

Article

Dynamical Relation between Quantum Squeezing and Entanglement in Coupled Harmonic Oscillator System

Lock Yue Chew ^{1,*} and Ning Ning Chung ²

¹ Division of Physics and Applied Physics, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 637371, Singapore

² Department of Physics, National University of Singapore, Singapore 117542, Singapore;
E-Mail: phycnn@nus.edu.sg

* Author to whom correspondence should be addressed; E-Mail: lockyue@ntu.edu.sg;
Tel.: +65-6316-2968; +65-6316-6984.

Received: 27 February 2014; in revised form: 14 April 2014 / Accepted: 18 April 2014 /

Published: 23 April 2014

Abstract: In this paper, we investigate into the numerical and analytical relationship between the dynamically generated quadrature squeezing and entanglement within a coupled harmonic oscillator system. The dynamical relation between these two quantum features is observed to vary monotonically, such that an enhancement in entanglement is attained at a fixed squeezing for a larger coupling constant. Surprisingly, the maximum attainable values of these two quantum entities are found to consistently equal to the squeezing and entanglement of the system ground state. In addition, we demonstrate that the inclusion of a small anharmonic perturbation has the effect of modifying the squeezing *versus* entanglement relation into a nonunique form and also extending the maximum squeezing to a value beyond the system ground state.

Keywords: quantum entanglement; squeezed state; coupled harmonic oscillators

Classification: PACS 03.65.Ge, 31.15.MD

1. Introduction

Entanglement is a fundamental resource for non-classical tasks in the field of quantum information [1]. It has been shown to improve communication and computation capabilities via the

notion of quantum dense coding [2], quantum teleportation [3], unconditionally secured quantum cryptographic protocols [4,5], and quantum algorithms for integer factorization [6]. For any quantum algorithm operating on pure states, it has been proven that the presence of multi-partite entanglement is necessary if the quantum algorithm is to offer an exponential speed-up over classical computation [7]. Note, however, that a non-zero value of entanglement might not be the necessary condition for quantum computational speed up of algorithm operating on mixed states [8]. In addition, in order to achieve these goals practically, it is necessary to maintain the entanglement within the quantum states which are fragile against the decohering environment. An approach would be to employ an entangled state with as large an entanglement as possible, and the idea is that the production of such entangled state could be tuned through the operation of quantum squeezing.

Indeed, the relation between quantum squeezing and quantum entanglement has been actively pursued in recent years [9–18]. Notably, the creation of entanglement is shown experimentally to be able to induce spin squeezing [9,10]. Such entanglement-induced squeezing has the important outcome of producing measuring instruments that go beyond the precision of current models. In addition, quantum squeezing is found to be able to induce, enhance and even preserve entanglement in decohering environments [11–13]. Previously, we have investigated the relation between the squeezing and entanglement of the ground state of the coupled harmonic oscillator system [16,17]. The ground state entanglement entropy was found to increase monotonically with an increase in quadrature squeezing within this system. When a small anharmonic perturbing potential is added to the system, a further enhancement in quadrature squeezing is observed. While the entropy-squeezing curve shifts to the right in this case, we realized that the entanglement entropy is still a monotonically increasing function in terms of quadrature squeezing.

In this paper, we have extended our earlier work discussed above by investigating into the dynamical relation between quadrature squeezing and entanglement entropy of the coupled harmonic oscillator system. Coupled harmonic oscillator system has served as useful paradigm for many physical systems, such as the field modes of electromagnetic radiation [19–21], the vibrations in molecular systems [22], and the formulation of the Lee model in quantum field theory [23]. It was shown that the coupled harmonic oscillator system possesses the symmetry of the Lorentz group $O(3, 3)$ or $SL(4, r)$ classically, and that of the symmetry $O(3, 2)$ or $Sp(4)$ quantum mechanically [24]. In addition, the physics of coupled harmonic oscillator system can be conveniently represented by the mathematics of two-by-two matrices, which have played a role in clarifying the physical basis of entanglement [25]. In Section 2 of this paper, we first described the coupled harmonic oscillator model. It is then followed by a discussion on the relation between the dynamically generated squeezing and entanglement of the coupled oscillator systems, which we have determined quantitatively via numerical computation. In Section 3 of the paper, we present analytical results in support of the numerical results obtained in Section 2. Here, we illustrate how the problem can be solved in terms of two-by-two matrices. Then, in Section 4 of the paper, we study how the inclusion of anharmonicity can influence the relation between the dynamically generated squeezing and entanglement. Finally, we give our conclusion in Section 5 of the paper.

2. Dynamical Relation of Quantum Squeezing and Entanglement in Coupled Harmonic Oscillator System

The Hamiltonian of the coupled harmonic oscillator system is described as follow:

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2}m_1\omega_1^2x_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2}m_2\omega_2^2x_2^2 + \lambda(x_2 - x_1)^2 \quad (1)$$

where x_1 and x_2 are the position co-ordinates, while p_1 and p_2 are the momenta of the oscillators. The interaction potential between the two oscillators is assumed to depend quadratically on the distance between the oscillators, and is proportional to the coupling constant λ . For simplicity, we have set $m_1 = m_2 = m$ and $\omega_1 = \omega_2 = \omega$. This Hamiltonian is commonly used to model physical systems such as the vibrating molecules or the squeezed modes of electromagnetic field. In fact, the model has been widely explored [26–28] and is commonly used to elucidate the properties of quantum entanglement in continuous variable systems [29–35].

