

Article

Knots in Art

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Abstract: We analyze applications of knots and links in the Ancient art, beginning from Babylonian, Egyptian, Greek, Byzantine and Celtic art. Construction methods used in art are analyzed on the examples of Celtic art and ethnical art of Tchokwe people from Angola or Tamil art, where knots are constructed as mirror-curves. We propose different methods for generating knots and links based on geometric polyhedra, suitable for applications in architecture and sculpture.

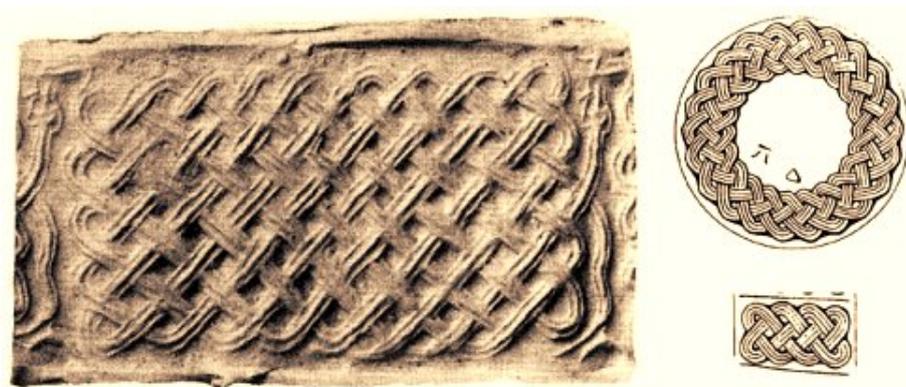
Keywords: knot; link; plait; mirror-curve; decorative knot; polyhedral knot

1. Knots in Ancient Art

Some of the oldest examples of monolinear curves and knots in rectangular grids associated with them were constructed using plates: Rectangular grids $RG[a, b]$ of dimensions $a, b \in N$, where a, b are coprime numbers. A snake eating its tail, the classical symbol of Ouroboros, the cylinder seal impression from Ur, Mesopotamia, ca. 2600–2500 B.C. showing a snake with interlacing coil placed in nine times five grid $RG[9, 5]$ (Figure 1) is an example of this construction. Similar examples with plates constructed as rectangular or circular grids are used as stamp seals from Anatolia, ca. 1700 B.C. We assume that

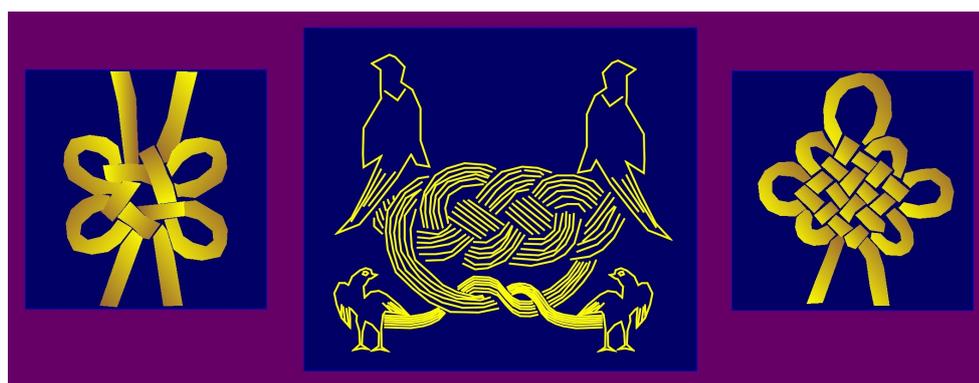
stamps, cylinders and seals with knots and links as their motifs appeared before proper writing is invented about 3500 B.C. [1].

Figure 1. (a) Cylinder seal, Ur, Mesopotamia, *ca.* 2600–2500 B.C.; (b) stamp seals, Anatolia, *ca.* 1700 B.C.



Due to the delicate nature of the medium, few examples of prehistoric Chinese knotting exist today. Some of the earliest evidence of knotting has been preserved on bronze vessels of the Warring States period (481–221 B.C.), Buddhist carvings of the Northern Dynasties period (317–581 A.D.) and on silk paintings during the Western Han period. Chinese knotting is a decorative handicraft art that began as a form of Chinese folk art in the Tang and Song Dynasty (960–1279 A.D.) in China. It was later popularized in the Ming). The art is also referred to as Chinese traditional decorative knots (Figure 2).

Figure 2. Chinese traditional decorative knots.



Some of the most beautiful legends related to knots appear in Greek mythology. The legend about the Gordian knot, associated with Alexander the Great, is often used as a metaphor for an intractable problem solved by a bold stroke (“cutting the Gordian knot”). Although nobody knows the type of the Gordian knot, the unknotting number, which is one of the simplest knot invariants but also very hard to compute, is also called “the Gordian number”.

The first study and description of the surgeon’s slings is written by a Greek physician named Heraklas, who lived during the first century A.D. In this essay Heraklas explains, giving step-by-step instructions, eighteen ways to tie orthopedic slings. A closure of each of them represents a knot or link (Figure 3).

Moreover, wonderful examples of Hercules knots (Figure 4) can be found in the Greek jewelry, beginning from the Minoan period. The reef knot was known to the ancient Greeks as the Hercules knot (*Herakleotikon hamma*), it was and is still used extensively in medicine as a binding knot. In his *Natural History* Pliny expresses the belief that wounds heal more quickly when bound with a Hercules knot.

Figure 3. Closure of the crossed noose that gives the knot 5_2 .

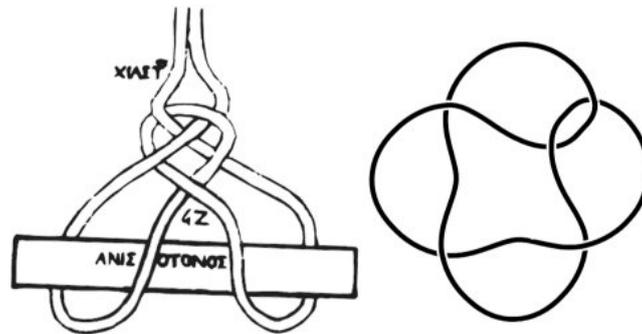
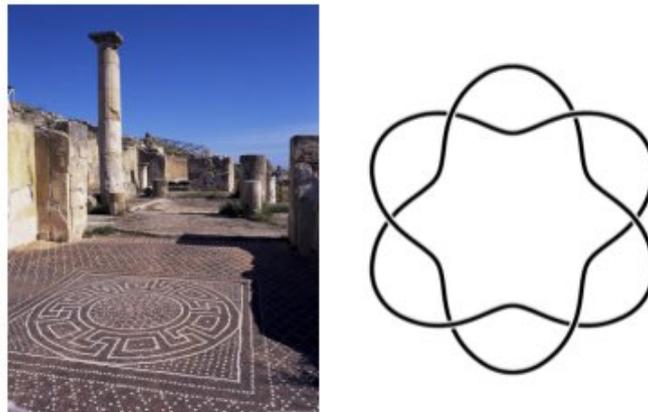


Figure 4. Hercules knot, Greek jewelry, Pontika (Ukraina), 300 B.C.



Beautiful examples of knots and links in art can be implicitly found in Greek mosaics. Decorations on Greek floor mosaics in the form of meanders are four-valent graphs that can be thought of as knot projections and easily transformed into the corresponding knots and links by adding the information which strand passes on top of the other. For example, the Greek mosaic (Figure 5a), viewed from knot theory point of view, represents a link 6_1^2 (Figure 5b).

Figure 5. Greek mosaic that can be transformed into the link 6_1^2 .



2. Mirror Curves

“The imitation of the three-dimensional arts of plaiting, weaving and basketry was the origin of interlaced and knotwork interlaced designs. There are few races that have not used it as a decoration of stone, wood and metal. Interlacing rosettes, friezes and ornaments are to be found in the art of most people surrounding the Mediterranean, the Black and Caspian Seas, Egyptians, Greeks, Romans, Byzantines, Moors, Persians, Turks, Arabs, Syrians, Hebrews and African tribes. Their highlights are Celtic interlacing knotworks, Islamic layered patterns and Moorish floor and wall decorations ” [2].

Celtic knots are one of the highlights of knot-art. Some researchers believe that the root of Celtic knot art is in knot designs in the 10th–11th century eastern mosaics patterns, especially Persian tiling. The very beginning of knotwork art has probably originated in mirror curves constructed from *plates*, which have been also recognized as the basis of all Celtic knotworks by the antiquarian J. Romilly Allen whose twenty years’ work is summarized in the book *Celtic Art in Pagan and Cristian Times* [3].

We give a brief outline of the mirror curves construction and details are given in Section 2.3.

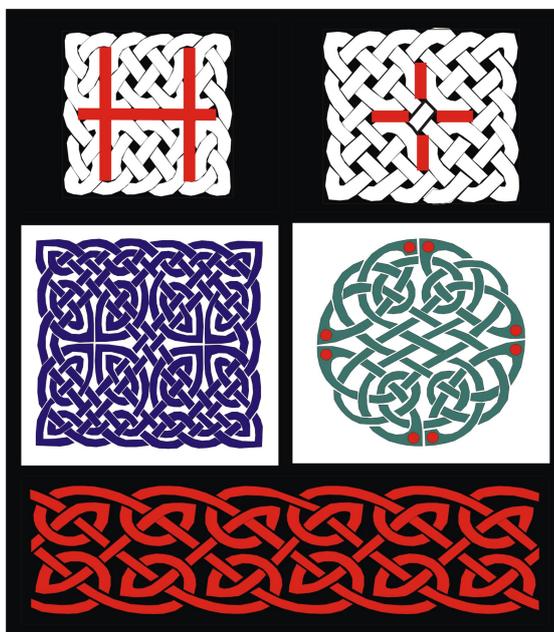
Basic construction uses a rectangular grid consisting of squares but can be generalized in two ways: To an arbitrary region of the plane consisting of an edge-to-edge tiling by polygons and by inserting internal two-sided mirrors. First, connect the midpoints of adjacent edges to obtain a 4-regular graph: Every vertex is incident to four edges, called *steps*. Every closed path in this graph, where each step appears only once, is called a *component*. A *mirror curve* is the set of all components. Since the graph is 4-valent, at each vertex we have three choices of edges to continue the path: To choose the left, middle, or right edge. If the middle edge is chosen the vertex is called a *crossing*. Every mirror curve can be converted into a knotwork design by introducing the relation “over-under”.

