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Article

Hidden Symmetries in Simple Graphs

Allen D. Parks

Electromagnetic and Sensor Systems Department, 18444 Frontage Road Suite 327, Naval Surface Warfare Center Dahlgren Division, Dahlgren, VA 22448-5161, USA; E-Mail: allen.parks@navy.mil

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Abstract: It is shown that each element σ in the normalizer of the automorphism group *Aut*(G) of a simple graph G with labeled vertex set V is an *Aut*(G) invariant isomorphism between G and the graph obtained from G by the σ permutation of V—*i.e.*, σ is a hidden permutation symmetry of G. A simple example illustrates the theory and the applied notion of system robustness for reconfiguration under symmetry constraint (*RUSC*) is introduced.

Keywords: graph theory; automorphism group; normalizer; hidden symmetry; symmetry measures

1. Introduction

The concept of hidden symmetries of an object was introduced by Weyl [1]. Underlying this is the notion that if X is an H-set, where H is a symmetry group (the group of *obvious symmetries*) acting on X, additional *hidden symmetries* associated with X may correspond to elements of a larger group which also acts upon X and contains H as a subgroup. Sophisticated approaches based upon Weyl's concept for finding hidden symmetries in physical systems have found application in solving and understanding a variety of problems of scientific interest (e.g., [2–5]), including numerous applications in computer science (see, for example, the survey [6] and the monograph [7]).

The primary objective of this paper is to show that each element σ in the normalizer of the automorphism group *Aut*(G) of a simple graph G with labeled vertex set V is an *Aut*(G) invariant isomorphism between G and the graph obtained from G by the σ relabeling of V (*i.e.*, σ is a *hidden permutation symmetry* of G). The remainder of this paper is organized as follows: the relevant topics in graph theory and group theory are summarized in the next section (for additional depth and clarification the reader is invited to consult such standard texts as [8] and [9]). The hidden permutation symmetries of a simple vertex labeled graph G are identified in Section 3. A simple example is presented in Section 4 to illustrate the theory. Closing remarks comprise the final section of this paper.

2. Preliminaries

A *simple graph* G is the pair $G = (V, E)$, where V is a finite set of at least two vertices and the edge set E is either a set of doubleton subsets of V or the empty set $\varnothing$. If $\{u, v\} \in E$, then u and v are *adjacent* in G. The *order of a graph* G is the cardinality $|V|$ of V, $|E|$ is its *size*, and G is a $(|V|, |E|)$ graph. G is *vertex labeled* when $V = \{1,2,3,\ldots,n\}$. A labeled graph which is relabeled by a permutation σ of it vertices is the graph where vertex i is relabeled as $\sigma(i)$. The *complement* G^c of G is the graph with vertex set V and edge set $E^c = \{\{u, v\}, u, v \in V\text{: } \{u, v\} \notin E\}$. Graph $G_1 = (V_1, E_1)$ is *isomorphic to* graph $G_2 = (V_2, E_2)$ if there is a bijection $\varphi : V_1 \to V_2$ such that $\{u, v\} \in E_1$ if and only if $\{\varphi(u), \varphi(v)\} \in E_2$. Thus, a graph isomorphism preserves adjacency. The bijection φ is the isomorphism between G_1 and G_2 and the associated graph isomorphism is denoted $\varphi : G_1 \to G_2$.

An *automorphism* of G is an isomorphism of G with itself. The set of all automorphisms of G under the operation "composition of functions" forms the automorphism (or symmetry) group $Aut(G)$ of G. When G is vertex labeled, then $Aut(G)$ is a subgroup of the *symmetric group* S_V of all permutations of V, denoted $Aut(G) \subset S_V$. Furthermore, $Aut(G) = Aut(G^c)$ and if G_1 and G_2 are isomorphic graphs, then $Aut(G_1)$ is isomorphic to $Aut(G_2)$, denoted $Aut(G_1) \sim Aut(G_2)$.