Next, let us discuss on the relation between the squeezing and entanglement of the lowest energy eigenstate of this coupled harmonic oscillator system. Note that

$$H |g\rangle = E_0 |g\rangle \quad (2)$$

with $|g\rangle$ being the ground state and E_0 being the lowest eigen-energy of the coupled oscillator system with Hamiltonian given by Equation (1). Entanglement between the two oscillators can be quantified by the von Neumann entropy:

$$S_{vN} = -\text{Tr} [\rho_l \ln \rho_l] \quad (3)$$

where ρ_l is the reduced density matrix. For squeezing parameter, we shall adopt the dimensionless definition:

$$S_x = -\ln \frac{\sigma_{x_1}}{\sigma_{x_1}^{(0)}} \quad (4)$$

with $\sigma_{x_1} = \sqrt{\langle x_1^2 \rangle - \langle x_1 \rangle^2}$ being the uncertainty associated with the first oscillator's position and the normalization constant $\sigma_{x_1}^{(0)} = \sqrt{\hbar/2m\omega}$ being the uncertainty associated with the harmonic oscillator's position. For simplicity, we shall evaluate only the position squeezing in the first oscillator.

Indeed, the position uncertainty squeezing and the entanglement entropy of the ground state of this oscillator have been solved analytically by previous studies [36,37] as follows:

$$S_x = -\ln \frac{\sqrt{\frac{\hbar}{2m\omega} \frac{1+\gamma}{2}}}{\sqrt{\frac{\hbar}{2m\omega}}} = -\ln \sqrt{\frac{1+\gamma}{2}} \quad (5)$$

where $\gamma = 1/\sqrt{1 + 4\lambda/m\omega^2}$; and

$$S_{vN} = \cosh^2 \left(\frac{\ln \gamma}{4} \right) \ln \left[\cosh^2 \left(\frac{\ln \gamma}{4} \right) \right] - \sinh^2 \left(\frac{\ln \gamma}{4} \right) \ln \left[\sinh^2 \left(\frac{\ln \gamma}{4} \right) \right] \quad (6)$$

As shown in Reference [17], by eliminating γ between Equations (5) and (6), the relation between the squeezing parameter and the von Neumann entropy of the ground state of the coupled harmonic oscillators is obtained as follow:

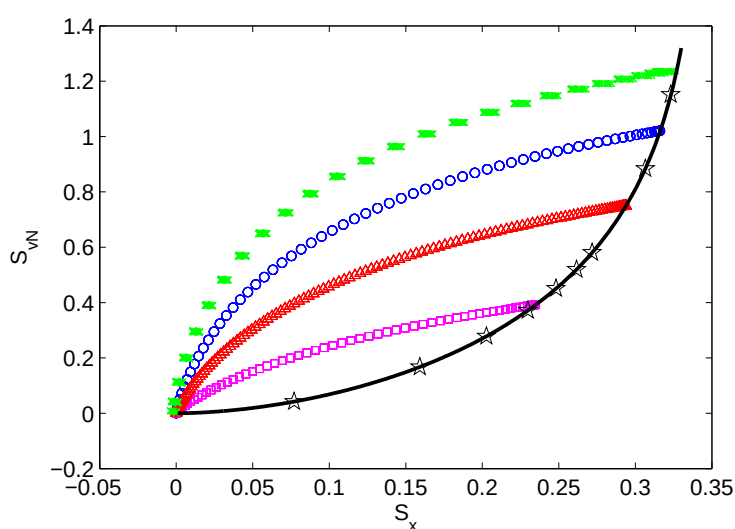
$$S_{vN} = \frac{(\xi + 1)^2}{4\xi} \ln \left(\frac{(\xi + 1)^2}{4\xi} \right) - \frac{(\xi - 1)^2}{4\xi} \ln \left(\frac{(\xi - 1)^2}{4\xi} \right) \quad (7)$$

with

$$\xi = \sqrt{2e^{-2S_x} - 1} \quad (8)$$

This relation is shown as a solid line in Figure 1.

Figure 1. A plot on the dynamical relation between entanglement and squeezing obtained numerically for coupled harmonic oscillator system with the coupling constant $\lambda = 0.75$ (squares), 2 (triangles), 3.75 (circles) and 6 (crosses). Note that the ground state entanglement-squeezing curve given by Equation (7) is plotted as a solid curve for comparison. In addition, the values of the maximum attainable squeezing and entanglement for various λ have been plotted as stars.