The name “mirror curves” can be justified by assuming that all of the outside edges of the region are mirrors, with the additional internal two-sided mirrors placed at the midpoints of all other edges of the polygons, colinear or perpendicular to the edge. For simplicity, assume that we have a rectangular grid consisting of squares, *i.e.*, plate, and that a ray of light is emitted from one edge-midpoint at an angle of 45° . It will travel through the grid and come to the starting point after a series of reflections. If the whole grid was not covered, we can repeat the process by starting at a different edge-midpoint, and continuing until the whole step graph is used, we trace a mirror curve.

The common geometrical construction principle of these designs, discovered by P. Gerdes, is the use of (two-sided) mirrors incident to the edges of a square, triangular or hexagonal regular plane tiling, or perpendicular to the edges in their midpoints [4,5]. In the ideal case, after the series of consecutive reflections, the ray of light reaches its beginning point, defining a single closed curve. In other cases, the result consists of several closed curves. For example, the following mirror-schemes (Figure 6) correspond to the Celtic designs from G. Bain's book *Celtic Art* [2].

The number of mirror curves in a design obtained from plates without internal mirrors is the greatest common divisor of the lengths of plates sides a and b , see Section 2.4. Hence, a single curve, *i.e.*, a knot, is obtained *iff* a, b are mutually prime numbers. It is interesting that all knots obtained from a single curve on a plate satisfy parametric equations using \cos functions known as Lissajous knots [6], and can be reparametrized to be viewed as billiard knots inside of a cube.

Figure 6. Celtic knot designs.



The infinite series of plates, obtained for an arbitrary $a \geq 3$ and $b = 2$, consists of rational knots and links, given by their Conway symbols [7] $313, 31213, 3121213, 312 \dots 213$ (Figure 7).

For an arbitrary $a \geq 3$ and $b = 3$ we obtain plates with polyhedral knots and links: For $a = 3$ we have 3-component link $8^*2 : 2 : 2 : 2$, for $a = 4$ the knot $1312^* : 20 :: 20.2.20$, *etc.* (Figure 8).

Figure 7. *KLs* obtained from $RG[a, 2]$ for $a = 3, 4, 5$.

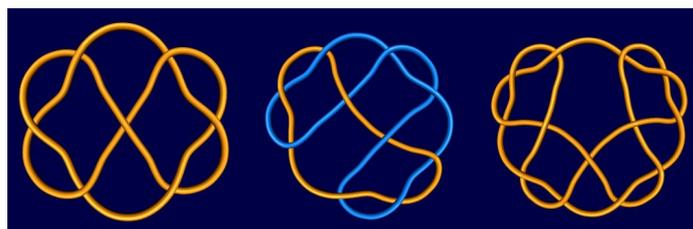
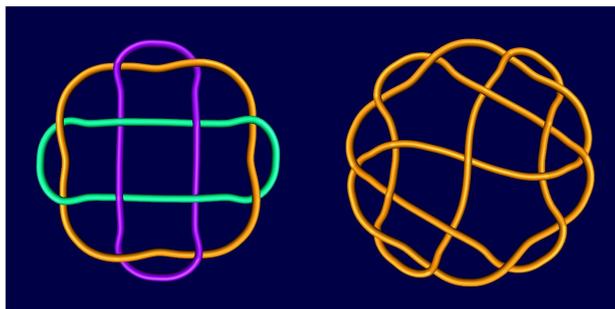


Figure 8. *KLs* obtained from $RG[a, 3]$ for $a = 3, 4$.

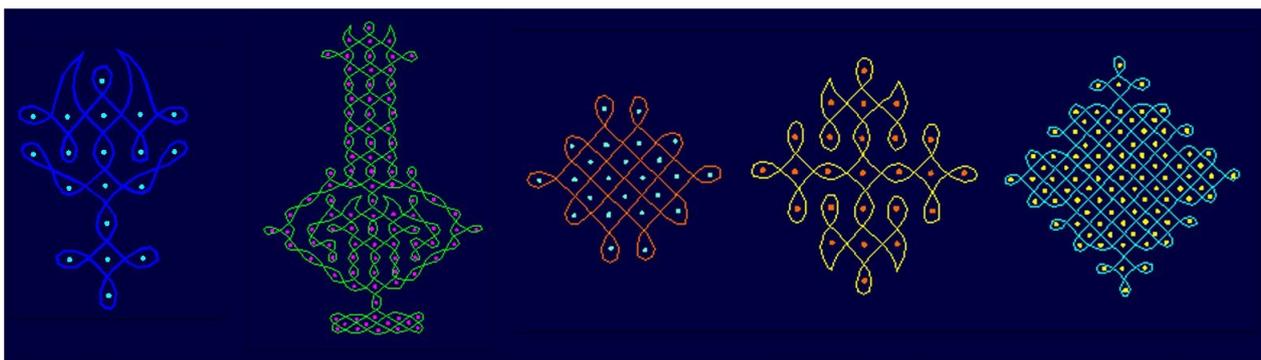


2.1. Tamil Threshold Designs

“During the harvest month of Margali (mid-December to mid-January), the Tamil women in South India used to draw designs in front of the thresholds of their houses every morning. Margali is the month in which all kinds of epidemics were supposed to occur. Their designs serve the purpose of appeasing the god Siva who presides over Margali. In order to prepare their drawings, the women sweep a small patch of about a yard square and sprinkle it with water or smear it with cow-dung. On the clean, damp surface they set out a rectangular reference frame of equidistant dots. Then the curve(s) forming the design is (are) made by holding rice-flour between the fingers and, by a slight movement of them, letting it fall out in a closed, smooth line, as the hand is moved in the desired directions. The curves are drawn in such a way that they surround the dots without touching them” [8].

The ideal design according to Tamil culture is composed of a single continuous line. Names given to designs formed of a single “never-ending” line are normally *pavitram*, meaning “ring” and *Brahma-mudi* or “Brahma’s knot” (Figure 9). The purpose of the *pavitram* is to scare giants, evil spirits, or devils away.

Figure 9. Tamil designs.



2.2. Tchokwe Sand Drawings

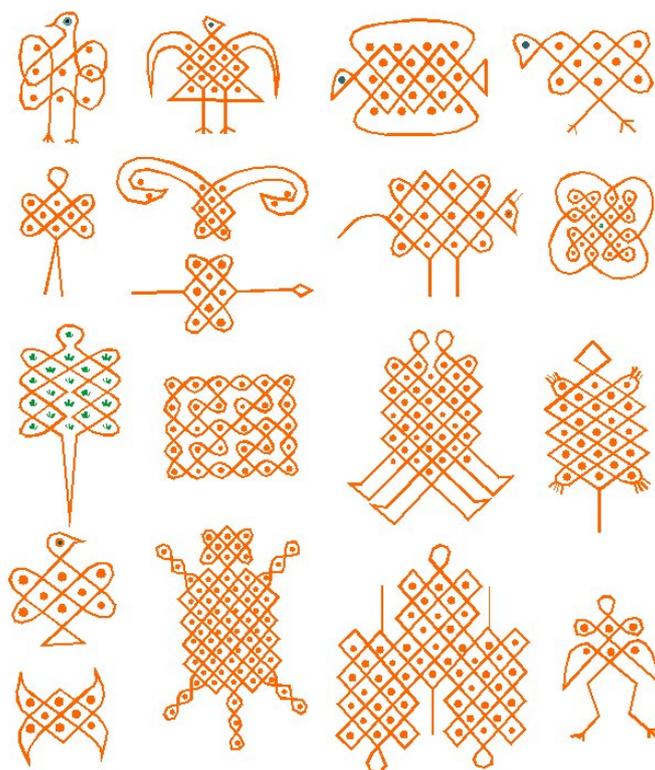
“The Tchokwe people of northeast Angola are well known for their beautiful decorative art. When they meet, they illustrate their conversations by drawings on the ground. Most of these drawings belong to a long tradition. They refer to proverbs, fables, games, riddles, *etc.* and play an important role in the transmission of knowledge from one generation to the other” [9].

“...Just like the Tamils of South India, the Tchokwe people invented a similar mnemonic device to facilitate the memorization of their standardized drawings. After cleaning and smoothing the ground,

they first set out with their fingertips an orthogonal net of equidistant points. The number of rows and columns depends on the motif to be represented. Applying their method, the Tchokwe drawing experts reduce the memorization of a whole design to that of mostly two numbers and a geometric algorithm. Most of their drawings display bilateral and/or rotational (90° or 180°) symmetries. The symmetry of their pictograms facilitates the execution of a drawing. This is important, as the drawings have to be executed smoothly and continuously. Any hesitation or stopping on the part of the drawer is interpreted by the audience as an imperfection and lack of knowledge, and assented with an ironic smile” [9].

Tchokwe sand drawings (Figure 10) called *sona* (singular: *lusona*) played an important role in transmitting knowledge and wisdom from one generation to the next. Young boys enjoyed making sand drawings with their fingers and hearing stories about them. They have learned how to make simple drawings and their meaning during the period of intensive schooling, the *mukanda* initiation rites. More complicated *sona* were only known by the story tellers, who were real *akwa kuta sona* (those who know how to draw), highly esteemed and forming a part of an elite in Tchokwe society [4].

