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Article

Squaring the Circle and Cubing the Sphere: Circular and Spherical Copulas

Michael D. Perlman * **and Jon A. Wellner**

Department of Statistics, University of Washington, Box 354322, Seattle, WA 98195-4322, USA;
E-Mail: jaw@stat.washington.edu

* Author to whom correspondence should be addressed; E-Mail: michael@stat.washington.edu;
Tel.: +1-206-543-7735; Fax: +1-206-685-7419.

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Abstract: Do there exist circular and spherical copulas in $\mathbb{R}^d$? That is, do there exist circularly symmetric distributions on the unit disk in $\mathbb{R}^2$ and spherically symmetric distributions on the unit ball in $\mathbb{R}^d$, $d \geq 3$, whose one-dimensional marginal distributions are uniform? The answer is yes for $d = 2$ and 3, where the circular and spherical copulas are unique and can be determined explicitly, but no for $d \geq 4$. A one-parameter family of elliptical bivariate copulas is obtained from the unique circular copula in $\mathbb{R}^2$ by oblique coordinate transformations. Copulas obtained by a non-linear transformation of a uniform distribution on the unit ball in $\mathbb{R}^d$ are also described, and determined explicitly for $d = 2$.

Keywords: bivariate distribution; multivariate distribution; unit disk; unit ball; circular symmetry; spherical symmetry; circular copula; spherical copula; elliptical copula

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1. Introduction

Do there exist spherically symmetric distributions on the closed unit ball B_d in $\mathbb{R}^d$ that have uniform one-dimensional marginal distributions on $[-1, 1]$? A distribution on B_d with this property may be said to "square the circle" when $d = 2$ and to "cube the sphere" when $d \geq 3$.

The cumulative distribution function (cdf) of a multivariate distribution on the unit cube $[0, 1]^d$ whose marginal distributions are uniform $[0, 1]$ is commonly called a *copula*; see Nelsen [1] for an accessible introduction to this topic. However, although it is customary to confine attention to distributions on the *unit* cube, our interest is in *spherically symmetric (= orthogonally invariant) distributions on* B_d *with uniform marginal distributions.* Therefore we take "copula" to mean a multivariate cdf on the *centered* cube $C_d := [-1,1]^d$ with uniform $[-1,1]$ marginals.

For $d = 2$ (resp., $d \geq 3$), such a copula, if it exists, will be called a *circular copula (resp., spherical copula)* if it is the cdf of a circularly symmetric (resp., spherically symmetric) distribution on the unit disk B_2 (resp., unit ball B_d).

It will be noted in Sections 2 and 3 that circular and spherical copulas are unique if they exist, but exist only for dimensions $d = 2$ and $d = 3$. The proof of non-existence for $d \geq 4$ is remarkably simple. Explicit expressions for these copulas are given in Sections 3 and 4 respectively.

In Section 5, a new one-parameter family of bivariate copulas called *elliptical copulas* is obtained from the unique circular copula in $\mathbb{R}^2$ by oblique coordinate transformations. Finally, in Section 6, copulas obtained by a non-linear transformation of a uniform distribution on the unit ball in $\mathbb{R}^d$ are described, and determined explicitly for $d = 2$.

2. Uniqueness and Existence of Circular and Spherical Copulas

Proposition 2.1. *Circular and spherical copulas are unique if they exist. (This result is well-known (e.g., Feller [2], pp. 31–33, who uses "random direction" to indicate the uniform distribution of* $U \in \partial B_3$*), and reappears frequently (e.g., Arellano-Valle [3], Theorem 3.1). The essence of the result goes back at least to Schoenberg [4].)*

Proof. If a circular or spherical copula exists on C_d, it is the cdf of a random vector $Z \equiv (Z_1, \ldots, Z_d)$ with a spherically symmetric distribution on B_d and with each $Z_i \sim \text{uniform}[-1,1]$. The latter implies that Z has no atom at the origin, *i.e.*, $P[Z = 0] = 0$, so we may consider the "polar coordinates" representation $Z = R \cdot U$, where $R = \|Z\| \leq 1$ and $U = Z/\|Z\|$. It is well known (e.g., Cambanis *et al.* [5], Lemmas 1 and 2) that the random unit vector $U \equiv (U_1, \ldots, U_d)$ is independent of R and is uniformly distributed on the unit sphere ∂B_d, which implies that each $U_i^2 \sim \text{Beta}(1/2, (d-1)/2)$. Since $Z_i = RU_i$, we have that

$$\log(Z_i^2) = \log(R^2) + \log(U_i^2). \tag{1}$$

Because R and U_i are independent, it follows that the characteristic function of $\log(R^2)$ is the quotient of the characteristic functions of $\log(Z_i^2)$ and $\log(U_1^2)$. Thus the distribution of $\log(R^2)$, and therefore that of R, is uniquely determined by the distributions of Z_i^2 and U_i^2, which are already specified above. Thus the the joint distribution of (R, U) is uniquely determined, hence so is the distribution of Z, hence so its cdf = copula. □

The existence of spherical copulas is easy to determine in three or more dimensions:

Proposition 2.2. *Spherical copulas do not exist for* $d \geq 4$*. For* $d = 3$*, the unique spherical copula is generated by the uniform distribution on the unit sphere* $\partial B_3 := \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$.

Proof. Let Z be as in the proof of Proposition 2.1. Then

$$\frac{1}{3} = E(Z_i^2) = E(R^2)E(U_i^2) \leq \frac{1}{d} \tag{2}$$

since $Z_i \sim \text{uniform}[-1, 1]$, $0 \leq R \leq 1$, and $U_i^2 \sim \text{Beta}(1/2,\,(d-1)/2)$. Thus $d \leq 3$, so a spherical copula cannot exist when $d \geq 4$.

Furthermore, if a spherical copula is to exist for $d = 3$, it follows from (2) that its generating random vector $Z \in B_3$ must satisfy $E(R^2) = 1$, hence $R = 1$ with probability one. This can occur only if Z is uniformly distributed on the unit sphere ∂B_3. But it is well known (this follows from the fact that the area of a spherical zone is proportion to its altitude—*cf.* Feller [2], Proposition (i), p. 30) that this distribution does indeed have uniform marginal distributions on $[-1, 1]$, hence generates the unique spherical copula for $d = 3$. $\square$

3. The Bivariate Case: The Unique Circular Copula

The following three questions constitute an engaging classroom exercise.

Question 1. Let (X, Y) be a random vector uniformly distributed on the unit disk (= ball) B_2 in $\mathbb{R}^2$. Find the marginal probability distributions of X and Y.

Answer: One can easily show that X has the "semi-circular" probability density function (pdf) given by

$$f(x) = \frac{2}{\pi}\sqrt{1 - x^2}, \quad -1 \leq x \leq 1. \tag{3}$$

(See Figure 1.) By symmetry, Y has the same pdf as X.

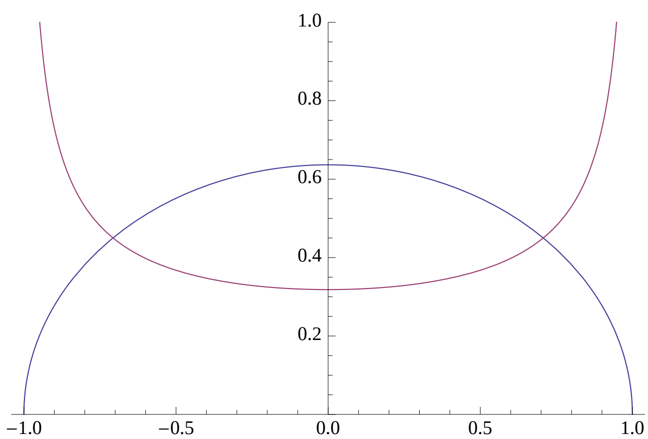
Question 2. Let (X, Y) be a random vector uniformly distributed on the unit circle ∂B_2 in $\mathbb{R}^2$. Find the marginal probability distributions of X and Y.

Answer: We can represent (X, Y) as $(\cos\Theta,\,\sin\Theta)$ where $\Theta \sim \text{uniform}[0, 2\pi)$. It follows readily that X has pdf

$$f(x) = \frac{1}{\pi\sqrt{1 - x^2}}, \quad -1 < x < 1. \tag{4}$$

(See Figure 1.) By symmetry, Y has the same pdf as X.

Figure 1. The densities (3) (lower, blue) and (4) (upper, purple).



In both cases, the joint distribution of (X, Y) is circularly symmetric, that is, invariant under all orthogonal transformations of $\mathbb{R}^2$. A comparison of the shapes of the pdfs in Figure 1 suggest a third question:

Question 3. Does a circularly symmetric bivariate distribution with uniform $[-1, 1]$ marginals exist on B_2? If so, it determines a circular copula on C_2, which is unique by Proposition 2.1. This also follows from uniqueness results for the Abel transform; see, e.g., Bracewell [6].

Answer: Optimistically, let's seek an absolutely continuous solution. That is, we seek a bivariate pdf on B_2 of the form

$$f(x, y) = g(x^2 + y^2)$$

such that the marginal pdf

$$f(x) \equiv \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y)dy, \qquad -1 < x < 1$$

is constant in x. Here g is a nonnegative function on $(0, 1)$ that must satisfy

$$2\pi \int_0^1 rg(r^2)dr = 1 \tag{5}$$

in order that $\iint_{B_2} f(x, y)dxdy = 1$ (transform to polar coordinates: $(x, y) \to (r, \theta)$).

To determine a suitable g, first set $h(t) = g(1 - t)$, then let $u = y/\sqrt{1 - x^2}$ to obtain

$$\begin{aligned} f(x) &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} h(1 - x^2 - y^2)dy = \sqrt{1 - x^2} \int_{-1}^{1} h((1 - u^2)(1 - x^2))du \\ &= 2\sqrt{1 - x^2} \int_0^1 h((1 - u^2)(1 - x^2))du. \end{aligned}$$

If we take $h(t) = ct^{-1/2}$ then clearly $f(x)$ does not depend on x, and choosing $c = 1/2\pi$ satisfies (5). Thus the bivariate pdf (see Figure 2)

$$f(x, y) = \frac{1}{2\pi\sqrt{1 - x^2 - y^2}}, \qquad x^2 + y^2 < 1 \tag{6}$$

determines a circularly symmetric bivariate distribution on B_2 and yields the desired circular copula.

