Symmetry **2012**, *4*, 344-378; doi:10.3390/sym4030344

OPEN ACCESS

symmetry

ISSN 2073-8994

www.mdpi.com/journal/symmetry

Article

Particle-Dependent Deformations of Lorentz Symmetry

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Received: 25 May 2012; in revised form: 21 June 2012 / Accepted: 26 June 2012 / Published: 3 July 2012

Abstract: I report results suggesting that it is possible to introduce laws of relativistic kinematics endowing different types of particles with suitably different deformed-Lorentz-symmetry properties. I also consider some possible applications of these results, among which I highlight those relevant for addressing a long-standing challenge in the description of composite particles, such as atoms, within quantum-gravity-inspired scenarios with Planck-scale deformations of Lorentz symmetry. Some of the new elements here introduced in the formulation of relativistic kinematics appear to also provide the starting point for the development of a correspondingly novel mathematical formulation of spacetime-symmetry algebras.

Keywords: Lorentz symmetry; quantum gravity; Hopf algebras

1. Introduction

One of the most active areas of quantum-gravity research over the last decade concerns the fate of Lorentz symmetry in the quantum-gravity realm (more precisely in the "quasi-Minkowski limit" of quantum gravity, the limit where quantum gravity should reproduce in first approximation particle physics and its special-relativistic properties). It is turning out to be particularly useful to divide all such studies into three categories: (i) cases where Lorentz/Poincaré symmetry remains unaffected; (ii) cases where there are departures from classical Lorentz/Poincaré symmetry such that a preferred-frame picture arises; (iii) cases where there are departures from classical Lorentz/Poincaré symmetry but the relativity of inertial frames is preserved. This third option, which was proposed in [1,2], is the one that challenges us more significantly for what concerns formalization. The case of "broken Lorentz symmetry" (with a preferred frame) is technically not much more challenging that a standard special-relativistic case, since it allows formalizations that are rather familiar, already relevant for example in the analysis of conventional propagation of light in certain material media. Instead formalizing "deformations of Lorentz symmetry", in the sense of the "DSR" ("doubly-special", or "deformed-special", relativity) proposal of [1,2], requires us to find ways of introducing departures from Lorentz/Poincaré symmetry while preserving the delicate balance that can assure the relativity of inertial frames. This DSR proposal focuses on the possibility of relativistic theories with two characteristic invariant scales, introducing a length/inverse-momentum scale ℓ with relativistic properties analogous to the familiar ones of the speed-of-light scale c. From a quantum-gravity perspective it would then be natural [1] to assume that the new relativistic-invariant scale ℓ be roughly of the order of the "Planck length", the inverse of the Planck scale.

At this point there is a rich literature on DSR-deformations of Lorentz symmetry, with several encouraging results (see, e.g., [1–8] and references therein). Most of these results concern DSR-relativistic formulations of the possibility of introducing relativistically some deformed on-shell relations and some associated deformations of the laws of composition of momenta. I here investigate issues which are important from a general "foundational" perspective, and are in particular relevant for one of the most significant residual challenges for such studies of DSR-relativistic deformations of on-shell (and momentum-conservation) relations, which concerns the description of composite particles, such as atoms. In some of the most studied attempts of formulating DSR-relativistic theories it emerges that "DSR-composite particles" should have relativistic properties different from those of their constituents. The simplest way to see that this might be the case is to consider a bunch of N ultrarelativistic particles all propagating along the 1 direction and each governed, say, by

$$p_0 \simeq p_1 + \frac{m^2}{2p_1} - \ell p_1^2 \tag{1}$$

We can then introduce some candidates for "total spatial momentum" and "total energy", given by $\mathcal{P}_0 = Np_0$ and $\mathcal{P}_1 = Np_1$, and observe that the validity of Equation (1) for each of the N particles implies

$$\mathcal{P}_0 \simeq \mathcal{P}_1 + \frac{\mu^2}{2\mathcal{P}_1} - \frac{\ell \mathcal{P}_1^2}{N} \tag{2}$$

where $\mu = Nm$ plays the role of rest energy of the N-particle system. The suppression by $1/N$ of the last term of this Equation (2) illustrates the issue: The nonlinearity of the DSR laws has nontrivial consequences for particle composites.

Much more than this simple-minded argument supports the concern that composite particles should have relativistic properties which are different from those of their constituents, and in particular the effects of the deformation should be more weakly felt by composites. I shall not review here these more sophisticated arguments, for which I refer my readers to [4,9] and, most notably [10]. Let me stress however that these technical arguments, based on the nonlinearity of the laws and the way it can affect the description of composites, also make sense physically: While an on-shell relation of type Equation (1) is certainly plausible for microscopic particles (at least if ℓ is indeed of the order of the inverse of the Planck scale), the same on-shell relation, even taking ℓ as the inverse of the Planck scale, is unacceptable for macroscopic bodies composed of very many micro-particles. The Planck-scale is huge by the standards of elementary particles but is actually a small scale ($\sim 10^{-5}g$) for macroscopic bodies, and as a result, unless there is a suppression of the type qualitatively shown in Equation (2), the DSR description of macroscopic bodies could be disastrous. In this study I shall set aside a possibility which in principle could be explored: By fine-tuning the form of the deformation beyond leading order one may try (see, e.g., [11]) to render the properties of macroscopic bodies compatible with the deformation. Such attempts should exploit the fact that the fundamental particles we observe have energies much below the Planck scale (and therefore only feel, if anything, the leading order corrections), whereas macroscopic bodies have energies much above the Planck scale (and therefore in such scenarios would be affected strongly by corrections beyond leading order).

So there is technical evidence of the fact that composite particles should have DSR-relativistic properties different from those of their constituents, with weaker deformation effects, and this is much welcome from the point of view of reproducing the observed properties of macroscopic bodies. But this encouraging correspondence between features for composite (particles and) bodies found on the theory side and our desiderata for the phenomenology of macroscopic bodies has also provided a formidable challenge for DSR research: If composite particles (and macroscopic bodies) have relativistic properties which are different from the ones of their constituents then these DSR-relativistic theories should be theories that do not prescribe "universal" laws of kinematics but rather particle-dependent ones!

Is that even possible?

Can a theory be fully relativistic and yet attribute different laws of kinematics to different particles?

These questions have remained so far unanswered, and it should be easily noticed that their significance goes well beyond the confines of DSR research: Establishing whether or not we can be sure that relativistic kinematics should be universal is a prototypical question relevant for the general foundations of physics.

I shall here report results suggesting that indeed it may be possible to have a non-universal relativistic theory, and specifically we could have logically consistent DSR-relativistic theories in which different particles (possibly "elementary" and "composite" particles) are governed by different laws of kinematics.

After a brief reminder, in the next Section 2, of the basic logical structure of DSR-relativistic theories, I set the stage for my analysis, in Section 3, by reviewing in some detail the relativistic kinematics of two much studied DSR setups. These are of course still standard "universal" DSR setups, and provide the starting point for the first group of new results, reported in Section 4, where I describe rules of relativistic kinematics such that one can combine particles with standard special-relativistic properties and particles with DSR-deformed relativistic properties. The key ingredient of these results is a "mixing composition law" suitable for writing a DSR-covariant law of conservation of momentum for processes involving different particles with different relativistic properties, and such that the covariance is assured by a suitably adapted action of boost generators on multiparticle systems.

Section 5 concerns the case of relativistic kinematics with two species of particles; one with some given DSR-deformed relativistic properties and the other with some other DSR-deformed relativistic properties. I introduce for this purpose a further generalization of the "mixing composition law" and of the laws of action of boost generators on multiparticle systems.

The new formulation of relativistic kinematics introduced in Sections 4 and 5 is most simply viewed from the perspective of applications to different types of particles (some "elementary" and some "composite"), but I also explore, in Section 6, the possibility that the laws characterized by weaker DSR deformation apply to a macroscopic body. I find preliminary encouragement for such a possible application.

Then, in Section 7, I consider the possibility of applications of the formulation of relativistic kinematics here introduced to the case of different types of elementary particles, establishing a few first points relevant for the phenomenology.

In the brief Section 8 I comment on the type of spacetime-symmetry algebra which could provide the formal/mathematical counterpart for the version of relativistic kinematics I here introduce. In this respect perhaps most notably I argue that a suitable generalization of the Hopf algebra notion of co-product, something of the sort of a "mixing co-product", could be inspired by the "mixing composition laws" I here introduced.

I work mostly at leading order in the deformation scale ℓ (with ℓ that can be both positive and negative, in the sense that both scenarios with $\ell/|\ell| = 1$ and scenarios with $\ell/|\ell| = -1$ are admissible). This keeps formulas at reasonably manageable level, sufficiently characterizes the new concepts, and would be fully sufficient for phenomenology if indeed the deformation scale is roughly of the order of the huge Planck scale (in which case a leading-order analysis should be all we need for comparison to data we could realistically imagine of gathering over the next few decades).

However, in Section 9, I do offer a small aside contemplating possible generalizations of my results to "all-order analyses".

Some speculations about possible future developments are offered in the brief closing Section 10.

I mostly focus on 1+1-dimensional cases, where all conceptual issues here relevant are already present and can be exposed more simply. Therefore my momenta will often have two components, $\{p_0, p_1\}$, and when I briefly switch to considering cases with more dimensions I use notation of the type $\{p_0, p_j\}$.

2. DSR-Deformations of Lorentz Symmetry

Before proceeding with the main part of the analysis, let me pause briefly, in this section, for summarizing the main points originally made in [1,2] concerning the consistency requirements that the relativity of inertial frames imposes on the relationship between the form of the on-shell(/dispersion) relation and the form of laws of energy-momentum conservation.

This is one of the most used DSR concepts, and plays a pivotal role in the analysis I report in the following sections.

This consistency between on-shell relation and laws of momentum conservation that follows from insisting on the relativity of inertial frames is also rather significant from the perspective of studies of the quantum-gravity problem, where in some cases one finds "preliminary theoretical evidence" of modifications of the on-shell relation but usually not accompanied so far by any information on whether or not there should also be modifications of the law of conservation of momentum. Indeed the idea of DSR-deformed Lorentz transformations was put forward [1,2] as a possible description of certain preliminary theory results suggesting that there might be violations of some special-relativistic laws in certain approaches to the quantum-gravity problem, most notably the ones based on spacetime noncommutativity and loop quantum gravity. The part of the quantum-gravity community interested in those results was interpreting them as a manifestation of a full breakdown of Lorentz symmetry, with the emergence of a preferred class of observers (an "ether"). But it was argued in [1] that departures from Special Relativity governed by a high-energy/short-distance scale may well be compatible with the Relativity Principle, the principle of relativity of inertial observers, at the cost of allowing some consistent modifications of the Poincaré transformations, and particularly of the Lorentz-boost transformations. And it was already observed in [1] that this in turn would require corresponding modifications of the laws of momentum conservation.

The DSR proposal could provide [1] a conceptual path for pursuing a broader class of scenarios of interest for fundamental physics, and in particular for quantum-gravity research, including the possibility of introducing the second observer-independent scale primitively in spacetime structure or primitively at the level of the (deformed) de Broglie relation between wavelength and momentum. However, the bulk of the preliminary results from quantum-gravity research concerns departures from the special-relativistic on-shell relation, and this in turn became the main focus of DSR research.

So let me consider a generic on-shell relation of the type

$$m^2 = p_0^2 - \mathbf{p}^2 + \Delta(E, \mathbf{p}; \ell) \tag{3}$$

where Δ is the deformation and ℓ is the deformation scale.

Evidently in general for $\Delta \neq 0$ such an on-shell relation Equation (3) is not Lorentz invariant. If we insist on this law and on the validity of classical (undeformed) Lorentz transformations between inertial observers we clearly end up with a preferred-frame picture, and the Principle of Relativity of inertial frames must be abandoned: The scale ℓ cannot be observer independent, and actually the whole form of Equation (3) may vary from one class of inertial observers to another.

From the alternative DSR perspective one would have to enforce the relativistic invariance of laws such as Equation (3), preserving the relativity of inertial frames, at the cost of modifying the action of boosts on momenta. Then in such theories both the velocity scale c (here mute only because of the choice of dimensions) and the length/inverse-momentum scale ℓ play the same role [1] of invariant scales of the relativistic theory which govern the form of boost transformations.

Several examples of boost deformations adapted in the DSR sense to modified on-shell relations have been analyzed in some detail (see e.g., [1–8] and references therein). Clearly these DSR-deformed boosts $\mathcal{N}_j$ must be such that

$$[\mathcal{N}_j, p_0^2 - \mathbf{p}^2 + \Delta(E, \mathbf{p}; \ell)] = 0 \tag{4}$$

This requirement Equation (4) of DSR-relativity is completely analogous to the corresponding ones of Galilean Relativity and Special Relativity: Of course in all these cases the on-shell relation is boost invariant (but respectively under Galilean boosts, Lorentz boosts, and DSR-deformed Lorentz boosts); for Special Relativity the action of boosts evidently must depend on the speed scale c and boosts must act non-linearly on velocities (since they must enforce observer-independence of c-dependent laws), and for DSR relativity the action of boosts evidently must depend on both the scale c and the scale ℓ, with boosts acting non-linearly both on velocities and momenta, since it must enforce observer-independence of c-dependent and ℓ-dependent laws.

Actually much of the logical structure of the conjectured transition from Special Relativity to a DSR theory can be understood in analogy with the transition from Galilean Relativity to Special relativity.

Famously, as the Maxwell formulation of electromagnetism, with an observer-independent speed scale "c", gained more and more experimental support (among which one should count the Michelson–Morley results) it became clear that Galilean relativistic symmetries could no longer be upheld. From a modern perspective we should see the pre-Einsteinian attempts to address that crisis (such as the ones of Lorentz) as attempts to "break Galilean invariance", *i.e.*, preserve the validity of Galilean transformations as laws of transformation among inertial observers, but renouncing to the possibility that those transformations be a symmetry of the laws of physics. The "ether" would be a preferred frame for the description of the laws of physics, and the laws of physics that hold in other frames would be obtained from the ones of the preferred frame via Galilean transformations. Those attempts failed.