Figure 10. Tchokwe sand drawings.



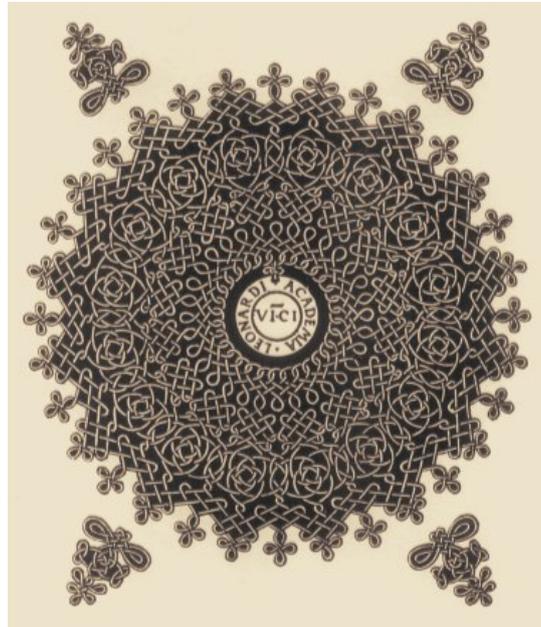
2.3. Construction of Mirror-Curves

“Leonardo spent much time in making a regular design of a series of knots so that the cord may be traced from one end to the other, the whole filling a round space...” [2].

Leonardo and Dürer, two of the greatest painters-mathematicians, were interested in constructing knot designs, closely related to mirror curves [2] (Figure 11). They knew and very effectively used the fact

that for a rectangular square grid $RG[a, b]$ of dimensions a, b , where a and b are relatively prime, mirror curve is always a single closed curve uniformly covering the rectangle.

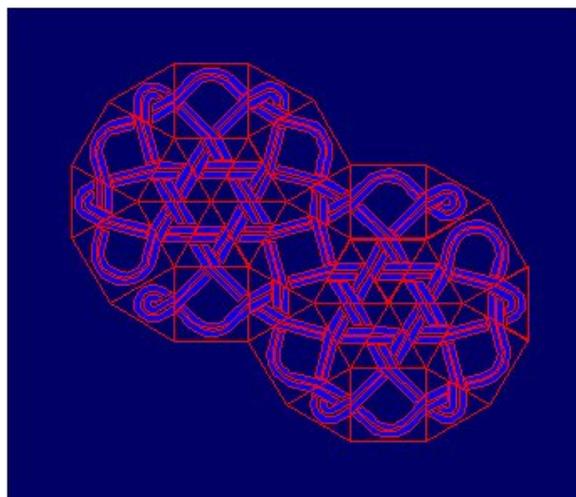
Figure 11. Leonardo's *Concatenation*.



The common geometrical construction principle of Celtic, Tchokwe, or Tamil knot designs, elaborated by P. Gerdes, is the use of (two-sided) mirrors incident to the edges of a square, triangular or hexagonal regular plane tiling, or perpendicular to the edges in their midpoints [4,5,8,9]. In the ideal case, after the series of consecutive reflections, the ray of light reaches its beginning point, defining a single closed curve. In other cases, the result consists of several closed curves. For example, the following mirror-schemes (Figure 6) correspond to the Celtic designs from G. Bain's book *Celtic Art* [2].

Gerdes' books [10,11] present examples and analysis of mirror curves in Ancient Egypt, Mesopotamia, India and African cultures, and [12,13] present an analysis of the relationship between mirror curves and matrices.

Moreover, there is one more beautiful geometrical property: Mirror curves can be obtained using only a few different prototiles. In particular, only three prototiles are sufficient for construction of all mirror curves with internal mirrors incident to the cell-edges of a regular triangular tiling, five for square, and 11 for hexagonal regular tiling (Figure 12) [14].

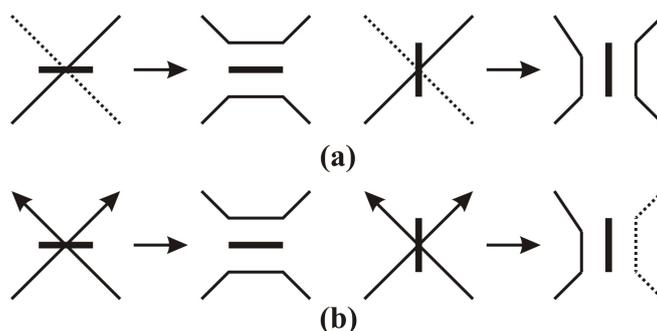
Figure 12. Knot design obtained from a uniform tiling.

Using the combinations of polygons from 11 uniform Archimedean tilings [15], or prototiles producing an impression of space structures and colored prototiles, we may obtain artistic interlacing patterns, examples of *modular design*: The use of a few initial elements (*modules*-prototiles) for creating an infinite collection of designs. The mirror curves obtained from Archimedean tilings resemble the optical phenomenon: Change in direction of a light ray which transfers from one to the other physical environment.

As we already stated, the ideal mirror curve was a monolinear one, consisting of a single curve and corresponding to a knot. Monolinear curves can be obtained from k curves, $k > 0$ covering a region of a plane tiling denoted by T by placing internal mirrors keeping in mind the effect it has on the curve. These moves are sometimes referred to as the “curve surgery”:

Mixed crossing Any mirror placed in a crossing point of two distinct curves connects them into one curve;

Self-crossing Depending on its position, a mirror placed into a self-crossing point of an (oriented) curve either preserves the number of curves, or breaks the curve in two (Figure 13).

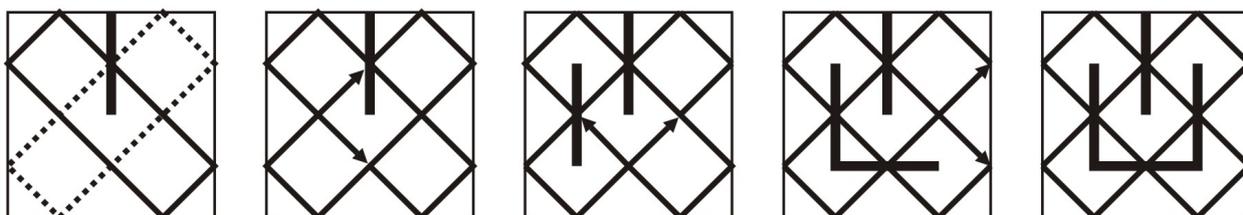
Figure 13. A mirror placed in a crossing point of (a) two different curves; (b) one curve.

Starting with k curves the minimal number of mirrors needed to transform them into a single curve is $k - 1$ and they need to be added in mixed crossing. This simple algorithm is obviously finite. If

$k = 1$, we have a single mirror curve, so no additional two-sided mirrors are needed. If the number of components is k ($k \geq 2$), we place first two-sided mirror in the crossing of two different curves, connect them and obtain $k - 1$ components. Continuing in the same way, a single mirror curve will be obtained after introducing $k - 1$ mirrors. Our game becomes more interesting if we allow adding mirrors in self-crossing points of the same component. This move can either preserve the number of curves or increase it by 1, so we can end up with a single or multi-component curve.

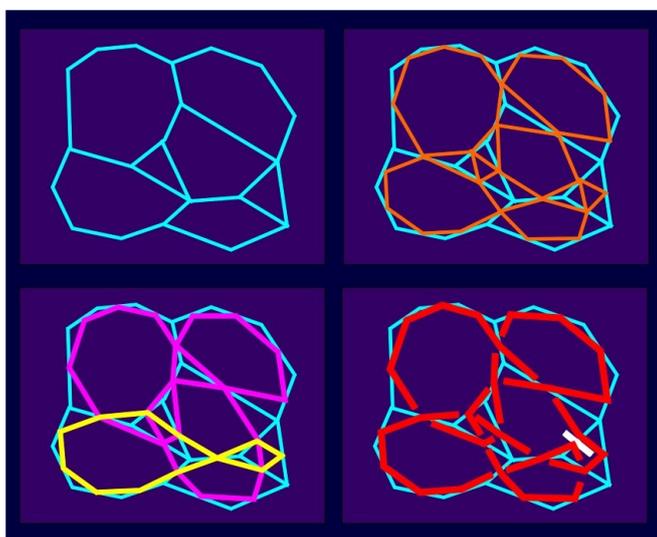
In the special case of a rectangular square grid $RG[a, b]$ of dimensions a, b , the initial number of curves obtained without internal mirrors is equal to the greatest common divisor $k = GCD(a, b)$, so in addition to the minimal number of mirrors needed to obtain a single curve, we know the maximal number $2ab - a - b$. In general, we can use these rules to modify any monolinear mirror curve without increasing the number of components (Figure 14).

Figure 14. The successive introduction of internal mirrors in the $RG[2, 2]$ that preserves a single curve.



Notice that the construction of mirror curves is independent from the metric properties or the geometry of the surface, so the same construction principle can be applied to any edge-to-edge tiling of an arbitrary surface, say sphere or a hyperbolic plane. Using the rules for adding two-sided mirrors, any mirror curve can be converted in a single mirror curve in a finite number of steps (Figure 15), as we have described above for plane tilings.

Figure 15. Construction of a single mirror curve from the tiling (a) by connecting edge mid-points (b); tracing components (c) and introducing a mirror (d).



2.4. Combinations of Mirror-Curves

Let us now describe four general rules for combining plate designs and/or mirror curves. The first three rules are given by P. Gerdes [5], and the fourth is proposed by S. Jablan. We will restrict our attention to mirror curves obtained from polyominoes with congruent square cells.