Question 4. Having determined the unique circularly symmetric distribution (6) on B_2 with uniform marginals, what is the corresponding cdf $F(x, y)$, that is, what is the corresponding circular copula?

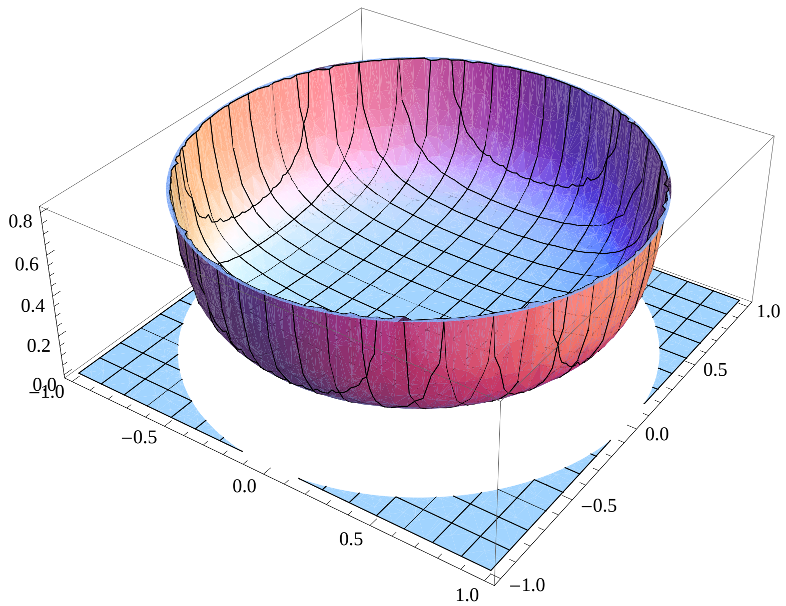
Answer (see Theorem 3.1): The circular symmetry of (X, Y) implies that its distribution is invariant under sign changes, *i.e.*, $(X, Y) \stackrel{d}{=} (\pm X, \pm Y)$. By the following lemma, the cdf $F(x, y) \equiv P[X \le x, Y \le y]$ on $C_2 \equiv [-1, 1]^2$ can be expressed in terms of $F_0(x, y)$, its truncation to the first quadrant:

$$F_0(x, y) \equiv P[0 \le X \le x, 0 \le Y \le y] \tag{7}$$

for $0 \le x, y \le 1$, and also in terms of the complementary cdf $\bar{F}(x, y) \equiv P[X > x, Y > y]$ for $0 \le x, y \le 1$. Because $(X, Y) \stackrel{d}{=} (\pm X, \pm Y)$ and has uniform $[-1, 1]$ marginals,

$$\begin{aligned} F_0(x, y) &= P[0 \le X \le 1, 0 \le Y \le 1] - P[X > x, 0 \le Y \le 1] \\ &\quad - P[0 \le X \le 1, Y > y] + P[X > x, Y > y] \\ &= \frac{1}{4} - \left(\frac{1 - x}{4}\right) - \left(\frac{1 - y}{4}\right) + \bar{F}(x, y) \\ &= \frac{x + y - 1}{4} + \bar{F}(x, y), \quad 0 \le x, y \le 1. \end{aligned} \tag{8}$$

Figure 2. Circularly symmetric bivariate density (6) on B_2.



Lemma 3.1. *Let (X, Y) be a bivariate random vector on C_2 with uniform $[-1, 1]$ marginal distributions and sign-change invariance, i.e., $(X, Y) \overset{d}{=} (\pm X, \pm Y)$. Then for $(x, y) \in C_2$,*

$$F(x,y) = \frac{x+y+1}{4} + \sigma(xy)\, F_0(|x|, |y|) \tag{9}$$

$$= \frac{x+y+1}{4} + \sigma(xy) \left[\frac{|x|+|y|-1}{4} + \bar{F}(|x|, |y|)\right] \tag{10}$$

where $\sigma(w) = \operatorname{sign}(w)$ if $w \neq 0$ and $\sigma(0) = 0$.

Proof. To obtain (9), consider four cases:

Case 1: $0 \le x, y \le 1$. Because (X, Y) is sign-change invariant and has uniform $[-1, 1]$ marginals,

$$\begin{aligned} F(x,y) &= P[0 < X \le x,\ 0 < Y \le y] + P[0 < X \le x,\ Y \le 0] \\ &\quad + P[X \le 0,\ 0 < Y \le y] + P[X \le 0,\ Y \le 0] \\ &= F_0(x,y) + \frac{x}{4} + \frac{y}{4} + \frac{1}{4} \\ &= \frac{x+y+1}{4} + \sigma(xy)\, F_0(|x|, |y|). \end{aligned}$$

Case 2: $-1 \le x \le 0 \le y \le 1$. Similarly,

$$\begin{aligned} F(x,y) &= P[X \le 0,\ 0 < Y \le y] - P[x < X \le 0,\ 0 \le Y \le y] \\ &\quad + P[X \le 0,\ Y \le 0] - P[x < X \le 0,\ Y \le 0] \\ &= \frac{y}{4} - F_0(-x, y) + \frac{1}{4} - \frac{(-x)}{4} \\ &= \frac{x+y+1}{4} + \sigma(xy)\, F_0(|x|, |y|). \end{aligned}$$

Case 3: $-1 \leq y \leq 0 \leq x \leq 1$. Similarly,

$$\begin{aligned} F(x, y) &= P[0 < X \leq x, Y \leq 0] - P[0 \leq X \leq x, y < Y \leq 0] \\ &\quad + P[X \leq 0, Y \leq 0] - P[X \leq 0, y < Y \leq 0] \\ &= \frac{x}{4} - F_0(x, -y) + \frac{1}{4} - \frac{(-y)}{4} \\ &= \frac{x+y+1}{4} + \sigma(xy)\, F_0(|x|, |y|). \end{aligned}$$

Case 4: $-1 \leq x, y \leq 0$. Similarly,

$$\begin{aligned} F(x, y) &= P[X \leq 0, Y \leq 0] - P[x < X \leq 0, Y \leq 0] \\ &\quad - P[X \leq 0, y < Y \leq 0] + P[x < X \leq 0, y < Y \leq 0] \\ &= \frac{1}{4} - \frac{(-x)}{4} - \frac{(-y)}{4} + F_0(-x, -y) \\ &= \frac{x+y+1}{4} + \sigma(xy)\, F_0(|x|, |y|). \end{aligned}$$

Finally, (10) follows from (9) by (8). □

Thus, to determine the circular copula $F(x, y)$ for the pdf (6), it suffices to determine the complementary cdf $\bar{F}(x, y)$ for $0 \leq x, y \leq 1$ and apply (10). Because $\bar{F}(x, y) = 0$ when $x^2 + y^2 \geq 1$, we need only consider the case where $x^2 + y^2 < 1$.

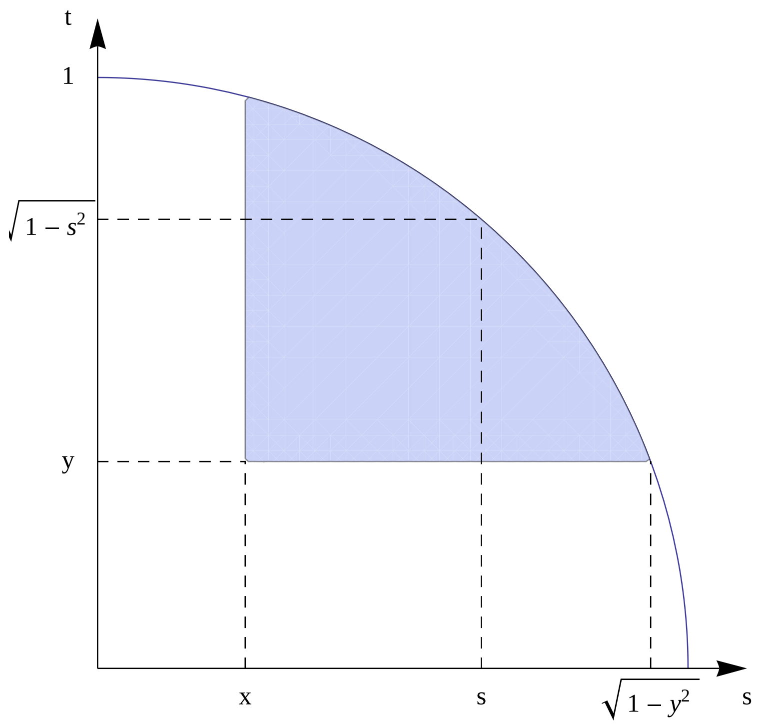
First approach: When $0 \leq x, y \leq 1$ and $x^2 + y^2 < 1$, $\bar{F}(x, y)$ can be expressed as follows. By using Figure 3 we find that

$$\begin{aligned} \bar{F}(x, y) &= \frac{1}{2\pi} \int_x^{\sqrt{1-y^2}} \left\{ \int_y^{\sqrt{1-s^2}} \frac{1}{\sqrt{1-s^2-t^2}} dt \right\} ds \\ &= \frac{1}{2\pi} \int_x^{\sqrt{1-y^2}} \left\{ \int_y^{\sqrt{1-s^2}} \frac{1}{\sqrt{(1-\frac{t^2}{1-s^2})}} \frac{dt}{\sqrt{1-s^2}} \right\} ds \\ &= \frac{1}{2\pi} \int_x^{\sqrt{1-y^2}} \left\{ \int_{\frac{y}{\sqrt{1-s^2}}}^{1} \frac{dv}{\sqrt{1-v^2}} \right\} ds \\ &= \frac{1}{2\pi} \int_x^{\sqrt{1-y^2}} \left[\frac{\pi}{2} - \arcsin\left(\frac{y}{\sqrt{1-s^2}} \right) \right] ds. \end{aligned} \tag{11}$$

However, we were unable to evaluate this integral directly.