What succeeded is completely complementary. Experimental evidence, and the analyses of Einstein (and Poincaré) led us to a "deformation of Galilean invariance": In Special Relativity the laws of transformation among observers still are a symmetry of the laws of physics (Special Relativity is no less relativistic then Galilean Relativity), but the special-relativistic transformation laws are a c-deformation of the Galilean laws of transformation with the noteworthy property of achieving the observer-independence of the speed scale c.

This famous c-deformation in particular replaces the Galilean on-shell relation $E = \mathrm{constant} + \mathbf{p}^2/(2m)$ with the special-relativistic version $E = \sqrt{c^2\mathbf{p}^2 + c^4m^2}$ and the Galilean composition of velocities $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v}$ with the much more complex special-relativistic law of composition of velocities.

This interplay between c-deformation of Galilean transformations and the associated deformations of the law of composition of velocities is analogous to the interplay between the DSR-type ℓ-deformation of Lorentz transformations and the associated deformations of the law of composition of momenta.

3. Two Known Examples of DSR Setups with Universality

I shall now give more tangibility to the brief review of DSR concepts contained in the previous section, by discussing explicitly two known examples of DSR-relativistic kinematics, and highlighting the connection between deformation of the on-shell relation and deformation of the laws of momentum conservation.

I shall keep notation consistent with the one adopted in [12].

The review work done in this section, for versions of DSR-relativistic kinematics that still are standard "universal DSR setups" (the deformation of Lorentz symmetry affects all particles in exactly the same way), sets the stage for the original results reported in the next sections: I will there take the DSR setups of this section as starting point for adding the possibility of "non-universal effects" (cases where the deformation of Lorentz symmetry affects different types of particles in different ways).

3.1. A Setup with Commutative Composition of Momenta

The first example I discuss adopts the on-shell relation

$$m^2 = p_0^2 - p_1^2 + 2\ell p_0 p_1^2 \tag{5}$$

and the following law of composition of momenta:

$$(k \oplus_\ell p)_1 = k_1 + p_1 + \ell k_0 p_1 + \ell p_0 k_1, (k \oplus_\ell p)_0 = k_0 + p_0 \tag{6}$$

For the composition of momenta of three particles I use, consistently with Equation (6), the following:

$$\begin{aligned}&[(k \oplus_\ell p) \oplus_\ell q]_1 = k_1 + p_1 + q_1 + \ell k_0(p_1 + q_1) + \ell p_0(k_1 + q_1) + \ell q_0(k_1 + p_1),\\ &[(k \oplus_\ell p) \oplus_\ell q]_0 = k_0 + p_0 + q_0\end{aligned} \tag{7}$$

From these one can also easily verify the associativity of the composition law $\oplus_\ell$.

Confirming the points highlighted in the brief review of DSR concepts offered in the previous section, we shall quickly see that the modification of the on-shell relation codified in Equation (5) can be introduced as a relativistic law only upon modifying the law of conservation of momenta by constructing such conservation laws using the composition law Equation (6). In the particular DSR setup I am considering, this consistency between modified on-shell relation and modified law of composition of momenta is achieved by introducing the following modified boost generator [12]

$$[N, p_0] = p_1 - \ell p_0 p_1, [N, p_1] = p_0 + \ell p_0^2 + \ell p_1^2 \tag{8}$$

I start by observing that the on-shell relation is invariant:

$$[N, p_0^2 - p_1^2 + 2\ell p_0 p_1^2] = 0 \tag{9}$$

And it is only slightly more tedious to check that the boost Equation (8) ensures the covariance of the laws of conservation of momentum based on the composition law Equation (6). I start checking this covariance for the conservation law

$$(k \oplus_\ell p) = 0 \tag{10}$$

Acting with the boost N one finds that

$$\begin{aligned}[N, (k \oplus_\ell p)_0] &= [N, (k_0 + p_0)] = k_1 - \ell k_0 k_1 + p_1 - \ell p_0 p_1\\ &= k_1 + p_1 + \ell p_0 k_1 + \ell k_0 p_1 - \ell(p_0 + k_0)(p_1 + k_1) = 0\end{aligned} \tag{11}$$

where on the right-hand side I of course used the conservation law itself (and took into account that I am working in leading order in ℓ). And similarly one finds that

$$\begin{aligned}[N, (k \oplus_\ell p)_1] &= [N, (k_1 + p_1 + \ell k_0 p_1 + \ell p_0 k_1)] = k_0 + p_0 + \ell p_0^2 + \ell k_0^2 + \ell p_1^2 + \ell k_1^2 + 2\ell k_1 p_1 + 2\ell k_0 p_0\\ &= k_0 + p_0 + \ell(p_0 + k_0)^2 + \ell(p_1 + k_1)^2 = 0\end{aligned} \tag{12}$$

where again on the right-hand side I used the conservation law itself.

The covariance of the conservation law $(k \oplus_\ell p) = 0$ under the boost Equation (8) is evidently confirmed by Equations (11,12).

Let me also check explicitly the covariance of the conservation law

$$(k \oplus_\ell p) \oplus_\ell q = 0 \tag{13}$$

Acting with the boost N on the 0 component of Equation (13) one finds that

$$\begin{aligned}[N,[(k\oplus_\ell p)\oplus_\ell q]_0] &= [N,(k_0+p_0+q_0)]\\ &= k_1-\ell k_0k_1+p_1-\ell p_0p_1+q_1-\ell q_0q_1\\ &= k_1+p_1+q_1+\ell p_0k_1\\ &+\ell k_0p_1+\ell q_0k_1+\ell k_0q_1+\ell p_0q_1+\ell q_0p_1-\ell(p_0+k_0+q_0)(p_1+k_1+q_1)\\ &=0\end{aligned}\tag{14}$$

And similarly for the 1 component one finds that

$$\begin{aligned}[N,((k\oplus_\ell p)\oplus_\ell q)_1] &= [N,(k_1+p_1+q_1+\ell k_0(p_1+q_1)+\ell p_0(k_1+q_1)+\ell q_0(k_1+p_1))]\\ &= k_0+p_0+q_0+\ell p_0^2+\ell k_0^2+\ell q_0^2+\ell p_1^2+\ell k_1^2\\ &+\ell q_1^2+2\ell k_1p_1+2\ell k_0p_0+2\ell k_1q_1+2\ell k_0q_0+2\ell p_1q_1+2\ell p_0q_0\\ &= k_0+p_0+q_0+\ell(p_0+k_0+q_0)^2+\ell(p_1+k_1+q_1)^2\\ &=0\end{aligned}\tag{15}$$

3.2. A κ-Poincaré Inspired Setup

The second example that I want to summarize for DSR-relativistic kinematics takes second in the order only because of its somewhat higher complexity, but actually is of particular interest from the DSR perspective. This DSR setup was analyzed from the perspective here relevant in the recent [12], and more preliminarily in several previous DSR studies (some comments on it were already in [1]). It is centered on a choice of on-shell relation and law of composition of momenta which became recently of interest [13,14] also in the study of the new proposal of "relative-locality momentum spaces" [15,16], and provides a set of rules for kinematics which can be naturally described from the viewpoint of the κ-Poincaré Hopf algebra [17,18].

The on-shell ("dispersion") relation is (focusing again on the $1+1$-dimensional case)

$$m^2 = p_0^2 - p_1^2 + \ell p_0 p_1^2 \tag{16}$$

and the law of composition of momenta is

$$(p\oplus_\ell p')_1 = p_1+p'_1+\ell p_0p'_1,\ (p\oplus_\ell p')_0 = p_0+p'_0 \tag{17}$$

The on-shell relation Equation (16) is invariant under the following action of a boost on the momentum of a particle:

$$[N,p_0]=p_1,\ [N,p_1]=p_0+\ell p_0^2+\frac{\ell}{2}p_1^2 \tag{18}$$

which indeed ensures

$$[N,p_0^2-p_1^2+\ell p_0p_1^2] = 2p_0p_1-2p_1(p_0+\ell p_0^2+\frac{\ell}{2}p_1^2)+\ell p_1^3+2\ell p_0^2p_1=0 \tag{19}$$

Once again the central feature is the consistency between the on-shell relation, in this case Equation (16), and the laws of conservation of momentum written in terms of the composition law Equation (17). And in this case where the momentum-composition law is, unlike the case of the previous subsection, noncommutative, we shall meet additional challenges (already discussed and addressed in [12]). Let me stress here that I am not advocating that the law of composition of momenta should be non-commutative; on the contrary one may well prefer [12] commutative laws of composition of momenta in a DSR setup. But part of the strength of the results I am here reporting in the following sections resides in the fact that I am able to generalize to the status of "non-universal DSR deformations" not merely the simplest DSR setups previously known with universality, but actually even some rather virulent DSR setups, which in particular may even involve a non-commutative law of composition of momenta.

Having stressed this, let me proceed with noticing the relativistic challenges produced by the non-commutativity of the composition law Equation (17). These challenges evidently manifest themselves when attempting to formulate the action of boosts on momenta obtained composing two or more single-particle momenta. In Special Relativity (and in DSR setups with commutative law of composition, such as the one in the previous subsection) it is possible to simply impose that the boost of a two-particle event $e_{p \oplus_\ell p'}$ be governed by

$$[N_{[p]} + N_{[p']}, p \oplus_\ell p']$$

where I decomposed the action of boosts on the composed momentum into two pieces, each given in terms of a boost acting exclusively on a certain momentum in the event.

This means that in Special Relativity one has a "total boost generator" obtained by combining trivially the boost generators acting on each individual particle. But with a noncommutative law of composition of momenta this simplicity is lost: The lack of symmetry under exchange of particles precludes, as one can easily verify [12], the possibility of adopting a "total boost generator" given by a trivial sum of single-particle boost generators. There is in particular no choice [12] of $N_{[p]}$ capable of ensuring that $[N_{[p]} + N_{[p']} + N_{[p'']}, (p \oplus_\ell p' \oplus_\ell p'')_\mu]$ vanishes whenever $(p \oplus_\ell p' \oplus_\ell p'')_\mu = 0$.

What does work, as shown in [12], is adopting a corresponding deformation of the "total-boost law"

$$N_{[p \oplus_\ell p']} = N_{[p]} + N_{[p']} + \ell p_0 N_{[p']} \tag{20}$$

and accordingly

$$N_{[p \oplus_\ell p' \oplus_\ell p'']} = N_{[p]} + N_{[p']} + N_{[p'']} + \ell p_0 N_{[p']} + \ell p_0 N_{[p'']} + \ell p'_0 N_{[p'']} \tag{21}$$

In [12] I verified in some detail that this prescription produces a fully consistent relativistic framework, with the needed compatibility between on-shell relation Equation (16) and law of composition of momenta Equation (17): The on-shell relation is invariant and the laws of conservation of momentum obtained from the composition law are covariant.

Let me here just review briefly the specific result of [12] concerning the covariance of the conservation law for a "trivalent process" with [19] $p \oplus_\ell p' \oplus_\ell p'' = 0$.

Checking that $N_{[p]} + N_{[p']} + N_{[p'']} + \ell p_0 N_{[p']} + \ell p_0 N_{[p'']} + \ell p'_0 N_{[p'']}$ does indeed ensure that the relativistic covariance of the conservation law $p \oplus_\ell p' \oplus_\ell p'' = 0$ is best done considering again separately the 0 ("time") component and the 1 ("spatial") component. For the 0 component one easily finds [12]

$$\begin{aligned}
&[N_{[p]} + N_{[p']} + N_{[p'']} + \ell p_0 N_{[p']} + \ell p_0 N_{[p'']} + \ell p'_0 N_{[p'']}, (p \oplus_\ell p' \oplus_\ell p'')_0] \\
&= [N_{[p]} + N_{[p']} + N_{[p'']} + \ell p_0 N_{[p']} + \ell p_0 N_{[p'']} + \ell p'_0 N_{[p'']}, p_0 + p'_0 + p''_0] \\
&= p_1 + p'_1 + p''_1 + \ell p_0 p'_1 + \ell p_0 p''_1 + \ell p'_0 p''_1 = (p \oplus_\ell p' \oplus_\ell p'')_1 = 0
\end{aligned} \tag{22}$$

where on the right-hand side I of course used the conservation law itself.

Similarly for the 1 component one easily finds that [12]

$$
\begin{aligned}
&[N_{[p]} + N_{[p']} + N_{[p'']} + \ell p_0 N_{[p']} + \ell p_0 N_{[p'']} + \ell p'_0 N_{[p'']}, (p \oplus_\ell p' \oplus_\ell p'')_1] \\
&\quad = [N_{[p]} + N_{[p']} + N_{[p'']} + \ell p_0 N_{[p']} + \ell p_0 N_{[p'']} + \ell p'_0 N_{[p'']}, p_1 + p'_1 + p''_1 + \ell p_0 p'_1 + \ell p_0 p''_1 + \ell p'_0 p''_1] \\
&\quad = p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2 + p'_0 + \ell p_0'^2 + \frac{\ell}{2} p_1'^2 + p''_0 + \ell p_0''^2 + \frac{\ell}{2} p_1''^2 + 2\ell p_0 p'_0 + 2\ell p_0 p''_0 + 2\ell p'_0 p''_0 + \ell p_1 p'_1 + \ell p_1 p''_1 + \ell p'_1 p''_1 \\
&\quad = p_0 + p'_0 + p''_0 + \ell(p_0 + p'_0 + p''_0)^2 + \frac{\ell}{2}(p_1 + p'_1 + p''_1)^2 = 0
\end{aligned}
\tag{23}
$$

where again on the right-hand side I used the conservation law $p \oplus_\ell p' \oplus_\ell p'' = 0$ itself, and I took again into account that I am working at leading order in ℓ.