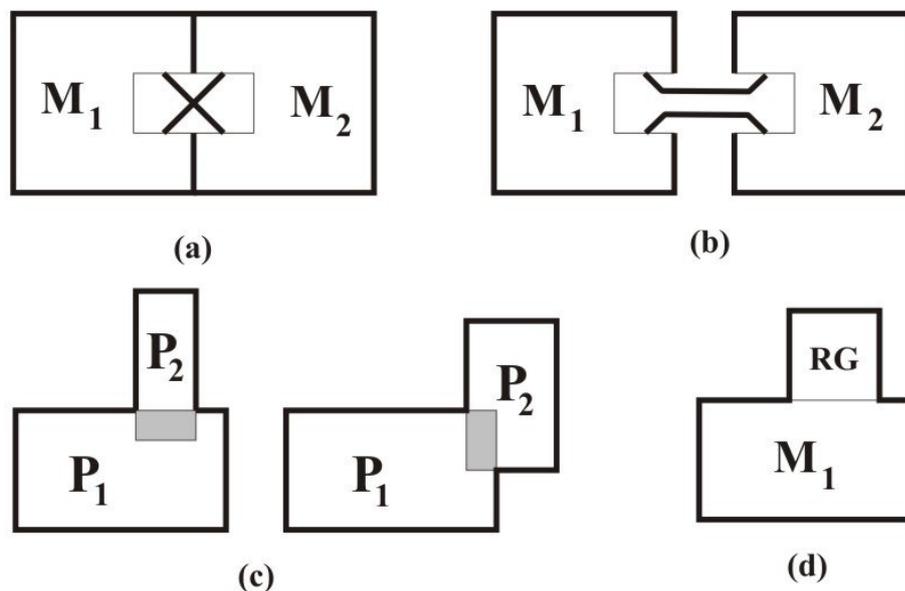
Construction rules:

1. The *first rule* describes how to combine two mirror curves that share one edge of an open cell on their borders (Figure 16a). Such a composition corresponds to the direct product of KL s, and it was probably one of the most exploited constructions in knotwork art. For given mirror curves M_1 and M_2 , we will call this kind of direct product \times -*direct product* and denote it by $M_1 \times M_2$. If we combine two mirror curves in this way, first with c_1 , and the other with c_2 components, the result is a new mirror curve with $c_1 + c_2 - 1$ components. Hence, the \times -direct product of two 1-component mirror curves is a new 1-component mirror curve. This idea was used, for example, in the Tchokwe designs and in many Celtic friezes.

As a particular application of the first rule, we can add a single square to the border of any monilinear mirror curve. This transformation corresponds to adding an external loop to a KL diagram. It does not change the number of components and can be repeated, since it has a decorative function in knotwork art. For example, the Tamil (unknot) design from Figure 9a is created by a series of external loop additions, beginning from the $RG[1, 1]$; the knot design from Figure 9b by adding loops to the $RG[4, 3]$; and the knot design from Figure 9c by adding loops to the $RG[5, 3]$. The same construction is used for Tchokwe designs (Figure 10a).

2. The *second rule* is the one defining the direct sum $K_1 \# K_2$ in knot theory (Figure 16b) (Notice that the first and second rule, *i.e.*, \times - and \parallel -product of mirror curves both correspond to the direct product of knots $K_1 \# K_2$, but the first with a nugatory crossing. Knots or links obtained by using \times - and \parallel -product of mirror curves are ambient isotopic to $K_1 \# K_2$, but not mutually isomorphic as knot or link diagrams.) In the language of mirror curves M_1 and M_2 , it means that we cut one external edge of each mirror-curve M_1 and M_2 , and reconnect them again to obtain a new mirror-curve that will be denoted by $M_1 \parallel M_2$.
3. The *third rule* is restricted to plate designs: Two monilinear plate designs whose overlap consists of exactly two cells will give a new monilinear plate design. The schematic interpretation of the third rule is given in Figure 16c.
4. *Adding* plate design P_1 to plate design P_2 is an edge-to-edge identification of their border cells belonging to rectilinear borders (Figure 16d). In the same way, we can add a plate design P_1 to some mirror curve M placed in some polyomino. This operation can be generalized to adding plate designs to polyominoes.

The fourth rule tells us that adding any design in a grid $RG[a, b]$, such that $b|a$ to any monilinear mirror curve M (or monilinear plate design P_2) along the edge b , produces a monilinear design (Figure 16d). In particular, any square RG added to a monilinear design gives a new monilinear design.

Figure 16. Rules for composing plate designs and mirror curves.

Moreover, adding a mirror curve M_2 to a monilinear mirror curve M_1 so that every curve contact point along edge b of the polyomino in which M_2 is placed belongs to a different component of M_2 . The new mirror curve $M_1 + M_2$ will produce a monilinear curve.

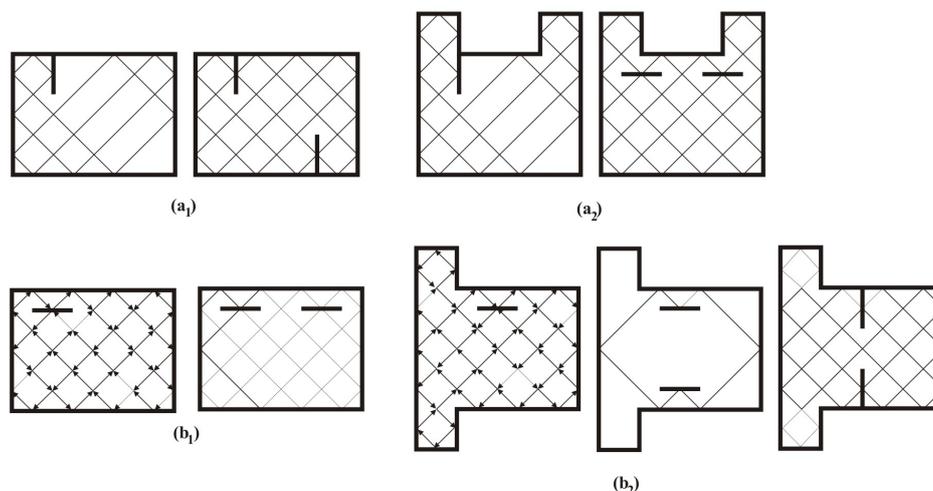
These four rules are sufficient for creating monilinear plate designs and extend the monilinearity from RG s to plate designs. Additional sophistication and variety comes from adding internal mirrors, illustrated in Figure 13. Symmetric mirror curves prevail in the knotwork art since symmetry is regarded as desirable visual property. This means that most of the mirror arrangements are not aesthetically pleasing: For example, only 8 out of 52 two-mirror arrangements from $RG[6, 3]$ are symmetrical.

We propose the following method for constructing symmetric mirror curves from a symmetric monilinear plate design P . First place an internal mirror in some crossing A of P and trace an oriented mirror curve M . Now we have two possibilities:

1. if our mirror curve M does not cover P completely, choose a mixed crossing point on M , symmetric to A and put a mirror symmetric to the mirror in A (Figure 17a₁). If a symmetric point with this property does not exist, rotate the mirror in A for 90° around its midpoint and then place the mirror symmetric to it (Figure 17a₂);
2. if P is completely covered by M , place a new mirror symmetric to the mirror in A (Figure 17b₁). If monilinearity is destroyed, rotate the mirror in A for 90° around its midpoint and then place the mirror symmetric to it (Figure 17b₂).

This approach is used to explain the construction of different mirror curves occurring in Tamil, Tchokwe and Celtic knotworks. We already explained and illustrated knotwork designs that obtained from a single monilinear plate design RG by adding a series of external loops, as well as designs obtained as direct products of monilinear plate designs (Rule 1, denoted by \times).

Figure 17. Algorithm for creating monilinear symmetric mirror curves.

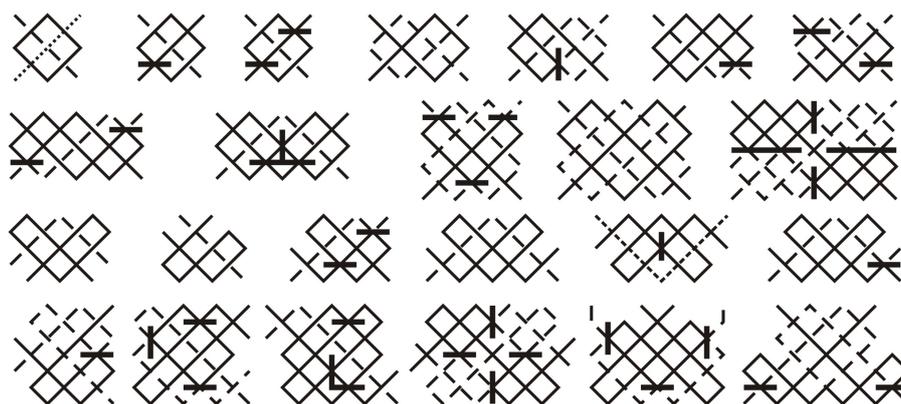


The first rule, \times -direct product, is very frequently used in Celtic knotwork art for the construction of friezes (or bordures), as well as the \parallel -direct product (Rule 2). Both of them are the standard tools for obtaining translational repetitive structures: Friezes or even plane symmetry groups.

Applying the second rule connects two monilinear *RGs* at their corners, *i.e.*, it is the \parallel -direct sum of the corresponding knots. Similar approach would be using (more or less) “open” plate designs *RGs* and the \parallel -direct sum of the corresponding mirror curves. Although resulting composite knots and links will be the same, obtained patterns will be visually very different.

In order to analyze Celtic knotworks based on \parallel -direct product first we need to insert some internal mirrors perpendicular to the edges in basic (monilinear) *RGs*, in order to obtain parts or “tangles” of Celtic knotworks with an appropriate placement of incoming and outgoing strands. The possible choices for their positions are two top (or bottom) corners of an elementary *RG*, two diagonal (ascending or descending) corners, or all four corners forming a tangle. In the first case we place internal mirrors perpendicular to “vertical” and “horizontal” edges of border cells, forming an L-shape form. Furthermore, cutting the long edge(s) of the design and reconnecting them, we obtain different frieze designs (direct products of basic *KLs*) with incoming and outgoing strands appropriately placed (Figure 18).

