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An Adaptive Certifying Semi-Algorithm for Darboux Integrating Factors of Rational Second-Order Ordinary Differential Equations

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Abstract

We present an adaptive certifying semi-algorithm for rational second-order ordinary differential equations (ODEs) in the Darboux-representative, non-degenerate subclass of the Liouvillian setting considered here. The method first computes a nonlocal symmetry and the associated polynomial vector field, and then uses auxiliary associated fields to obtain linear equations for the polynomial data involved in a Darboux integrating factor. The adaptive step factors the residual coefficient space, tests the resulting factors as Darboux polynomials, and reinserts the certified factors into the next linear search. For fixed degree bounds, the procedure uses only linear algebra, factorization, and Darboux-polynomial tests; its correctness does not depend on the adaptive choices, but on a final symbolic certification of the returned integrating factor and first integral. The method is therefore a certifying search procedure for the stated subclass, not a completeness result for all rational second-order ODEs with Liouvillian first integrals. Five certified computational examples, implemented in Maple, illustrate substantial reductions in the number of undetermined coefficients and document the final symbolic verification step.

Keywords: Darboux integrability; rational second-order ordinary differential equations; Liouvillian first integrals; nonlocal symmetries; adaptive algorithms; symbolic computation

MSC: 34A05; 34A34; 37C10; 68W30

1. Introduction

The symbolic computation of first integrals and integrating factors for nonlinear ordinary differential equations is a central problem in the interface between differential equations, computer algebra, and applied mathematics. For rational second order ordinary differential equations (rational 2ODEs), Liouvillian first integrals are commonly approached through Lie symmetries, hidden and nonlocal symmetries, λ -symmetries, Darboux theory, and extensions of the Prelle–Singer method [1–13]. These approaches are powerful, but difficult examples arise when the equation has no Lie point symmetries, when the relevant Darboux polynomials have high degree, or when a direct three-dimensional Darboux search produces a large number of undetermined coefficients.

The S-function method introduced in [14], and later improved in [15], offers an alternative route: it constructs an associated rational first order equation whose implicit general solution is a first integral of the original 2ODE. Equivalently, this amounts to computing a nonlocal symmetry, or a λ -symmetry, without splitting the determining



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equation into a large overdetermined system of partial differential equations. In difficult cases, however, the associated first order equation may still be hard to solve by standard computer-algebra routines.

Nonlinear ordinary differential equations also arise as reductions of nonlinear wave models in applied settings, including soliton dynamics. Recent examples include cavitation-driven soliton-like mechanisms in microrobotic launching [16] and data-driven studies of multicolor soliton formation and switching in fiber lasers [17]. The present paper is not a study of these physical models, but such examples illustrate why symbolic access to first integrals and integrating factors remains useful in nonlinear dynamics.

The goal of this paper is computational. We use the nonlocal symmetry not as the final reduction step, but as a device for building linear algebraic problems whose solutions reveal the Darboux factors of an integrating factor. More precisely, the nonlocal symmetry determines three associated planar polynomial vector fields that share the first integral and an integrating factor of the rational 2ODE. We then construct auxiliary planar fields for which this integrating factor is a first integral. The identities relating the associated fields and the auxiliary fields are linear in the unknown polynomial components.

The resulting method is an adaptive certifying semi-algorithm for the Darboux-representative, non-degenerate subclass specified below. For prescribed degree bounds, it forms polynomial candidates, solves linear systems, factors the remaining coefficient polynomials, tests the resulting factors for the Darboux property, and then adapts the next candidate by inserting the verified Darboux factors already found. The word adaptive is used in this operational sense: the next search space depends on the information extracted from the previous residual linear space. No random or probability-based claim is made. Correctness is instead obtained by certification: every integrating factor and first integral returned by the method is checked by direct symbolic substitution in the defining identities. Thus, the method is reliable when it returns a certified answer, but it is not a decision procedure for the existence of Liouvillian first integrals in the unrestricted rational 2ODE class.

This separation between search and certification is important. The method is reliable when it returns an answer, but it is not claimed to terminate in complete generality, since no general a priori upper bound is known for the degrees of the auxiliary polynomial candidates. Thus, failure means failure for the prescribed Darboux representative, degree bounds, and adaptive refinements, not nonexistence of a Darboux integrating factor.