The results Equations (22,23) establish that indeed the boosts Equations (18,20,21), besides admitting the on-shell relation Equation (16) as invariant, also admit $p \oplus_\ell p' \oplus_\ell p'' = 0$ as a covariant law.

4. Simplest Scenarios for a DSR Setup without Universality

The previous sections only provided the preliminaries for the main analysis that I report in this manuscript, which is contained in this and the next section. It was a rather bulky effort on preliminaries, but I felt this might be beneficial since the results I am reporting are to a large extent surprising/unexpected and it might be helpful for my readers to be equipped with a nearly self-contained summary of the previous results which provide the starting point for the analysis I am here reporting. As announced, I am going to show that there are logically consistent DSR-relativistic theories in which different types of particles are governed by different laws of kinematics.

4.1. Simplest Case with Commutative Composition of Momenta

In order to exhibit a first (and simplest) example of non-universal DSR setup I find it convenient to adopt for this subsection the notation p (or p' or p'' ...) for the momenta of a type of particles affected by the DSR ℓ-deformation of Lorentz symmetry discussed in the previous Subsection 3.1 so that in particular

$$
m^2 = p_0^2 - p_j^2 + 2\ell p_0 p_j^2 \tag{24}
$$

and

$$
(p \oplus_\ell p')_1 = p_1 + p'_1 + \ell p_0 p'_1 + \ell p'_0 p_1, (p \oplus_\ell p')_0 = p_0 + p'_0 \tag{25}
$$

And then I shall consistently denote with k (or k' or k'' ...) the momenta of particles of a second type, a type governed by undeformed Lorentz symmetry, so that in particular

$$
\mu^2 = k_0^2 - k_j^2 \tag{26}
$$

and

$$
(k \oplus_\star k')_1 = k_1 + k'_1, (k \oplus_\star k')_0 = k_0 + k'_0 \tag{27}
$$

For the first type ("p-type") of particles I shall insist again on

$$
[N_{[p]}, p_0] = p_1 - \ell p_0 p_1, [N_{[p]}, p_1] = p_0 + \ell p_0^2 + \ell p_1^2 \tag{28}
$$

while naturally, for the second type ("k-type") of particles, I take

$$[N_{[k]}, k_0] = k_1, [N_{[k]}, k_1] = k_0 \tag{29}$$

The key point will be to show that there are laws of conservation of momentum allowing momentum to be exchanged between the two types of particles in a consistently relativistic manner. [If the two types of particles could not interact then they would actually not "coexist": The "Universe" of ℓ-deformed p-type particles would remain decoupled from (undetectable and irrelevant for) the "Universe" of special-relativistic k-type particles, and *vice versa*.]

It is easy to see that the main challenge resides in finding a suitable way to compose momenta of different types, *i.e.*, finding some "mixing composition law", of the type $p \oplus_{\ell\star} k$, while insisting that conservation laws written in terms of such a composition law would be covariant under a consistent prescription for the action of boosts. A possibility that does work is

$$(p \oplus_{\ell\star} k)_j = p_j + k_j + \frac{\ell}{2} p_0 k_1 + \frac{\ell}{2} k_0 p_1, (p \oplus_{\ell,\star} k)_0 = p_0 + k_0 \tag{30}$$

In order to convince my readers that this does work let me start slowly, focusing at first on "bi-valent processes" in this theory with two types of particles. Some of these bi-valent processes, the ones without "mixing", will pose no challenge: Since also the deformed law of composition of momenta is commutative the adoption of standard total-boost actions will suffice, so that for processes with conservation law $k \oplus_\star k' = 0$ will be covariant under

$$N_{[k \oplus_\star k']} = N_{[k]} + N_{[k']} \tag{31}$$

while processes with conservation law $p \oplus_\ell p' = 0$ will be covariant under

$$N_{[p \oplus_\ell p']} = N_{[p]} + N_{[p']} \tag{32}$$

A class of bi-valent processes which involves "mixing" is the one of processes with conservation law of the type $p \oplus_{\ell\star} k = 0$. Again the fact that in this subsection also the particles subjected to deformed-Lorentz-symmetry properties have commutative momentum composition allows me to rely on a standard total-boost action:

$$N_{[p \oplus_{\ell\star} k]} = N_{[p]} + N_{[k]} \tag{33}$$

Let us verify that indeed this total-boost action ensures the covariance of the conservation law $p \oplus_{\ell\star} k = 0$. I start again from the 0 component of $p \oplus_{\ell\star} k = 0$ for which I find

$$\begin{aligned}
[N_{[p]} + N_{[k]}, k_0 + p_0] &= p_1 + k_1 - \ell p_0 p_1 \\
&= p_1 + k_1 + \frac{\ell}{2} p_0 k_1 + \frac{\ell}{2} k_0 p_1 - \frac{\ell}{2} p_0 (p_1 + k_1) - \frac{\ell}{2} p_1 (p_0 + k_0) \\
&= (p \oplus_{\ell\star} k)_1 = 0
\end{aligned} \tag{34}$$

where I of course used again the conservation law itself and took again into account that I am working at leading order in ℓ. [Working at leading order in ℓ one finds, e.g., $\ell p_0[(p \oplus_{\ell\star} k)_1] \simeq \ell p_1[p_1 + k_1]$.].

Similarly for the 1 component of $p \oplus_{\ell\star} k = 0$ I find

$$[N_{[p]} + N_{[k]}, p_1 + k_1 + \frac{\ell}{2}p_0 k_1 + \frac{\ell}{2}k_0 p_1] = p_0 + \ell p_0^2 + \ell p_1^2 + k_0 + \ell p_1 k_1 + \ell p_0 k_0$$
$$= p_0 + k_0 + \ell p_0(p_0 + k_0) + \ell p_1(p_1 + k_1)$$
$$= 0 \tag{35}$$

So the composition laws $\oplus_{\ell\star}, \oplus_\ell, \oplus_\star$ when combined with a simple prescription for boosts does ensure the covariance of conservation laws for all "bi-valent processes".

I can now move on to the case of tri-valent processes. The covariance of the kinematics of processes with $k \oplus_\star k' \oplus_\star k'' = 0$ and $p \oplus_\ell p' \oplus_\ell p'' = 0$ is assured respectively by ordinary Lorentz symmetry and by the results reviewed in the previous section. Of course the only potentially troublesome tri-valent processes are the ones whose conservation laws involve "mixing", which leads me to focus on the cases $p \oplus_\ell p' \oplus_{\ell\star} k = 0$ and $p \oplus_{\ell\star} k \oplus_\star k' = 0$.

In order to show that these "tri-valent mixing conservation laws" are covariant under the action of the boosts

$$N_{[p]} + N_{[p']} + N_{[k]}$$

let me start from the 0 component of $p \oplus_\ell p' \oplus_{\ell\star} k = p_0 + p_0' + k_0 = 0$, for which I easily find the expected result

$$[N_{[p]} + N_{[p']} + N_{[k]}, p_0 + p_0' + k_0] = p_1 + p_1' + k_1 - \ell p_0 p_1 - \ell p_0' p_1'$$
$$= p_1 + p_1' + k_1 + \ell p_0 p_1' + \ell p_1 p_0' + \frac{\ell}{2}(p_1 + p_1')k_0 + \frac{\ell}{2}(p_0 + p_0')k_1$$
$$- \frac{\ell}{2}(p_0 + p_0')(p_1 + p_1' + k_1) - \frac{\ell}{2}(p_1 + p_1')(p_0 + p_0' + k_0)$$
$$= (p \oplus_\ell p' \oplus_{\ell\star} k)_1 = 0 \tag{36}$$

Also successful is the verification for the 1 component, $(p \oplus_\ell p' \oplus_{\ell\star} k)_1 = p_1 + p_1' + k_1 + \ell p_0 p_1' + \ell p_1 p_0' + \frac{\ell}{2}(p_1 + p_1')k_0 + \frac{\ell}{2}(p_0 + p_0')k_1 = 0$, which progresses as follows:

$$[N_{[p]} + N_{[p']} + N_{[k]}, p_1 + p_1' + k_1 + \ell p_0 p_1' + \ell p_1 p_0' + \frac{\ell}{2}(p_1 + p_1')k_0 + \frac{\ell}{2}(p_0 + p_0')k_1]$$
$$= p_0 + \ell p_0^2 + \ell p_1^2 + p_0' + \ell p_0'^2 + \ell p_1'^2 k_0 + 2\ell p_1 p_1' + 2\ell p_0 p_0' + \ell(p_1 + p_1')k_1 + \ell(p_0 + p_0')k_0$$
$$= p_0 + p_0' + k_0 + \ell(p_0 + p_0')(p_0 + p_0' + k_0) + \ell(p_1 + p_1')(p_1 + p_1' + k_1) = 0 \tag{37}$$

Having done this it is not hard to adapt the results I obtained for $p \oplus_\ell p' \oplus_{\ell\star} k = 0$ to the only slightly different case of $p \oplus_{\ell\star} k \oplus_\star k' = 0$. In order to establish also the covariance of $p \oplus_{\ell\star} k \oplus_\star k' = 0$ let me start again with its 0 component, for which I easily find a satisfactory result:

$$[N_{[p]} + N_{[k]} + N_{[k']}, p_0 + k_0 + k_0'] = p_1 + k_1 + k_1' - \ell p_0 p_1$$
$$= p_1 + k_1 + k_1' + \frac{\ell}{2}p_0(k_1 + k_1') + \frac{\ell}{2}p_1(k_0 + k_0')$$
$$- \frac{\ell}{2}p_1(p_0 + k_0 + k_0') - \frac{\ell}{2}p_0(p_1 + k_1 + k_1') = 0 \tag{38}$$

And equally satisfactory is the corresponding verification for the 1 component, $(p \oplus_{\ell\star} k \oplus_\star k')_1 = p_1 + k_1 + k'_1 + \frac{\ell}{2}p_0(k_1 + k'_1) + \frac{\ell}{2}p_1(k_0 + k'_0) = 0$, which progresses as follows:

$$\begin{aligned}
&[N_{[p]} + N_{[k]} + N_{[k']}, p_1 + k_1 + k'_1 + \frac{\ell}{2}p_0(k_1 + k'_1) + \frac{\ell}{2}p_1(k_0 + k'_0)] \\
&= p_0 + \ell p_0^2 + \frac{\ell}{2}p_1^2 + k_0 + k'_0 + \ell p_1(k_1 + k'_1) + \ell p_0(k_0 + k'_0) \\
&= p_0 + k_0 + k'_0 + \ell p_1(p_1 + k_1 + k'_1) + \ell p_0(p_0 + k_0 + k'_0) = 0
\end{aligned} \tag{39}$$

So I did show that there are examples of kinematics for "mixing interactions" which satisfy the demands of the relativity of inertial frames, producing rules of relativistic kinematics with a consistent description of interactions among particles with different relativistic properties: The results Equations (36–39) confirm that the boosts I introduced in Equations (28,29,31–33), besides admitting the on-shell relations Equations (24,26) as invariants, also admit the conservation laws $p \oplus_\ell p' \oplus_\ell p'' = 0$ and $p \oplus_{\ell\star} k \oplus_\star k' = 0$ as covariant laws.

4.2. Simplest κ-Poincaré-Inspired Case

I next want to exhibit the possibility of constructing relativistic kinematics analogous to the one just shown in the previous subsection, but this time involving the κ-Poincaré-inspired DSR setup of Subsection 3.2. So, for this subsection, I shall consistently denote with p (or p' or p'' ...) the momenta of the type of particles affected by the κ-Poincaré-inspired DSR ℓ-deformation of Lorentz symmetry discussed in Subsection 3.2, so that in particular

$$m^2 = p_0^2 - p_j^2 + \ell p_0 p_j^2 \tag{40}$$

$$[N_{[p]}, p_0] = p_1, [N_{[p]}, p_1] = p_0 + \ell p_0^2 + \frac{\ell}{2}p_1^2 \tag{41}$$

And I shall consistently denote with k (or k' or k'' ...) the momenta of particles of the second type, the type governed by undeformed Lorentz symmetry, so that in particular

$$\mu^2 = k_0^2 - k_j^2 \tag{42}$$

$$[N_{[k]}, k_0] = k_1, [N_{[k]}, k_1] = k_0 \tag{43}$$

Naturally the composition law for the special-relativistic momenta ("k-type") is again standard,

$$(k \oplus_\star k')_j = k_j + k'_j, (k \oplus_\star k')_0 = k_0 + k'_0 \tag{44}$$

whereas for "p-type particles" I use the prescription discussed for the κ-Poincaré-inspired DSR ℓ-deformation of Lorentz symmetry discussed in Subsection 3.2:

$$(p \oplus_\ell p')_j = p_j + p'_j + \ell p_0 p'_j, (p \oplus_\ell p')_0 = p_0 + p'_0 \tag{45}$$

Again the main challenge resides in finding some suitable way to compose momenta of different types, *i.e.*, finding some "mixing composition law" of the type $p \oplus_{\ell\star} k$. And accomplishing this task is less obvious in the case where one (or more) type of particles is governed by a noncommutative law of composition of momenta. Equipped with no theorem but rather the findings of a lengthy "trial and error exercise", I can simply exhibit an example of such a "mixing composition law" which does work [20]:

$$(p \oplus_{\ell\star} k)_j = p_1 + k_1 + \frac{\ell}{2}(p_0 + k_0)k_1 + \frac{\ell}{2}k_0 p_1, (p \oplus_{\ell,\star} k)_0 = p_0 + k_0 \tag{46}$$