Figure 18. Celtic tangles.



Another possibility is creating “tangles” from *RGs* (Figure 18) and composing them into chains (or closed circles) (Figures 19 and 20). In the case of Tchokwe sand drawings a similar strategy was used in order to obtain “open” *RGs* that can be further composed by \parallel -direct sum or \times -direct product (Figure 10b) to form larger knotworks.

Figure 19. Celtic knots, friezes and plane knotwork ornaments.

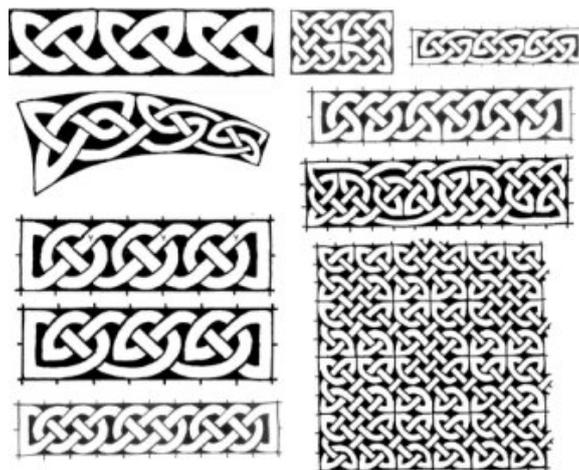
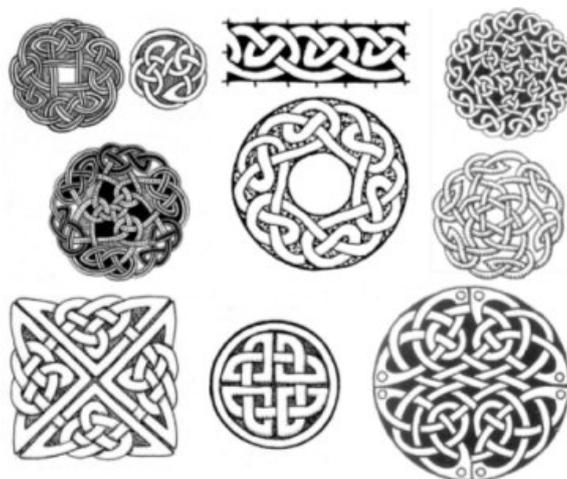


Figure 20. Celtic circle and square knot designs.



The third rule was one of the favorite rules in the construction of Tchokwe sand drawings. A whole series of “social” monolinear plate designs representing a leopard with cubs, a design called *kambava wamulivwe* that represents an animal called *kambava* that died inside a rock (Figure 21b), or *lusona* drawing called *tambwe* that represents a lion (Figure 21c) are composed in this way.

The fourth rule offers the highest degree of freedom with respect to the shape of the plate, often producing symmetric plates in knotwork art (Figure 22). Hence, we can construct a perfect monolinear curves in any shape obtained by adding different plates.

Figure 21. (a) \parallel -direct product in the Tchokwe design; (b,c) the application of Rule 3 in Tchokwe designs.

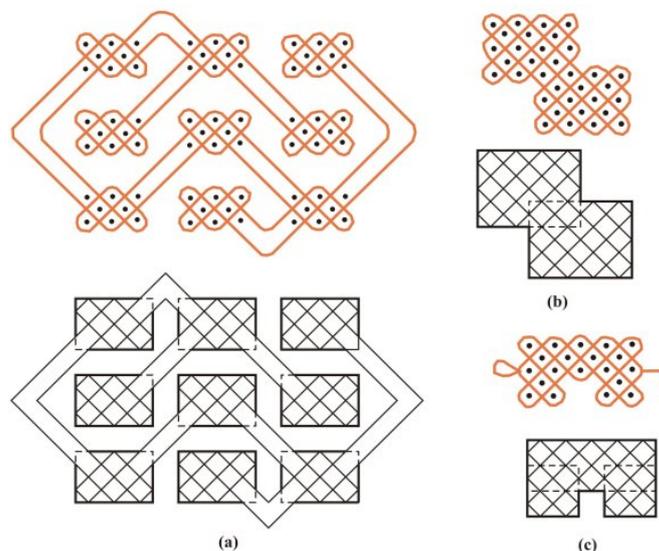
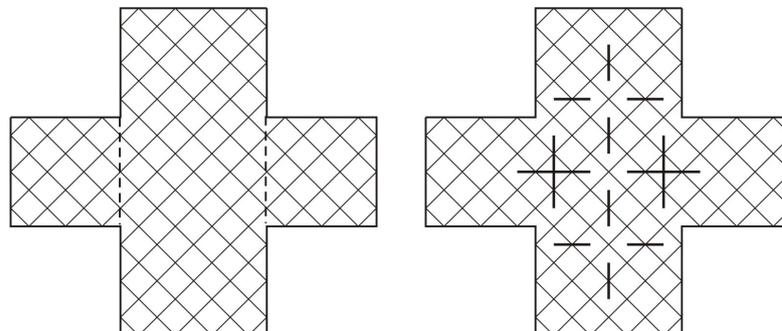


Figure 22. Derivation of Celtic monolinear cross knot design from plate design obtained using the Rule 4.



Creating a variety of monolinear plate designs opens the door to artistic creativity and play: There is a huge number of ways for introducing internal edge-incident and edge-perpendicular mirrors while preserving monolinearity (Figures 23–26).

Figure 23. Celtic monolinear knot cross design.

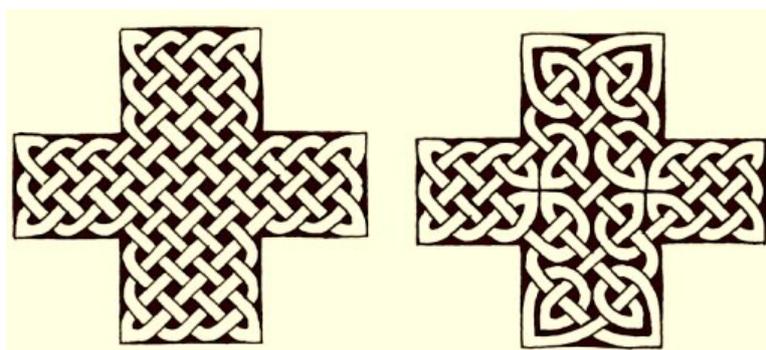


Figure 24. Celtic mirror curves.

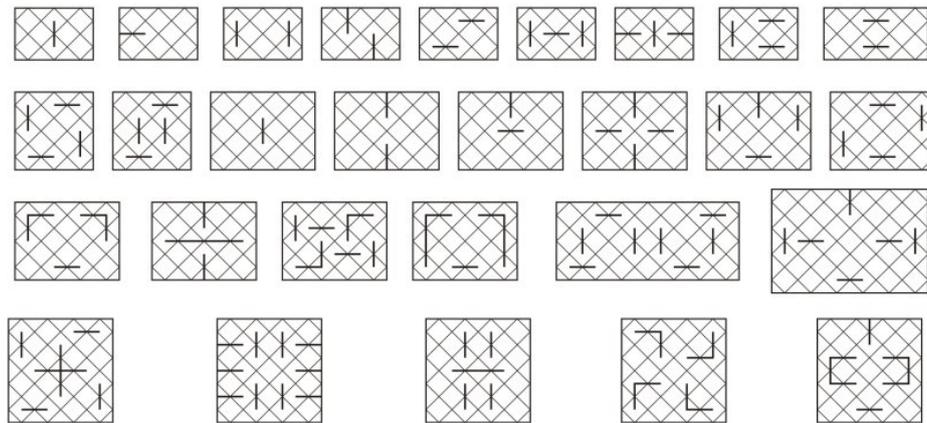


Figure 25. Celtic monolinear design with broken symmetry.

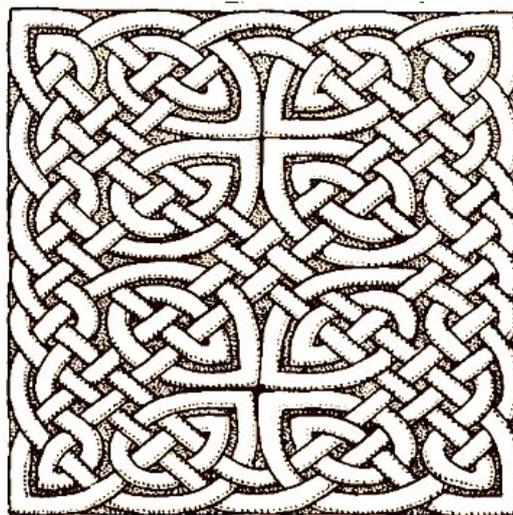
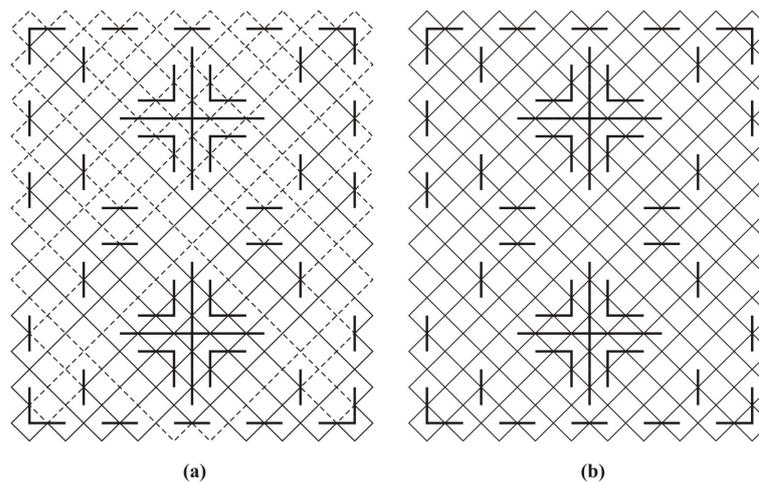


Figure 26. Construction of Celtic monolinear knot design (b) by breaking the symmetry of the two-component symmetric design (a).