Second approach: Fortunately, we have found a solution in the molecular biology and optics literatures, where the problem of finding the area of the intersection of two spherical caps on the unit sphere ∂B_3 has been addressed. The following general result is due to Tovchigrechko and Vakser [7] and also appears in Oat and Sander [8].

Figure 3. Region of integration, 2-dimensional case.



Lemma 3.2. *Let S_1 and S_2 be spherical caps on ∂B_3. Let r_1 and r_2 denote their angular radii and let d denote the angular distance between their centers ($0 < d \leq \pi$). Assume that $0 < r_1, r_2 \leq \pi/2$ and $d \leq r_1+r_2$, so that the intersection $S_1 \cap S_2 \neq \emptyset$ and consists of a single "diangle"; (see Figures 4 and 5.) Then Area$(S_1 \cap S_2)$ is given by*

$$\begin{aligned} A(r_1, r_2; d) &= 2\pi - 2\pi\cos(r_1) - 2\pi\cos(r_2) - 2\arccos\left(\frac{\cos(d) - \cos(r_1)\cos(r_2)}{\sin(r_1)\sin(r_2)}\right) \\ &+ 2\cos(r_1)\arccos\left(\frac{\cos(d)\cos(r_1) - \cos(r_2)}{\sin(d)\sin(r_1)}\right) \\ &+ 2\cos(r_2)\arccos\left(\frac{\cos(d)\cos(r_2) - \cos(r_1)}{\sin(d)\sin(r_2)}\right). \end{aligned} \tag{12}$$

This result can be applied to obtain our desired circular copula as follows.

If (X, Y, Z) is uniformly distributed on ∂B_3, then the event $\{X > x, Y > y\}$ corresponds to the intersection of the two spherical caps $\{X > x\}$ and $\{Y > y\}$, so $P[X > x, Y > y]$ is given by the area $A(x, y)$ of this intersection divided by the total area of ∂B_3, *i.e.*, by 4π. (See Figures 4 and 5.) Also, the joint distribution of (X, Y) is circularly symmetric on the unit disk B_2 and has uniform marginals, so must be the unique such bivariate distribution, namely the distribution with pdf (6).

Figure 4. Intersection of two spherical caps.

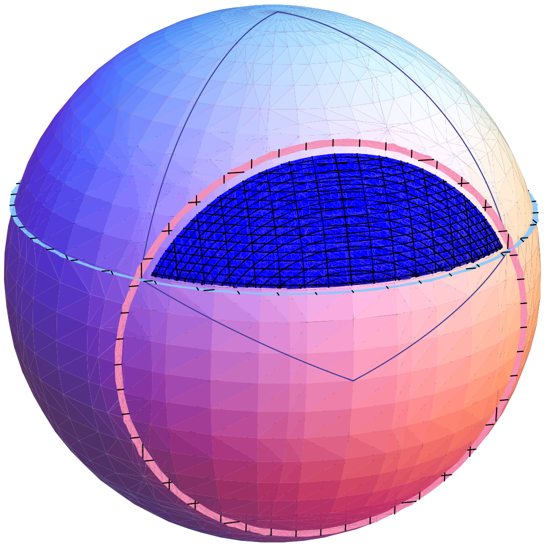
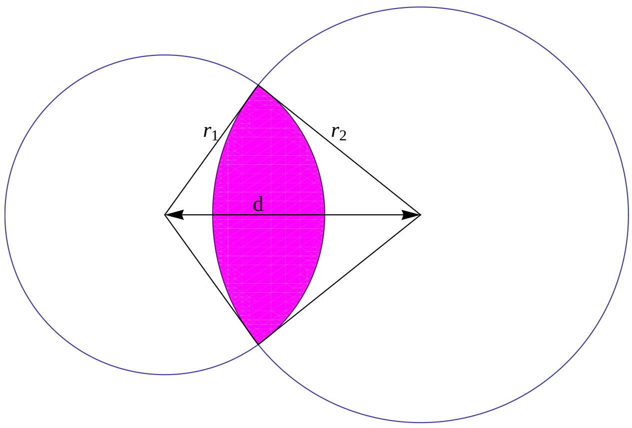


Figure 5. Intersection of two spherical caps, circular representation (modified from Tovchigrechko and Vakser [7]).



Thus, for $0 \leq x, y \leq 1$ and $x^2 + y^2 < 1$, our desired complementary cdf is given by

$$\begin{aligned}
\bar{F}(x,y) &= \frac{1}{4\pi}A(x,y) && (13)\\
&= \frac{1}{4\pi}A(\arccos(x),\ \arccos(y);\ \pi/2) && (14)\\
&= \frac{1}{2} - \frac{x}{2} - \frac{y}{2} - \frac{1}{2\pi}\arccos\left(-\frac{xy}{\sqrt{(1-x^2)(1-y^2)}}\right)\\
&\quad + \frac{x}{2\pi}\arccos\left(\frac{-y}{\sqrt{1-x^2}}\right) + \frac{y}{2\pi}\arccos\left(\frac{-x}{\sqrt{1-y^2}}\right)\\
&\equiv \frac{1-x-y}{4} + \alpha(x,y) && (15)
\end{aligned}$$

where for $0 \le x, y \le 1$ and $x^2 + y^2 < 1$,

$$\begin{aligned}\alpha(x,y) &= \frac{1}{2\pi}\left[x \arcsin\left(\frac{y}{\sqrt{1-x^2}}\right) + y \arcsin\left(\frac{x}{\sqrt{1-y^2}}\right)\right. \\ &\qquad \left. - \arcsin\left(\frac{xy}{\sqrt{(1-x^2)(1-y^2)}}\right)\right]. \end{aligned} \tag{16}$$

Theorem 3.1. *The unique circular copula on C_2 is given by*

$$F(x,y) = \frac{x+y+1}{4} + \alpha(x,y) \tag{17}$$

where $\alpha(x,y)$ is defined by (16) for $x^2 + y^2 < 1$ and by

$$\alpha(x,y) = \sigma(xy) \cdot \left(\frac{|x| + |y| - 1}{4}\right) \tag{18}$$

for $x^2 + y^2 \ge 1$. Note that Equations (16) and (18) agree when $x^2 + y^2 = 1$ and both are sign-change equivariant on C_2: for all $(x,y) \in C_2$ and all $\epsilon, \delta = \pm 1$,

$$\alpha(\epsilon x, \delta y) = \epsilon\delta \cdot \alpha(x,y). \tag{19}$$

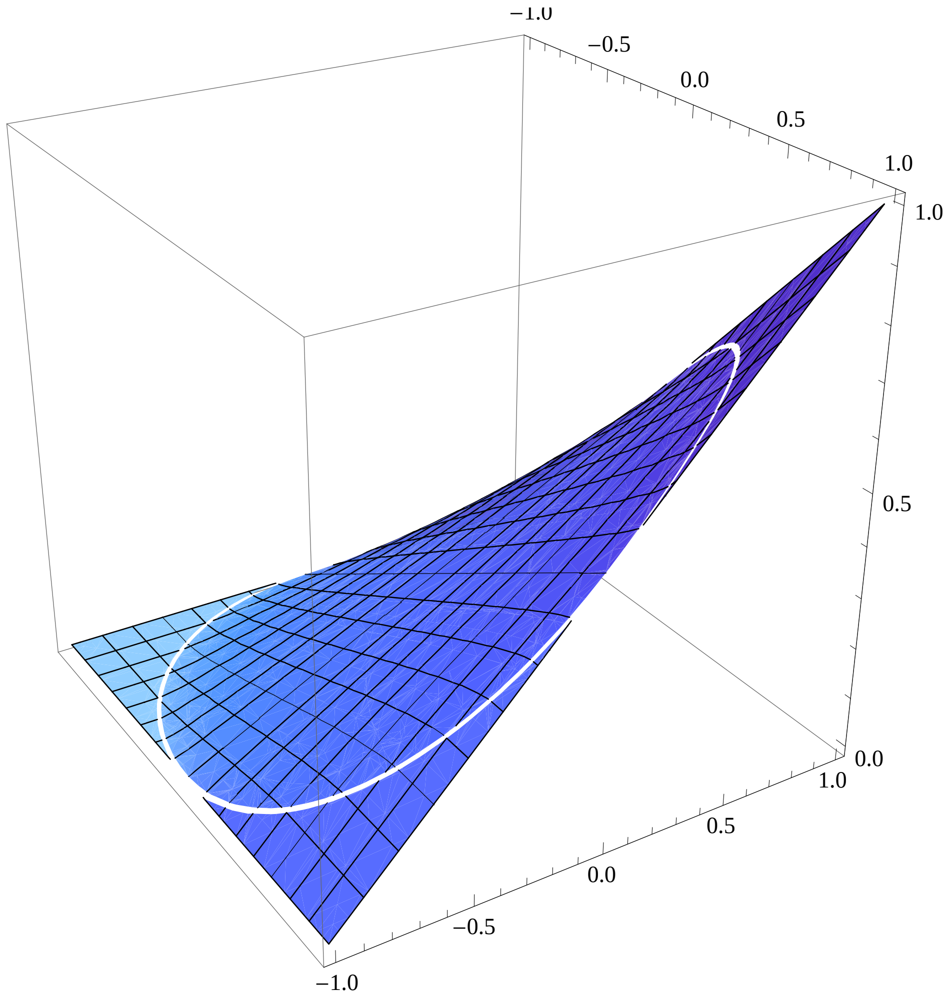
Proof. From Equations (10) and (15), when $x^2 + y^2 < 1$ we have

$$\begin{aligned} F(x,y) &= \frac{x+y+1}{4} + \sigma(xy)\, \alpha(|x|,|y|) && (20) \\ &= \frac{x+y+1}{4} + \alpha(x,y) && (21) \end{aligned}$$

by (19). When $x^2 + y^2 \ge 1$, $\bar{F}(|x|,|y|) = 0$ so (17) again holds by (10) and (18). □

See Figure 6 for a plot of the resulting copula (on $[-1,1]^2$).

Figure 6. The copula (17) in Theorem 3.1.



4. The Trivariate Case: the Unique Spherical Copula

Question 5. Having determined the unique spherically symmetric distribution on B_3 with uniform marginals, namely, the uniform distribution on the unit sphere ∂B_3, what is the corresponding cdf $F(x, y, z)$ on C_3, *i.e.*, the unique spherical copula?