The *order of a group* X is $|X|$ and the *order of* $x \in X$ is the least positive integer m such that $x^m = e$, where e is the identity element in X. If $X \subset Y$ and $yXy^{-1} = X$ for every $y \in Y$, then X is a *normal subgroup* of Y, denoted $X \lhd Y$. Here $y^{-1} \in Y$ is the inverse of y. The *normalizer* $N(Aut(G))$ of $Aut(G)$ in S_V is the group defined by

$$N(Aut(G)) = \{\sigma \in S_V : \sigma Aut(G)\sigma^{-1} = Aut(G)\} \tag{1}$$

and is the largest subgroup in S_V for which $Aut(G) \lhd N(Aut(G))$.

3. Hidden Symmetries of G

The automorphisms of the symmetry group $Aut(G)$ of G are the obvious symmetries of G. The objective of this section is to show that each $\sigma \in N(Aut(G))$ is a hidden permutation symmetry of G—*i.e.*, it is an $Aut(G)$ invariant graph isomorphism between G and the graph obtained from G by the application of σ to G's vertex labels (thus, $\sigma \in Aut(G)$ is both a G automorphism and a hidden permutation symmetry of G). The next two lemmas are required to prove this.

Lemma 3.1 *Let* $G = (V, E)$ *be a simple vertex labeled graph. If* $\sigma \in S_V$ *and* G_σ *is the graph obtained by relabeling the vertices of* G *as prescribed by* σ*, then* $\sigma : G \to G_\sigma$ *is an isomorphism.*

Proof. The relabeling of G's vertices is specified by the permutation $\sigma : V \to V$ so that the associated relabeled edges are the set $E_\sigma = \{\{\sigma(i), \sigma(j)\}\text{: } \{i, j\} \in E\}$. Now let $V_\sigma = V$, define $G_\sigma = (V_\sigma, E_\sigma)$, and observe that $\sigma : V \to V_\sigma$ is a bijection with the property that $\{i, j\} \in E$ if and only if $\{\sigma(i), \sigma(j)\} \in E_\sigma$. Thus, $\sigma : G \to G_\sigma$ is an isomorphism.

Lemma 3.2 *Let* $G = (V, E)$ *be a simple vertex labeled graph,* $\sigma \in S_V$*, and* $G_\sigma = (V_\sigma, E_\sigma)$ *be the graph obtained by the* σ *relabeling of* G*'s* vertices. *If* $\alpha \in Aut(G)$*, then* $\sigma\alpha\sigma^{-1} \in Aut(G_\sigma)$.

Proof. Since $\sigma : G \to G_\sigma$ is an isomorphism (Lemma 3.1), then so is $\sigma^{-1}: G_\sigma \to G$ and diagram (2) commutes, where "$\dashrightarrow$" denotes that the diagram is completed by the map $\beta = \sigma\alpha\sigma^{-1}$. But β is an isomorphism because it is a composition of the isomorphisms σ, α, and σ^{-1}. Therefore, $\beta = \sigma\alpha\sigma^{-1} \in Aut(G_\sigma)$ since it is the isomorphism $\beta : G_\sigma \to G_\sigma$.

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & V \\ \downarrow \sigma & & \downarrow \sigma \\ V_\sigma & \overset{\beta}{\dashrightarrow} & V_\sigma \end{array} \tag{2}$$

Theorem 3.3 (Hidden Permutation Symmetries) *Let $G = (V, E)$ be a simple vertex labeled graph and G_σ be the graph obtained by the σ relabeling of G's vertices. If $\sigma \in N(Aut(G))$, then $\sigma : G \to G_\sigma$ is an isomorphism for which $Aut(G_\sigma) = Aut(G)$.*

