b(X) \to \mathcal{F}_b(X)$ be the associated Hutchinson operator. The following are equivalent:*

(i) F *is continuous,*

(ii) f *is boundedly uniformly continuous.*

Proof. The implication (ii) $\Rightarrow$ (i) follows at once from Theorem 1. We shall prove (i) $\Rightarrow$ (ii).

A contrario, suppose that f is not boundedly uniformly continuous, though F is continuous. Then there exist bounded sequences $x_n, y_n \in X$ and $\eta > 0$ such that $d(x_n, y_n) \to 0$ and $d(f(x_n), f(y_n)) \geq \eta$. Passing if necessary to a subsequence we can assume that $d(x_n, y_n) \downarrow 0$ monotonically. Making use of the Efremovic lemma ([11], (3.3.1, p. 92)) we can also assume that $d(f(x_n), f(y_m)) \geq \eta/8$ for all indices n, m.

Now observe that x_n and consequently y_n do not have accumulation points. Otherwise, $x_{n_k} \to x_*$ for some subsequence n_k and $y_{n_k} \to x_*$. That would imply $f(x_{n_k}) \to f(x_*)$, $f(y_{n_k}) \to f(x_*)$ against η-separation. Thus x_n and y_n form discrete sets.

Put

$$S_n := \{x_l\}_{l=1}^{\infty} \cup \{y_m\}_{m=n}^{\infty}$$

$S := \{x_l\}_{l=1}^{\infty}$. We have $S_n, S \in \mathcal{F}_b(X)$ and

$$d_H(S_n, S) \leq \sup_{m \geq n} d(x_m, y_m) = d(x_n, y_n) \to 0$$

Nevertheless $F(S_n) \not\to F(S)$, because $d_H(f(S_n), f(S)) \geq \eta/8$. This violates the continuity of F. $\square$

Having established a criterion for the continuity of F we are ready to give an example of a continuous map inducing a discontinuous Hutchinson operator.

Example 1. *Let X be an infinite dimensional normed space. Let $r > 0$. Let x_n be an r-separated sequence, i.e., $d(x_n, x_m) \geq r$ for $n \neq m$, which is bounded. Moreover, let y_n be a sequence disjoint from x_n with the property that $d(x_n, y_n) < r/3n$ for all n. Such sequences x_n, y_n always exist. The sets $S := \{x_n\}_{n=1}^{\infty}$, $Y := \{y_n\}_{n=1}^{\infty}$ are discrete closed subsets of X. Put $f(x) := x_1$ for $x \in S$, $f(y) := y_1$ for $y \in Y$. Since $S \cup Y$ is discrete, the function $f : S \cup Y \to X$ is continuous. Since $S \cup Y$ is closed, the Dugundji-Tietze theorem ([14], (1.3, p. 2)) yields an extension of f on the whole X. Recapitulating: $d(x_n, y_n) \to 0$, $d(f(x_n), f(y_n)) = d(x_1, y_1) > 0$. We see that f is not boundedly uniformly continuous. Consequently, the associated operator F cannot be continuous on $\mathcal{F}_b(X)$.*

5. Attractors and Continuity

Let $F : \mathcal{P}(X) \to \mathcal{P}(X)$ be the Hutchinson operator induced by an IFS. By F^n we denote the n-fold composition of F.

A compact nonempty set $A \subset X$ is a *strict attractor*, when there exists an open neighbourhood $U \supset A$ such that $F^n(S) \to A$ for all $S \in \mathcal{K}(U)$ (the limit being taken with respect to d_H), *cf.* [1,2].

One easily sees that if $F : \mathcal{K}(X) \to \mathcal{K}(X)$ is well-defined and continuous (for instance F associated with the IFS of continuous maps), and A is a strict attractor of F, then $F(A) = A$. Indeed,

$$A \leftarrow F^{n+1}(A) = F(F^n(A)) \to F(A)$$

We extend this observation to a class of discontinuous systems.

Proposition 3. *A strict attractor A of the IFS given by an upper semicontinuous multifunction $\varphi : X \to \mathcal{P}(X)$ is invariant, i.e., $F(A) = A$, where $F : \mathcal{P}(X) \to \mathcal{P}(X)$ is the Hutchinson operator generated by φ.*