In this paper, we have gone beyond the static relation between squeezing and entanglement based on the stationary ground state. In particular, we have explored numerically into the dynamical generation of squeezing and entanglement via the quantum time evolution, with the initial state being the tensor product of the vacuum states ($|0, 0\rangle$) of the oscillators. Note that the obtained results hold true for any initial coherent states ($|\alpha_1, \alpha_2\rangle$) since the entanglement dynamics of the coupled harmonic oscillator system is independent of initial states [38]. In general, the system dynamics is either two-frequency periodic or quasi-periodic depending on whether the ratio of the two frequencies, $f_1 = 1$ and $f_2 = \sqrt{1 + 4\lambda}$, are rational or irrational. By yielding the values of the squeezing parameter and the entanglement entropy at the same time point within their respective dynamical evolution, we obtained the dynamical relations between the squeezing and entanglement for different coupling constants $\lambda = 0.75, 2, 3.75$ and 6, as shown in Figure 1. Interestingly, the results show a smooth monotonic increase of the dynamically generated entanglement entropy as the quadrature squeezing increases for each λ . In addition, the dynamically generated entanglement entropy is observed to be larger for a fixed squeezing as λ increases. It is surprising that the maximum attainable values of these two quantum entities determined dynamically are found to fall consistently on the system ground states' squeezing and entanglement relation as given by Equations (7) and (8) for all values of λ . More importantly, this relation also serve as a bound to the entanglement entropy and squeezing that are generated dynamically.

3. Analytical Derivation on the Dynamical Relation between Quantum Squeezing and Entanglement

In this section, we shall perform an analytical study on the dynamical relationship between quantum squeezing and the associated entanglement production. We first yield the second quantized form of the Hamiltonian of the coupled harmonic oscillator system as follow:

$$H = a_1^\dagger a_1 + a_2^\dagger a_2 + 1 + \frac{\lambda}{2} \{ (a_1^\dagger + a_1) - (a_2^\dagger + a_2) \}^2 \quad (9)$$

Then, the time evolution of the annihilation operator a_j (as well as the creation operator $a_j^\dagger$) can be determined according to the following Heisenberg equation of motion:

$$\frac{d}{dt} a_j = \frac{1}{i} [a_j, H] \quad (10)$$

From this, we obtain:

$$\frac{d}{dt} \tilde{a} = A \tilde{a} \quad (11)$$

with $\tilde{a} = (a_1 \ a_1^\dagger \ a_2 \ a_2^\dagger)^T$ and

$$A = \begin{pmatrix} B & C \\ C & B \end{pmatrix} \quad (12)$$

Note that

$$B = i \begin{pmatrix} -(1 + \lambda) & -\lambda \\ \lambda & 1 + \lambda \end{pmatrix} \quad (13)$$

and

$$C = i \begin{pmatrix} \lambda & \lambda \\ -\lambda & -\lambda \end{pmatrix} \quad (14)$$

Due to the symmetry in the coupled oscillator system, the matrix A is symmetric in the form of a two-by-two matrix although it is not symmetric in its full four-by-four matrix form. This symmetric property enables a simple evaluation of the time dependent annihilation and creation operators of the oscillators:

$$\tilde{a}(t) = F \tilde{a}(0) \quad (15)$$

where

$$F = \frac{1}{2} \begin{pmatrix} J e^{D_1 t} J - K e^{D_2 t} K^{-1} & J e^{D_1 t} J + K e^{D_2 t} K^{-1} \\ J e^{D_1 t} J + K e^{D_2 t} K^{-1} & J e^{D_1 t} J - K e^{D_2 t} K^{-1} \end{pmatrix} \quad (16)$$

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (17)$$

$$D_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad (18)$$

$$D_2 = \begin{pmatrix} i\Omega & 0 \\ 0 & -i\Omega \end{pmatrix} \quad (19)$$

and

$$K = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \quad (20)$$

with $\Omega = f_2 = \sqrt{1 + 4\lambda}$ and $\beta = (1 + \Omega)/(1 - \Omega)$. We then have:

$$a_1(t) = \left(\frac{1}{2}e^{-it} - \eta_1 + \eta_2 \right) a_1(0) + \eta_3 a_1^\dagger(0) + \left(\frac{1}{2}e^{-it} + \eta_1 - \eta_2 \right) a_2(0) - \eta_3 a_2^\dagger(0) \quad (21)$$

$$a_1^\dagger(t) = -\eta_3 a_1(0) + \left(\frac{1}{2}e^{it} - \eta_1^* + \eta_2^* \right) a_1^\dagger(0) + \eta_3 a_2(0) + \left(\frac{1}{2}e^{it} + \eta_1^* - \eta_2^* \right) a_2^\dagger(0) \quad (22)$$

$$a_2(t) = \left(\frac{1}{2}e^{-it} + \eta_1 - \eta_2 \right) a_1(0) - \eta_3 a_1^\dagger(0) + \left(\frac{1}{2}e^{-it} - \eta_1 + \eta_2 \right) a_2(0) + \eta_3 a_2^\dagger(0) \quad (23)$$

$$a_2^\dagger(t) = \eta_3 a_1(0) + \left(\frac{1}{2}e^{it} + \eta_1^* - \eta_2^* \right) a_1^\dagger(0) - \eta_3 a_2(0) + \left(\frac{1}{2}e^{it} - \eta_1^* + \eta_2^* \right) a_2^\dagger(0) \quad (24)$$

where

$$\begin{aligned} \eta_1 &= \frac{(1 - \Omega)^2}{8\Omega} e^{i\Omega t} \\ \eta_2 &= \frac{(1 + \Omega)^2}{8\Omega} e^{-i\Omega t} \\ \eta_3 &= \frac{i(1 - \Omega)(1 + \Omega)}{4\Omega} \sin(\Omega t) \end{aligned}$$