Proof. The fact that $\sigma : G \to G_\sigma$ is an isomorphism is established by Lemma 3.1. Recall from Lemma 3.2 that—*since* $\sigma \in N(Aut(G)) \subset S_V$—for each $\alpha \in Aut(G)$ there is a distinct $\beta = \sigma\alpha\sigma^{-1} \in Aut(G_\sigma)$. However, because $\sigma \in N(Aut(G))$, then by definition (1) it is also the case that $\beta \in Aut(G)$ so that $Aut(G_\sigma) \subseteq Aut(G)$. Furthermore, $\beta \in Aut(G)$ implies $\beta = \sigma\alpha\sigma^{-1}$ for some $\alpha \in Aut(G)$ and $\sigma \in N(Aut(G))$. Consequently, $\beta \in Aut(G_\sigma)$ so that $Aut(G) \subseteq Aut(G_\sigma)$. Thus, $Aut(G_\sigma) = Aut(G)$.

Note that in general $Aut(G_\sigma) \sim Aut(G)$ when $\sigma \in S_V$. However, when $\sigma \in N(Aut(G))$ the group isomorphism is the identity map.

Corollary 3.4 $\sigma \in N(Aut(G))$ *is a hidden permutation symmetry for* G^c.

Proof. Since $Aut(G^c) = Aut(G)$, then it must be the case that $N(Aut(G^c)) = N(Aut(G))$ so that $\sigma \in N(Aut(G))$ if and only if $\sigma \in N(Aut(G^c))$. It follows from Theorem 3.3 that $\sigma : G^c \to G^c{}_\sigma$ is an isomorphism for which $Aut(G^c{}_\sigma) = Aut(G^c)$.

4. Example: Hidden Symmetries of a Simple Vertex Labeled (4, 5) Graph

Let $G = (V, E)$, where $V = \{1,2,3,4\}$ and $E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{2,4\}\}$.

4.1. The Automorphism and Normalizer Groups for G

By inspection it is found that

$$Aut(G) = \{i, \alpha_1, \alpha_2, \alpha_3\}$$

where—when expressed in Cayley cycle notation—$i = (1)(2)(3)(4)$, $\alpha_1 = (13)(2)(4)$, $\alpha_2 = (24)(1)(3)$, and $\alpha_3 = (13)(24)$ (here, i is clearly the group identity element). The Cayley table for $Aut(G)$ is easily determined from these and is given by Table 1.

Table 1. The Cayley table for *Aut(G)*.

	i	α_1	α_2	α_3
i	i	α_1	α_2	α_3
α_1	α_1	i	α_3	α_2
α_2	α_2	α_3	i	α_1
α_3	α_3	α_2	α_1	i

It is interesting to note that up to (group) isomorphism there are only two groups of order four—the cyclic group $\mathbf{Z}_4$ and the *Viergruppe* **V** of Felix Klein. Inspection of Table 1 reveals that $Aut(G) \nsim \mathbf{Z}_4$ because there is no fourth order element in $Aut(G)$. Thus, it must be the case that $Aut(G) \sim \mathbf{V}$ (this is further corroborated from the table by the facts that $Aut(G)$ is an abelian group and that every $Aut(G)$ element is order two—which are properties of **V**).

In order to find $N(Aut(G))$ it is necessary to apply definition Equation (1) to the elements of S_V. Trial and error yields

$$N(Aut(G)) = \{i,\alpha_1,\alpha_2,\alpha_3,\sigma_1,\sigma_2,\sigma_3,\sigma_4\}$$

where $\sigma_1 = (1234)$, $\sigma_2 = (1432)$, $\sigma_3 = (14)(23)$, and $\sigma_4 = (12)(34)$. The Cayley table for $N(Aut(G))$ is presented as Table 2. As an aside—observe from Table 2 that $N(Aut(G))$ is a nonabelian group. Consequently, $N(Aut(G))$ must be isomorphic to either the quaternion group **Q** or the dihedral group $\mathbf{D}_4$ since these are the only nonabelian groups of order eight. It is also seen from a closer examination of Table 2 that $N(Aut(G))$ is generated by σ_1 and α_1 which satisfy the relations $(\sigma_1)^4 = i$, $(\alpha_1)^2 = i$, and $\alpha_1\sigma_1\alpha_1 = \sigma_2 = \sigma_1^{-1}$. Since these are precisely the generators and relations that define $\mathbf{D}_4$ then it must be the case that $N(Aut(G)) \sim \mathbf{D}_4$.