Proof. Fix $\varepsilon > 0$. By Proposition 2 in [15] we know that for some $\delta > 0$

$$\varphi(N_\delta A) \subset N_{\varepsilon/2}\varphi(A)$$

so

$$F(N_\delta A) \subset N_\varepsilon F(A) \tag{1}$$

From the definition of a strict attractor there exists n_0 such that $h(F^n(A), A) < \delta$ for $n \geq n_0$, so

$$F^n(A) \subset N_\delta A \tag{2}$$

Combining Equations (1) and (2) gives for $n \geq n_0$

$$F^{n+1}(A) \subset F(N_\delta A) \subset N_\varepsilon F(A)$$

Thus

$$A = \lim_{n\to\infty} F^{n+1}(A) \subset N_{2\varepsilon} F(A)$$

and since ε was arbitrary $A \subset F(A)$. Due to $\subset$-monotonicity of F we now have

$$A \subset F(A) \subset \ldots \subset F^n(A) \to A = \overline{\bigcup_{n=1}^{\infty} F^n(A)}$$

which means $F(A) = A$. $\square$

6. Vietoris Continuity

We have seen that the Hutchinson operator behaves badly on hyperspaces other than $\mathcal{K}(X)$. We claim that the problem comes from the peculiarity of the Hausdorff metric topology. This peculiarity disappears for the Vietoris topology. Note, however, that some researchers feel that the Vietoris topology is too stringent for fractal geometry and other applications, e.g., [11] (Chapter 2.2, p. 49) and [12].

Let X be a Hausdorff topological space. We distinguish yet another hyperspace $\mathcal{F}(X)$—the collection of all nonempty closed subsets of X. We endow $\mathcal{F}(X)$ with the Vietoris topology.

Denote for $V \subset X$

$$V^+ := \{C \in \mathcal{F}(X) : C \subset V\}$$
$$V^- := \{C \in \mathcal{F}(X) : C \cap V \neq \emptyset\}$$

The *Vietoris topology* in $\mathcal{F}(X)$ is generated by subbasic sets of the form V^+ and V^-, where V runs through open subsets of X, *cf.* [11], (Definition 2.2.4, p. 47).

Theorem 3. *Let X be a normal topological space. Let $\varphi : X \to \mathcal{F}(X)$ be a Vietoris continuous multifunction. Then the associated Hutchinson operator $F : \mathcal{F}(X) \to \mathcal{F}(X)$ is Vietoris continuous.*

Proof. Let $V \subset X$ be open and $S \in \mathcal{F}(X)$ such that $F(S) \in V^+$. Let us shrink V to an open set W such that $F(S) \subset W$, $\overline{W} \subset V$. By Vietoris continuity of φ for each $s \in S$ there exists an open $U_s \ni s$ such that $\varphi(U_s) \subset W$. Put $U := \bigcup_{s\in S} U_s$. Then $S \in U^+$ and U is open. We have to check that $F(C) \in V^+$ for all $C \in U^+$. Indeed, if $C \subset U$, then

$$F(C) \subset \overline{\varphi(U)} \subset \overline{W} \subset V$$

Let $V \subset X$ be open, $S \in \mathcal{F}(X)$ and $F(S) \in V^-$. Thus

$$\emptyset \neq \overline{F(S) \cap V} = \overline{\varphi(S) \cap V}$$

so $\varphi(s) \cap V \neq \emptyset$ for some $s \in S$. By Vietoris continuity of φ there exists an open $U \ni s$ such that $\varphi(u) \cap V \neq \emptyset$ for all $u \in U$. We have to verify that $F(C) \in V^-$ for all $C \in U^-$. If $C \cap U \neq \emptyset$, then $u \in U$ for some $u \in C$. Therefore

$$F(C) \cap V \supset \varphi(u) \cap V \neq \emptyset$$

□

Although we were unaware of a direct statement of Theorem 3 in the literature, it can be deduced from a combination of Theorems 5.10.1, 5.7.2 and 5.3.1 in [16]. For $\mathcal{K}(X)$ the normality of X can be weakened to mere Hausdorff separation, *cf.* [5] (Proposition 3.1.3, p. 64).

Theorem 4 (Kieninger)**.** *Let X be a Hausdorff topological space. Let $\varphi : X \to \mathcal{K}(X)$ be a Vietoris continuous multifunction with compact values. Then the Hutchinson operator $F : \mathcal{K}(X) \to \mathcal{K}(X)$ induced by φ is Vietoris continuous.*

Let us remark that our Theorem 1 is a particular case of Theorem 4. This is because for a metric space X the Hausdorff metric topology and Vietoris topology coincide on $\mathcal{K}(X)$ ([11] (Exercises 3.2.9 and 3.2.10, p. 90)).

7. Infinite Systems

Here we develop ideas contained in [5,17,18].