With these results, we are now ready to determine the analytical expressions of both the quantum entanglement and squeezing against time. For entanglement, we shall employ the criterion developed by Duan *et al.* [39] for quantification since it leads to simplification of the analytical expression while remaining valid as a measure of entanglement in coupled harmonic oscillator systems. According to this criterion, as long as

$$S_D = 2 - (\Delta u)^2 - (\Delta v)^2 > 0 \quad (25)$$

the state of the quantum system is entangled. Note that $u = x_1 + x_2$ and $v = p_1 - p_2$ are two EPR-type operators, whereas Δu and Δv are the corresponding quantum fluctuation. This allows us to express the entanglement measure S_D as follow:

$$\begin{aligned} S_D(t) &= 2(\langle a_1^\dagger a_1 \rangle - \langle a_1^\dagger \rangle \langle a_1 \rangle + \langle a_2^\dagger a_2 \rangle - \langle a_2^\dagger \rangle \langle a_2 \rangle + \\ &\quad \langle a_1^\dagger a_2^\dagger \rangle - \langle a_1^\dagger \rangle \langle a_2^\dagger \rangle + \langle a_1 a_2 \rangle - \langle a_1 \rangle \langle a_2 \rangle) \end{aligned} \quad (26)$$

Note that the short form $\langle O \rangle$ used in Equation (26) implies $\langle \alpha_1, \alpha_2 | O(t) | \alpha_1, \alpha_2 \rangle$, where $|\alpha_1, \alpha_2\rangle$ represents a tensor product of arbitrary initial coherent states. Recall that the subsequent results are independent of the initial states as mentioned in the last section. After substituting Equations (21)–(24) into Equation (26), we obtain the analytical expression of entanglement against time:

$$S_D(t) = (\Omega^2 - 1) \sin^2 \Omega t \quad (27)$$

In coupled harmonic oscillator systems, S_D has a unique monotonic relation with S_{vN} (see Figure 2). For squeezing, we have

$$\begin{aligned} S_x(t) &= -\ln \sqrt{\frac{\langle x_1^2 \rangle - \langle x_1 \rangle^2}{0.5}} \\ &= -\ln \sqrt{\langle a_1^{\dagger 2} \rangle - \langle a_1^\dagger \rangle^2 + \langle a_1^2 \rangle - \langle a_1 \rangle^2 + \langle a_1^\dagger a_1 \rangle - \langle a_1^\dagger \rangle \langle a_1 \rangle + \langle a_1 a_1^\dagger \rangle - \langle a_1 \rangle \langle a_1^\dagger \rangle} \end{aligned} \quad (28)$$

Then, by substituting Equations (21)–(24) into Equation (28) as before, we obtain the analytical expression of squeezing against time:

$$S_x(t) = -\ln \sqrt{1 - \frac{\Omega^2 - 1}{2\Omega^2} \sin^2 \Omega t} \quad (29)$$

We can also obtain an analytical expression between S_D and S_x by substituting Equation (27) into Equation (29) with some rearrangement:

$$S_D = 2\Omega^2 (1 - e^{-2S_x}) \quad (30)$$

It is important to note that S_x can only span a range of values $0 \leq S_x \leq S_x^{(m)}$, where $S_x^{(m)} = -\ln \sqrt{(\Omega^2 + 1)/2\Omega^2}$. Furthermore, for a coupled harmonic oscillator system with a fixed value of λ , the dynamically generated squeezing can be higher than the squeezing in the system's ground state. The analytical result given by Equation (30) is plotted in Figure 3 for $\lambda = 0.75, 2, 3.75, 6$ and 10 , with each curve begins at $S_x = 0, S_D = 0$ and ends at $S_x = S_x^{(m)}, S_D = S_D^{(m)} = \Omega^2 - 1$. In fact, the set of end points given by $S_x = S_x^{(m)}, S_D = S_D^{(m)}$ gives rise to the solid curve in Figure 3. Specifically, the maximum entanglement and the maximum squeezing parameter relates as follow:

$$S_D^{(m)} = \frac{1 - \xi^2}{\xi^2} \quad (31)$$

with

$$\xi = \sqrt{2e^{-2S_x^{(m)}} - 1} \quad (32)$$

Note that Equation (32) is the same as Equation (8), and Equation (31) corresponds to the ground state solid curve of Figure 1. This allows us to deduce the monotonic relation between S_D and S_{vN} , which is performed by evaluating the relation between S_D of the maximum entangled state and S_{vN} of the ground state at equal amount of squeezing. Indeed, the resulting derived relationship shown as solid line in Figure 2 is valid due to the fact that the link between $S_D(t)$ and $S_{vN}(t)$ is found to be expressible by precisely the same curve. Thus, we have concretely affirmed the one to one correspondence between S_D and S_{vN} through this relationship. More importantly, we have clearly demonstrated that the maximum entanglement attained dynamically is the same as the degree of entanglement of a ground state with the same squeezing.