Answer: As in Section 3, let (X, Y, Z) be uniformly distributed on ∂B_3, so that $F(x, y, z) = P[X \leq x, Y \leq y, Z \leq z]$. Again we first determine the complementary cdf $\bar{F}(x, y, z) \equiv P[X > x, Y > y, Z > z]$ for $0 \leq x, y, z \leq 1$ and $x^2 + y^2 + z^2 < 1$, the intersection of the first octant of C_3 with the interior of B_3. Here the event $\{X > x, Y > y, Z > z\}$ corresponds to the intersection of the three spherical caps $\{X > x\}$, $\{Y > y\}$, and $\{Z > z\}$ on ∂B_3, so $\bar{F}(x, y, z)$ is the area $A(x, y, z)$ of this intersection divided by the total area 4π of ∂B_3.

Recall that two approaches were proposed in Section 3 to obtain the area $A(x, y)$ of the intersection of *two* circular caps $\{X > x\}$ and $\{Y > y\}$. The first approach led to the integral (11) that we were unable to evaluate explicitly, so we adopted a second approach based on the geometric Lemma 3.2 of Tovchigrechko and Vakser [7]. Andrey Tovchigrechko has kindly suggested a method for extending Lemma 3.2 to the case of three spherical caps in general position, which if carried out would yield an explicit expression for $A(x, y, z)$. However, we have found that because the axes of our three caps are mutually orthogonal, the two approaches just mentioned for the bivariate case can be combined to obtain $\bar{F}(x, y, z) \equiv \frac{1}{4\pi} A(x, y, z)$ directly for the trivariate case, as now described.

We begin by extending (11) to obtain an integral expression for $\bar{F}(x, y, z)$ when $0 \le x, y, z \le 1$ and $x^2 + y^2 + z^2 < 1$. We require the fact that

$$0 \le a, b \le 1 \;\text{ and }\; a^2 + b^2 = 1 \;\text{ implies }\; \arcsin(a) + \arcsin(b) = \pi/2. \tag{22}$$

Lemma 4.1. *If* $0 \le x, y, z \le 1$ *and* $x^2 + y^2 + z^2 < 1$*, then*

$$\bar{F}(x, y, z) = \frac{1}{4\pi} \int_x^{\sqrt{1-y^2-z^2}} \left[\frac{\pi}{2} - \arcsin\left(\frac{y}{\sqrt{1-s^2}}\right) - \arcsin\left(\frac{z}{\sqrt{1-s^2}}\right) \right] ds. \tag{23}$$

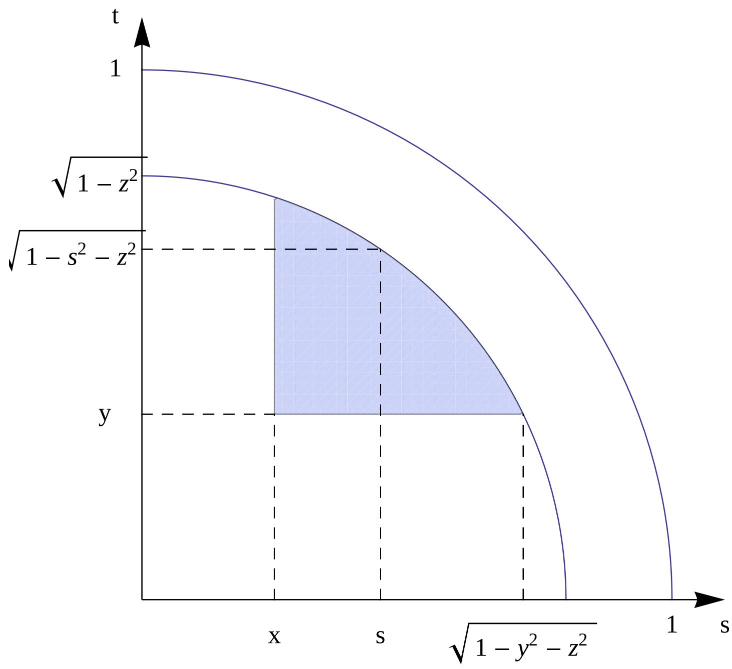
Because (X, Y, Z) *is exchangeable,* (23) *remains valid under any permutation of* x, y, z *on the right-hand side.*

Proof. Since $X^2 + Y^2 + Z^2 = 1$ and $(X, Y, Z) \stackrel{d}{=} (X, Y, -Z)$, it follows from (6) by using Figure 7 that

$$\begin{aligned}
&P[X > x,\, Y > y,\, Z > z] \\
&= \frac{1}{2} P[X > x,\, Y > y,\, X^2 + Y^2 < 1 - z^2] \\
&= \frac{1}{4\pi} \int_x^{\sqrt{1-y^2-z^2}} \left\{ \int_y^{\sqrt{1-s^2-z^2}} \frac{1}{\sqrt{1-s^2-t^2}} dt \right\} ds \\
&= \frac{1}{4\pi} \int_x^{\sqrt{1-y^2-z^2}} \left\{ \int_y^{\sqrt{1-s^2-z^2}} \frac{1}{\sqrt{(1-\frac{t^2}{1-s^2})}} \frac{dt}{\sqrt{1-s^2}} \right\} ds \\
&= \frac{1}{4\pi} \int_x^{\sqrt{1-y^2-z^2}} \left\{ \int_{\frac{y}{\sqrt{1-s^2}}}^{\sqrt{\frac{1-s^2-z^2}{1-s^2}}} \frac{dv}{\sqrt{1-v^2}} \right\} ds \\
&= \frac{1}{4\pi} \int_x^{\sqrt{1-y^2-z^2}} \left[\arcsin\left(\sqrt{\frac{1-s^2-z^2}{1-s^2}}\right) - \arcsin\left(\frac{y}{\sqrt{1-s^2}}\right) \right] ds.
\end{aligned}$$

Now apply (22) to obtain (23). □

Figure 7. Region of integration, 3-dimensional case, Lemma 4.1.



As noted above, the integral in (23) appears difficult to evaluate explicitly but the following indirect argument succeeds. Recall from (11) and (15) that when $0 \le x, y \le 1$ and $x^2 + y^2 < 1$,

$$\begin{aligned} \bar{F}(x,y) &= \frac{1}{2\pi}\int_x^{\sqrt{1-y^2}} \left[\frac{\pi}{2} - \arcsin\left(\frac{y}{\sqrt{1-s^2}}\right)\right] ds \\ &= \frac{1-x-y}{4} + \alpha(x,y) \end{aligned}$$

where $\alpha(x,y)$ is given by (16). Because $z \le \sqrt{1-x^2-y^2} \le \sqrt{1-y^2}$ when $0 \le x, y, z \le 1$ and $x^2 + y^2 + z^2 < 1$, it follows that

$$\begin{aligned} &\frac{1}{2\pi}\int_z^{\sqrt{1-x^2-y^2}} \left[\frac{\pi}{2} - \arcsin\left(\frac{y}{\sqrt{1-s^2}}\right)\right] ds \\ &= \frac{\sqrt{1-x^2-y^2}-z}{4} + \alpha(z,y) - \alpha(\sqrt{1-x^2-y^2},\, y). \end{aligned} \tag{24}$$

Therefore from (23) and (24), if $0 \le x, y, z \le 1$ and $x^2 + y^2 + z^2 < 1$ then

$$\begin{aligned}
&4\pi \bar{F}(x,y,z) \\
&= \int_z^{\sqrt{1-x^2-y^2}} \left[\frac{\pi}{2} - \arcsin\left(\frac{x}{\sqrt{1-s^2}}\right)\right] ds \\
&\quad + \int_z^{\sqrt{1-x^2-y^2}} \left[\frac{\pi}{2} - \arcsin\left(\frac{y}{\sqrt{1-s^2}}\right)\right] ds - \frac{\pi}{2}\left[\sqrt{1-x^2-y^2} - z\right] \\
&= \frac{\pi}{2}(\sqrt{1-x^2-y^2} - z) + 2\pi\left[\alpha(z,x) - \alpha(\sqrt{1-x^2-y^2},\, x)\right] \\
&\quad + \frac{\pi}{2}(\sqrt{1-x^2-y^2} - z) + 2\pi\left[\alpha(z,y) - \alpha(\sqrt{1-x^2-y^2},\, y)\right] \\
&\quad - \frac{\pi}{2}(\sqrt{1-x^2-y^2} - z) \\
&= \frac{\pi}{2}(\sqrt{1-x^2-y^2} - z) \\
&\quad + 2\pi\left[\alpha(z,x) + \alpha(z,y) - \alpha(\sqrt{1-x^2-y^2},\, x) - \alpha(\sqrt{1-x^2-y^2},\, y)\right] \\
&= \frac{\pi}{2}(\sqrt{1-x^2-y^2} - z) \\
&\quad + x \arcsin\left(\frac{z}{\sqrt{1-x^2}}\right) + z \arcsin\left(\frac{x}{\sqrt{1-z^2}}\right) - \arcsin\left(\frac{xz}{\sqrt{(1-x^2)(1-z^2)}}\right) \\
&\quad + y \arcsin\left(\frac{z}{\sqrt{1-y^2}}\right) + z \arcsin\left(\frac{y}{\sqrt{1-z^2}}\right) - \arcsin\left(\frac{yz}{\sqrt{(1-y^2)(1-z^2)}}\right) \\
&\quad - x \arcsin\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2}}\right) - \sqrt{1-x^2-y^2} \arcsin\left(\frac{x}{\sqrt{x^2+y^2}}\right) \\
&\quad + \arcsin\left(\frac{x\sqrt{1-x^2-y^2}}{\sqrt{(1-x^2)(x^2+y^2)}}\right) - y \arcsin\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-y^2}}\right) \\
&\quad - \sqrt{1-x^2-y^2} \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \arcsin\left(\frac{y\sqrt{1-x^2-y^2}}{\sqrt{(1-y^2)(x^2+y^2)}}\right).
\end{aligned}$$