Together with the remarkable example of a monolinear cross knot design (Figure 23), interesting examples of a complex monolinear design are shown in Figures 25 and 26. Because the symmetric version of the same design (Figure 26a) is a two-component knot design, the Celtic master modified it into an almost symmetric monolinear design by breaking symmetry (Figure 26b).

3. Symmetry and Classification of Knotwork Designs

P. Cromwell [16] used symmetry of mirror curves for the classification of the Celtic frieze designs, and P. Gerdes [8] for the reconstruction of Tamil designs. At the first glance it seems that symmetry is the mathematical basis for the construction and possible classification of perfect curves [8,14,16], but the existence of asymmetric curves asks for alternative approaches.

If we treat them as purely geometric object we can impose the following equality relation:

Two mirror curves are equal *iff* there is a similarity transforming one into the other.

In other words, if one mirror curve can be obtained from the other by a combined action of proportionality and isometry, they are considered to be the same. This relation can be studied on the level of mirror arrangements, in a very similar way.

A transformation S of the Euclidean n -dimensional space E^n is called *isometry* if for every two points X, Y and their images $X' = S(X), Y' = S(Y)$ holds $XY \cong X'Y'$.

A *figure* f is any non-empty subset of points of space.

A figure is called *invariant* with regard to a transformation S if $S(f) = f$, and S is called a symmetry of f .

Symmetries of a figure f form a group called the symmetry group of f and denoted G_f (see, e.g., [15,17,18]).

Isometric symmetry groups of the space E^n can be classified according to a sequence of maximal proper (sub)spaces invariant with respect to the action of transformations of the groups in question. Symmetry groups of friezes G_{21} , bands G_{321} , plane ornaments G_2 , and layers G_{32} can be used for the classification of knot-work patterns. Isometric symmetry groups will be denoted according to the *crystallographic notation* (or *Hermann and Maugin notation*).

There exist exactly 7 symmetry groups of friezes, 31 symmetry groups of bands, 17 symmetry groups of plane ornaments, and 80 symmetry groups of layers (see, e.g., [15,17–20]).

The *fundamental region* of a symmetry group of an object or pattern is the smallest part of the pattern, which, based on the symmetry, determines the whole object or pattern.

In all symmetry-oriented classifications of interlaced patterns, *i.e.*, infinite knotwork patterns (e.g., [16] or in [15]), symmetry is used as the only criterion for the classification. Linear knotwork patterns are classified according to 7 symmetry groups of friezes, or 31 symmetry groups of bands [15,19,21] without taking into account their topological or knot-theoretical properties. In the same way, plane symmetry patterns are classified according to 17 symmetry groups of ornaments or 80 symmetry groups of layers. In all these cases we have an asymmetric fundamental region multiplied by symmetries belonging to the symmetry group, without taking into consideration that the fundamental region can be any asymmetric tangle with its particular knot-theoretical properties. For example, according to the symmetry oriented classification, two bands shown in Figure 27 will be considered as

equivalent because their symmetry group is $p1a1$, in spite of the fact that the first is based on the tangle $2\ 1$ giving as the numerator closure Hopf link 2_1^2 (2), and the other is the direct product of knots $2\ 1\ 1\ 2$.

Figure 27. Celtic friezes with the same symmetry group of bands $p1a1$.



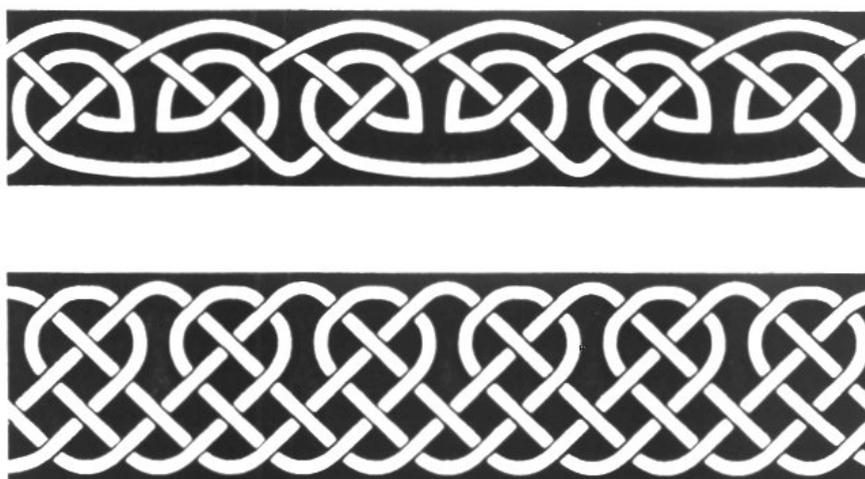
For the classification of infinite symmetric interlaced patterns we propose the following two criteria:

1. isometric symmetry group of the pattern;
2. tangle inscribed in the fundamental region.

These criteria are not always sufficient, and we also need to consider other knot-theoretical properties such as whether a pattern represents prime or composite KL arrangement.

As in the case of an n -tangle ($n = 2, 3, \dots$) in general, we may consider all possible closures of tangles contained in the fundamental region as a base for further classification. For example, by joining left and right ends of the tangles (Figure 28) that are the basic elements of two friezes with the same symmetry $p121$ we obtain links $3\#2$ and $2\ 1\ 2, 2, 2+$, respectively.

Figure 28. Celtic friezes with the same symmetry group of bands $p121$.



This classification, proposed by S. Jablan and Lj. Radović in 2001, is similar to the approach proposed earlier by I. Emery [22].

4. Knot Mosaics and Mirror Curves

Modularity is very effectively and widely used in the history of ornamental art [18]. It is a manifestation of the principle of economy or simply the optimization problem of how to minimize the number of basic elements (modules) while maximizing the number of different ornaments (structures) obtained by their recombination. A typical example of modular elements are prototiles for different tilings, e.g., a Truchet tile or “kites” and “darts” as the basic elements of Penrose tilings. A lesser known fact is that knot tilings are also modular: Five types of basic elements in a regular square grid are sufficient to construct shadows of all possible knots and links. A set of such elements named *KnotTiles* (Figure 29) is proposed by the first author [23], and a similar set is produced by R. Fathauer [24]. The second set of elements is used by S. Lomonaco and L. Kauffman to construct knot mosaics and discuss their connection with quantum computing [25]. The Lomonaco–Kauffman Conjecture that knot mosaics are equivalent to the tame knot theory is proved by T. Kyria [26] (Figures 30 and 31). In his book *Knots: Mathematics with a Twist* [27] A. Sossinsky noticed that smoothings of crossings, effectively used in skein relations for computing knot polynomials and for obtaining Kauffman states of knots and links, are nothing else than the placements of two-sided mirrors in mirror-curves, used by Celtic knot masters to construct knots decorating their burial stones, menhirs, or illuminations in manuscripts. In *Mirror-curves and Knot Mosaics* [28] we show the equivalence of knot mosaics, mirror curves, and grid (arc) representations of knots and links. Moreover, we introduce codes for mirror-curves suitable for computer applications, and use them for computation the Kauffman bracket polynomial and L-polynomial of knots and links.

Figure 29. *KnotTiles* by S. Jablan.

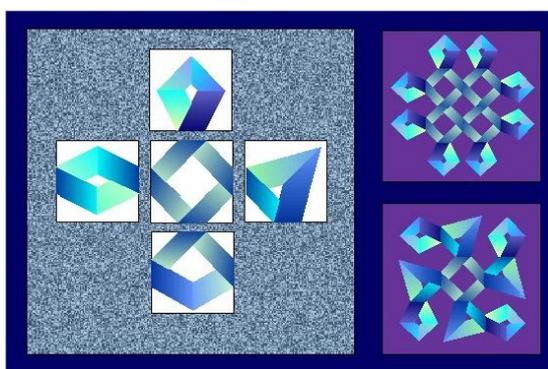


Figure 30. Figure-eight knot and Borromean rings as knot mosaics [25].

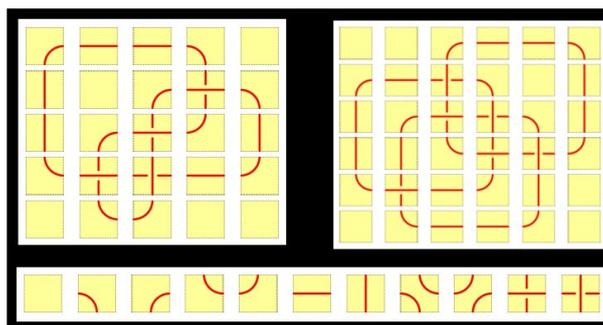


Figure 31. Figure-eight knot and Borromean rings from [25] represented as mirror-curves[28].