In verifying that this does work let me start again slowly, focusing at first on "bi-valent processes". And again the bi-valent processes without "mixing" will pose no challenge: Processes with conservation law $k \oplus_\star k' = 0$ will have covariance ensured by standard total-boost actions

$$N_{[k\oplus_\star k']} = N_{[k]} + N_{[k']} \tag{47}$$

while processes with conservation law $p \oplus_\ell p' = 0$ will have covariance ensured by the total-boost actions discussed in Subsection 3.2

$$N_{[p\oplus_\ell p']} = N_{[p]} + N_{[p']} + \ell p_0 N_{[p']} \tag{48}$$

The case of even just bi-valent processes that involve "mixing", with conservation law of the type $p \oplus_{\ell\star} k = 0$, does instead require some reasoning, particular in this case with noncommutative law of composition of momenta. What I found, and this should not be too surprising in light of the discussion offered in Subsection 3.2, is that a relativistically consistent description of such "mixing bi-valent processes" is obtained in terms of a total-boost action which itself involves a sort of "mixing":

$$N_{[p\oplus_{\ell\star} k]} = N_{[p]} + N_{[k]} + \frac{\ell}{2}(p_0 + k_0)N_{[k]} + \frac{\ell}{2}k_0 N_{[p]} \tag{49}$$

Let us verify that indeed this total-boost action ensure the covariance of the conservation law $p \oplus_{\ell\star} k = 0$. I start again from the 0 component of $p \oplus_{\ell\star} k = 0$ for which I find

$$\begin{aligned}[N_{[p]} + N_{[k]} + \frac{\ell}{2}(p_0 + k_0)N_{[k]} + \frac{\ell}{2}k_0 N_{[p]}, k_0 + p_0] &= 7p_1 + k_1 + \frac{\ell}{2}(p_0 + k_0)k_1 + \frac{\ell}{2}k_0 p_1 \\ &= (p \oplus_{\ell\star} k)_1 = 0\end{aligned} \tag{50}$$

Similarly for the 1 component of $p \oplus_{\ell\star} k = 0$ I find

$$\begin{aligned}&[N_{[p]} + N_{[k]} + \frac{\ell}{2}(p_0 + k_0)N_{[k]} + \frac{\ell}{2}k_0 N_{[p]}, p_1 + k_1 + \frac{\ell}{2}(p_0 + k_0)k_1 + \frac{\ell}{2}k_0 p_1] \\ &= p_0 + \ell p_0^2 + \frac{\ell}{2}p_1^2 + k_0 + \frac{\ell}{2}(p_1 + k_1)k_1 + \ell(p_0 + k_0)k_0 + \frac{\ell}{2}p_1 k_1 + \ell p_0 k_0 \\ &= p_0 + k_0 + \ell(p_0 + k_0)^2 + \frac{\ell}{2}(p_1 + k_1)^2 = 0\end{aligned} \tag{51}$$

where again on the right-hand side I used the conservation law $p \oplus_{\ell\star} k = 0$ and took again into account that I am working at leading order in ℓ.

So I did find that my description of boosts and of the composition laws $\oplus_{\ell\star}, \oplus_\ell, \oplus_\star$ ensures the covariance of conservation laws for all "bi-valent processes". Moving on to the case of "tri-valent processes" let me first notice again that for the cases without mixing, $k \oplus_\star k' \oplus_\star k'' = 0$ and $p \oplus_\ell p' \oplus_\ell p'' = 0$, a relativistic description is already ensured by results summarized in the previous section. But I must again proceed cautiously with tri-valent processes whose conservation laws involve "mixing", such as the cases with $p \oplus_\ell p' \oplus_{\ell\star} k = 0$ and $p \oplus_{\ell\star} k \oplus_\star k' = 0$.

For the "tri-valent mixing conservation law" $p \oplus_\ell p' \oplus_{\ell\star} k = 0$ I can easily show that it is indeed covariant under the action of a suitably mixed composition of boosts, given by

$$N_{[p]} + N_{[p']} + N_{[k]} + \ell p_0 N_{[p']} + \frac{\ell}{2}(p_0 + p_0' + k_0)N_{[k]} + \frac{\ell}{2}k_0(N_{[p]} + N_{[p']})$$

Indeed for the 0 component I find

$$\begin{aligned}
&[N_{[p]} + N_{[p']} + N_{[k]} + \ell p_0 N_{[p']} + \frac{\ell}{2}(p_0 + p_0' + k_0)N_{[k]} + \frac{\ell}{2}k_0(N_{[p]} + N_{[p']}), p_0 + p_0' + k_0] \\
&= p_1 + p_1' + k_1 + \ell p_0 p_1' + \frac{\ell}{2}(p_0 + p_0' + k_0)k_1 + \frac{\ell}{2}k_0(p_1 + p_1') \\
&= (p \oplus_\ell p' \oplus_{\ell\star} k)_1 = 0
\end{aligned} \tag{52}$$

Also successful is the verification for the 1 component of $p \oplus_\ell p' \oplus_{\ell\star} k = 0$, which progresses as follows:

$$\begin{aligned}
&[N_{[p]} + N_{[p']} + N_{[k]} + \ell p_0 N_{[p']} + \frac{\ell}{2}(p_0 + p_0' + k_0)N_{[k]} + \frac{\ell}{2}k_0(N_{[p]} + N_{[p']}), p_1 + p_1' + k_1 + \ell p_0 p_1' + \frac{\ell}{2}(p_0 + p_0' + k_0)k_1 + \frac{\ell}{2}k_0(p_1 + p_1')] \\
&= p_0 + \ell p_0^2 + \frac{\ell}{2}p_1^2 + p_0' + \ell {p_0'}^2 + \frac{\ell}{2}{p_1'}^2 + k_0 + \ell p_1 p_1' + 2\ell p_0 p_0' + \frac{\ell}{2}(p_1 + p_1' + k_1)k_1 + \ell(p_0 + p_0' + k_0)k_0 + \ell(p_0 + p_0')k_0 \\
&= p_0 + p_0' + k_0 + \ell(p_0 + p_0' + k_0)^2 + \frac{\ell}{2}(p_1 + p_1' + k_1)^2 = 0
\end{aligned} \tag{53}$$

And I find that one can proceed analogously for the slightly different case of the conservation law $p \oplus_{\ell\star} k \oplus_\star k' = 0$, which is covariant under the action of a suitably mixed composition of boosts, given by

$$N_{[p]} + N_{[k]} + N_{[k']} + \frac{\ell}{2}(p_0 + k_0 + k_0')(N_{[k]} + N_{[k']}) + \frac{\ell}{2}(k_0 + k_0')N_{[p]}$$

For the 0 component the verification proceeds as follows

$$\begin{aligned}
&[N_{[p]} + N_{[k]} + N_{[k']} + \frac{\ell}{2}(p_0 + k_0 + k_0')(N_{[k]} + N_{[k']}) + \frac{\ell}{2}(k_0 + k_0')N_{[p]}, p_0 + k_0 + k_0'] \\
&= p_1 + k_1 + k_1' + \frac{\ell}{2}(p_0 + k_0 + k_0')(k_1 + k_1') + \frac{\ell}{2}(k_0 + k_0')p_1 = (p \oplus_{\ell\star} k \oplus_\star k')_1 = 0
\end{aligned} \tag{54}$$

And the verification for the 1 component of $p \oplus_{\ell\star} k \oplus_\star k' = 0$ is analogous:

$$\begin{aligned}
&[N_{[p]} + N_{[k]} + N_{[k']} + \frac{\ell}{2}(p_0 + k_0 + k_0')(N_{[k]} + N_{[k']}) + \frac{\ell}{2}(k_0 + k_0')N_{[p]}, p_1 + k_1 + k_1' + \frac{\ell}{2}(p_0 + k_0 + k_0')(k_1 + k_1') + \frac{\ell}{2}(k_0 + k_0')p_1] \\
&= p_0 + \ell p_0^2 + \frac{\ell}{2}p_1^2 + k_0 + k_0' + \frac{\ell}{2}(p_1 + k_1 + k_1')(k_1 + k_1') + \ell(p_0 + k_0 + k_0')(k_0 + k_0') + \frac{\ell}{2}p_1(k_1 + k_1') + \ell p_0(k_0 + k_0') \\
&= p_0 + p_0' + k_0 + \ell(p_0 + p_0' + k_0)^2 + \frac{\ell}{2}(p_1 + p_1' + k_1)^2 = 0
\end{aligned} \tag{55}$$

So I got a consistently relativistic picture also for kinematics involving both ordinarily special-relativistic particles and particles whose kinematics is governed by the κ-Poincaré-inspired DSR-deformation of Lorentz symmetry discussed in Subsection 3.2.

5. A More General Scenario without Universality

The new results reported in the previous section establish a possibility for coexistence of DSR-relativistic and ordinarily special-relativistic particles. I am now going to report results that further generalize the realm of possibilities, allowing for the case of two types of particles, both with DSR-relativistic (non-special-relativistic) properties, but different DSR-relativistic properties. I find that this can be done for several different combinations of DSR-relativistic properties, and in particular also cases with two types of particles with different DSR-relativistic properties but both governed by a commutative law of composition of momentum. I shall be here satisfied exhibiting explicitly a single example of this sort, but an example of rather high complexity (so that readers will feel reassured that, since I manage to make it work for a case of this complexity, of course also other simpler cases can be made to work). Specifically in this section I consider the case of two types of particles with both types governed by the κ-Poincaré-inspired DSR-deformation of Lorentz symmetry discussed in Subsection 3.2, but one type of particle has DSR deformation scale ℓ while the other type of particle has DSR deformation scale λ, denoting with p (or p' or p'' ...) the momenta of the type of particles governed by

$$m^2 = p_0^2 - p_j^2 + \ell p_0 p_j^2 \tag{56}$$

and

$$[N_{[p]}, p_0] = p_1, [N_{[p]}, p_1] = p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2 \tag{57}$$

and denoting with k (or k' or k'' ...) the momenta of the type of particles governed by

$$\mu^2 = k_0^2 - k_j^2 + \lambda k_0 k_j^2 \tag{58}$$

and

$$[N_{[k]}, k_0] = k_1, [N_{[k]}, k_1] = k_0 + \lambda k_0^2 + \frac{\lambda}{2} k_1^2 \tag{59}$$

Also for the "laws of composition without mixing", $p \oplus_\ell p'$ and $k \oplus_\lambda k'$, everything is clear from the onset of the analysis:

$$(p \oplus_\ell p')_j = p_j + p'_j + \ell p_0 p'_j, (p \oplus_\ell p')_0 = p_0 + p'_0 \tag{60}$$

$$(k \oplus_\lambda k')_j = k_j + k'_j + \lambda k_0 k'_j, (k \oplus_\lambda k')_0 = k_0 + k'_0 \tag{61}$$

In particular, this ensures again that processes involving only "laws of composition without mixing" will be described in consistent relativistic manner by enforcing the "laws of action of boosts without mixing" which I already used above:

$$N_{[p \oplus_\ell p']} = N_{[p]} + N_{[p']} + \ell p_0 N_{[p']}$$

$$N_{[k \oplus_\lambda k']} = N_{[k]} + N_{[k']} + \lambda k_0 N_{[k']}$$

So once again the main challenge resides in finding a consistent way to compose momenta of different types of particles, *i.e.*, finding some "mixing composition law" $p \oplus_{\ell,\lambda} k$, while insisting that conservation laws written in terms of such a composition law would be covariant under a consistent prescription for the action of boosts. An example of such a "mixing composition law" which does work is the following

$$(p \oplus_{\ell,\lambda} k)_j = p_j + k_j + \frac{\ell + \lambda}{2} p_0 k_j, (p \oplus_{\ell,\lambda} k)_0 = p_0 + k_0 \tag{62}$$

And I shall also show that the needed counterpart taking the shape of a "mixing composition of boosts" that leads to a consistently relativistic description is given by

$$N_{[p \oplus_{\ell,\lambda} k]} = N_{[p]} + N_{[k]} + \frac{\ell + \lambda}{2} p_0 N_{[k]}$$

With these characterizations I have completed the specification of kinematics needed for the results being reported in this section. I shall now proceed to deriving these results.

Let us verify that indeed my proposed boost actions ensure compatibility with the conservation laws obtained from the composition laws $\oplus_\ell$, $\oplus_\lambda$, and $\oplus_{\ell,\lambda}$. Since indeed for processes involving either exclusively $\oplus_\ell$ or exclusively $\oplus_\lambda$ the compatibility with my deformed boosts was already verified in Section 3, I can focus on cases which involve at least one $\oplus_{\ell,\lambda}$. Looking first at "bi-valent processes" I consider $p \oplus_{\ell,\lambda} k = 0$. For the 0 component of $p \oplus_{\ell,\lambda} k = 0$ one finds

$$[N_{[p]} + N_{[k]} + \frac{\ell+\lambda}{2} p_0 N_{[k]}, k_0 + p_0] = p_1 + k_1 + \frac{\ell+\lambda}{2} p_0 k_1 = (p \oplus_{\ell,\lambda} k)_1 = 0 \tag{63}$$

where on the right-hand side I of course used again the conservation law itself.

Similarly for the 1 component of $p \oplus_{\ell,\lambda} k = 0$ one finds

$$\begin{aligned}
&[N_{[p]} + N_{[k]} + \frac{\ell+\lambda}{2} p_0 N_{[k]}, p_1 + k_1 + \frac{\ell+\lambda}{2} p_0 k_1] = p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2 + k_0 + \lambda k_0^2 + \frac{\lambda}{2} k_1^2 + \frac{\ell+\lambda}{2} p_1 k_1 + (\ell+\lambda) p_0 k_0 \\
&= p_0 + k_0 + \frac{\ell-\lambda}{2}(p_0^2 - k_0^2) + \frac{\ell-\lambda}{4}(p_1^2 - k_1^2) + \frac{\ell+\lambda}{2}(p_0^2 + k_0^2 + 2k_0 p_0) + \frac{\ell+\lambda}{4}(p_1^2 + k_1^2 + 2k_1 p_1) \\
&= p_0 + k_0 + \frac{\ell-\lambda}{2}(p_0 - k_0)(p_0 + k_0) + \frac{\ell-\lambda}{4}(p_1 - k_1)(p_1 + k_1) + \frac{\ell+\lambda}{2}(p_0 + k_0)^2 + \frac{\ell+\lambda}{4}(p_1 + k_1)^2 = 0
\end{aligned} \tag{64}$$

where again on the right-hand side I used the conservation law $p \oplus_{\ell,\lambda} k = 0$ and took again into account that I am working at leading order in ℓ, λ. [Working at leading order in ℓ, λ one finds $(\ell+\lambda)[(p \oplus_{\ell,\lambda} k)_1]^2 \simeq (\ell+\lambda)(p_1 + k_1)^2$ and $(\ell-\lambda)(p_1 - k_1)(p \oplus_{\ell,\lambda} k)_1 \simeq (\ell-\lambda)(p_1 - k_1)(p_1 + k_1)$.]