Let X be a normal topological space and I be a compact Hausdorff space. An infinite IFS $(X, f_i : i \in I)$ can be turned into a parametric form $\Phi : I \times X \to X$, where $\Phi(i, x) := f_i(x)$ for $x \in X$. We extend this to infinite multivalued IFSs and consider multivalued system $\Phi : I \times X \to \mathcal{K}(X)$ over the alphabet I. The Vietoris topology is taken in $\mathcal{K}(X)$, while $I \times X$ has the product topology. We assume that Φ is Vietoris continuous. Naturally, we set $F : \mathcal{K}(X) \to \mathcal{K}(X)$,

$$F(S) := \overline{\Phi(I \times S)} = \Phi(I \times S), \text{ for } S \in \mathcal{K}(X)$$

to be the Hutchinson operator induced by Φ.

Theorem 5. *Let X be a normal topological space and I be compact. Let $\Phi : I \times X \to \mathcal{K}(X)$ be Vietoris continuous. Then the induced Hutchinson operator $F : \mathcal{K}(X) \to \mathcal{K}(X)$ is Vietoris continuous.*

Proof. Denote $\varphi_i : X \to \mathcal{K}(X)$, $\varphi_i(x) := \Phi(i, x)$ for $x \in X$, $i \in I$. We will show that $\varphi : X \to \mathcal{K}(X)$, $\varphi(x) := \bigcup_{i \in I} \varphi_i(x) = \Phi(I \times \{x\})$ for $x \in X$, is Vietoris continuous. Then one calls Theorem 3 to finish the proof. From the continuity of φ_i we know that there exists an open $U \ni x$ such that $V \cap \varphi_i(u) \neq \emptyset$ for all $u \in U$. In particular, $V \cap \varphi(u) \neq \emptyset$ for $u \in U$.

Let V be an open set with $V \supset \varphi(x) \neq \emptyset$. Fix $i \in I$, $x \in X$. Let $J_{i,x} \times U_{i,x} \ni (i, x)$ be an open rectangle in $I \times X$ such that $\Phi(j, u) \subset V$ for all $(j, u) \in J_{i,x} \times U_{i,x}$. Continuity of Φ makes this possible. Take a finite subcover $\bigcup_{i \in I_x} J_{i,x} \supset I$; $I_x \subset I$ finite. Define $U := \bigcap_{i \in I_x} U_{i,x}$, an open neighbourhood of x. All of this leads to the inclusion $\varphi(u) = \Phi(I \times \{u\}) \subset V$ for every $u \in U$. $\square$

Now, let X be a *k-space*, *i.e.*, a Hausdorff space where the set $C \subset X$ is closed if and only if $K \cap C$ is closed for every $K \in \mathcal{K}(X)$ ([19], (p. 152)). Such spaces are prevalent in analysis and include first countable, locally compact and Fréchet spaces among others. We additionally assume that X is a normal space ([19] (3.3.16, p. 151, 3.3.24, p. 153)).

We introduce the *space of multifunctions* as a function space $\mathcal{M}(X, X) := \mathcal{C}(X, \mathcal{K}(X))$ endowed with the compact-open topology. The *compact-open topology* has subbasic sets of the form

$$[K, \mathcal{O}] := \{\varphi \in \mathcal{M}(X, X) : \varphi(K) \subset \mathcal{O}\}$$

where $K \subset X$ is compact and $\mathcal{O} \subset \mathcal{K}(X)$ is open [19].

Theorem 6. *Let X be a normal k-space. Let $\{\varphi_i\}_{i \in I} \subset \mathcal{M}(X, X)$ be a collection of multifunctions such that*

(a) each φ_i is Vietoris continuous and has compact values,
(b) the entire $\{\varphi_i\}_{i \in I}$ is compact with respect to the compact-open topology.

Let $\varphi : X \to \mathcal{P}(X)$, $\varphi(x) = \bigcup_{i \in I} \varphi_i(x)$ for $x \in X$. Then

(i) φ constitutes a Vietoris continuous multifunction with compact values,
(ii) the Hutchinson operator $F : \mathcal{K}(X) \to \mathcal{K}(X)$ corresponding to φ is Vietoris continuous.

Proof. Let us identify I with $\{\varphi_i\}_{i \in I}$. We topologize I by pulling the compact-open topology from $\{\varphi_i\}_{i \in I}$. Define the evaluation mapping $\Phi : I \times X \to \mathcal{K}(X)$ by $\Phi(i, x) := \varphi_i(x)$ where φ_i corresponds to i via identification. Since I is compact, the evaluation is continuous; *cf.* [19] (2.6.11, p. 110; 3.4.3, p. 158; 3.4.20, p. 163). Thus, we are in position to apply Theorem 5 and complete the proof. $\square$

Theorem 6 (i) can be viewed as an improvement upon a known property concerning the continuity of unions of multifunctions, *cf.* [11], (Exercise 6.2.5, p. 197).

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Author Contributions

M.F.B. conceived the study; K.L. prepared the references; M.F.B. and K.L. formulated theorems and examples, devised proofs and wrote the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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