Figure 2. This plot shows the monotonic relation between S_D and S_{vN} in coupled harmonic oscillator systems. $S_D(t)$ and $S_{vN}(t)$ are plotted as squares ($\lambda = 0.75$), triangles ($\lambda = 2$), circles ($\lambda = 3.75$) and crosses ($\lambda = 6$). The relation between the ground state von Neuman entropy given by $S_{vN} = \frac{(\xi+1)^2}{4\xi} \ln \left(\frac{(\xi+1)^2}{4\xi} \right) - \frac{(\xi-1)^2}{4\xi} \ln \left(\frac{(\xi-1)^2}{4\xi} \right)$ and the maximum dynamically generated entanglement given by $S_D^{(m)} = \frac{1-\xi^2}{\xi^2}$ is plotted as solid curve. Note that both S_{vN} and $S_D^{(m)}$ are functions of the squeezing parameter S_x and $\xi = \sqrt{2e^{-2S_x} - 1}$.

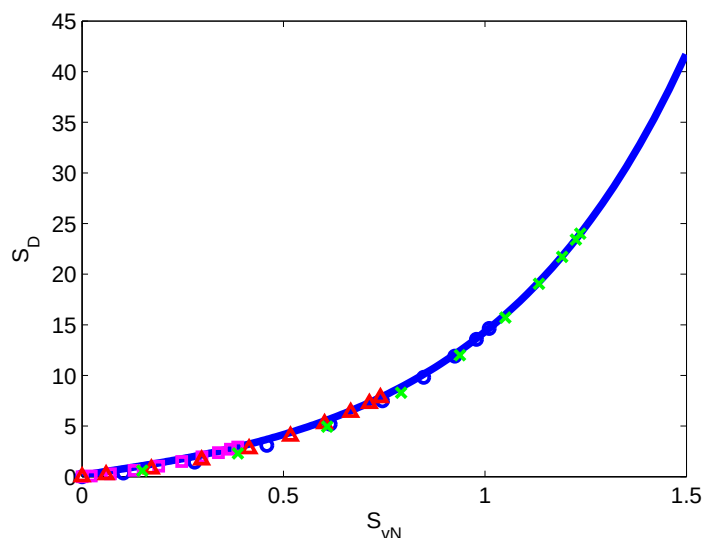
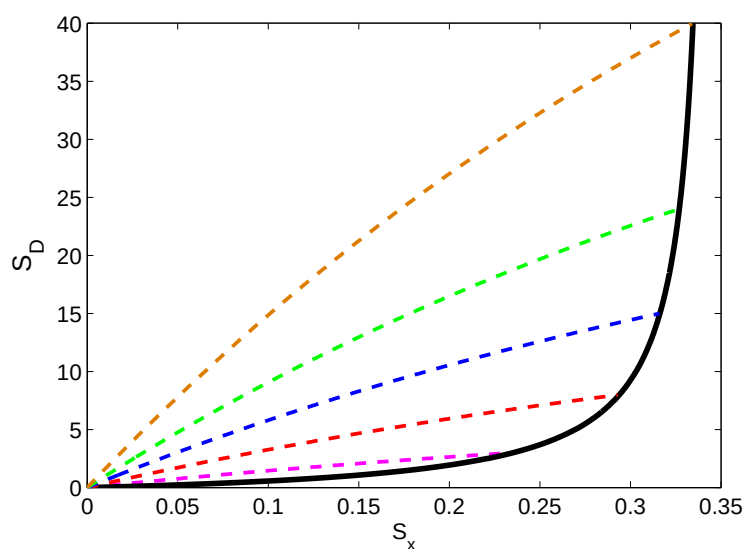


Figure 3. A plot on the dynamical relation between entanglement and squeezing given by Equation (30) for coupled harmonic oscillator system. The relation is dependent on λ and the curves from top to bottom are with respect to $\lambda = 10, 6, 3.75, 2$, and 0.75 respectively. Note that the thick solid curve represents the values of the maximum attainable squeezing and entanglement for the range $0 < \lambda < 10$.



When projected into the $x_1 - p_2$ or $x_2 - p_1$ plane, the initial coherent state can be represented by a circular distribution with equal uncertainty in both x and p direction. During the time evolution, the circular distribution is being rotated and squeezed. As a result, squeezing and entanglement are generated such that the distribution becomes elliptical in the $x_1 - p_2$ or $x_2 - p_1$ plane with rotation of the ellipse's major axis away from the x - or p -axis which creates entanglement. The generation of squeezing and entanglement reaches their maximum values at the same time when the major axis of the elliptical distribution has rotated 45° away from the x - or p -axis. Note that at this point, squeezing is merely in the collective modes. On the other hand, as discussed in Reference [37], the ground state wave function of the coupled harmonic oscillator system is separable in their collective modes. In both cases, entanglement and squeezing relates uniquely as given by Equation (7) and (31).

4. Quantum Squeezing and Entanglement in Coupled Anharmonic Oscillator Systems

Next, let us investigate the effect of including an anharmonic potential on the dynamical relation between squeezing and entanglement through the following Hamiltonian systems:

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2}m_1\omega_1^2x_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2}m_2\omega_2^2x_2^2 + \lambda(x_2 - x_1)^2 + \epsilon(x_1^4 + x_2^4) \quad (33)$$

For simplicity, we consider only the quartic perturbation potential. For previous studies of entanglement in coupled harmonic oscillators with quartic perturbation, see Reference [40] and the references therein. Again, we choose the initial state to be the tensor product of the vacuum states. We then evolve the state numerically through the Hamiltonian given by Equation (33). For the numerical simulation, we consider only a small anharmonic perturbation, *i.e.*, $\epsilon = 0.1$ and 0.2 . Note that we have truncated the basis size at $M = 85$ at which the results are found to converge.