We also isolate a structural point which explains the scope of the Darboux-representative assumption. The planar Singer–Christopher theorem gives Darboux integrating factors for the associated planar fields, while the original integrating factor R is also an integrating factor for those fields. The quotient of two such integrating factors is a first integral of the corresponding associated field. Since each associated field has one passive coordinate, this quotient may a priori depend on both the common first integral I and the passive coordinate. However, the quotient is not an arbitrary first integral when the planar integrating factor is Darboux; it belongs to a Darboux-relative subclass. We formulate below the resulting quotient-rigidity problem, since a positive answer would recover a stronger version of the original Darboux-representative theorem.

Main Results and Originality

The contribution of the paper is not a replacement for the S-function method or for the general Darboux–Prelle–Singer theory. It is a certifying computational refinement for the subclass in which the associated Darboux quotient can be represented within the polynomial ansatz used by the algorithm. Its originality lies in combining three ingredients in a single adaptive workflow: the nonlocal-symmetry construction of associated planar

fields, linear systems for the auxiliary polynomial components and quotient Y , and a factor-test-refine loop in which only factors passing Darboux tests are reinserted into later linear searches.

The main results are as follows. First, we give a non-degenerate Darboux-representative formulation under which the quotient $Y = R/\mathcal{R}$ is polynomial for the selected representative. Secondly, we describe ALDIF as a certifying semi-algorithm whose search stage uses linear algebra, factorization, and Darboux-polynomial tests, while correctness is established only by final symbolic verification. Thirdly, we provide five Maple-certified examples, including cases where the direct S-function route did not return the desired solution under the stated computational limits. These examples are not claimed to prove global superiority; they document a reproducible reduction mechanism and identify the remaining theoretical obstruction, namely the quotient-rigidity problem.

In this article, we construct an adaptive linear semi-algorithm to compute the Darboux polynomials (DPs) present in the integrating factor of a rational 2ODE (The problem of determining, from a polynomial vector field, a Darboux integrating factor (or determining that such a factor does not exist) is old and difficult. Since a Darboux integrating factor is formed by Darboux polynomials (DPs), a previous problem consists of, given a polynomial vector field, determining the invariant algebraic curves/hypersurfaces that define the DPs. For an overview see [18–33] and references therein.). The paper is organized as follows. Section 2 recalls the algebraic computation of a nonlocal symmetry in the class L_S and fixes the notation used later. Section 3.1 contains the main construction: first, the nonlocal symmetry is used to define three associated planar polynomial vector fields sharing the first integral and an integrating factor of the original 2ODE; then auxiliary planar fields are introduced whose first integral is the Darboux integrating factor. This gives linear equations for the unknown polynomial components and leads to the adaptive semi-algorithm ALDIF. Section 4.1 presents five certified computational tests, compares the procedure with the S-function method in the same computer-algebra environment, records the reproducibility data available from the Maple pre-implementation, and describes the present limitations and possible extensions of the method.

2. Materials and Methods

Finding a Nonlocal Symmetry

In this section, we present an algorithm to determine a symmetry of a rational 2ODE that has an LFI belonging to the class L_S defined below. The procedure builds on the method developed in [14]; in the present paper, it is used as the first stage of a certifying Darboux-factor search rather than as a stand-alone claim of general superiority over the S-function method. First, we will make some definitions and establish some results in a context more appropriate to what we will present.

Consider a rational 2ODE

$$z' = \frac{M(x, y, z)}{N(x, y, z)} = \phi(x, y, z), \quad (z \equiv y'), \quad (1)$$

where M and N are coprime polynomials in $\mathbb{C}[x, y, z]$. A first integral I of the 2ODE (1) is a function that is constant over the solutions of (1).

Definition 1. Let L be a Liouvillian field extension (For a formal definition of Liouvillian field extension see [34].) of $\mathbb{C}(x, y, z)$. A function $I(x, y, z) \in L$ is said to be a **Liouvillian first integral (LFI)** of the rational 2ODE (1) if $\mathfrak{X}(I) = 0$, where $\mathfrak{X} \equiv N\partial_x + zN\partial_y + M\partial_z$ is the **polynomial vector field associated (or Darboux operator associated)** with the 2ODE (1).

Definition 2. Let Y be a polynomial vector field over a polynomial ring and let p be a nonconstant polynomial in that ring. The polynomial p is said to be a **Darboux polynomial (DP)** of Y if $Y(p) = q p$, where q is a polynomial called the **cofactor** of p with respect to Y .

Definition 3. Let $I(x, y, z)$ be an LFI of the rational 2ODE (1) and consider that its derivatives can be written as $\partial_i(I) = R P_i$ ($i \in \{1, 2, 3\}$, $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$) where P_i are coprime polynomials in $\mathbb{C}[x, y, z]$ and R is a Liouvillian function of (x, y, z) . In this case, we say that I is a member of the set L_S and that R is an **integrating factor** associated with I .