So I did find that my description of boosts (and of the composition laws $\oplus_{\ell,\lambda}, \oplus_\ell, \oplus_\lambda$) ensures the covariance of conservation laws for "bi-valent processes".

I can now move on to the case of tri-valent processes, verifying that the formulation of boost transformations I adopted affords me the covariance also of the cases with $p \oplus_\ell p' \oplus_{\ell,\lambda} k = 0$ and $p \oplus_{\ell,\lambda} k \oplus_\lambda k' = 0$.

Let me start from the 0 component of $p \oplus_\ell p' \oplus_{\ell,\lambda} k = 0$ for which I easily find the expected result

$$\begin{aligned}
&[N_{[p]} + N_{[p']} + N_{[k]} + \ell p_0 N_{[p']} + \frac{\ell+\lambda}{2}(p_0 + p'_0) N_{[k]}, p_0 + p'_0 + k_0] \\
&= p_1 + p'_1 + k_1 + \ell p_0 p'_1 + \frac{\ell+\lambda}{2}(p_0 + p'_0) k_1 \\
&= (p \oplus_\ell p' \oplus_{\ell,\lambda} k)_1 = 0
\end{aligned} \tag{65}$$

Also successful, but slightly more tedious, is the verification of the covariance for the 1 component of $p \oplus_\ell p' \oplus_{\ell,\lambda} k = 0$, which progresses as follows:

$$\begin{aligned}
&[N_{[p]} + N_{[p']} + N_{[k]} + \ell p_0 N_{[p']} + \frac{\ell+\lambda}{2}(p_0 + p'_0) N_{[k]}, p_1 + p'_1 + k_1 + \ell p_0 p'_1 + \frac{\ell+\lambda}{2}(p_0 + p'_0) k_1] \\
&= p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2 + p'_0 + \ell p_0'^2 + \frac{\ell}{2} p_1'^2 + k_0 + \lambda k_0^2 + \frac{\lambda}{2} k_1^2 + \\
&+ \ell p_1 p'_1 + 2\ell p_0 p'_0 + \frac{\ell+\lambda}{2}(p_1 + p'_1) k_1 + (\ell+\lambda)(p_0 + p'_0) k_0 \\
&= p_0 + p'_0 + k_0 + \frac{\ell-\lambda}{2}(p_0^2 + p_0'^2 - k_0^2) + \frac{\ell-\lambda}{4}(p_1^2 + p_1'^2 - k_1^2) + (\ell-\lambda) p_0 p'_0 + \frac{\ell-\lambda}{2} p_1 p'_1 + \\
&+ \frac{\ell+\lambda}{2}(p_0^2 + p_0'^2 + k_0^2) + \frac{\ell+\lambda}{4}(p_1^2 + p_1'^2 + k_1^2) + (\ell+\lambda)(p_0 p'_0 + p_0 k_0 + p'_0 k_0) + \frac{\ell+\lambda}{2}(p_1 p'_1 + p_1 k_1 + p'_1 k_1) \\
&= p_0 + p'_0 + k_0 + \frac{\ell-\lambda}{2}(p_0 + p'_0 - k_0)(p_0 + p'_0 + k_0) + \frac{\ell-\lambda}{4}(p_1 + p'_1 - k_1)(p_1 + p'_1 + k_1) + \\
&+ \frac{\ell+\lambda}{2}(p_0 + p'_0 + k_0)^2 + \frac{\ell+\lambda}{4}(p_1 + p'_1 + k_1)^2 = 0
\end{aligned} \tag{66}$$

Having done this it is not hard to adapt the results I obtained for $p \oplus_\ell p' \oplus_{\ell,\lambda} k = 0$ to the only slightly different case of $p \oplus_{\ell,\lambda} k \oplus_\lambda k' = 0$. Let me start again with its 0 component, for which I easily find a satisfactory result:

$$\begin{aligned}
&[N_{[p]} + N_{[k]} + N_{[k']} + \frac{\ell+\lambda}{2} p_0 (N_{[k]} + N_{[k']}) + \lambda k_0 N_{[k']}, p_0 + k_0 + k'_0] \\
&= p_1 + k_1 + k'_1 + \frac{\ell+\lambda}{2} p_0 (k_1 + k'_1) + \lambda k_0 k'_1 \\
&= (p \oplus_{\ell,\lambda} k \oplus_\lambda k')_1 = 0
\end{aligned} \tag{67}$$

And equally satisfactory (but again slightly more tedious) is the corresponding verification for the 1 component of $p \oplus_{\ell,\lambda} k \oplus_\lambda k' = 0$ which progresses as follows:

$$\begin{aligned}
&[N_{[p]} + N_{[k]} + N_{[k']} + \frac{\ell+\lambda}{2} p_0 (N_{[k]} + N_{[k']}) + \lambda k_0 N_{[k']}, p_1 + k_1 + k'_1 + \frac{\ell+\lambda}{2} p_0 (k_1 + k'_1) + \lambda k_0 k'_1] \\
&= p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2 + k_0 + \lambda k_0^2 + \frac{\lambda}{2} k_1^2 + k'_0 + \lambda {k'_0}^2 + \frac{\lambda}{2} {k'_1}^2 + \\
&+ \frac{\ell+\lambda}{2} p_1 (k_1 + k'_1) + (\ell+\lambda) p_0 (k_0 + k'_0) + \lambda k_1 k'_1 + 2\lambda k_0 k'_0 \\
&= p_0 + k_0 + p'_0 + \frac{\lambda-\ell}{2} (k_0^2 + {k'_0}^2 - p_0^2) + \frac{\lambda-\ell}{4} (k_1^2 + {k'_1}^2 - p_1^2) + (\lambda-\ell) k_0 k'_0 + \frac{\lambda-\ell}{2} k_1 k'_1 + \\
&+ \frac{\ell+\lambda}{2} (p_0^2 + k_0^2 + {k'_0}^2) + \frac{\ell+\lambda}{4} (p_1^2 + k_1^2 + {k'_1}^2) + (\ell+\lambda)(p_0 k_0 + p_0 k'_0 + k'_0 k_0) + \frac{\ell+\lambda}{2} (p_1 k_1 + p_1 k'_1 + k'_1 k_1) \\
&= p_0 + k_0 + p'_0 + \frac{\lambda-\ell}{2} (k_0 + k'_0 - p_0)(k_0 + k'_0 + p_0) + \frac{\lambda-\ell}{4} (k_1 + k'_1 - p_1)(k_1 + k'_1 + p_1) + \\
&+ \frac{\ell+\lambda}{2} (p_0 + k_0 + k'_0)^2 + \frac{\ell+\lambda}{4} (p_1 + k_1 + k'_1)^2 = 0
\end{aligned} \tag{68}$$

So I did find consistent rules of kinematics allowing meaningful (interacting) theories that are fully relativistic in spite of allowing for the coexistence of a type of particle whose relativistic properties are governed by a DSR ℓ-deformation of Lorentz symmetry and of a type of particle whose relativistic properties are governed by a DSR λ-deformation of Lorentz symmetry.

6. Composite Particles and Potential Implications for Macroscopic Bodies

The main result I am here announcing is contained in the previous two sections. In the remainder of this manuscript my only objective is to show that the new class of DSR-relativistic scenarios introduced in the previous two sections may have application in quite a few different physical pictures:

- **(i)** It could evidently be used to describe pictures in which different "elementary/fundamental" particles have different relativistic properties.
- **(ii)** It could also be used to describe pictures in which all "elementary/fundamental" particles have the same DSR-deformed relativistic properties, but "composite microscopic particles" such as atoms (because of the known mechanisms mentioned in Section I) have different relativistic properties, with weaker deformation of special-relativistic properties than the fundamental particles that compose them.
- **(iii)** And perhaps it could also be used to describe pictures in which microscopic particles have DSR-deformed relativistic properties, but macroscopic bodies (again because of the known mechanisms mentioned in Section I) have ordinary special-relativistic properties.

For cases (**i**)and (**ii**), assuming indeed $|\ell|^{-1}$ is of the order of the Planck scale the new class of DSR-relativistic theories introduced in the previous two sections certainly provides plausible physical pictures, since for microscopic particles (even for atoms) the Planck scale is a gigantic scale and all effects of DSR deformation amount to small corrections.

The physical picture of case (**iii**) instead does not look too promising: Even for this case (**iii**) the fact that I have shown here how different relativistic properties can coexist is a significant step forward, but for macroscopic bodies the Planck scale is actually a small energy scale and there is therefore the risk of predicting hugely unrealistic effects. It is also for this reason that so far the most popular way to handle macroscopic bodies in DSR research has been (see, e.g., [4,10]) the one of renouncing to the introduction of direct interactions between macroscopic and microscopic particles: The interaction with a macroscopic particle could be of course also described in terms of the microscopic interactions involving the constituents of the macroscopic body.

It may well be the case that in spite of the option I managed to produce in the previous two sections one should still proceed in this way. But it is no longer so obvious that this should be necessary: In this section I am going to report a simple derivation which provides encouragement for the possibility of allowing direct interactions between microscopic particles and macroscopic bodies (without needing to decompose such interactions in terms of the microscopic interactions between the microscopic particle and the constituents of the macroscopic body).

The simple derivation I am reporting does not establish anything of general validity, but it does show that at least some of the derivations which could turn pathological with a DSR description of macroscopic bodies actually do not.

This simple derivation concerns an elastic collision between an elementary particle and a macroscopic body

$$e^- + X \to e^- + X$$

where e^- stands for any elementary particle (e.g., an electron) governed by a DSR-relativistic description of the type in Section 3

$$p_0 = p + \frac{m^2}{2p} - \frac{\ell}{2}p^2 \tag{69}$$

while X is a macroscopic body of large mass M ($M > |\ell|^{-1}$) in a nonrelativistic regime but possibly with large spatial momentum k ($|\ell|^{-1} < k \ll M$)

$$k_0 = M + \frac{k^2}{2M} \tag{70}$$

Since I am considering $M > |\ell|^{-1}$ (and $k > |\ell|^{-1}$) it is natural to fear that this elastic scattering might give pathological results, such as a huge transfer of momentum (of order, e.g., ℓk) from the macroscopic body to the micro particle. But this is not what the formalization I developed in the previous two sections predicts.

In reaching this conclusion the nontrivial point of the derivation is of course the "mixing composition law" which I introduced in Section 4. The process I am here considering has two incoming and two outgoing particles, so it must be written in terms of two antipodes (see [12] and references therein)

$$(\ominus_\ell p') \oplus_\ell p \oplus_{\ell\star} (\ominus_\star k') \oplus_\star k = 0 \tag{71}$$

Since $\ominus_\star$ is the undeformed composition law one has that $(\ominus_\star k)_\mu = -k_\mu$, whereas for the $\oplus_\ell$ composition law the antipode is such that still $(\ominus_\ell p)_0 = -p_0$ but $(\ominus_\ell p)_j = -p_j + \ell p_0 p_j$ (in fact $[(\ominus_\ell p) \oplus_\ell p]_j = -p_j + \ell p_0 p_j + p_j + \ell(-p_0)p_j = 0$).

So for the process I am considering one has

$$-p'_1 + \ell p'_0 p'_1 + p_1 - \ell p'_0 p_1 - k'_1 + k_1 + \frac{\ell}{2}(p_0 - p'_0)(k_1 - k'_1) = 0 \tag{72}$$

and

$$-p'_0 + p_0 - k'_0 + k_0 = 0 \tag{73}$$

From Equation (72) one finds that

$$p'_1 - p_1 \simeq k_1 - k'_1 + \frac{\ell}{2}(p_0 + p'_0)(p'_1 - p_1) \simeq k_1 - k'_1 + \ell p_1(p'_1 - p_1) \tag{74}$$

where on the right-hand side, consistently with the fact that I am working throughout at leading order, I used zero-th order properties in a reexpression of the first-order term.

Then from Equation (73) one has that

$$p'_1 + \frac{m^2}{2p'_1} - \frac{\ell}{2}p'^2_1 + M + \frac{k'^2_1}{2M} = p_1 + \frac{m^2}{2p_1} - \frac{\ell}{2}p_1^2 + M + \frac{k_1^2}{2M} \tag{75}$$

which gives

$$p'_1 - p_1 = \frac{1}{2M}(k_1^2 - k'^2_1) + \frac{m^2}{2p_1} - \frac{m^2}{2p'_1} - \frac{\ell}{2}(p_1^2 - p'^2_1) \simeq \frac{1}{2M}(k_1^2 - k'^2_1) + \frac{m^2}{2p_1} - \frac{m^2}{2p'_1} \tag{76}$$

where on the right-hand side I used again zero-th order properties in a reexpression of the first-order term.

Combining Equations (74,76) one sees that, at least in the specific context of an elastic collision between a DSR-deformed micro particle and special-relativistic macroscopic body no pathology arises: Neither in Equation (74) nor in Equation (76) one finds pathological correction terms of the type ℓk or ℓM.

If it turned out to be possible to extend this observation to a wider class of phenomena we might also have an exciting opportunity for the description of "observers" in DSR-relativistic theories of this sort. In any relativistic theory an observer is to be identified with a macroscopic device (or, idealizing, a network of such devices), and so far studies of this type of on-shell-relation-centered DSR scenarios have kept such observers in a sort of "limbo", protected artificially (by hand) from the implications of the deformation of relativistic symmetries. If it turned out to be possible to generalize the observation I reported in this section those artificial abstractions could be eliminated in favor of a more satisfactory picture of DSR observers.