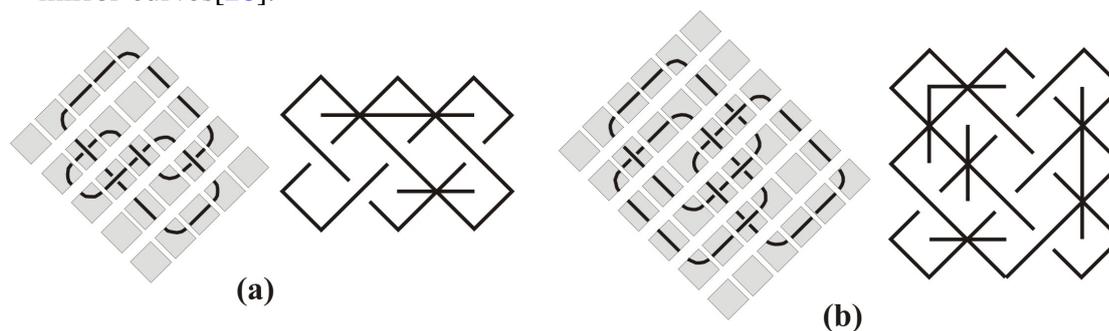
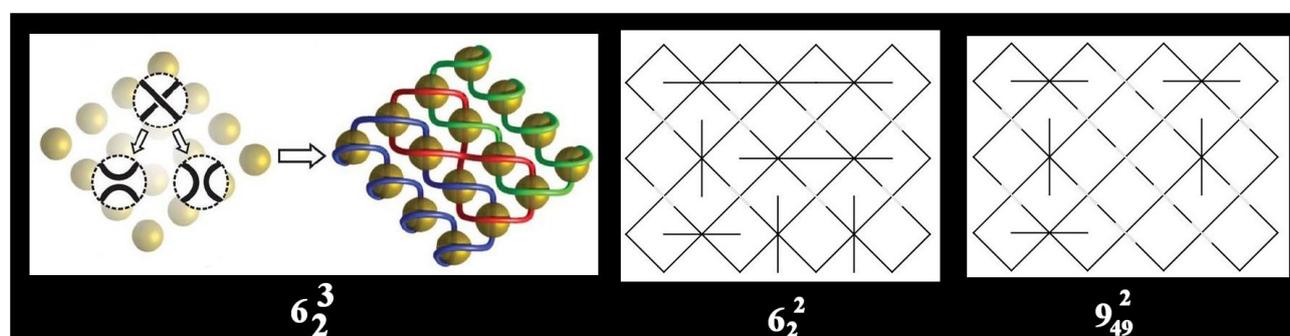


Figure 32. Borromean rings 6_2^3 synthesized in chiral nematic colloids [29] and missing links 6_2^2 and 9_{49}^2 .



The scientific relevance of mirror curves has been most recently confirmed in physics paper *Reconfigurable Knots and Links in Chiral Nematic Colloids* [29] (Figure 32). The authors describe a method for configuring so-called Saturn rings (single defect loops around colloid particles) in nematic colloids using laser tweezers. This control over the configurations of the Saturn rings allows them to classify all nematic braids in a $p \times q$ colloidal array, and all knots and links obtained in the 4×4 particle array. Knots and links obtained in this way represent a special class of mirror-curves, containing only positive crossing that correspond to the particles. The authors have synthesized several examples like the Hopf link, trefoil and the Borromean rings, but then managed to distinguish almost 40 different knot and link types (with the links 6_2^2 and 9_{49}^2 omitted) supported in the 4×4 . They conclude that such a large diversity of topological objects suggests that it is possible to design any knot or link on a sufficiently large colloidal array. In the attempt to extend these results we generate all knots and links from arrays of dimensions $p \times q$ for $p \leq 6$, $q \leq 6$ satisfying specific conditions mentioned above. We distinguish more than 1000 of different knot and link types. However, we are still not able to prove that all knots and links can be realized as the mirror-curves satisfying the mentioned specific requirements on crossings and matrix columns, although we know that mirror-curves are equivalent to the tame knot theory, *i.e.*, all knots and links can be realized as mirror-curves in a sufficiently large grid.

5. Knots in Modern Sculpture and Architecture

Although knots are usually viewed as one-dimensional objects in 3-dimensional space they can also be viewed on a two-dimensional surface. In particular, to every knot we can assign a special kind of surface

whose boundary is the knot itself, called the Seifert surface. Knots, links, self-avoiding curves derived from them, and Seifert surfaces are very successfully used in sculpture beginning from 1930s, e.g., in the works by Naum Gabo, John Robinson, Charles O. Perry, Carlo Sequin, and Brent Collins (Figure 33). In the contemporary art typical examples are sculptures by Bathsheba Grossman [30] created by the program Seifert View by J.J. van Wijk [31], or sculptures by A. Bulatov [32].

Figure 33. Sculptures by Charles O. Perry, John Robinson, and Brent Collins.



Indonesian weaving and Tamari balls inspired the original tensegrity researchers B. Fuller and K. Snelson to use knots and links in architecture. Their approach became more popular during the last decades, with the use of light-weight materials, tensegrity and computer design tools (CAD/CAM). Some recent architectural projects, like Water cube in Beijing, based on the Waire–Phela structure that offers the better solution of the Kelvin conjecture about optimal partition of the space into cells of equal volume with the least possible area of surface between them (*i.e.*, what is the most efficient soap bubble foam), or UNStudio Möbius house and Mercedes-Benz Museum, Stuttgart, inspired by a Möbius strip and interpretation of a trefoil knot as a boundary of its Seifert surface opened up the possibility to think about the other worlds, different from Cartesian grids, polyhedra, or uniformly spaced grid systems that dominated architecture for centuries [33]. These projects were initiated using ideas from topology, minimal surfaces, knots, links, or non-uniform space tessellations.

From mathematical point of views, diagrams representing knots or links are planar 4-valent graphs embedded in a sphere or plane with extra information determining the over-under relation of the strands. In architecture, such diagrams on a sphere can be easily transformed, by choosing a basic face, to the semi-spherical constructions of the form suitable for the construction of flexible roofs or cupolas. Embedding 4-valent graphs on non-planar surfaces different than spheres is an interesting mathematical problem with a huge potential for applications in architecture.

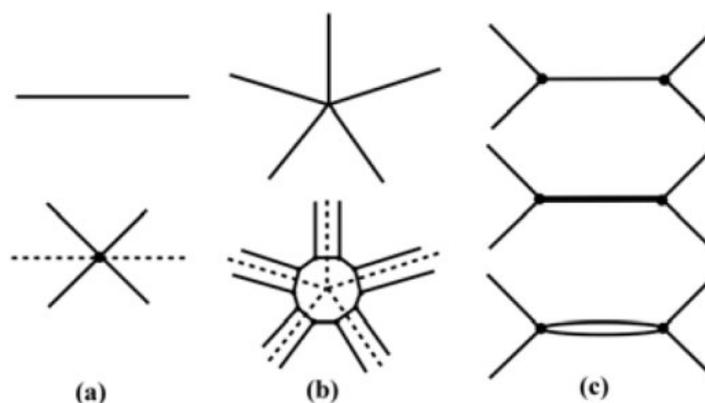
5.1. Polyhedral Knots and Links

Because of their regularity and symmetry, polyhedra attracted the attention of mathematicians, natural scientists, and architects for centuries. We propose several constructions of knots and links originating from regular, uniform (Archimedean), and other polyhedra. Knots and links obtained in this way

preserve important symmetry properties of the original polyhedron and represent the basis for the construction of polyhedral knotted structures from nano- to the macro-level.

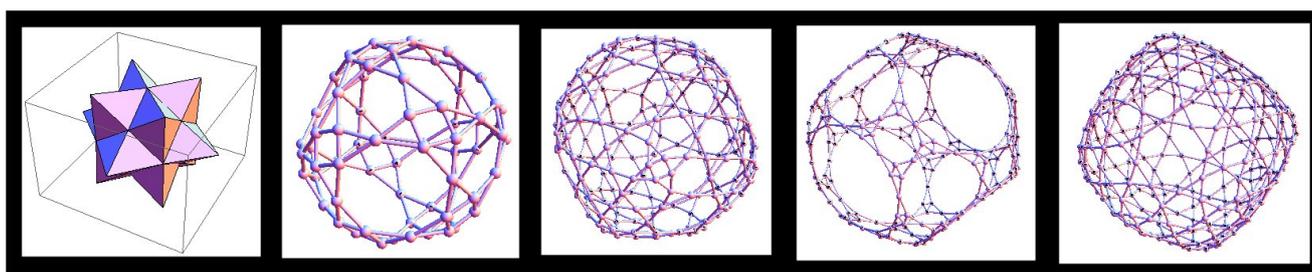
The term “polyhedral links” stands for knots and links derived from different geometric polyhedra by the mid-edge construction (Figure 34a) or cross-curve and double-line covering (Figure 34b), as well as the alternating source links derived from 3-valent geometric polyhedra by edge doubling (Figure 34c).

Figure 34. (a) Mid-edge construction; (b) cross-curve and double-line covering construction; (c) edge doubling construction. Broken lines denote edges of the original graph deleted during the construction.



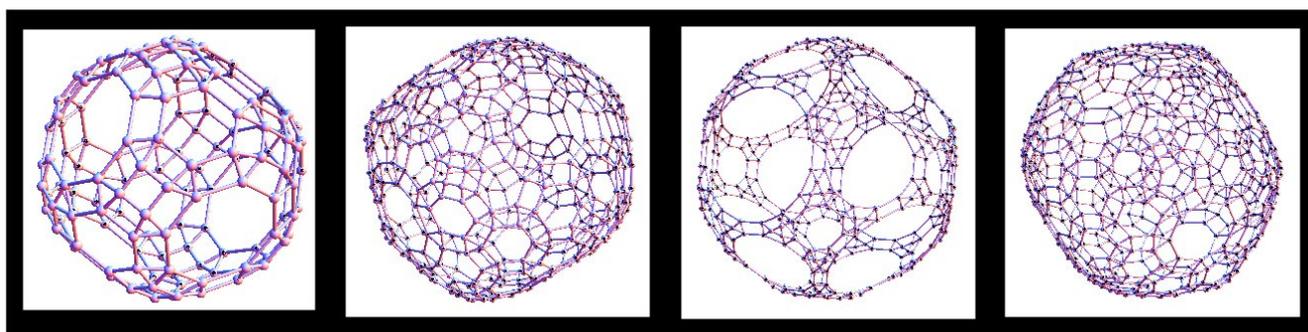
The middle graph $M(G)$ of a polyhedral graph G , is obtained by connecting the mid-edge points of G (Figure 35a) belonging to the adjacent edges of G . The resulting $M(G)$ is always a 4-valent graph, *i.e.*, a basic polyhedron. On the other hand, every at least 3-vertex connected basic polyhedron is the middle graph of some geometrical polyhedron. Let D_L be a link diagram obtained from $M(G)$ by turning 4-valent vertices into over-crossings and under-crossings in an alternating manner. Notice that the initial graph G is the underlying graph of the link diagram D_L , *i.e.*, $G(D_L) = G$. Hence the middle graph construction and turning a diagram into a graph by forgetting the over-under information are in some sense mutually inverse or dual. If the original graph G does not contain digons as the faces, the same holds for its middle graph.