Definition 4. Let ω be a polynomial 1-form defined by $\omega \equiv \sum_i P_i dx_i$. A function G is said to be an **integrating factor** for the 1-form ω if the 1-form $G \omega$ is exact.

Remark 1. From Definitions 3 and 4, we have that R is an integrating factor for the 1-form ω .

Definition 5. Let $I \in L_S$ be as above. We define $\mathfrak{J} \equiv \sum_i P_i \partial_i$.

The italic symbol I denotes the scalar first integral, whereas $\mathfrak{J}$ denotes the polynomial vector field built from the components P_i . The notation is deliberately related through $\nabla I = R(P_1, P_2, P_3)$, but the two objects are distinct throughout the paper.

Remark 2. The condition $\mathfrak{X}(I) = 0$ is equivalent to $\langle \mathfrak{X} | \mathfrak{J} \rangle = N P_1 + z N P_2 + M P_3 = 0$. (In what follows, the operators $\nabla(\cdot)$, $\langle \nabla | \cdot \rangle$, $\nabla \wedge \cdot$ stand for **grad**, **div**, **curl**, respectively.)

The vector field $\mathfrak{J}$ can be used to build a symmetry for the 2ODE (1):

Theorem 1. Let $I \in L_S$ be a first integral of the rational 2ODE (1) as above. Then, in evolutionary form, 2ODE (1) admits the nonlocal symmetry generated by

$$\mathfrak{S} = e^{\int_x (-P_2/P_3)} \partial_y, \quad (2)$$

where $\int_x$ denotes a local inverse of the total derivative $D_x \equiv \mathfrak{X}/N$ along the solutions, up to an additive first integral.

Proof. For the vector field $\mathfrak{S} = e^{\int_x (-P_2/P_3)} \partial_y$, its first prolongation along the rational 2ODE (1), then its first prolongation is given by

$$\mathfrak{S}^{(1)} = e^{\int_x (-P_2/P_3)} \partial_y + D_x \left(e^{\int_x (-P_2/P_3)} \right) \partial_z = e^{\int_x (-P_2/P_3)} (\partial_y - (P_2/P_3) \partial_z),$$

and, so, $\mathfrak{S}^{(1)}(I) = e^{\int_x (-P_2/P_3)} (I_y - (P_2/P_3) I_z)$. Since $I \in L_S$, $I_y = R P_2$ and $I_z = R P_3$. Hence $I_y - \frac{P_2}{P_3} I_z = 0$, and therefore $\mathfrak{S}^{(1)}(I) = 0$. By the standard first-integral criterion for generalized symmetries of a scalar second order ODE, this gives the stated nonlocal symmetry. $\square$

Therefore, if we find the vector field $\mathfrak{J}$ (or at least two of its components) then we have the symmetry $\mathfrak{S}$. We first show that the conditions the polynomials P_i must satisfy can be written as first order partial differential equations (1PDEs) of Riccati type.

Lemma 1. Let $I \in L_S$ and $\mathfrak{J}$ be as above. Then the vector field defined by $\mathfrak{L} \equiv \nabla \wedge \mathfrak{J}$ obeys the condition $\langle \mathfrak{L}, \mathfrak{J} \rangle = 0$.

Proof. From the hypotheses of the lemma $I_i = R P_i$. So, we can write $\mathfrak{J}$ as $\frac{\nabla(I)}{R}$ implying that

$$\mathfrak{L} = \nabla \wedge \mathfrak{J} = \frac{1}{R} \underbrace{\nabla \wedge \nabla(I)}_{=0} + \nabla \left(\frac{1}{R} \right) \wedge \nabla(I).$$

$$\text{So, } \langle \mathfrak{L}, \mathfrak{J} \rangle = \left\langle \nabla \left(\frac{1}{R} \right) \wedge \nabla(I), \frac{\nabla(I)}{R} \right\rangle = 0. \quad \square$$

Lemma 2. Let $I \in L_S$, $\mathfrak{X}$, $\mathfrak{J}$, and $\mathfrak{L}$ be defined as above. Then $\mathfrak{X} \wedge \mathfrak{L} = \frac{\mathfrak{X}(R)}{R} \mathfrak{J}$.