With a small anharmonic perturbation, the dynamically generated entanglement entropy is no longer a smooth monotonically increasing function of the quadrature squeezing as before (see Figure 4). This implies that for coupled anharmonic oscillator systems, the dynamically generated degree of entanglement cannot be characterized through a measurement of the squeezing parameter. In addition, when the anharmonic potential is included, the maximum attainable squeezing is much enhanced. This effect is clearly shown in Figure 4, where we observe that the maximum dynamical squeezing extends far beyond the largest squeezing given by the coupled anharmonic oscillator system's ground state at different λ . In addition, as we increase the anharmonic perturbation from 0.1 to 0.2 , we found that the maximum attainable squeezing continues to grow with extension going further beyond the largest squeezing given by the ground state of the coupled anharmonic oscillator system.

Figure 4. The effect of anharmonicity ($\epsilon = 0.1$) on the dynamical relation between quadrature squeezing and entanglement. Note that we have employed the following parameter: (a) $\lambda = 0.75$; (b) $\lambda = 2$; (c) $\lambda = 3.75$; and (d) $\lambda = 6$. We have plotted the ground state entanglement-squeezing curve of the coupled anharmonic oscillator system with $\epsilon = 0.1$ as solid curve for comparison.

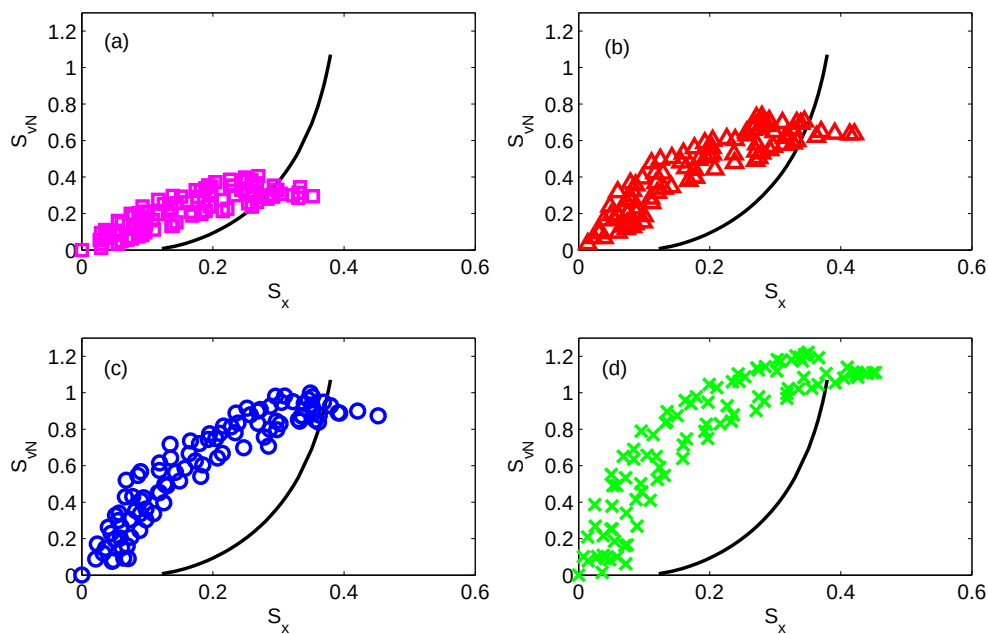
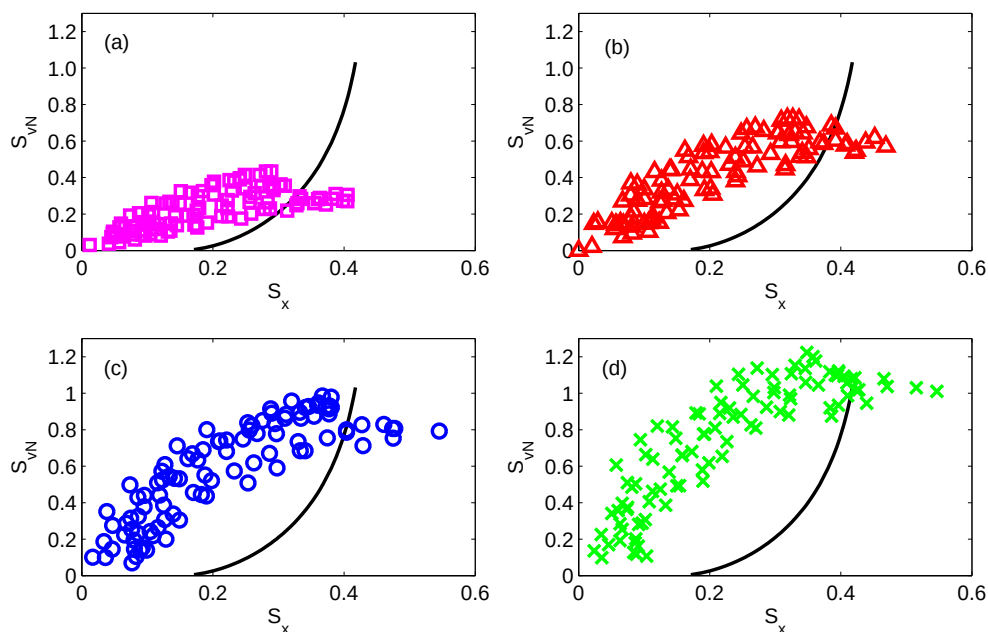