By (22), however,

$$\begin{aligned}
\sqrt{1-x^2-y^2} \arcsin\left(\frac{x}{\sqrt{x^2+y^2}}\right) \;&+\; \sqrt{1-x^2-y^2} \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right) \\
&= \sqrt{1-x^2-y^2}\left(\frac{\pi}{2}\right),
\end{aligned}$$

so if we define $h(x,y)$ by

$$\begin{aligned}
h(x,y) \;=\; &\arcsin\left(\frac{xy}{\sqrt{(1-x^2)(1-y^2)}}\right) + \arcsin\left(\frac{x\sqrt{1-x^2-y^2}}{\sqrt{1-x^2}\sqrt{x^2+y^2}}\right) \\
&+ \arcsin\left(\frac{y\sqrt{1-x^2-y^2}}{\sqrt{1-y^2}\sqrt{x^2+y^2}}\right)
\end{aligned} \tag{25}$$

for $0 \le x, y \le 1$ and $x^2 + y^2 < 1$, then

$$\begin{aligned}
&4\pi\bar{F}(x,y,z)\\
&= -\frac{\pi}{2}z + h(x,y) - \arcsin\left(\frac{xy}{\sqrt{(1-x^2)(1-y^2)}}\right)\\
&\quad + x\arcsin\left(\frac{z}{\sqrt{1-x^2}}\right) + z\arcsin\left(\frac{x}{\sqrt{1-z^2}}\right) - \arcsin\left(\frac{xz}{\sqrt{(1-x^2)(1-z^2)}}\right)\\
&\quad + y\arcsin\left(\frac{z}{\sqrt{1-y^2}}\right) + z\arcsin\left(\frac{y}{\sqrt{1-z^2}}\right) - \arcsin\left(\frac{yz}{\sqrt{(1-y^2)(1-z^2)}}\right)\\
&\quad - x\arcsin\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2}}\right) - y\arcsin\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-y^2}}\right)\\
&= -\frac{\pi}{2}z + h(x,y) + \alpha(x,y)\\
&\quad - x\arcsin\left(\frac{y}{\sqrt{1-x^2}}\right) - y\arcsin\left(\frac{x}{\sqrt{1-y^2}}\right)\\
&\quad + x\arcsin\left(\frac{z}{\sqrt{1-x^2}}\right) + z\arcsin\left(\frac{x}{\sqrt{1-z^2}}\right) - \arcsin\left(\frac{xz}{\sqrt{(1-x^2)(1-z^2)}}\right)\\
&\quad + y\arcsin\left(\frac{z}{\sqrt{1-y^2}}\right) + z\arcsin\left(\frac{y}{\sqrt{1-z^2}}\right) - \arcsin\left(\frac{yz}{\sqrt{(1-y^2)(1-z^2)}}\right)\\
&\quad - x\arcsin\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2}}\right) - y\arcsin\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-y^2}}\right)
\end{aligned}$$

where $\alpha(x,y)$ is given by (16). Now (22) gives

$$x\arcsin\left(\frac{y}{\sqrt{1-x^2}}\right) + x\arcsin\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2}}\right) = x\left(\frac{\pi}{2}\right)$$

$$y\arcsin\left(\frac{x}{\sqrt{1-y^2}}\right) + y\arcsin\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-y^2}}\right) = y\left(\frac{\pi}{2}\right)$$

so the above simplifies to

$$4\pi\bar{F}(x,y,z) = -\frac{\pi}{2}(x+y+z) + h(x,y) + 2\pi\Delta(x,y,z) \tag{26}$$

where

$$\Delta(x,y,z) = \alpha(x,y) + \alpha(x,z) + \alpha(y,z) \tag{27}$$

a symmetric function of (x,y,z). By (22), however,

$$\begin{aligned}
h(x,y) &= \frac{\pi}{2} - \arcsin\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{(1-x^2)(1-y^2)}}\right) + \arcsin\left(\frac{x\sqrt{1-x^2-y^2}}{\sqrt{1-x^2}\sqrt{x^2+y^2}}\right)\\
&\quad + \arcsin\left(\frac{y\sqrt{1-x^2-y^2}}{\sqrt{1-y^2}\sqrt{x^2+y^2}}\right)\\
&\equiv \frac{\pi}{2} - \gamma + \alpha + \beta
\end{aligned}$$

and

$$
\begin{aligned}
&\sin(\alpha+\beta) \\
&= \sin\alpha\cos\beta+\cos\alpha\sin\beta \\
&= \frac{x\sqrt{1-x^2-y^2}}{\sqrt{1-x^2}\sqrt{x^2+y^2}}\frac{x}{\sqrt{1-y^2}\sqrt{x^2+y^2}}+\frac{y}{\sqrt{1-x^2}\sqrt{x^2+y^2}}\frac{y\sqrt{1-x^2-y^2}}{\sqrt{1-y^2}\sqrt{x^2+y^2}} \\
&= \frac{\sqrt{1-x^2-y^2}}{\sqrt{(1-x^2)(1-y^2)}} \\
&= \sin\gamma
\end{aligned}
$$

so $\alpha+\beta=\gamma$, hence $h(x,y)\equiv\frac{\pi}{2}$ identically in (x,y). Therefore we conclude that

$$\bar{F}(x,y,z)=\frac{1-x-y-z}{8}+\frac{\Delta(x,y,z)}{2} \tag{28}$$

for $0\le x,y,z\le 1$ and $x^2+y^2+z^2<1$.

We now apply (28) to obtain the cdf $F(x,y,z)$ for all $(x,y,z)\in C_3$. For this, extend the definition of Δ in (27) to all $(x,y,z)\in C_3$ by means of (16) and (18).

Theorem 4.1. *The unique spherical copula $F(x,y,z)$ on C_3 is given as follows: for $x^2+y^2+z^2<1$,*

$$F(x,y,z)=\begin{cases}\dfrac{1+x+y+z}{8}+\dfrac{\Delta(x,y,z)}{2}, & \text{if } x^2+y^2+z^2<1\\[2mm] \dfrac{1+x+y+z}{8}+\dfrac{\Delta(x,y,z)}{2} & \\ +\,\sigma(xyz)\left[\dfrac{1-|x|-|y|-|z|}{8}+\dfrac{\Delta(|x|,|y|,|z|)}{2}\right], & \text{if } x^2+y^2+z^2\ge 1.\end{cases}$$

Proof. See [9]. □

5. A One-Parameter Family of Elliptical Copulas

Let $(X,Y)\sim f(x,y)$ in (6), the unique circularly symmetric distribution on the unit disk B_2 with uniform $[-1,1]$ marginals. For any angle $\gamma\in(-\pi/2,\pi/2)$, consider the transformed variables

$$U=X,\quad V_\gamma=X\sin\gamma+Y\cos\gamma. \tag{29}$$

By the circular symmetry of (X,Y), $V_\gamma\overset{d}{=}Y\sim$ uniform$[-1,1]$, so the random vector (U,V_γ) again generates a copula on the centered square C_2. Denote the pdf and cdf of (U,V_γ) by $f_\gamma(u,v)$ and $F_\gamma(u,v)$ respectively. Then $\{F_\gamma\mid\gamma\in(-\pi/2,\pi/2)\}$ is a one-parameter family of *elliptical copulas*, so-called because the support of (U,V_γ) is the ellipse

$$E_\gamma:=\{(u,v)\mid u^2+v^2-2uv\sin\gamma\le\cos^2\gamma\}. \tag{30}$$

(Note that $E_0=B_2$.) From (29), the correlation coefficient of U and V_γ is given simply by

$$\rho(U,V_\gamma)=\sin\gamma \tag{31}$$

so γ indicates the degree of linear dependence between U and V_γ.

Proposition 5.1. *The pdf of* (U, V_γ) *is given by*

$$f_\gamma(u,v) = \frac{1}{2\pi\sqrt{\cos^2\gamma - u^2 - v^2 + 2uv\sin\gamma}} 1_{E_\gamma}(u,v). \tag{32}$$

Proof. The pdf can be obtained by a standard Jacobian computation. From (29),

$$x(u,v) = u, \quad y(u,v) = \frac{v - u\sin\gamma}{\cos(\gamma)} \tag{33}$$

so

$$\begin{aligned} \frac{\partial x(u,v)}{\partial u} &= 1, & \frac{\partial x(u,v)}{\partial v} &= 0, \\ \frac{\partial y(u,v)}{\partial u} &= -\tan\gamma, & \frac{\partial y(u,v)}{\partial v} &= \frac{1}{\cos\gamma}. \end{aligned}$$

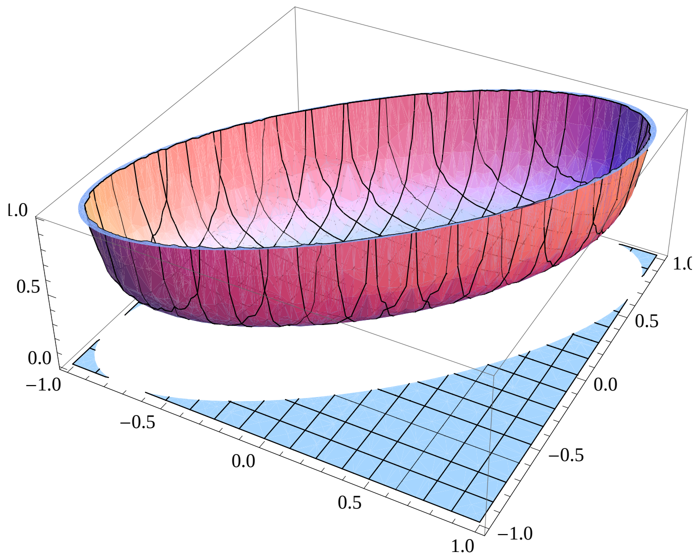
Thus the Jacobian of the transformation is $J = 1/\cos\gamma$, so from (6) we obtain

$$\begin{aligned} f_\gamma(u,v) &= f(x(u,v), y(u,v)) \cdot J \\ &= \frac{1}{2\pi\sqrt{1 - u^2 - (v - u\sin\gamma)^2/\cos^2\gamma}} 1_{B_2}(x(u,v), y(u,v)) \cdot \frac{1}{\cos\gamma} \\ &= \frac{\cos\gamma}{2\pi\sqrt{(1-u^2)\cos^2\gamma - (v - u\sin\gamma)^2}} \frac{1}{\cos\gamma} \cdot 1_{B_2}(x(u,v), y(u,v)) \\ &= \frac{1}{2\pi\sqrt{\cos^2\gamma - u^2 - v^2 + 2uv\sin\gamma}} 1_{E_\gamma}(u,v). \end{aligned}$$

□

Figure 8 shows the density $f_\gamma(u,v)$ with $\gamma = \pi/4$.