Table 2. The Cayley table for *N(Aut(G))*.

	i	α_1	α_2	α_3	σ_1	σ_2	σ_3	σ_4
i	i	α_1	α_2	α_3	σ_1	σ_2	σ_3	σ_4
α_1	α_1	i	α_3	α_2	σ_3	σ_4	σ_1	σ_2
α_2	α_2	α_3	i	α_1	σ_4	σ_3	σ_2	σ_1
α_3	α_3	α_2	α_1	i	σ_2	σ_1	σ_4	σ_3
σ_1	σ_1	σ_4	σ_3	σ_2	α_3	i	α_1	α_2
σ_2	σ_2	σ_3	σ_4	σ_1	i	α_3	α_2	α_1
σ_3	σ_3	σ_2	σ_1	σ_4	α_2	α_1	i	α_3
σ_4	σ_4	σ_1	σ_2	σ_3	α_1	α_2	α_3	i

4.2. The Hidden Permutation Symmetries of G

In order to illustrate Theorem 3.3, first note that i, α_1, α_2, and α_3 either fix vertex labels *2* and *4* or permutes them, whereas σ_1, σ_2, σ_3, and σ_4 relabel *2* and *4* as *1* and *3*, or vice versa. Thus—as automorphisms—i, α_1, α_2, and α_3 must preserve adjacency by mapping edge $\{2,4\}$ in G to edge $\{2,4\}$ in the associated relabeled graphs and—as isomorphisms—σ_1, σ_2, σ_3, and σ_4 must preserve adjacency by mapping edge $\{2,4\}$ in G to edge $\{1,3\}$ in the associated relabeled graphs. This is evidenced in Table 3 which lists the $N(Aut(G))$ image of each edge in G in the associated relabeled graph. There the bold face first column lists the edges in G and the bold face first row lists the elements of $N(Aut(G))$. The table entries are the $N(Aut(G))$ images of G edges in the corresponding relabeled graphs. For example, the image of edge $\{2,3\}$ in G under the map α_3 is the edge $\{1,4\}$ in the graph with vertices relabeled by α_3. It is obvious from this table that $\sigma : G \to G_\sigma$, $\sigma \in N(Aut(G))$, is an isomorphism because $\{i,j\} \in E$ if and only if $\{\sigma(i),\sigma(j)\} \in E_\sigma$ (*i.e.*, $\sigma : V \to V_\sigma = V$ is an edge preserving bijection).

Table 3. The *N(Aut(G))* images of *E*.

	i	α_1	α_2	α_3	σ_1	σ_2	σ_3	σ_4
{1,2}	{1,2}	{2,3}	{1,4}	{3,4}	{2,3}	{1,4}	{3,4}	{1,2}
{2,3}	{2,3}	{1,2}	{3,4}	{1,4}	{3,4}	{1,2}	{2,3}	{1,4}
{3,4}	{3,4}	{1,4}	{2,3}	{1,2}	{1,4}	{2,3}	{1,2}	{3,4}
{1,4}	{1,4}	{3,4}	{1,2}	{2,3}	{1,2}	{3,4}	{1,4}	{2,3}
{2,4}	{2,4}	{2,4}	{2,4}	{2,4}	{1,3}	{1,3}	{1,3}	{1,3}