7. Implications for Particle-Physics Processes

Having discussed a possible application of the results I reported in Sections 4 and 5 for the inclusion of special-relativistic macroscopic bodies in an otherwise DSR-relativistic framework, in this section I go to the opposite extreme of the range of possible applications of the results I reported in Sections 4 and 5, by considering the possibility that different microscopic (possibly "fundamental") particles be governed by different DSR-relativistic properties.

As I already stressed above (this was characterized as case (**i**) in the previous section) this possible application is clearly viable if one assumes indeed that $|\ell|^{-1}$ is of the order of the Planck scale (or some other ultralarge momentum scale), since for microscopic particles (even for atoms) the Planck scale is gigantic and all effects of DSR deformation amount to small corrections.

While this is evident, it is nonetheless valuable for me to stress that, in spite of the apparently "invasive" prescription of different DSR-relativistic properties for different types of particles, the deformation is still a very smooth deformation of special-relativistic symmetries. I shall do this by showing that some pathologies, which are expected in cases where different particles have different relativistic properties (if this is the result of a full breakdown of relativistic symmetries, with emergence of a preferred frame), are not present when this feature is introduced in the DSR-compatible manner I proposed in this manuscript.

As examples of cases where these differences between full breakdown of relativistic symmetries and DSR-deformations of relativistic symmetries, I use, for illustrative purposes, some processes which were of interest in discussions of the relativistic anomaly for relativity of neutrinos tentatively reported by the OPERA collaboration [21] (note however [22]). Discussions of possible non-universal departures from Lorentz symmetry naturally came under consideration [23–28] in contemplating the preliminary OPERA reports, since for photons of, say, about 20 GeV the agreement of the speed law with the special-relativistic prescription is confirmed at the level of 1 part in 10^{18} (see, e.g., [29–31]), whereas taking the OPERA first announcement at face value one should assume that $\sim$ 20 GeV neutrinos experience departures from the special-relativistic speed law at the much higher level of a few parts in 10^5.

Universal (particle-type independent) anomalous effects on propagation speeds could be described in a DSR-relativistic picture, as shown in particular in [14,32–35]. A preliminary attempt of generalizing such results to cases with non-universal departures from Lorentz symmetry was reported in [24]. It was there observed that one might well have a universal DSR deformation and yet have "effectively particle-dependent effects": The illustrative example adopted in [24] is centered on an on-shell relation of the type

$$p_0^2 = p^2 + m^2 + 2\ell \frac{p_0^2 p^2}{m} \tag{77}$$

universally for all particles, but yet effectively attributing stronger departures from special relativity to lighter particles, because of the explicit dependence on the mass (a relativistic invariant) in the correction term. [And notice that, while Equation (77) could not be applied to massless particles, similar features are found adopting for example $p_0^2 = p^2 + m^2 + \ell^2 p^6/(p_0^2 - p^2)$, which is applicable to massless particles.]

I have provided in this manuscript a possible alternative by showing that one can have a consistent DSR-relativistic framework even endowing different types of particles with genuinely different DSR-relativistic properties. It is now possible to contemplate genuinely "non-universal" DSR-relativistic pictures.

In order to illustrate some of the implications of my results for particle-physics processes I adopt in this section the scenario worked out in Section 5, with two types of particles, a type governed by

$$p_0 = p + \frac{m_\nu^2}{2p} - \frac{\ell}{2} p^2 \tag{78}$$

and a type governed by

$$k_0 = k + \frac{m_x^2}{2k} - \frac{\lambda}{2}k^2 \tag{79}$$

and for definiteness I assume $|\lambda| \leq |\ell|$.

7.1. Absence of Cherenkov-like $Y \to Y + X$ Processes

In-vacuo Cherenkov-like processes $Y \to Y + X$ (where X may also be, e.g., an electron-positron pair) are forbidden in special relativity, but in theories that violate/break relativistic invariance (with associated emergence of a preferred "ether" frame), they can be allowed if the energy of the incoming Y particle is above a certain threshold value. (This was in particular of interest [36–38] when contemplating the anomaly tentatively reported by OPERA.)

Reference [24] (on the basis of results already discussed, for other purposes, in [39,40]) already observed that in standard "universal" DSR-relativistic descriptions such in-vacuo Cherenkov-like processes necessarily remain forbidden.

The notion of "non-universal" DSR-relativistic framework, which I here introduced, also fully preserves the relativity of inertial frames, so in-vacuo Cherenkov-like processes must also be forbidden [12,39] in this new class of relativistic theories. But it is still valuable to see this in an explicit calculation. So let me analyze the relativistic kinematics of $Y \to Y + X$ in the novel setup of Section 5.

Within the framework here introduced in Section 5 such a process should be described in terms of the "mixing conservation law"

$$(\ominus_\ell p') \oplus_\ell p \oplus_{\ell\lambda} k = 0 \tag{80}$$

where p is the (four-)momentum of the outgoing Y-particle, k is the momentum of the outgoing X, and $\ominus_\ell p'$ is the $\oplus_\ell$-antipode of the momentum of the incoming Y.

In light of the properties specified for $\oplus_\ell$ in Sections 3–5, one finds that the 0 component of this conservation law is

$$0 = -p'_0 + p_0 + k_0 = -p'_1 - \frac{m^2}{2p'_1} + \frac{\ell}{2}p_1'^2 + p_L + \frac{m^2 + p_T^2}{2p_L} - \frac{\ell}{2}p_L^2 + k_L + \frac{m_x^2 + p_T^2}{2k_L} - \frac{\ell}{2}k_L^2 \tag{81}$$

where I also used Equations (78,79) with m the mass of the incoming and outgoing Y, and m_x the rest energy of X. Also notice that I am assuming that the incoming Y is ultrarelativistic (in particular $p \gg m_x$) and the momenta transverse to the direction of the incoming Y are small: Along that transverse direction both the outgoing Y and the outgoing X have transverse momenta of roughly the same magnitude, here denoted with p_T, which is very small compared to the momenta these outgoing particles have along the direction of the incoming Y.

For the momenta (with index $_L$) along that direction of the incoming Y one obtains from Equation (80)

$$-p' + \ell p'_0 p' + p_L - \ell p'_0 p_L + k_L + \frac{\ell + \lambda}{2}(p_0 - p'_0)k_L = 0 \tag{82}$$

i.e.,

$$p' = p_L + k_L + \ell p'_0 k_L - \frac{\ell + \lambda}{2}k_0 k_L \tag{83}$$

where I used again in the leading-order correction some properties established at 0-th order.

Combining Equation (81) with Equation (83) one obtains

$$\frac{p_T^2}{2p_L}+\frac{p_T^2}{2k_L} = \frac{m^2}{2p_1'}-\frac{m^2}{2p_L}-\frac{m_x^2}{2k_L}+\ell p_0' k_L-\frac{\ell+\lambda}{2}k_0 k_L-\frac{\ell}{2}p_1'^2+\frac{\ell}{2}p_L^2+\frac{\ell}{2}k_L^2 \simeq \frac{m^2}{2p_1'}-\frac{m^2}{2p_L}-\frac{m_x^2}{2k_L} \tag{84}$$

where on the right-hand side I observed that

$$\ell p_0' k_L-\frac{\ell+\lambda}{2}k_0 k_L-\frac{\ell}{2}p_1'^2+\frac{\ell}{2}p_L^2+\frac{\ell}{2}k_L^2 \simeq 0$$

using properties valid at 0-th order in rearranging this leading-order correction.

Equation (84) is the main result of this subsection. For scenarios with breakdown of Lorentz symmetry, with emergence of a preferred frame, the key observation concerning in-vacuo Cherenkov-like processes is that the analog of Equation (84) contains strong corrections [36] from Lorentz-symmetry-breaking terms such that then real values of p_T can be allowed. One way to see that in-vacuo Cherenkov-like processes are forbidden in special relativity is through the fact that it would require imaginary values of p_T. And in my DSR-relativistic analysis I found, as codified in Equation (84), that, independently of the value of the momentum of the incoming Y, the p_T is necessarily imaginary; in-vacuo Cherenkov-like processes are indeed still forbidden in my DSR-relativistic framework.

7.2. Absence of Anomalies for Particle Decays

Another known "pathology" for theories that violate/break relativistic invariance (with associated emergence of a preferred "ether" frame) is the possibility that certain particle decays, such as the pion-decay channel $\pi \to \mu + \nu$, end up having the phase space available for the decay severely reduced at high energies. (This too was of interest [41–43] when contemplating the anomaly tentatively reported by OPERA.)

Also with respect to this other broken-Lorentz-symmetry pathology the notion of "non-universal" DSR-relativistic framework, which I here introduced, turns out to be immune, mainly as a result of the fact that it fully preserves the relativity of inertial frames. I shall be here satisfied analyzing the relativistic kinematics of the specific particle-decay process $\pi \to \mu + \nu$ within the novel setup of Section 5, so that in particular

$$p_{\nu 0} = p_{\nu L} + \frac{m_\nu^2 + p_T^2}{2p_{\nu L}} - \frac{\ell}{2}p_{\nu L}^2 \tag{85}$$

$$k_{\pi 0} = k_\pi + \frac{m_\pi^2}{2k_\pi} - \frac{\lambda}{2}k_\pi^2 \tag{86}$$

$$k_{\mu 0} = k_{\mu L} + \frac{m_\mu^2 + p_T^2}{2k_{\mu L}} - \frac{\lambda}{2}k_{\mu L}^2 \tag{87}$$

I am evidently again focusing on the case that the incoming particle (this time the pion) is ultrarelativistic ($p_\pi \gg m_\pi$) and the momenta transverse to the direction of the incoming pion are small: Along that transverse direction both the outgoing neutrino and the outgoing muon have transverse momenta of roughly the same magnitude, here denoted again with p_T, which is very small compared to the momenta these outgoing particles have along the direction of the incoming pion.

On the basis of the findings here reported in Section 5 the process $\pi \to \mu + \nu$ should be described in terms of the "mixing conservation law"

$$p_\nu \oplus_{\ell\lambda} [k_\mu \oplus_\lambda (\ominus_\lambda k_\pi)] = 0 \tag{88}$$

where $\ominus_\lambda k_\pi$ is the $\oplus_\lambda$-antipode of the momentum of the incoming pion.

For the 0 component this gives

$$0 = -k_{\pi\,0} + p_{\nu\,0} + k_{\mu\,0} = -k_\pi - \frac{m_\pi^2}{2k_\pi} + \frac{\lambda}{2}k_\pi^2 + p_{\nu L} + \frac{m_\nu^2 + p_T^2}{2p_{\nu L}} - \frac{\ell}{2}p_{\nu L}^2 + k_{\mu L} + \frac{m_\mu^2 + p_T^2}{2k_{\mu L}} - \frac{\lambda}{2}k_{\mu L}^2 \tag{89}$$

where I also used Equations (85–87).

For the momenta (with index $_L$) along the direction of the incoming pion one obtains from Equation (88):

$$p_{\nu L} + k_{\mu L} - k_\pi + \lambda k_{\pi 0} k_\pi - \lambda k_{\mu 0} k_\pi + \frac{\ell + \lambda}{2} p_{\nu 0}(k_{\mu L} - k_\pi) = 0 \tag{90}$$

i.e.,

$$k_\pi = p_{\nu L} + k_{\mu L} + +\ell p_{\nu 0} k_\pi - \frac{\ell + \lambda}{2} p_{\nu 0} p_{\nu L} \tag{91}$$

where again on the right-hand side I used properties of the 0-th order result in rearranging the leading-order correction.

Combining Equation (89) with Equation (91) one obtains

$$\frac{p_T^2}{2p_{\nu L}} + \frac{p_T^2}{2k_{\mu L}} = \frac{m_\pi^2}{2k_\pi} - \frac{m_\nu^2}{2p_{\nu L}} - \frac{m_\mu^2}{2k_{\mu L}} + \ell p_{\nu 0} k_\pi - \frac{\ell + \lambda}{2} p_{\nu 0} p_{\nu L} + \frac{\ell}{2} p_{\nu L}^2 + \frac{\lambda}{2} k_{\mu L}^2 - \frac{\lambda}{2} k_\pi^2 \simeq \frac{m_\pi^2}{2k_\pi} - \frac{m_\nu^2}{2p_{\nu L}} - \frac{m_\mu^2}{2k_{\mu L}} \tag{92}$$

where on the right-hand side I observed that

$$\ell p_{\nu 0} k_\pi - \frac{\ell + \lambda}{2} p_{\nu 0} p_{\nu L} + \frac{\ell}{2} p_{\nu L}^2 + \frac{\lambda}{2} k_{\mu L}^2 - \frac{\lambda}{2} k_\pi^2 \simeq 0$$

using properties valid at 0-th order in rearranging this leading-order correction.

Equation (92) is the main result of this subsection. For scenarios with breakdown of Lorentz symmetry and emergence of a preferred frame, the key observation concerning $\pi \to \mu + \nu$ is that the analog of Equation (92) contains strong corrections [41–43] from Lorentz-symmetry-breaking terms, such that the combinations of $p_{\nu L}$ and $k_{\mu L}$ which satisfy the requirement $p_T^2 > 0$ may only amount to a very small phase space. Instead in my DSR-relativistic analysis I found Equation (92), in which all correction terms canceled each other out, so that the phase space available for the decay in the DSR case is identical to the phase space available for the decay in the standard special-relativistic case.

I should stress that this result holds in a leading-order analysis, and it is therefore reliable only for $p_\pi \ll |\ell|^{-1}$. For $p_\pi \sim |\ell|^{-1}$ one might have sizable modifications of the phase space even in a DSR case.