Figure 5. The effect of anharmonicity ($\epsilon = 0.2$) on the dynamical relation between quadrature squeezing and entanglement. Note that we have employed the following parameter: (a) $\lambda = 0.75$, (b) $\lambda = 2$, (c) $\lambda = 3.75$, and (d) $\lambda = 6$. We have plotted the ground state entanglement-squeezing curve of the coupled anharmonic oscillator system with $\epsilon = 0.2$ as solid curve for comparison.



5. Conclusions

We have studied into the dynamical generation of quadrature squeezing and entanglement for both coupled harmonic and anharmonic oscillator systems. Our numerical and analytical results show that the quantitative relation that defines the dynamically generated squeezing and entanglement in coupled harmonic oscillator system is a monotonically increasing function. Such a monotonic relation vanishes, however, when a small anharmonic potential is added to the system. This result implies the possibility of characterizing the dynamically generated entanglement by means of squeezing in the case of coupled harmonic oscillator system. In addition, we have uncovered the unexpected result that the maximum attainable entanglement and squeezing obtained dynamically matches exactly the entanglement-squeezing relation of the system's ground state of the coupled harmonic oscillators. When an anharmonic potential is included, we found that the dynamically generated squeezing can be further enhanced. We perceive that this result may provide important insights to the construction of precision instruments that attempt to beat the quantum noise limit.

Acknowledgements

L. Y. Chew would like to thank Y. S. Kim for the helpful discussion on this work during the ICSSUR 2013 conference held in Nuremberg, Germany.

Author Contributions

All authors contribute equally to the theoretical analysis, numerical computation, and writing of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

References

1. Nielson, M.A.; Chuang, I.L. *Quantum Computation and Quantum Information*; Cambridge University Press: Cambridge, UK, 2000.
2. Bennett, C.H.; Wiesner, S.J. Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states. *Phys. Rev. Lett.* **1992**, *69*, 2881–2884.
3. Bennett, C.H.; Brassard, G.; Crépeau, C.; Jozsa, R.; Peres, A.; Wootters, W.K. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Phys. Rev. Lett.* **1993**, *70*, 1895–1899.
4. Bennett, C.H.; Brassard, G. Quantum cryptography: Public key distribution and coin tossing. In Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing, IEEE Computer Society, New York, NY, USA, 1984; pp. 175–179.
5. Ekert, A.K. Quantum cryptography based on Bell's theorem. *Phys. Rev. Lett.* **1991**, *67*, 661–663.
6. Shor, P.W. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM J. Comput.* **1997**, *26*, 1484–1509.