Figure 8. The density $f_\gamma(u,v)$ in (32) (with $\gamma = \pi/4$).



To describe the family of elliptical copulas F_γ, we extend the definitions (16) and (18) as follows. First, for $(u,v) \in E_\gamma$ define

$$\begin{aligned} \alpha_\gamma(u,v) &= \frac{1}{2\pi}\left[u \arcsin\left(\frac{v - u\sin\gamma}{\cos\gamma\sqrt{1-u^2}}\right) + v\arcsin\left(\frac{u - v\sin\gamma}{\cos\gamma\sqrt{1-v^2}}\right)\right. \\ &\qquad \left. - \arcsin\left(\frac{uv - \sin\gamma}{\sqrt{(1-u^2)(1-v^2)}}\right)\right]. \end{aligned} \tag{34}$$

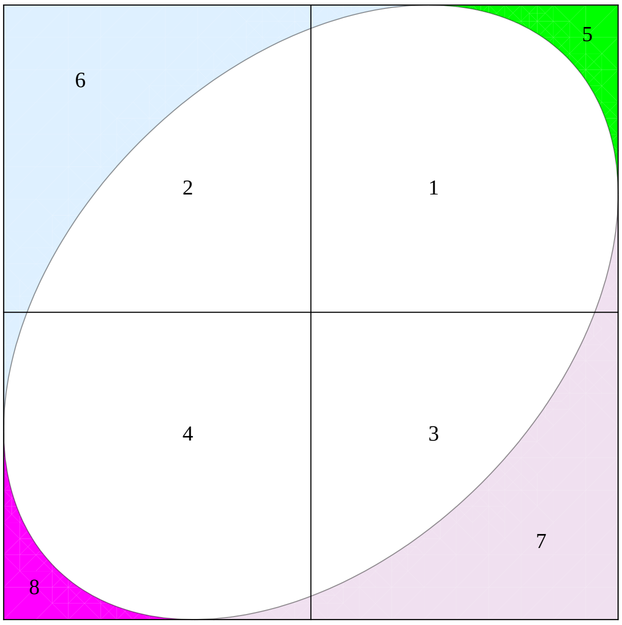
Note that α_γ reduces to α in (16) when $\gamma = 0$, *i.e.*, when $V_\gamma = Y$. From (12),

$$A(\arccos u,\ \arccos v;\ \frac{\pi}{2} - \gamma) = (1 - u - v)\pi + 4\pi\alpha_\gamma(u, v). \tag{35}$$

Next, extend the definition of $\alpha_\gamma(u, v)$ to $C_2 \setminus E_\gamma$ as follows (see Figure 9):

$$\alpha_\gamma(u, v) = \begin{cases} \dfrac{u+v-1}{4} & \text{if } (u, v) \in R_5(\gamma) := (C_2 \setminus E_\gamma) \cap \{(u, v) \mid u + v > 1 + \sin\gamma\}, \\ \dfrac{u-v+1}{4} & \text{if } (u, v) \in R_6(\gamma) := (C_2 \setminus E_\gamma) \cap \{(u, v) \mid v - u > 1 - \sin\gamma\}, \\ \dfrac{-u+v+1}{4} & \text{if } (u, v) \in R_7(\gamma) := (C_2 \setminus E_\gamma) \cap \{(u, v) \mid v - u < \sin\gamma - 1\}, \\ \dfrac{-u-v-1}{4} & \text{if } (u, v) \in R_8(\gamma) := (C_2 \setminus E_\gamma) \cap \{(u, v) \mid u + v < -\sin\gamma - 1\}. \end{cases} \tag{36}$$

Figure 9. Eight regions $R_1(\gamma) - R_8(\gamma)$ for an elliptical copula (with $\gamma = \pi/8$).



Note that (34) and (36) agree on ∂E_γ, *i.e.*, when $u^2 + v^2 - 2uv\sin\gamma = \cos^2\gamma$. Also note that (36) reduces to α in (18) when $\gamma = 0$. The following lemma will be useful for the proof of Theorem 5.1.

Lemma 5.1. *Let (U, V) be a bivariate random vector in C_2 with uniform $[-1, 1]$ marginals that satisfies $(U, V) \overset{d}{=} (-U, -V)$. Then the cdf $F(u, v)$ satisfies*

$$F(u, v) = \frac{u+v}{2} + F(-u, -v), \quad (u, v) \in C_2. \tag{37}$$

Proof. By the symmetry condition,

$$\begin{aligned} F(u, v) &= P[-U \le u,\ -V \le v] = P[U \ge -u,\ V \ge -v] \\ &= 1 - P[U < -u] - P[V < -v] + P[U < -u,\ V < -v] \\ &= 1 - \left(\frac{-u+1}{2}\right) + \left(\frac{-v+1}{2}\right) + P[U \le -u,\ V \le -v] \\ &= \frac{u+v}{2} + F(-u,\ -v). \end{aligned}$$

□

Theorem 5.1. *The cdf ≡ copula of* (U, V_γ) *is given by (see Figure 12)*

$$F_\gamma(u, v) = \frac{u + v + 1}{4} + \alpha_\gamma(u, v), \qquad (u, v) \in C_2. \tag{38}$$

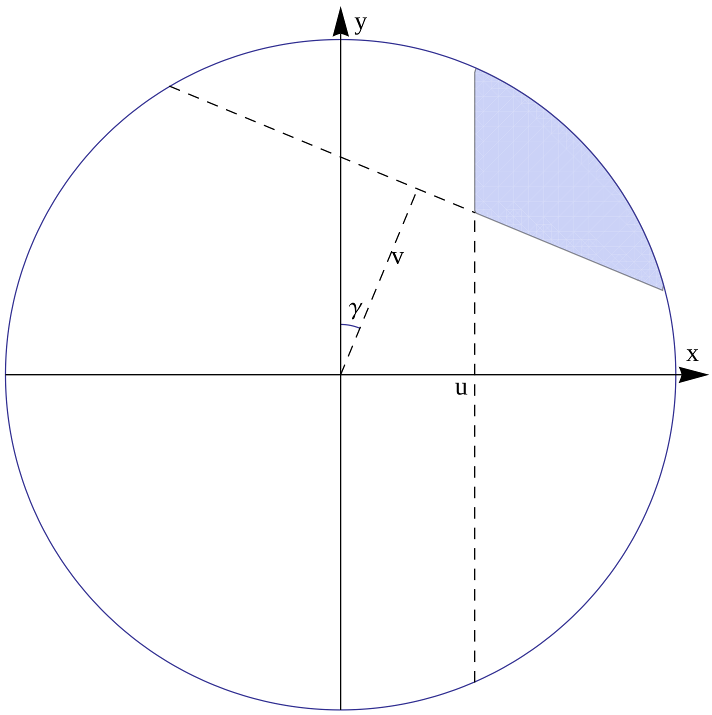
Proof. To find $F_\gamma(u, v)$ we again use the formula (12) for the area of the intersection of two spherical caps on ∂B_3. Here, unlike (14), the axes of the two caps are not necessarily perpendicular. The single formula (38) is obtained by considering the partition $C_2 = \cup_{i=1}^{8} R_i(\gamma)$, where $R_5(\gamma) - R_8(\gamma)$ are defined in (36) and (see Figure 9)

$$\begin{aligned} R_1(\gamma) &= E_\gamma \cap \{(u, v) \mid 0 \le u, v \le 1\}, \\ R_2(\gamma) &= E_\gamma \cap \{(u, v) \mid -1 \le u \le 0 \le v \le 1\}, \\ R_3(\gamma) &= E_\gamma \cap \{(u, v) \mid -1 \le v \le 0 \le u \le 1\}, \\ R_4(\gamma) &= E_\gamma \cap \{(u, v) \mid -1 \le u, v \le 0\}. \end{aligned}$$

Case 1: $(u, v) \in R_1(\gamma)$. By using Figure 10,

$$\begin{aligned} F_\gamma(u, v) &= P[U \le u,\ V_\gamma \le v] \\ &= 1 - P[U > u] - P[V_\gamma > v] + P[U > u,\ V_\gamma > v] \\ &= 1 - \left(\frac{1-u}{2}\right) - \left(\frac{1-v}{2}\right) + P[X > u,\ X \sin\gamma + Y \cos\gamma > v] \\ &= \frac{u+v}{2} + \frac{1}{4\pi} A(\arccos u, \arccos v;\ \pi/2 - \gamma) \\ &= \frac{u+v+1}{4} + \alpha_\gamma(u, v) \qquad \text{[by (35)]}. \end{aligned}$$

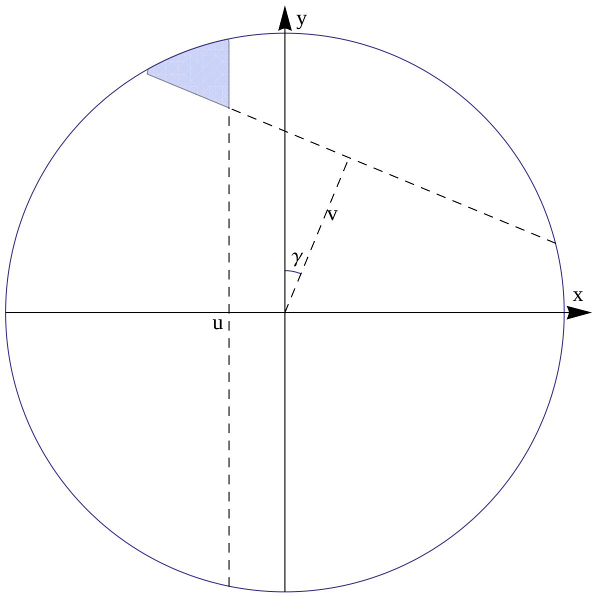
Figure 10. The region $[X > u, X\sin(\gamma) + Y\cos(\gamma) > v]$ for Case 1 (with $\gamma = \pi/8$).