7.3. Aside on Quantum Gravity and the Planck Scale

The observations reported in the previous two subsections provide some of the characterizations of the differences, at the level of observable predictions, between broken and deformed Lorentz symmetry, for the case of non-universal effects. One should expect that the most promising applications of these results will be in the context where DSR-deformations of Lorentz symmetry were first conceived, *i.e.*, studies of the quantum-gravity problem, for which the deformation scale is roughly of the order of the Planck scale.

There were some studies (see, e.g., [44–46]) proposing mechanisms by which quantum-gravity/quantum-spacetime effects could produce departures from Lorentz symmetry of different magnitude for different particles. The results I reported in this manuscript show that such scenarios do not necessarily have to "break" Lorentz invariance, producing a preferred "ether" frame. One could attempt to discuss such scenarios in terms of the particle-type-dependent DSR-deformations of Lorentz symmetry I here proposed. And then the results reported in the previous two subsections could be valuable assets for the corresponding "Planck-scale phenomenology".

8. Aside on "Hopf-Hopf Algebras with Mixing Co-Products"

I have here introduced some new scenarios for (DSR-)relativistic kinematics. I expect that in order to achieve a full empowerment of my proposal it will be necessary to identify a symmetry-algebra counter-part to the novel type of relativistic kinematics. This balance is rather visible in special relativity, whose full understanding requires combining the Poincaré symmetry algebra and Einstein kinematics.

And this "balance of powers" appears to be preserved also in the illustrative example of "universal" DSR-relativistic theory which I here took as starting point, in Subsection 3.2: For the specific example of DSR deformation of relativistic kinematics here reviewed in Subsection 3.2 one can find numerous points of contact with the structure of the κ-Poincaré Hopf algebra [17,18]. In particular, the most crucial aspect of that construction, the composition law

$$(p \oplus_\ell p')_1 = p_1 + p'_1 + \ell p_0 p'_1, (p \oplus_\ell p')_0 = p_0 + p'_0 \tag{93}$$

can be placed in correspondence with the law of "co-product" that characterizes the κ-Poincaré Hopf algebra in the Majid–Ruegg basis [17], which in leading order reads

$$\Delta(P_1) \simeq P_1 \otimes 1 + 1 \otimes P_1 + \frac{1}{\kappa} P_0 \otimes P_1, \Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 \tag{94}$$

In order to establish a similar connection between relativistic kinematics and symmetry algebras for the novel scenarios for (DSR-)relativistic kinematics I here introduced in Sections 4 and 5, it would seem necessary to introduce on the algebra side enough structure to accommodate at least 3 laws of coproduct: Two different laws of the type Equation (94)

$$\Delta_\ell(P_1) \simeq P_1 \otimes 1 + 1 \otimes P_1 + \ell P_0 \otimes P_1 \,, \Delta_\ell(P_0) = P_0 \otimes 1 + 1 \otimes P_0$$
$$\Delta_\lambda(P_1) \simeq P_1 \otimes 1 + 1 \otimes P_1 + \lambda P_0 \otimes P_1 \,, \Delta_\lambda(P_0) = P_0 \otimes 1 + 1 \otimes P_0$$

and a novel "mixing coproduct" of a type illustrated by

$$\Delta_{\ell\lambda}(P_1) \simeq P_1 \otimes 1 + 1 \otimes P_1 + \frac{\ell + \lambda}{2} P_0 \otimes P_1, \Delta_{\ell\lambda}(P_0) = P_0 \otimes 1 + 1 \otimes P_0 \tag{95}$$

I am not aware of any work on symmetry algebras (some sort of "$\kappa\kappa$-Poincaré algebra") already providing these structures, but it appears natural to expect that such a construction should be possible.

And in addition to looking for a "symmetry-algebra counterpart", it would be desirable to have also (one form or another of) a "spacetime picture". I shall not speculate much about this here. I expect it may require several striking steps of abstraction, and some time to mature. Even for universal DSR deformation we are still in the "digestion process" for some of the striking new features that the associate spacetime pictures typically introduce, such as the relativity of spacetime locality [33,34]. We must expect features possibly even more virulent (but again not necessarily in conflict with established experimental facts) to be typical for the novel non-universal DSR deformations I am here introducing.

Hoping to offer a useful contribution to the study of these issues I venture to formulate here only a preliminary speculation. This concerns the fact that for the illustrative example of "universal" DSR-relativistic theory which I here took as starting point, in Section 3, there is an established "formal link" to the so-called "κ-Minkowski spacetime" (see, e.g., [17,18,47]), in which the same scale κ of Equation (94) appears in a form of spacetime noncommutativity

$$[\hat{x}_j, \hat{t}] = i\frac{1}{\kappa}\hat{x}_j \ , \quad [\hat{x}_j, \hat{x}_k] = 0 \tag{96}$$

The core feature of this formal link is visible already in how the composition law Equation (17)

$$(p \oplus_\ell p')_1 = p_1 + p'_1 + \ell p_0 p'_1 \tag{97}$$

emerges in cases where certain ordering prescriptions are applied in analyses of κ-Minkowski noncommutativity, such as in

$$e^{ip_j\hat{x}^j} e^{ip_0\hat{t}} e^{ip'_j\hat{x}^j} = e^{ip_j\hat{x}^j} e^{ie^{\ell p_0}p'_j\hat{x}^j} e^{ip_0\hat{t}} \simeq e^{i[p_j+p'_j+\ell p_0 p'_j]\hat{x}^j} e^{ip_0\hat{t}} \tag{98}$$

where I used Equation (96) for $\ell \equiv 1/\kappa$.

Inspired by this observation, one may consider attempting to provide a formal spacetime picture for a non-universal DSR-relativistic scenario of the type in Section 5 by seeking a generalization/deformation of κ-Minkowski spacetime, suitable for accommodating some "mixing composition law" of the type illustrated in Equation (62) of Section 5 as

$$(p \oplus_{\ell,\lambda} k)_j = p_j + k_j + \frac{\ell+\lambda}{2} p_0 k_j \tag{99}$$

or similarly [48]

$$(p \oplus_{\ell,\lambda} k)_j = p_j + k_j + \frac{\lambda}{2} p_0 k_j - \frac{\ell}{2} k_0 p_j \tag{100}$$

I feel that one tempting possibility would be the one of contemplating a notion of quantum spacetime in which the coordinates of different types of particles have different noncommutativity properties, which taking as starting point κ-Minkowski spacetime may lead one to consider a generalization suitable for being labeled as "$\kappa\kappa$-Minkowski spacetime".

Postponing a more in depth analysis of possible alternatives to future work, I just want to notice, concerning this $\kappa\kappa$-Minkowski speculation, that structures such as the one found in Equations (99,100) might be naturally encountered in cases where suitable ordering prescriptions are applied in analyses of a scenario such that my "p-particles" have coordinates with noncommutativity

$$[\hat{x}_j, \hat{t}] = i\ell\hat{x}_j, [\hat{x}_j, \hat{x}_k] = 0 \tag{101}$$

while my "k-particles" have coordinates with noncommutativity

$$[\hat{x}'_j, \hat{t}] = i\lambda \hat{x}'_j, [\hat{x}'_j, \hat{x}'_k] = 0 \tag{102}$$

One could then produce terms of the form $\lambda p_0 k_j$ from applying suitable ordering prescriptions in such a $\kappa\kappa$-Minkowski spacetime, as shown by

$$e^{ip_j\hat{x}^j} e^{ip_0\hat{t}} e^{ik_j\hat{x}^j} = e^{ip_j\hat{x}^j} e^{ie^{\lambda p_0} k_j \hat{x}'^j} e^{ip_0\hat{t}} \simeq e^{i[p_j\hat{x}^j + (k_j+\lambda p_0 k_j)\hat{x}'^j]} e^{ip_0\hat{t}} \tag{103}$$

and one could produce terms of the form $\ell k_0 p_j$ from other applications of suitable ordering prescriptions in such a $\kappa\kappa$-Minkowski spacetime, as shown by

$$e^{ik_j\hat{x}'^j} e^{ik_0\hat{t}} e^{ip_j\hat{x}^j} = e^{ik_j\hat{x}'^j} e^{ie^{\ell k_0} p_j \hat{x}^j} e^{ip_0\hat{t}} \simeq e^{i[p_j\hat{x}^j + (k_j+\lambda p_0 k_j)\hat{x}'^j]} e^{ik_0\hat{t}} \tag{104}$$

9. Aside Beyond Leading Order

I have worked throughout this manuscript at leading order in the deformation scale. As stressed above this is the only reasonable choice since in quantum-gravity/Planck-scale phenomenology it would be already very fortunate to ever uncover leading-order effects, and therefore, at least presently, going beyond leading order is unjustified. As I stressed already in [1,4], enforcing some sort of requirement of mathematical consistency beyond leading order not only is inappropriate because of the limitations of the expected experimental sensitivities, but also may be inappropriate in light of the complexity of the quantum-gravity problem: Even if DSR-deformations of Lorentz symmetry do end up being actually relevant for quantum gravity (and of course this is only a remote hypothesis) it may well be the case that only their "leading-order formulation" makes sense physically. This is because the availability of a description in terms of DSR deformations essentially still assumes a rather standard picture of spacetime, novel enough to include striking new features such as a relativity of spacetime locality [15,33,34], but conventional enough to allow a description to a large extent still consistent with the standard role of spacetime in physics. However, the nature of the quantum-gravity problem is such that it would not be surprising if, as the characteristic energy scales get closer to the Planck scale (just as the leading-order analysis starts to be insufficient), at some point there would be the onset of a completely foreign regime of the laws of physics, not even affording us the luxury of the abstraction of an (however exotic) spacetime formulation.

So I do not view the development of DSR pictures beyond leading order as an important priority. It is nonetheless conceptually intriguing and provides amusing challenges. I shall not dwell much here on this issue, but let me nonetheless at least exhibit, in this section, partial results that provide some encouragement for the idea that such "all-order particle-type-dependent DSR-deformations" are indeed possible.

The first ingredient I introduce for this purpose is an "all-order generalization" of the type of particle described in Subsection 3.2. This generalization replaces Equations (16–18,20) with

$$\cosh(\ell m) = \cosh(\ell p_0) - \frac{\ell^2}{2} e^{-\ell p_0} p_1^2 \tag{105}$$

$$(p \oplus_\ell p')_1 = p_1 + e^{\ell p_0} p'_1 (p \oplus_\ell p')_0 = p_0 + p'_0 \tag{106}$$

$$[N, p_0] = p_1, [N, p_1] = \frac{e^{2\ell p_0} - 1}{2\ell} + \frac{\ell}{2} p_1^2 \tag{107}$$

$$N_{[p \oplus_\ell p']} = N_{[p]} + e^{\ell p_0} N_{[p']} \tag{108}$$

These formulas may be viewed as a kinematical counterpart from some of the structures, including the mentioned co-products, of one of the descriptions [17] of the κ-Poincaré Hopf algebra.

In this section I refer to particles governed by Equations (105–108) as "p-particles" and denote their momenta consistently with p (or p' or p'' ...).

It is easy to see that Equations (105–108) ensure relativistic consistency for the description of such particles. In particular,

$$[N, \cosh(\ell p_0) - \frac{\ell^2}{2} e^{-\ell p_0} p_1^2] = 0 \tag{109}$$

and interactions among "p-particles", with conservation law $p \oplus_\ell p' \oplus_\ell p'' = 0$, evidently admit consistent relativistic description, as shown by the following two Equations:

$$[N_{[p]} + e^{\ell p_0} N_{[p']} + e^{\ell(p_0 + p'_0)} N_{[p'']}, p_0 + p'_0 + p''_0] = p_1 + e^{\ell p_0} p'_1 + e^{\ell(p_0 + p'_0)} p''_1 = (p \oplus_\ell p' \oplus_\ell p'')_1 = 0 \tag{110}$$

$$\begin{aligned}
&[N_{[p]} + e^{\ell p_0} N_{[p']} + e^{\ell(p_0 + p'_0)} N_{[p'']}, p_1 + e^{\ell p_0} p'_1 + e^{\ell(p_0 + p'_0)} p''_1] \\
&= \left(\frac{e^{2\ell p_0} - 1}{2\ell} + \frac{\ell}{2} p_1^2 \right) + e^{2\ell p_0} \left(\frac{e^{2\ell p'_0} - 1}{2\ell} + \frac{\ell}{2} {p'_1}^2 \right) + e^{2\ell(p_0 + p'_0)} \left(\frac{e^{2\ell p''_0} - 1}{2\ell} + \frac{\ell}{2} {p''_1}^2 \right) \\
&+ \ell e^{\ell p_0} p_1 p'_1 + \ell e^{\ell(p_0 + p'_0)} p_1 p''_1 + \ell e^{\ell(2p_0 + p'_0)} p'_1 p''_1 = \frac{\ell}{2} \left(p_1 + e^{\ell p_0} p'_1 + e^{\ell(p_0 + p'_0)} p''_1 \right)^2 \\
&+ \left(\frac{e^{2\ell(p_0 + p'_0 + p''_0)} - 1}{2\ell} \right) = 0
\end{aligned} \tag{111}$$

where for both of these results I of course enforced on the right-hand side the conservation law $p \oplus_\ell p' \oplus_\ell p'' = 0$ itself.

Note that from Equations (110,111) it also follows that, for the conservation law $p \oplus_\ell p' = 0$, one has

$$[N_{[p]} + e^{\ell p_0} N_{[p']}, p_0 + p'_0] = 0$$

and

$$[N_{[p]} + e^{\ell p_0} N_{[p']}, p_1 + e^{\ell p_0} p'_1] = 0$$

So I have a fully consistent (and consistent to all orders in ℓ) DSR-relativistic description of the kinematics of p-particles, propagating (with $p \oplus_\ell p' = 0$ conservation) and interacting among themselves (with $p \oplus_\ell p' \oplus_\ell p'' = 0$ conservation).