Figure 35. Escher’s solid from the *Mathematica* data base of polyhedra [34] and basic polyhedra obtained from it and its transforms (truncated, stellated and godesated Escher solid) by mid-edge construction.



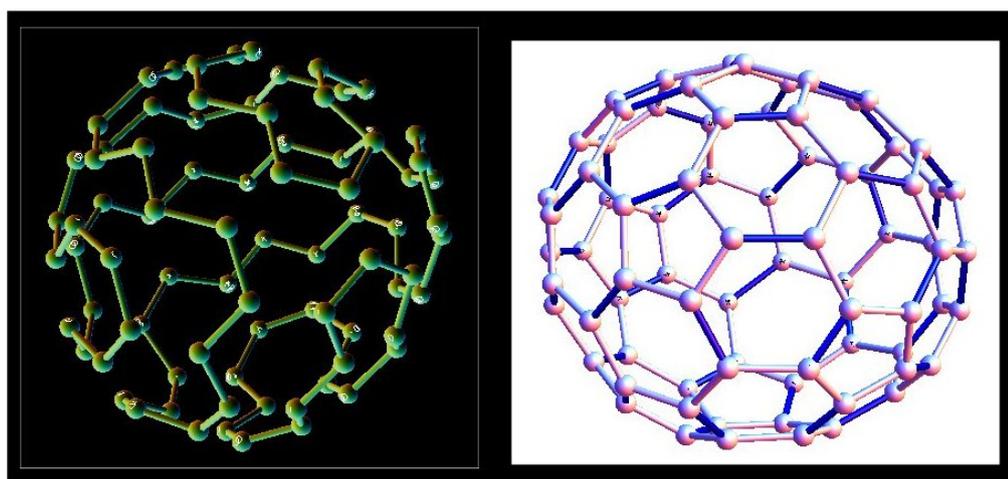
A new approach to the understanding the construction of polyhedral links has been developed in the papers [35,36]. These new methods involve the “cross-curve and double-line covering” construction based on the Platonic and Archimedean solids (Figure 36). The cross-curve and edge doubling construction are needed in order to obtain 4-valent graphs. The cross-curve construction consist of replacing n -valent vertices of a polyhedral graph by n -cross-curves, and then joining the adjacent loose ends to obtain a double-line covering.

Figure 36. Basic polyhedra obtained from Escher’s solid and its transforms by cross-curve and double-line covering.



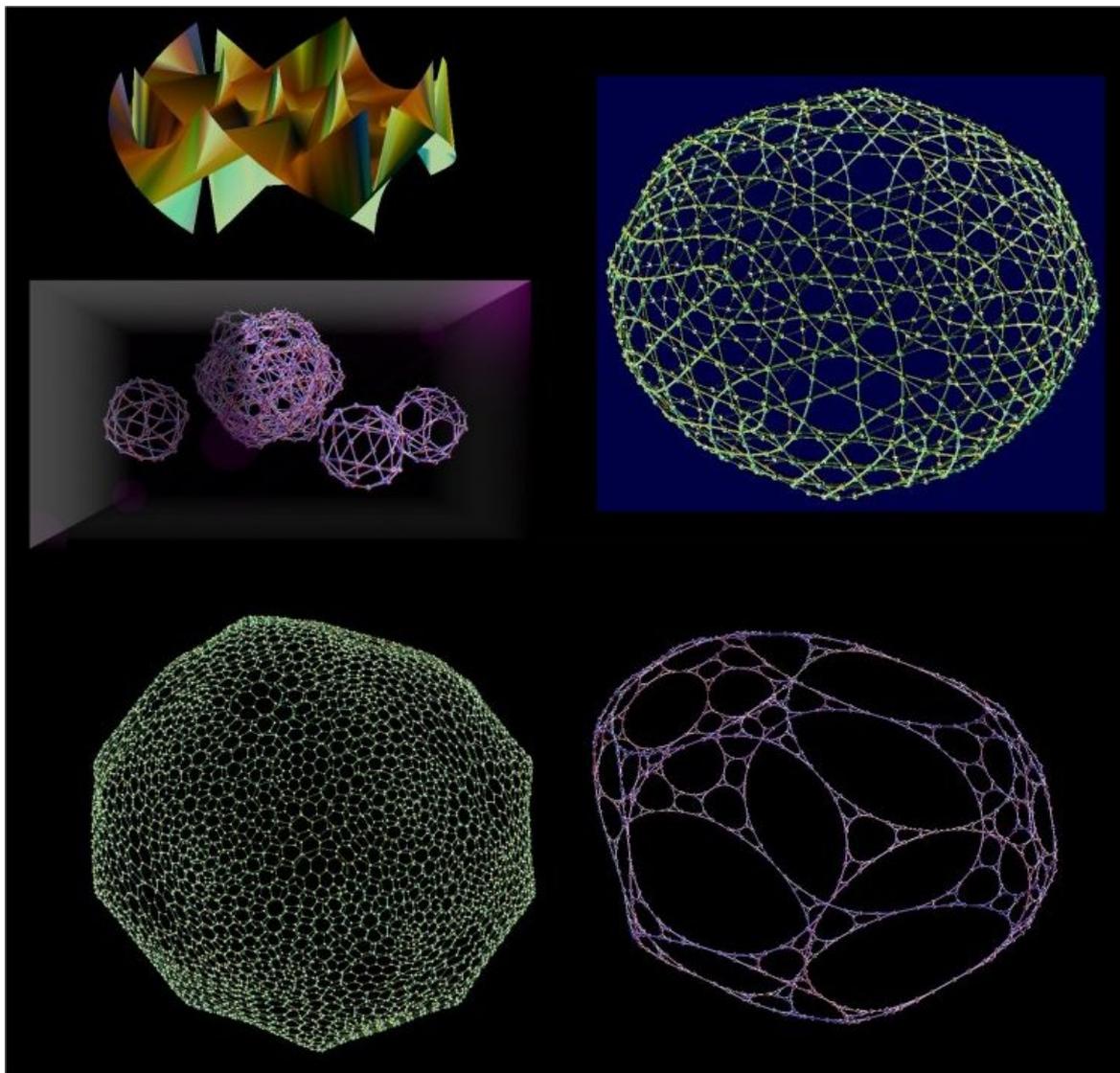
Edge doubling construction (Figure 37b) can be used for obtaining 4-regular graphs from 3-regular graphs: One of the three edges of a 3-regular graph is replaced by a double edge (digon) [37]. In particular, the edge doubling construction can be applied to graphs of truncated polyhedra of fullerene graphs, which are always 3-valent, yielding various polyhedral source links.

Figure 37. Hamiltonian cycle on fullerene C_{60} and edge doubling applied to C_{60} .



With the advance of digital techniques and availability of ultra-light materials, ground plans representing shadows of knots and links now become 3D structures made from flexible elastic rods (Figure 38). Instead of rigid bodies, buildings of the future can be transformable entities arising from flat 4-valent nets to their 3D embeddings, similar to the NODUS-structures proposed by D. Kozlov [38]. Frequently used geodesic domes based mostly on rigid 3-valent graphs can be replaced by 4-valent flexible geodesic domes or complex knots and links resembling the structure of natural sponges [39].

Figure 38. A Seifert surface corresponding to knot 9_{35} and different polyhedral knots and links.



6. Conclusions

The discovery of knots probably predates that of fire or wheel. Ropes, cords and the knots that are needed to secure them played an important role in the early technological development. The main reason for the lack of discovery of such artifacts is that they have been made from organic materials (vegetable fibres, sinews, thongs, hair, *etc.*) and are thus subject to decay. The indirect testimony for an early use of cordage and knots are perforated objects, beads or pendants, dating some 300,000 years ago, and spherical stones found in Africa and China (about 500,000 years old), probably used as bola weights in hunting. More recently bows and arrows that required well-made cordage and secure knots, as well as Paleolithic figurine in soft limestone from Kostenki (Russia, 24, 000 B.C.) show belts made from multiple twined flexible elements. Few knots from Neolith are preserved in North Zealand and Denmark. Sophisticated plaits made with strips of date palm leaf originate from Ancient Egypt. Arrangements of knots served as a basis for mathematical recording systems in the Peruvian *quipus* or Zuñi knots in New

Mexico, where knots were used as symbolic and mnemonic devices. Various examples of knot-art can be found in all ancient civilizations, in Japanese and Chinese art, Celtic art, ethnic Tamil and Tchokwe art, in Arabian, Greek or Smyrnian laces... The extensive use of knot-work pictures created for decorative and religious purposes in Celtic art required a high level of mathematics. Fine examples include knotted curves even with zoomorphic ornaments. Sophisticated knot constructions can be found in the works of two most prominent Renaissance artists and mathematicians: Leonardo and Dürer, as well as in Michelangelo's drawings. In modern art, this approach is represented in creations of Naum Gabo and other constructivist sculptors.

We have barely touched numerous applications of knots in art, and discussed only few in detail, for example those based on mirror-curves or polyhedral structures. Some other examples of knot art, like Persian mosaics (and knots in mosaics in general), lace, textile, macrame, *etc.*, remain out of the scope of this paper, together with knots in computer art. One of the most remarkable examples where mathematics (knot theory), computer science and art meet is computer software *KnotPlot* [40] by Rob Scharein—it will be a subject of our future paper focusing on the role of knots in art and science.

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