Case 2: $(u, v)) \in R_2(\gamma)$. Because $(X, Y) \stackrel{d}{=} (-X, Y)$ and using Figure 11

$$\begin{aligned} F_\gamma(u, v) &= P[U \leq u] - P[U \leq u,\ V_\gamma > v] \\ &= \frac{u+1}{2} - P[X \leq u,\ X \sin\gamma + Y\cos\gamma > v] \\ &= \frac{u+1}{2} - P[-X \leq u,\ -X \sin\gamma + Y\cos\gamma > v] \\ &= \frac{u+1}{2} - P[X \geq -u,\ X \sin(-\gamma) + Y\cos(-\gamma) > v] \\ &= \frac{u+1}{2} - \frac{1}{4\pi} A(\arccos(-u), \arccos v;\ \pi/2 + \gamma) \\ &= \frac{u+v+1}{4} - \alpha_{-\gamma}(-u, v) \qquad \text{[by (35)]} \\ &= \frac{u+v+1}{4} + \alpha_\gamma(u, v) \qquad \text{[by (34)]}. \end{aligned}$$

Figure 11. The region $[X < u, X\sin(\gamma) + Y\cos(\gamma) > v]$ for Case 2 (with $\gamma = \pi/8$).



Case 3: $(u, v) \in R_3(\gamma)$. Then $(-u, -v) \in R_2(\gamma)$, so by Lemma 5.1 and Case 2,

$$\begin{aligned} F_\gamma(u, v) &= \frac{u+v}{2} + F_\gamma(-u, -v) \\ &= \frac{u+v}{2} + \frac{-u-v+1}{4} + \alpha_\gamma(-u, -v) \\ &= \frac{u+v+1}{4} + \alpha_\gamma(u, v) \qquad \text{[by (34)]}. \end{aligned}$$

Case 4: $(u, v) \in R_4(\gamma)$. Then $(-u, -v) \in R_1(\gamma)$, so by Lemma 5.1 and Case 1, the argument for Case 3 applies verbatim.

Case 5: $(u, v) \in R_5(\gamma)$.

$$
\begin{aligned}
F_\gamma(u,v) &= 1 - P[U > u] - P[V_\gamma > v] \\
&= 1 - \left(\frac{1-u}{2}\right) - \left(\frac{1-v}{2}\right) \\
&= \frac{u+v+1}{4} + \alpha_\gamma(u,v) \qquad\qquad [\text{by } (36)].
\end{aligned}
$$

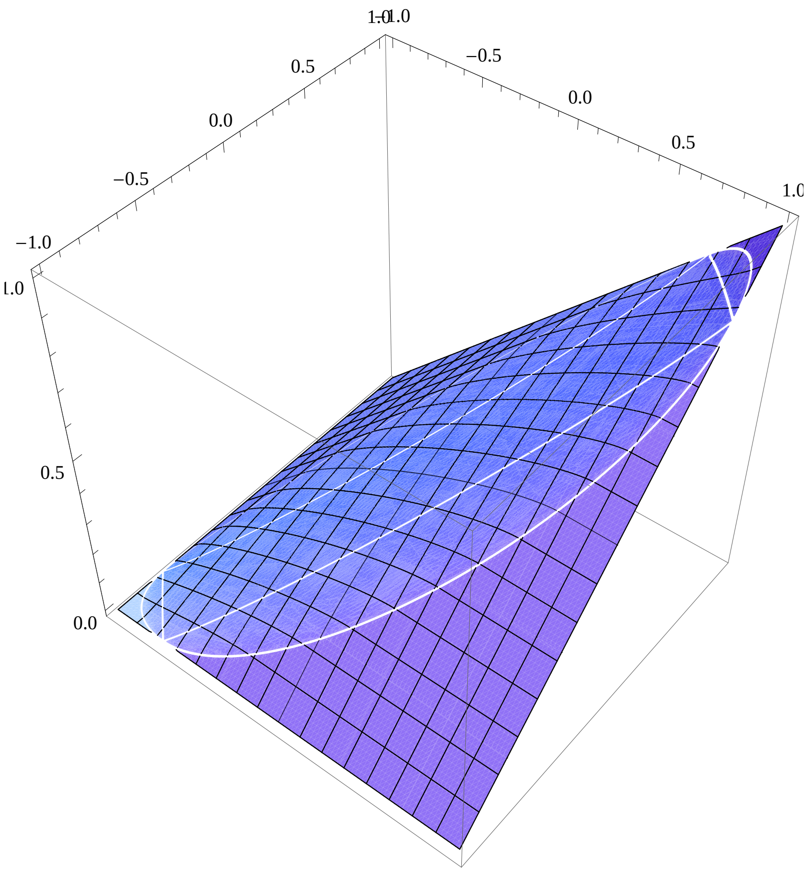
Case 6: $(u, v) \in R_6(\gamma)$.

$$
\begin{aligned}
F_\gamma(u,v) &= P[U \le u] \\
&= \frac{u+1}{2} \\
&= \frac{u+v+1}{4} + \alpha_\gamma(u,v) \qquad\qquad [\text{by } (36)].
\end{aligned}
$$

Case 7: $(u, v) \in R_7(\gamma)$. Then $(-u, -v) \in R_6(\gamma)$, so by Lemma 5.1 and Case 6, the argument for Case 3 applies verbatim.

Case 8: $(u, v) \in R_8(\gamma)$. Then $(-u, -v) \in R_5(\gamma)$, so by Lemma 5.1 and Case 5, the argument for Case 3 applies verbatim. □

Figure 12. The copula $F_\gamma(u, v)$ in (38) (with $\gamma = \pi/4$).



6. Copulas Derived from the Uniform Distribution on the Unit Ball

Up to now we have addressed the question of whether copulas can be generated by means of linear functions of a circularly symmetric or spherically symmetric random vector. Now we ask whether non-linear functions of such random vectors can generate copulas. We shall restrict attention to random vectors uniformly distributed over the unit ball B_d and produce relatively simple non-linear functions that generate copulas on C_d.

We begin with the bivariate case. Suppose that (X, Y) is distributed uniformly on the unit disk $B_2 = \{(x, y) \in \mathbb{R}^2 \mid \; x^2 + y^2 \leq 1\}$. Because

$$\begin{aligned} && X \mid Y &\sim \text{uniform}[-\sqrt{1-Y^2},\, \sqrt{1-Y^2}] \\ \text{and} && Y \mid X &\sim \text{uniform}[-\sqrt{1-X^2},\, \sqrt{1-X^2}] \end{aligned}$$

it follows that the random variables

$$U := \frac{X}{\sqrt{1-Y^2}}, \qquad V := \frac{Y}{\sqrt{1-X^2}} \tag{39}$$

satisfy

$$\begin{aligned} U \mid Y &\sim \text{uniform}[-1, 1] \\ V \mid X &\sim \text{uniform}[-1, 1]. \end{aligned}$$

Thus, U and Y are independent, V and X are independent, and unconditionally,

$$\begin{aligned} U &\sim \text{uniform}[-1, 1] \\ V &\sim \text{uniform}[-1, 1] \end{aligned}$$

so the joint distribution of (U, V) generates a copula $F(u, v)$ on the centered cube $C_2 \equiv [-1, 1]^2$. Note that U and V are not linear functions of (X, Y).

Question 6: Are U and V independent, and if not, what is the nature of their dependence?

Answer: Clearly U and V are uncorrelated, since $E(U) = E(V) = 0$ and

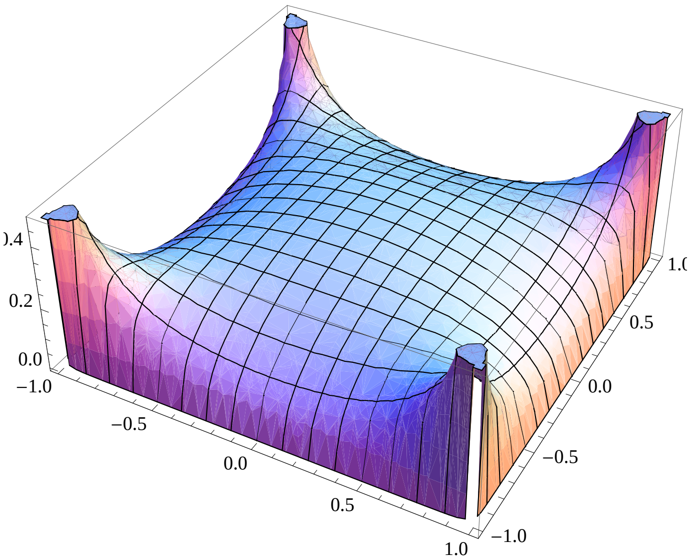
$$E(UV) = E\left(\frac{XY}{\sqrt{(1-X^2)(1-Y^2)}}\right) = 0$$

all by the circular symmetry of (X, Y). However, the joint pdf and cdf of (U, V) derived below show that they are not independent.

Proposition 6.1. *The joint density of (U, V) is given by (see Figure 13)*

$$f(u, v) = \frac{1}{\pi} \frac{\sqrt{(1-u^2)(1-v^2)}}{(1-u^2v^2)^2} 1_{C_2}(u, v). \tag{40}$$

Figure 13. Joint density $f(u, v)$ of (U, V) in (40).