My next task is to introduce a second type of particles, with different DSR-relativistic properties. For these "k-particles" (whose momenta I shall consistently denote with k or k'...) I take the following on-shell relation

$$\cosh(\ell\mu) = \cosh(\ell k_0) - \frac{\ell^2}{2} k_1^2 \tag{112}$$

and boost such that

$$[N_{[k]}, k_0] = k_1, [N_{[k]}, k_1] = \frac{\sinh(\ell k_0)}{\ell} \tag{113}$$

which indeed is compatible with the on-shell relation

$$[N_{[k]}, \cosh(\ell k_0) - \frac{\ell^2}{2} k_1^2] = 0 \tag{114}$$

Notice that I am now considering a case where also the DSR-deformation of the second type of particle is characterized by the same deformation scale ℓ, but the form of the laws of transformation that apply to the two types of particles are very significantly different.

For this section of "aside beyond leading order" I do not go as far as introducing a proper composition law for k-particles and/or a proper "p-particle/k-particle mixing composition law" to be used as general rules applicable to all sorts of processes. I do not see any in-principle obstruction for deriving mixing conservation laws from mixing composition laws also in such analyses beyond leading order, but surely the nonlinearities render that task more cumbersome. Considering the rather exploratory tone of this section I therefore opt for the simplest option of providing directly the mixing conservation laws, relevant for certain specific processes, verifying that they do admit a fully relativistic description.

I will also not elaborate on the ways to construct such conservation laws, but the careful reader will notice that their structure can be seen as inspired by either one of two ways of characterizing equivalently the covariance of my choice of conservation law for "k-particle propagation processes", which is the undeformed one:

$$k_0 + k_0' = 0, k_1 + k_1' = 0 \tag{115}$$

This evidently is compatible with a correspondingly undeformed law of "composition of boosts" $N_{[k]}$:

$$[N_{[k]} + N_{[k']}, k_0 + k_0'] = k_1 + k_1' = 0, [N_{[k]} + N_{[k']}, k_1 + k_1'] = \frac{\sinh(\ell k_0)}{\ell} + \frac{\sinh(\ell k_0')}{\ell} = 0 \tag{116}$$

where I of course also used the conservation law $k_0 + k_0' = 0$, $k_1 + k_1' = 0$ itself.

The anticipated interesting alternative way to characterize this conservation law for "k-particle propagation processes" is centered on noticing that Equation (115) can be rewritten equivalently as

$$k_0 + k_0' = 0, e^{-\ell k_0'/2} k_1 + e^{\ell k_0/2} k_1' = 0 \tag{117}$$

whose compatibility with the $N_{[k]}$ boost could be described as follows

$$[e^{-\ell k_0'/2} N_{[k]} + e^{\ell k_0/2} N_{[k']}, k_0 + k_0'] = e^{-\ell k_0'/2} k_1 + e^{\ell k_0/2} k_1' = 0 \tag{118}$$

$$[e^{-\ell k_0'/2} N_{[k]} + e^{\ell k_0/2} N_{[k']}, e^{-\ell k_0'/2} k_1 + e^{\ell k_0/2} k_1'] = e^{-\ell k_0'/2} \frac{\sinh(\ell k_0)}{\ell} + e^{\ell k_0/2} \frac{\sinh(\ell k_0')}{\ell} = 0 \tag{119}$$

I do not provide any other conservation law for processes involving exclusively k-particles (as such I provide a picture for k-particles that do not self-interact). But I do provide "mixing conservation laws", for processes involving both k-particles and p-particles, with structure which may be viewed as inspired either by the form of Equations (115,116) or by the form of Equations (117–119).

The first such "mixing composition law" which I exhibit is for simple "oscillation processes"

$$k_0 + p_0 = 0, k_1 + e^{\ell k_0/2} p_1 = 0$$

and is compatible with the following law of "mixing of boosts"

$$N_{[k]} + e^{\ell k_0/2} N_{[p]}$$

This is easily verified from the following two observations:

$$[N_{[k]} + e^{\ell k_0/2} N_{[p]}, k_0 + p_0] = k_1 + e^{\ell k_0/2} p_1 = 0$$

and

$$\begin{aligned}
&[N_{[k]} + e^{\ell k_0/2} N_{[p]}, k_1 + e^{\ell k_0/2} p_1] \\
&= \frac{\sinh(\ell k_0)}{\ell} + e^{\ell k_0} \left(\frac{e^{2\ell p_0} - 1}{2\ell} + \frac{\ell}{2} p_1^2 \right) + \frac{\ell}{2} e^{\ell k_0/2} k_1 p_1 \\
&= \frac{\ell}{2} e^{\ell k_0/2} p_1 \left(k_1 + e^{\ell k_0/2} p_1 \right) + \left(\frac{e^{\ell k_0 + 2\ell p_0} - e^{-\ell k_0}}{2\ell} \right) \\
&= 0
\end{aligned}$$

A first example of consistently relativistic conservation law for interactions among k-particles and p-particles is the following:

$$k_0 + p_0 + p'_0 = 0, k_1 + e^{\ell k_0/2} p_1 + e^{\ell k_0/2} e^{\ell p_0} p'_1 = 0$$

which is for one k-particle interacting with two p-particles and is compatible with the following law of "kpp mixing of boosts"

$$N_{[k]} + e^{\ell k_0/2} N_{[p]} + e^{\ell k_0/2} e^{\ell p_0} N_{[p']}$$

This is easily verified as follows:

$$[N_{[k]} + e^{\ell k_0/2} N_{[p]} + e^{\ell k_0/2} e^{\ell p_0} N_{[p']}, k_0 + p_0 + p'_0] = k_1 + e^{\ell k_0/2} p_1 + e^{\ell k_0/2} e^{\ell p_0} p'_1 = 0$$

$$\begin{aligned}
&[N_{[k]} + e^{\ell k_0/2} N_{[p]} + e^{\ell k_0/2} e^{\ell p_0} N_{[p']}, k_1 + e^{\ell k_0/2} p_1 + e^{\ell k_0/2} e^{\ell p_0} p'_1] \\
&= \frac{\sinh(\ell k_0)}{\ell} + e^{\ell k_0} \left(\frac{e^{2\ell p_0} - 1}{2\ell} + \frac{\ell}{2} p_1^2 \right) + e^{\ell k_0} e^{2\ell p_0} \left(\frac{e^{2\ell p'_0} - 1}{2\ell} + \frac{\ell}{2} {p'_1}^2 \right) \\
&+ \frac{\ell}{2} e^{\ell k_0/2} k_1 p_1 + \frac{\ell}{2} e^{\ell k_0/2} e^{\ell p_0} k_1 p'_1 + \ell e^{\ell k_0} e^{\ell p_0} p_1 p'_1 \\
&= \frac{\ell}{2} \left(e^{\ell k_0/2} p_1 + e^{\ell k_0/2} e^{\ell p_0} p'_1 \right) \left(k_1 + e^{\ell k_0/2} p_1 + e^{\ell k_0/2} e^{\ell p_0} p'_1 \right) + \left(\frac{e^{\ell k_0 + 2\ell p_0 + 2\ell p'_0} - e^{-\ell k_0}}{2\ell} \right) = 0
\end{aligned}$$

Finally I also exhibit a consistently relativistic conservation law for interactions involving two k-particles and one p-particle:

$$k_0 + k'_0 + p_0 = 0 \, , e^{-\ell k'_0} e^{-\ell k_0/2} k_1 + e^{-\ell k'_0/2} k'_1 + p_1 = 0$$

which is compatible with the following law of "kkp mixing of boosts"

$$e^{-\ell k'_0} e^{-\ell k_0/2} N_{[k]} + e^{-\ell k'_0/2} N_{[k']} + N_{[p]}$$

as one can again easily verify:

$$[e^{-\ell k_0'} e^{-\ell k_0/2} N_{[k]} + e^{-\ell k_0'/2} N_{[k']} + N_{[p]}, k_0 + k_0' + p_0] = e^{-\ell k_0'} e^{-\ell k_0/2} k_1 + e^{-\ell k_0'/2} k_1' + p_1 = 0$$

$$\begin{aligned}
&[e^{-\ell k_0'} e^{-\ell k_0/2} N_{[k]} + e^{-\ell k_0'/2} N_{[k']} + N_{[p]}, e^{-\ell k_0'} e^{-\ell k_0/2} k_1 + e^{-\ell k_0'/2} k_1' + p_1] \\
&= e^{-2\ell k_0'} e^{-\ell k_0} \frac{\sinh(\ell k_0)}{\ell} + e^{-\ell k_0'} \frac{\sinh(\ell k_0')}{\ell} + \left(\frac{e^{2\ell p_0} - 1}{2\ell} + \frac{\ell}{2} p_1^2\right) - \frac{\ell}{2} e^{-2\ell k_0'} e^{-\ell k_0} k_1^2 - \frac{\ell}{2} e^{-\ell k_0'} k_1'^2 - \ell e^{-3\ell k_0'/2} e^{-\ell k_0/2} k_1 k_1' \\
&= \frac{\ell}{2} \left(p_1 - e^{-\ell k_0'} e^{-\ell k_0/2} k_1 - e^{-\ell k_0'/2} k_1'\right) \left(p_1 + e^{-\ell k_0'} e^{-\ell k_0/2} k_1 + e^{-\ell k_0'/2} k_1'\right) + \left(\frac{e^{2\ell p_0} - e^{-2\ell k_0 - 2\ell k_0'}}{2\ell}\right) = 0
\end{aligned}$$

10. Closing Remarks

In closing let me first return to the "foundational" significance of the study I have here reported. One can hardly imagine a more foundational question than the one I contemplated: Does relativistic invariance necessarily require us to assume universal relativistic properties for all particles?

I have here shown that, at least within the confines of relativistic kinematics and of the typically most significant tests of relativistic covariance, it appears to be possible to have relativistic theories that do not assign universal relativistic properties for all particles.

Narrowing the perspective to the context of the ongoing studies of DSR-relativistic proposals, the progress I here reported for the construction of scenarios with "non-universal" deformation of Lorentz symmetry (with particle-type-dependent deformations of relativistic kinematics) can be valuable for addressing some long-standing issues in DSR research, here mentioned in Section 1, and can also be the starting point for several further developments and further generalizations.

Some of these future studies could be directed toward the understanding of the associated relativity of spacetime locality. It is established that already for some "universal" DSR-deformation schemes spacetime locality becomes relative [33–35]. The case of "non-universal" DSR-deformation here introduced should have subtle implications also for the characterization of the relativity of spacetime locality. And it is established that for some schemes of "universal" DSR deformation it is possible to provide a formulation within the "relative-locality framework" [15,16], centered on the geometry of momentum space. It would therefore be interesting to attempt to generalize the relative-locality-framework formulation also to the case of the "non-universal" DSR deformations here introduced. But this raises some intriguing questions: Could then one describe the different types of particles on the same geometry of momentum space? or should one rather seek a formulation based on different momentum-space geometries for different particles? And in that case which sort of geometric requirement could codify the feature here formulated in terms of a "mixing composition law"?

One more intriguing question I want to mention concerns the types of different relativistic properties which can be compatible. The illustrative examples of "non-universal DSR deformations" I here analyzed only involved relatively mild differences of relativistic properties: I managed to "mix" particles governed by quantitatively very different deformations of relativistic kinematics, but all members of a class of related such deformations. Having established that this can be done consistently, one may now ask whether it is possible to "mix" in a single consistent relativistic framework particles with more profoundly different relativistic properties. Ideally one would like to establish some sort of theorem characterizing the types of different relativistic properties which can be made compatible in the sense I here introduced. Such a theorem appears to be extremely challenging, but even gaining some expertise on the basis of a few "trial-and-error exercises" might be valuable.

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19. Throughout this manuscript I write conservation laws at a process with conventions such that all momenta intervening in the process are incoming into the process, so that indeed a trivalent process would be characterized by a conservation law of the type $p \oplus_\ell p' \oplus_\ell p'' = 0$. The case of one (or two) of the momenta that is outgoing from the process, say the momentum p, is recovered by simply substituting for p the "antipode" of the momentum of that outgoing particle, with the antipode $\ominus p$ defined so that $(\ominus p) \oplus p = 0$. [For the composition law Equation (17) the antipode is such that $(\ominus_\ell p)_j = -p_j + \ell p_0 p_j$ and $(\ominus_\ell p)_0 = -p_0$.]
20. Notice that in this specific example the limit of $p \oplus_{\ell\star} k$ as $p \to 0$ is not k. So scenarios such as this require a profound distinguishability between a single-particle system and a multi-particle system, a feature which is plausible but surprising. I should stress that this feature is not a general aspect of my "mixing composition laws", and indeed in the example Equation (30) one does find that $p \oplus_{\ell\star} k \to k$ as $p \to 0$. Also notice that in this subsection I could have opted for adopting as mixing composition law the limit $\lambda \to 0$ of the composition law Equation (62) from the next section, and in that case the limit $p \to 0$ does give k. As a matter of fact I opted for showing here Equation (46) rather than the limit $\lambda \to 0$ of Equation (62), just so that the content of this subsection would not simply amount to the $\lambda \to 0$ of the content of the next section, thereby showing some evidence of the variety of possibilities that could be considered (including the possibility of having $p \oplus_{\ell\star} k \neq k$ in the limit $p \to 0$).
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48. Note that a conservation law of the type $p_0 + k_0 = 0, p_j + k_j + (\ell + \lambda)p_0 k_j/2 = 0$ is equivalent to $p_0 + k_0 = 0$, $p_j + k_j + \lambda p_0 k_j/2 - \ell k_0 p_j/2 = 0$ since (for $p_0 + k_0 = 0$ and working at leading order) one has that from $p_j + k_j + \frac{\ell+\lambda}{2} p_0 k_j = 0$ it follows that $(1 + \ell k_0/2)[p_j + k_j + \frac{\ell+\lambda}{2} p_0 k_j] = 0$ and in turn $p_j + k_j + \frac{\lambda}{2} p_0 k_j - \frac{\ell}{2} k_0 p_j = 0$.