Proof. From Lemma 1, we have that $\mathfrak{L} = \nabla \left(\frac{1}{R} \right) \wedge \nabla(I)$. So (Using the identity $A \wedge (B \wedge C) = \langle A, C \rangle B - \langle A, B \rangle C$)

$$\mathfrak{X} \wedge \mathfrak{L} = \mathfrak{X} \wedge \left(\nabla \left(\frac{1}{R} \right) \wedge \nabla(I) \right) = \underbrace{\langle \mathfrak{X} | \nabla(I) \rangle}_{\mathfrak{X}(I)} \nabla \left(\frac{1}{R} \right) - \langle \mathfrak{X} | \nabla \left(\frac{1}{R} \right) \rangle \nabla(I).$$

$$\text{Since } \mathfrak{X}(I) = 0, \quad \mathfrak{X} \wedge \mathfrak{L} = -\langle \mathfrak{X} | -\frac{\nabla(R)}{R^2} \rangle \nabla(I) = \frac{\mathfrak{X}(R)}{R} \frac{\nabla(I)}{R} = \frac{\mathfrak{X}(R)}{R} \mathfrak{J}. \quad \square$$

Remark 3. What was stated in Lemma 1 is the well known integrability condition $\omega \wedge d\omega = 0$ for the 1-form ω . The point of writing the Lemmas 1 and 2 using the language of vector fields is simply because it is more suitable to obtain some algebraic conclusions that we are going to present.

Proposition 1. Let $I \in L_S$, $\mathfrak{X}$, $\mathfrak{J}$ be as above. Then the polynomials P_2, P_3 obey the following equations:

$$P_2 \frac{\mathfrak{X}(R)}{R} + N \partial_y \left(\frac{M}{N} \right) P_3 + \mathfrak{X}(P_2) = 0, \quad (3)$$

$$P_3 \frac{\mathfrak{X}(R)}{R} + N P_2 + N \partial_z \left(\frac{M}{N} \right) P_3 + \mathfrak{X}(P_3) = 0. \quad (4)$$

Proof. We indicate the algebraic reduction. From the hypotheses of the theorem, we have that $\langle \mathfrak{X} | \mathfrak{J} \rangle = N P_1 + z N P_2 + M P_3 = 0$. Solving it for P_1 , and substituting the result in $\mathfrak{X} \wedge \mathfrak{L} = \frac{\mathfrak{X}(R)}{R} \mathfrak{J}$ (Lemma 2) one obtains, after rearranging terms, the conditions expressed by 1PDEs Equations (3) and (4). $\square$

Remark 4. There are analogous conditions for the other two pairs $\{P_1, P_3\}$ and $\{P_1, P_2\}$ that result from solving the equation $N P_1 + z N P_2 + M P_3 = 0$ for P_2 and P_3 , respectively, and substituting the results into the equation $\mathfrak{X} \wedge \mathfrak{L} = \frac{\mathfrak{X}(R)}{R} \mathfrak{J}$.

Remark 5. We can eliminate the term $\frac{\mathfrak{X}(R)}{R}$ from Equations (3) and (4) and obtain an equation for the polynomials P_2 and P_3 . Analogously, we can eliminate it from the pairs of equations for $\{P_1, P_3\}$ and $\{P_1, P_2\}$, obtaining another two equations (see Theorem 2 below).

Theorem 2. Let $I \in L_S$, $\mathfrak{X}$, $\mathfrak{J}$ be defined as above. In the field $\mathbb{C}(x, y, z)$, and for $P_k \neq 0$ and $F_i \neq 0$, the following equations hold:

$$\mathfrak{X} \left(\frac{P_j}{P_k} \right) = F_i \partial_k \left(\frac{F_j}{F_i} \right) \left(\frac{P_j}{P_k} \right)^2 + F_i \left(\partial_k \left(\frac{F_k}{F_i} \right) - \partial_j \left(\frac{F_j}{F_i} \right) \right) \left(\frac{P_j}{P_k} \right) - F_i \partial_j \left(\frac{F_k}{F_i} \right), \quad (5)$$

where $F_1 = N$, $F_2 = z N$, $F_3 = M$, and (i, j, k) is any permutation of the set $\{1, 2, 3\}$ (i.e., there is no summation over the repeated indices in Equation (5)).

Proof. From the hypotheses of the theorem, we have the following conditions: $\langle \mathfrak{X} | \mathfrak{J} \rangle = \langle \mathfrak{L} | \mathfrak{J} \rangle = 0$. As in the proof of Proposition 1 (to avoid tedious calculations), we only indicate the path: solve the equation $\langle \mathfrak{X} | \mathfrak{J} \rangle = 0$ for one of the polynomials P_i and substitute the solution in the equation $\langle \mathfrak{L} | \mathfrak{J} \rangle = 0$. We obtain (depending on which polynomial we choose) the condition expressed by (5) for the two other polynomial functions. $\square$

Remark 6. Equation (8) are quadratic in the rational functions $\left(\frac{P_i}{P_k}\right)$, making their resolution a difficult computational problem.