7. Jozsa, R.; Linden, N. On the role of entanglement in quantum-computational speed-up. *Proc. R. Soc. Lond. A* **2003**, *459*, 2011–2032.
8. Lanyon, B.P.; Barbieri, M.; Almeida, M.P.; White, A.G. Experimental quantum computing without entanglement. *Phys. Rev. Lett.* **2008**, *101*, 200501:1–200501:4.
9. Sørensen, A.; Duan, L.M.; Cirac, J.I.; Zoller, P. Many-particle entanglement with Bose-Einstein condensates. *Nature* **2001**, *409*, 63–66.
10. Bigelow, N. Squeezing Entanglement. *Nature* **2001**, *409*, 27–28.
11. Furuichi, S.; Mahmoud, A.A. Entanglement in a squeezed two-level atom. *J. Phys. A Math. Gen.* **2001**, *34*, 6851–6857.
12. Xiang, S.; Shao, B.; Song, K. Quantum entanglement and nonlocality properties of two-mode Gaussian squeezed states. *Chin. Phys. B* **2009**, *18*, 418–425.
13. Galve, F.; Pachón, L.A.; Zueco, D. Bringing entanglement to the high temperature limit. *Phys. Rev. Lett.* **2010**, *105*, doi:10.1103/PhysRevLett.105.180501.
14. Ulam-Orgikh, D.; Kitagawa, M. Spin squeezed and decoherence limit in Ramsey spectroscopy. *Phys. Rev. A* **2001**, *64*, doi:10.1103/PhysRevA.64.052106.
15. Wolf, M.M.; Eisert, J.; Plenio, M.B. Entangling power of passive optical elements. *Phys. Rev. Lett.* **2003**, *90*, 047904:1–047904:4.
16. Chung, N.N.; Er, C.H.; Teo, Y.S.; Chew, L.Y. Relation of the entanglement entropy and uncertainty product in ground states of coupled anharmonic oscillators. *Phys. Rev. A* **2010**, *82*, doi:10.1103/PhysRevA.82.014101.
17. Chew, L.Y.; Chung, N.N. Quantum entanglement and squeezing in coupled harmonic and anharmonic oscillators systems. *J. Russ. Laser Res.* **2011**, *32*, 331–337.
18. Er, C.H.; Chung, N.N.; Chew, L.Y. Threshold effect and entanglement enhancement through local squeezing of initial separable states in continuous-variable systems. *Phys. Scripta* **2013**, *87*, doi:10.1088/0031-8949/87/02/025001.
19. Han, D.; Kim, Y.S.; Noz, M.E. Linear canonical transformations of coherent and squeezed states in the Wigner phase space. *Phys. Rev. A* **1988**, *37*, 807–814.
20. Han, D.; Kim, Y.S.; Noz, M.E. Linear canonical transformations of coherent and squeezed states in the Wigner phase space. II. Quantitative analysis. *Phys. Rev. A* **1989**, *40*, 902–912.
21. Han, D.; Kim, Y.S.; Noz, M.E. Linear canonical transformations of coherent and squeezed states in the Wigner phase space. III. Two-mode states. *Phys. Rev. A* **1990**, *41*, 6233–6244.
22. Wilson, E.B.; Decius, J.C.; Cross, P.C. *Molecular Vibration*; McGraw-Hill: New York, NY, USA, 1955.
23. Schweber, S.S. *An Introduction to Relativistic Quantum Field Theory*; Row-Peterson: New York, NY, USA, 1961.
24. Han, D.; Kim, Y.S.; Noz, M.E. O(3,3)-like symmetries of coupled harmonic-oscillators. *J. Math. Phys.* **1995**, *36*, 3940–3954.
25. Kim, Y.S.; Noz, M.E. Coupled oscillators, entangled oscillators, and Lorentz-covariant harmonic oscillators. *J. Opt. B Quantum Semiclass. Opt.* **2005**, *7*, S458–S467.
26. Eisert, J.; Plenio, M.B.; Bose, S.; Hartley, J. Towards quantum entanglement in nanoelectromechanical devices. *Phys. Rev. Lett.* **2004**, *93*, 190402:1–190402:4.

27. Joshi, C.; Jonson, M.; Öhberg, P.; Andersson, E. Constructive role of dissipation for driven coupled bosonic modes. *Phys. Rev. A* **2013**, *87*, 062304:1–062304:4.
28. Joshi, C.; Hutter, A.; Zimmer, F.E.; Jonson, M.; Andersson, E.; Öhberg, P. Quantum entanglement of nanocantilevers. *Phys. Rev. A* **2010**, *82*, doi:10.1103/PhysRevA.82.043846.
29. Ikeda, S.; Fillaux, F. Incoherent elastic-neutron-scattering study of the vibrational dynamics and spin-related symmetry of protons in the KHCO_3 crystal. *Phys. Rev. B* **1999**, *59*, 4134–4145.
30. Fillaux, F. Quantum entanglement and nonlocal proton transfer dynamics in dimers of formic acid and analogues. *Chem. Phys. Lett.* **2005**, *408*, 302–306.
31. Audenaert, K.; Eisert, J.; Plenio, M.B.; Werner, R.F. Symmetric qubits from cavity states. *Phys. Rev. A* **2002**, *66*, 042327:1–042327:6.
32. Martina, L.; Soliani, G. Hartree-Fock approximation and entanglement. Available online: <http://arxiv.org/abs/0704.3130> (accessed on 18 April 2014).
33. Chung, N.N.; Chew, L.Y. Energy eigenvalues and squeezing properties of general systems of coupled quantum anharmonic oscillators. *Phys. Rev. A* **2007**, *76*, doi:10.1103/PhysRevA.76.032113.
34. Chung, N.N.; Chew, L.Y. Two-step approach to the dynamics of coupled anharmonic oscillators. *Phys. Rev. A* **2009**, *80*, doi:10.1103/PhysRevA.80.012103.
35. Jellal, A.; Madouri, F.; Merdaci, A. Entanglement in coupled harmonic oscillators studied using a unitary transformation. *J. Stat. Mech.* **2011**, doi:10.1088/1742-5468/2011/09/P09015.
36. McDermott, R.M.; Redmount, I.H. Coupled classical and quantum oscillators. Available online: <http://arxiv.org/abs/quant-ph/0403184> (accessed on 18 April 2014).
37. Han, D.; Kim, Y.S.; Noz, M.E. Illustrative example of Feymann’s rest of the universe. *Am. J. Phys.* **1999**, *67*, 61–66.
38. Chung, N.N.; Chew, L.Y. Dependence of entanglement dynamics on the global classical dynamical regime. *Phys. Rev. E* **2009**, *80*, 016204:1–016204:7.
39. Duan, L.M.; Giedke, G.; Cirac, J.I.; Zoller, P. Inseparable criterion for continuous variable systems. *Phys. Rev. Lett.* **2000**, *84*, 2722–2725.
40. Joshi, C.; Jonson, M.; Andersson, E.; Öhberg, P. Quantum entanglement of anharmonic oscillators. *J. Phys. B At. Mol. Opt. Phys.* **2011**, *44*, doi:10.1088/0953-4075/44/24/245503.