Proof. This pdf is again obtained via the Jacobian method. It follows from (39) that

$$u^2(1-y^2) = x^2, \qquad \text{and} \qquad v^2(1-x^2) = y^2.$$

Substitution of the second expression for y^2 into the left side of the first relation and vice versa yields

$$x^2 = \frac{u^2(1-v^2)}{1-u^2v^2}, \qquad y^2 = \frac{v^2(1-u^2)}{1-u^2v^2}$$

so, since x and u (y and v) have the same signs by (39), we obtain

$$x \equiv x(u,v) = \frac{u\sqrt{1-v^2}}{\sqrt{1-u^2v^2}}, \qquad y \equiv y(u,v) = \frac{v\sqrt{1-u^2}}{\sqrt{1-u^2v^2}}. \tag{41}$$

Thus

$$\begin{aligned}\frac{\partial x}{\partial u} &= \sqrt{1-v^2}\left[\frac{1}{\sqrt{1-u^2v^2}} + u(1-u^2v^2)^{-3/2}(uv^2)\right] \\ &= \frac{\sqrt{1-v^2}}{\sqrt{1-u^2v^2}}\left\{1 + \frac{u^2v^2}{1-u^2v^2}\right\} \\ &= \frac{\sqrt{1-v^2}}{(1-u^2v^2)^{3/2}},\end{aligned}$$

$$\begin{aligned}\frac{\partial x}{\partial v} &= u\left[\frac{(1-v^2)^{-1/2}(-v)}{\sqrt{1-u^2v^2}} + \sqrt{1-v^2}(1-u^2v^2)^{-3/2}(u^2v)\right] \\ &= \frac{uv}{\sqrt{1-v^2}\sqrt{1-u^2v^2}}\left\{-1 + \frac{u^2(1-v^2)}{1-u^2v^2}\right\} \\ &= -\frac{uv(1-u^2)}{\sqrt{1-v^2}(1-u^2v^2)^{3/2}}.\end{aligned}$$

By symmetry it follows that the Jacobian is given by

$$J = \begin{pmatrix} \dfrac{\sqrt{1-v^2}}{(1-u^2v^2)^{3/2}} & -\dfrac{uv(1-u^2)}{\sqrt{1-v^2}(1-u^2v^2)^{3/2}} \\ -\dfrac{uv(1-v^2)}{\sqrt{1-u^2}(1-u^2v^2)^{3/2}} & \dfrac{\sqrt{1-u^2}}{(1-u^2v^2)^{3/2}} \end{pmatrix}$$

and hence the determinant of J is given by

$$|J| = \frac{\sqrt{(1-u^2)(1-v^2)}}{(1-u^2v^2)^2}.$$

Because the pdf of (X, Y) is $f(x, y) = \frac{1}{\pi}1_{B_2}(x, y)$, the result (40) follows. $\square$

For $0 \leq u, v \leq 1$, $(u, v) \neq (1, 1)$, let $E_1(u)$ and $E_2(v)$ be the ellipses

$$E_1(u) = \left\{(x, y) \;\middle|\; \frac{x^2}{u^2} + y^2 \leq 1\right\} \tag{42}$$

$$E_2(v) = \left\{(x, y) \;\middle|\; x^2 + \frac{y^2}{v^2} \leq 1\right\}. \tag{43}$$

The next lemma leads to the cdf $F(u, v)$ corresponding to the pdf (40).

Lemma 6.1.

$$\mathrm{Area}(E_1(u) \cap E_2(v)) = 2u \arcsin\left(\frac{v\sqrt{1-u^2}}{\sqrt{1-u^2v^2}}\right) + 2v \arcsin\left(\frac{u\sqrt{1-v^2}}{\sqrt{1-u^2v^2}}\right).$$

Proof. Define the points o, a, b, c, d, d, f, g as follows: see Figure 14,

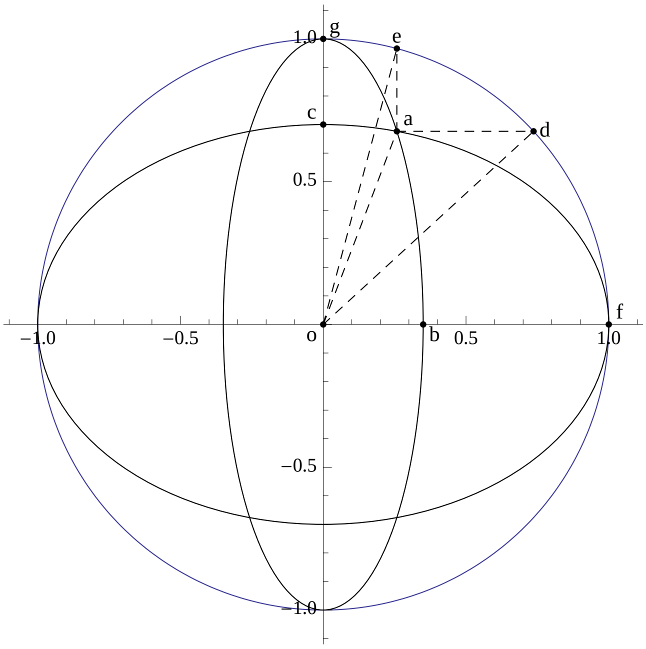
$$\begin{aligned}
o &= (0, 0) \\
a &= (x(u, v),\, y(u, v)) = \left(\frac{u\sqrt{1-v^2}}{\sqrt{1-u^2v^2}}, \frac{v\sqrt{1-u^2}}{\sqrt{1-u^2v^2}}\right) \\
b &= (u, 0) \\
c &= (0, v) \\
d &= (\sqrt{1-y^2(u, v)},\, y(u, v)) \\
e &= (x(u, v),\, \sqrt{1-x^2(u, v)}) \\
f &= (1, 0) \\
g &= (0, 1).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{1}{4}\mathrm{Area}(E_1(u) \cap E_2(v)) &= \mathrm{Area}(oab) + \mathrm{Area}(oac) \\
&= u\,\mathrm{Area}(odf) + v\,\mathrm{Area}(oeg) \\
&= \frac{u}{2}\arcsin(y(u, v)) + \frac{v}{2}\arcsin(x(u, v))
\end{aligned}$$

from which the result follows. $\square$

Figure 14. Integration regions for Lemma 6.2.



Theorem 6.1. *The copula (= cdf) corresponding to the pdf* (40) *is given by (see Figure 15)*

$$F(u,v) = \frac{u+v+1}{4} + \frac{u}{2\pi}\arcsin\left(\frac{v\sqrt{1-u^2}}{\sqrt{1-u^2v^2}}\right) + \frac{v}{2\pi}\arcsin\left(\frac{u\sqrt{1-v^2}}{\sqrt{1-u^2v^2}}\right), \quad (u,v) \in C_2.$$

Proof. Because (U,V) is sign-change invariant and has uniform $[-1,1]$ marginals, it follows from (7) and (9) in Lemma 3.1 and from (39) that for $(u,v) \in C_2$,

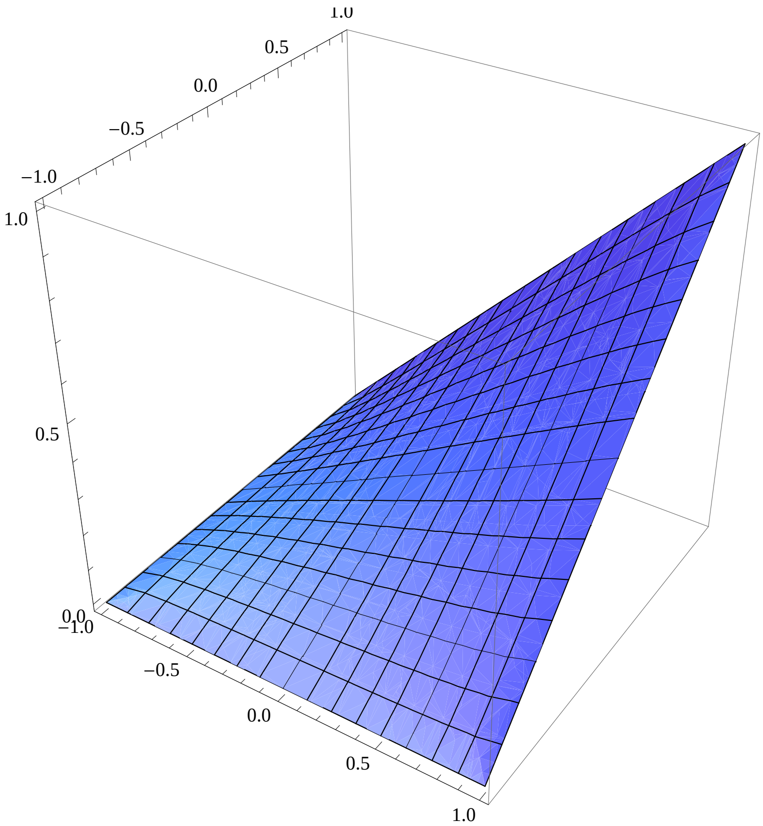
$$\begin{aligned} F(u,v) &= \frac{u+v+1}{4} + \sigma(uv)\, P[0 \le U \le |u|,\, 0 \le V \le |v|] \\ &= \frac{u+v+1}{4} + \frac{\sigma(uv)}{4} P[U^2 \le u^2,\, V^2 \le v^2] \\ &= \frac{u+v+1}{4} + \frac{\sigma(uv)}{4} P[(X,Y) \in E_1(u) \cap E_2(v)] \\ &= \frac{u+v+1}{4} + \frac{\sigma(uv)}{4\pi}\mathrm{Area}(E_1(|u|) \cap E_2(|v|)). \end{aligned}$$

The result now follows from Lemma 6.1. □

Remark: The construction (39) extends readily to generate a copula on C_d. For $d = 3$, for example, let (X,Y,Z) be uniformly distributed on the unit ball B_3 and define

$$U := \frac{X}{\sqrt{1-Y^2-Z^2}}, \quad V := \frac{Y}{\sqrt{1-X^2-Z^2}}, \quad W := \frac{Z}{\sqrt{1-X^2-Y^2}}.$$

Figure 15. Nonlinear transformation copula $F(u, v)$ in Theorem 6.1.



Then the marginal distributions of U, V, and W are each uniform $[-1, 1]$ so the cdf $G(u, v, w)$ is a copula on C_3. To find this copula one would need to determine $\mathrm{Volume}(E_1(u) \cap E_2(v) \cap E_3(w))$, where now, for $0 \leq u, v, w \leq 1$, $E_1(u)$, $E_2(v)$, and $E_3(w)$ are the ellipsoids

$$\begin{aligned} E_1(u) &= \left\{(x, y, z) \,\middle|\, \frac{x^2}{u^2} + y^2 + z^2 \leq 1\right\} \\ E_2(v) &= \left\{(x, y, z) \,\middle|\, x^2 + \frac{y^2}{v^2} + z^2 \leq 1\right\} \\ E_3(w) &= \left\{(x, y, z) \,\middle|\, x^2 + y^2 + \frac{y^2}{w^2} \leq 1\right\}. \end{aligned}$$

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