To avoid the difficulty pointed out in the above observation, we can make use of the following result:

Proposition 2. Let $I \in L_S$ and let $\mathfrak{J}$ be defined as above. If P_3 and $P_1 + zP_2$ are coprime polynomials, then $P_3 = cN$ and $P_1 + zP_2 = -cM$ for some $c \in \mathbb{C}^*$. After the normalization described above, one may take $P_3 = N$.

Proof. From the hypotheses, $I_i = R P_i$ and therefore

$$\phi(x, y, z) = -\frac{I_x + z I_y}{I_z} = -\frac{R P_1 + z R P_2}{R P_3} = -\frac{P_1 + z P_2}{P_3} = \frac{M}{N}. \quad (6)$$

Since M and N are coprime, and since P_3 and $P_1 + zP_2$ are coprime, equality of the two reduced rational functions implies $P_3 = cN$ and $P_1 + zP_2 = -cM$ for some $c \in \mathbb{C}^*$. Rescaling the triple (P_1, P_2, P_3) by c^{-1} and R by c gives the normalization $P_3 = N$. $\square$

Remark 7. The only case where $P_3 \neq N$ can occur is if the polynomials P_3 and $P_1 + zP_2$ have a non-constant polynomial factor in common. We will deal with this ‘degenerate’ case in another paper.

Remark 8. The non-degenerate setting used in this paper also excludes identically zero associated-field components and non-coprime component pairs at the stages where the Darboux-representative normalization is invoked. These exclusions are structural, not numerical: in such cases, one may need a different representative, a preliminary transformation, or a separate degenerate analysis. Repeated Darboux factors and linearly dependent cofactors, by contrast, are not excluded by the procedure itself. The factorization step records repeated irreducible factors with their multiplicities, and each retained factor is tested independently by the Darboux-polynomial identity; the final output is accepted only after the integrating factor and first integral are certified by direct substitution.

Corollary 1. Under the normalization $P_3 = N$ supplied by Proposition 2, P_2 obeys the equation

$$\mathfrak{X}(P_2) = P_2^2 + (N_x + z N_y + M_z) P_2 + M N_y - M_y N. \quad (7)$$

Proof. Choosing $i = 1, j = 2, k = 3$ in (5), we have that

$$\mathfrak{X}\left(\frac{P_2}{N}\right) = N \partial_z \left(\frac{z N}{N}\right) \left(\frac{P_2}{N}\right)^2 + N \left(\partial_z \left(\frac{M}{N}\right) - \partial_y \left(\frac{z N}{N}\right)\right) \left(\frac{P_2}{N}\right) - N \partial_y \left(\frac{M}{N}\right). \quad (8)$$

Multiplying both sides of (8) by N^2 and isolating the term $\mathfrak{X}(P_2)$ on the left hand side, we arrive at Equation (7). $\square$

Equation (7) already indicates a good algorithm for finding the nonlocal symmetry: we could construct a polynomial candidate with undetermined coefficients p_i , substitute it into (7), and try to solve the resulting algebraic system for the undetermined coefficients. However, the presence of the term P^2 implies that the algebraic system is quadratic in the p_i , which can be very difficult to solve (computationally speaking) if the degree of the 2ODE is relatively high. In order to make the process more efficient, we will use another corollary to Proposition 2.

Corollary 2. Assume that the hypotheses of Proposition 2 hold and that the normalization $P_3 = N$ has been chosen. If no monomial term of $z P_2$ cancels with a monomial term of P_1 in the identity $M = -(P_1 + z P_2)$, then every monomial of P_2 occurs in M_z .

Proof. The conclusion follows directly from the fact that $M = -(P_1 + z P_2)$. $\square$

Therefore, if the conditions of corollary 2 hold, we can just use the monomials of M_z to construct the candidate P_c .

In this form, Algorithm 1 is a finite symbolic procedure once the candidate monomials have been fixed. The non-cancellation hypothesis explains why the monomials of M_z are sufficient in the case considered here; when cancellations occur, a preliminary transformation may be used as in Remark 10. This is why we refer to NLS as an algorithm for a fixed candidate regime, while ALDIF below is called a semi-algorithm in the broader adaptive setting.

Algorithm 1: NLS, non-cancellation case.

Input: coprime polynomials $M, N \in \mathbb{C}[x, y, z]$ defining the rational 2ODE $z' = M/N$, and a degree/search regime for the class L_S .