To see that *Aut*(*G*) is the automorphism group for each graph relabeled by $\sigma \in \{\sigma_1,\sigma_2,\sigma_3,\sigma_4\} = N(Aut(G)) - Aut(G)$ (*i.e.*, that each such isomorphism $\sigma: G \to G_\sigma$ is *Aut*(*G*) invariant), observe that the automorphisms of *Aut*(*G*) are the only bijective vertex maps which preserve adjacency in each G_σ and map edge {1,3} in each G_σ to itself. For example, the set of edges in the graph relabeled by σ_2 (the sixth column in Table 3) is bijectively mapped in an adjacency preserving manner onto itself by $\alpha_2 \in Aut(G)$ according to the mappings given by (3) (the associated vertex maps appear in parentheses). Similar results also hold for *i*, α_1, and α_3 so that *Aut*(*G*) is the automorphism group for this σ_2 relabeled graph, *i.e.*, *Aut*(*G*) is invariant under the isomorphism σ_2.

$$\begin{array}{l} \{1,4\} \mapsto \{1,2\}\ldots\ (1 \mapsto 1 \text{ and } 4 \mapsto 2) \\ \{1,2\} \mapsto \{1,4\}\ldots\ (1 \mapsto 1 \text{ and } 2 \mapsto 4) \\ \{2,3\} \mapsto \{3,4\}\ldots\ (2 \mapsto 4 \text{ and } 3 \mapsto 3) \\ \{3,4\} \mapsto \{2,3\}\ldots\ (3 \mapsto 3 \text{ and } 4 \mapsto 2) \\ \{1,3\} \mapsto \{1,3\}\ldots\ (1 \mapsto 1 \text{ and } 3 \mapsto 3) \end{array} \tag{3}$$

5. Closing Remarks

Although every permutation relabeling σ of the vertex labels of a simple graph *G* defines an isomorphic copy G_σ of *G* with an automorphism group that is *isomorphic to* that of *G*, only those permutations in the normalizer of *G*'s automorphism group yield G_σ's with automorphism groups *identical to* that of *G*. These special permutations define automorphism group invariant isomorphisms of *G*—*i.e.*, they are the hidden (permutation) symmetries of *G*. Thus, each hidden permutation symmetry of *G* specifies a way in which *G* can be relabeled without changing its underlying fundamental (obvious) symmetry.

Various real complex systems of recent interest—such as biochemical processes, global trading patterns, and scientific collaborations—can be modeled as simple labeled graphs. Many of these systems are surprisingly highly symmetric (*i.e.*, they possess large numbers of obvious symmetries). Within the context of complex systems the hidden permutation symmetries of the labeled graph representing a system identify the system's *robustness for reconfiguration under symmetry constraint* (*RUSC*), *i.e.*, the ability to reconfigure the system without changing its fundamental symmetry.

In order to better understand symmetry and its affect on system properties, effort has been devoted in recent years to developing simple measures which quantify system symmetry in terms of the automorphism group of the system's graph model (e.g., [10,11]). The most direct measure of (obvious) symmetry in a graph *G* is the quantity $\alpha_G = |\, Aut(G)\, |$. An analogous extension of this to a measure which includes the hidden permutation symmetries in *G* that are not in *Aut*(*G*) is the *RUSC* number.

$$\rho_G \equiv |\, N(Aut(G)) \,|$$

This quantity counts the total number of ways G (*i.e.*, the system) can be relabeled (*i.e.*, reconfigured) without changing the automorphism group $Aut(G)$ (*i.e.*, the fundamental symmetry of the system). The difference $\delta_G = \rho_G - \alpha_G$ and the ratio $\eta_G \equiv \delta_G/\rho_G$ also provide additional measures of a system's *RUSC*.

For a system represented by the above (*4*,*5*) graph, $\alpha_G = 4$, $\rho_G = 8$, $\delta_G = 4$, and $\eta_G = ½$. Thus, there are *8* system configurations which have identical fundamental symmetries. Four of these reconfigurations are defined by permutations in the set $N(Aut(G)) - Aut(G)$ and comprise half of the total number of possible reconfigurations.

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