Output: either a polynomial P_2 defining the nonlocal symmetry $\mathfrak{S} = e^{\int_x (-P_2/N)} \partial_y$, or failure for the chosen search regime.

1: Select the monomials from M_z : $[mon_1, mon_2, \dots, mon_k]$.

2: Construct a polynomial candidate P_c with undetermined coefficients p_i : $P_c := \sum_{i=1}^k p_i mon_i$.

3: Substitute P_c in equation $\mathfrak{X}(P_2) - P_2^2 - (N_x + z N_y + M_z) P_2 - M N_y + M_y N = 0$ and collect the resulting polynomial equation in the variables (x, y, z) obtaining a finite algebraic system for the coefficients p_i .

4: Solve this system and substitute each admissible solution in the candidate P_c to obtain a candidate P_2 .

5: Certify the candidate by substituting it back into the Riccati-type equation. If the identity is satisfied, return P_2 and the symmetry $\mathfrak{S} = e^{\int_x (-P_2/N)} \partial_y$; otherwise report failure for the chosen candidate set.

Example 1. Consider the 2ODE

$$z' = -\frac{x(-2x^2yz^3 + 3xy^2z^2 + x^3z - 2xyz + 2y^2)}{x^5 + y^4}. \quad (9)$$

Applying procedure NLS to 2ODE (9):

- The monomials of M_z are: $[x^3yz^2, x^2y^2z, x^4, x^2y]$.
- $P_c = p_1 x^3yz^2 + p_2 x^2y^2z + p_3 x^4 + p_4 x^2y$.
- From steps 3 and 4 we obtain the solution:
 $\{p_1 = -2, p_2 = 0, p_3 = 0, p_4 = -2\}$.
 Substituting it on P_c , we obtain: $P_2 = -2yx^2(xz^2 + 1)$.

Remark 9. The algorithm NLS spent 0.05 s of CPU time and approximately 0 MB of memory use (These data (referring to CPU time and memory use) were established in a test of the algorithm NLS implemented in Maple 17 running on a notebook with an Intel i5 processor—4 GB-8th Gen.).

Remark 10. In the case (rare) where some of the monomial terms of $z P_2$ and P_1 cancel, we can follow a strategy similar to the one used in the blow up technique (which is used in desingularization

of degenerate singular points of planar vector fields—see [35] and references therein). The idea is to perform a transformation of variables that changes the monomials of z P_2 and P_1 in a disproportionate way, avoiding cancellation.

Let us see how we can deal with the situation described in Remark 10 (in practice) in the next example:

Example 2. Consider the 2ODE

$$z' = \frac{z(x^2z^5 - x^2z^4 + 2xyz^3 - xz^4 - 2xyz^2 - yz^3 + xz^2 + y^2z - 2yz^2 - y^2 - yz)}{(xz^2 - xz + yz + y)(xz^2 - y)}. \quad (10)$$

If we apply the algorithm NLS, we will see that it fails to determine the symmetry. However, applying the transformation $\{x = x^2, y = y\}$ to the 2ODE, we obtain the following transformed 2ODE:

$$\begin{aligned} z' = & -z(2x^3z^4 - z^5x^2 + 16x^3yz^2 - 8x^2yz^3 + 32x^3y^2 - 8x^3z^2 - \\ & 16x^2y^2z + 2x^2z^3 + xz^4 + 8x^2yz + 16xyz^2 + 2yz^3 + 16xy^2 + \\ & 8zy^2)/(x(-z^2 + 4y)(2zx^2 - xz^2 - 4xy - 2yz)) \end{aligned} \quad (11)$$

This time, applying the algorithm NLS to the transformed 2ODE, we get

$$P_2 = -(xz^4 + 8xyz^2 + 8zx^2 + 16xy^2 - 4xz^2 - 16xy - 8yz)xz, \quad (12)$$

Remark 11. The CPU time and memory use for the procedure NLS applied to the transformed 2ODE (12) was 0.08 s and 0 MB approximately.

3. Construction of the Method

3.1. Using a Nonlocal Symmetry to Determine a Darboux Integrating Factor

In this section, we use a nonlocal symmetry to construct a Darboux integrating factor for the 2ODE (1):

- In the first subsection, we use the nonlocal symmetry to construct three associated planar polynomial vector fields (with the remaining variable acting as a parameter) such that they ‘share’ the first integral and an integrating factor with the 2ODE.
- In the second subsection, we introduce three further associated planar polynomial vector fields whose first integral is the Darboux integrating factor of the 2ODE. These vector fields can be found together with the Darboux polynomials present in the integrating factor of the 2ODE by solving **linear systems of indeterminates**.
- Finally, we propose an adaptive certifying semi-algorithm to determine a Darboux integrating factor for the rational 2ODE and present an example.

3.2. Planar Vector Fields Associated with the 2ODE

Theorem 3. Let $I \in L_S$, $\mathfrak{X}$ and $\mathfrak{J}$ be defined as above. Then, the following statements hold:

- (a) The polynomial vector fields defined by $\mathfrak{X}_i \equiv -\sum_{j,k} \epsilon_{ijk} P_j \partial_k$, ($i, j, k \in \{1, 2, 3\}$), admit I as a first integral, i.e., $\mathfrak{X}_i(I) = 0$.
- (b) $\mathfrak{X}_i(R) = -R \langle \nabla | \mathfrak{X}_i \rangle$.

Proof. (a) Statement (a) follows directly from the definition:

$$\begin{aligned} \mathfrak{X}_i(I) &= \langle \mathfrak{X}_i | \mathfrak{J} \rangle = \langle -\sum_{j,k} \epsilon_{ijk} P_j \partial_k | \sum_a P_a \partial_a \rangle = -\sum_{j,k} \epsilon_{ijk} P_j P_k = 0. \\ (b) \text{ We have that } \nabla \wedge \nabla(I) &= \nabla \wedge (R \mathfrak{J}) = \nabla(R) \wedge \mathfrak{J} + R \nabla \wedge \mathfrak{J} = 0. \end{aligned}$$

$$\begin{aligned}\nabla(R) \wedge \mathfrak{I} &= \sum_{j,k} \epsilon_{ijk} \partial_j(R) P_k \partial_i = - \sum_{j,k} \epsilon_{ijk} P_j \partial_k(R) \partial_i = \sum_i \mathfrak{X}_i(R) \partial_i. \\ R \nabla \wedge \mathfrak{I} &= R \sum_{j,k} \epsilon_{ijk} \partial_j(P_k) \partial_i = R \sum_i \langle \nabla | \mathfrak{X}_i \rangle \partial_i. \quad \square\end{aligned}$$

Remark 12. Note that the associated vector fields $\mathfrak{X}_i$ (see statement (a) of Theorem 3) present I as a Liouvillian first integral (the same LFI admitted by the vector field $\mathfrak{X}$). Therefore, by the results of Singer [19] and Christopher [21], they admit a Darboux integrating factor.

Remark 13. Note also that, in view of statement (b) (Theorem 3), the integrating factor R of the vector field $\mathfrak{X}$ is also an integrating factor for the vector fields $\mathfrak{X}_i$.

The preceding remarks motivate the Darboux-representative setting used below:

Remark 14. In what follows, we work in the Darboux-representative case: the integrating factor R associated with the first integral $I \in L_S$ is assumed to admit a Darboux representation, in the sense made precise in Definition 8. This is the class used by the construction and by all certified examples below. The planar Singer–Christopher theorem motivates the use of Darboux integrating factors for the auxiliary planar vector fields $\mathfrak{X}_i$, but it is not used here as an unconditional proof that every rational 2ODE with a first integral in L_S yields a global Darboux integrating factor for the three-dimensional field $\mathfrak{X}$. Consequently, ALDIF should be read as a certifying semi-algorithm for the Darboux-representative, non-degenerate cases that pass the final symbolic verification.

Remark 15. The preceding restriction is connected with a quotient-rigidity issue. Let R_i be a Darboux integrating factor of one of the planar associated fields $\mathfrak{X}_i$, whose existence is supplied by the planar Singer–Christopher theorem. Since R is also an integrating factor of $\mathfrak{X}_i$, the quotient

$$G_i := \frac{R_i}{R}$$

satisfies $\mathfrak{X}_i(G_i) = 0$. This is immediate from the fact that the quotient of two integrating factors of the same planar field is a first integral of that field. Nevertheless, because $\mathfrak{X}_i$ has one passive coordinate x_i , the conclusion $\mathfrak{X}_i(G_i) = 0$ only implies, locally and away from singular sets, that

$$G_i = H_i(I, x_i),$$

not necessarily that $G_i = \mathcal{F}_i(I)$. The extra information is that R_i is Darboux: hence G_i is not an arbitrary first integral of $\mathfrak{X}_i$, but a quotient between a Darboux integrating factor of the planar field and the common integrating factor R .

Proposition 3. Let $I \in L_S$, let R be the integrating factor associated with I , and let R_i be a Darboux integrating factor of an associated planar field $\mathfrak{X}_i$. Put $G_i = R_i/R$. Suppose that the Darboux-relative first integral G_i contains a nonconstant factor depending only on I , in the sense that there exists a nonconstant one-variable function Φ such that $\Phi(I)$ is a Darboux function. Then the three-dimensional field $\mathfrak{X}$ admits a Darboux first integral and, consequently, a Darboux integrating factor.

Proof. If $J = \Phi(I)$ is a Darboux function and Φ is nonconstant, then

$$\mathfrak{X}(J) = \Phi'(I)\mathfrak{X}(I) = 0$$

wherever the expression is defined. Thus J is a Darboux first integral of $\mathfrak{X}$. Moreover

$$dJ = \Phi'(I) dI = \Phi'(I) R (P_1 dx + P_2 dy + P_3 dz).$$

Hence the corresponding integrating factor is $\Phi'(I)R$. In the Darboux case under consideration, this belongs to the Darboux class, except at the usual singular sets of the chosen representative. $\square$

Remark 16. Proposition 3 gives a useful dichotomy. If the Darboux-relative quotient G_i has an essential Darboux dependence on the common first integral I , then a Darboux first integral of $\mathfrak{X}$ is already obtained. If no such Darboux dependence on I is possible, then the remaining possible dependence of G_i is confined to the passive coordinate of the planar field, up to factors that can be absorbed in the choice of representative. A complete rigidity statement of this type, together with a compatibility argument for the three associated fields, would recover a stronger form of the original Darboux-representative assertion. We leave this as the structural issue behind the present Darboux-representative hypothesis, rather than using it as an unproved completeness theorem.

3.3. The Vector Fields Associated with $\mathfrak{X}_i$

In [33,36], the authors proposed a new way of determining Darboux polynomials present in the integrating factor of plane polynomial vector fields: The basic idea was to construct another polynomial vector field such that its first integral is a Darboux integrating factor of the original vector field. In this section, we will show how to adapt this idea to the current problem.

3.3.1. Some Important Results

To begin with, we will present some results obtained in [33] (unless by Proposition 4).

Definition 6. Let $X_0 \equiv f_0 \partial_x + g_0 \partial_y$ (f_0 and g_0 are coprime polynomials in $\mathbb{C}[x, y]$) be a polynomial vector field presenting a Liouvillian first integral I_0 and, consequently, a Darboux integrating factor R_0 . Let $X_1 \equiv f_1 \partial_x + g_1 \partial_y$ (where f_1 and g_1 are coprime polynomials in $\mathbb{C}[x, y]$) be another polynomial vector field such that $X_1(R_0) = 0$. We call X_1 an **associated vector field through the integrating factor R_0** .

Remark 17. Since any function of the first integral I_0 (which is invariant under the flow of the vector field X_0) multiplied by the integrating factor R_0 is itself an integrating factor $\bar{R}_0$ ($\bar{R}_0 \equiv \mathcal{G}(I_0) R_0$), $\bar{X}_1(\bar{R}_0) = 0$ defines an equivalence class of vector fields: $[\bar{X}_1]$.

To prove the following theorem, first we need to prove that there is always a polynomial vector field such that $X_1(R_0) = 0$.

Proposition 4. Let X_0 be a polynomial vector field defined as above, and let R_0 be a Darboux integrating factor for X_0 . Then there exists a polynomial vector field X_1 such that $X_1(R_0) = 0$.

Proof. Write

$$R_0 = e^{Z_0} \prod_j p_j^{n_j},$$

where $Z_0 \in \mathbb{C}(x, y)$, the $p_j \in \mathbb{C}[x, y]$ are irreducible, and the n_j are constants. Hence the logarithmic derivatives

$$A := \partial_x \log R_0 = \partial_x Z_0 + \sum_j n_j \frac{\partial_x p_j}{p_j}, \quad B := \partial_y \log R_0 = \partial_y Z_0 + \sum_j n_j \frac{\partial_y p_j}{p_j}$$

are rational functions. If R_0 is constant, any non-zero polynomial vector field may be chosen. Suppose therefore that R_0 is nonconstant. If $B \neq 0$, write the rational function $-A/B$ as a/b , with $a, b \in \mathbb{C}[x, y]$ coprime. Then

$$X_1 := b \partial_x + a \partial_y$$

is polynomial and

$$X_1(R_0) = R_0(bA + aB) = 0.$$

If $B \equiv 0$, then $A \not\equiv 0$ and $\partial_y R_0 = 0$, so the choice $X_1 = \partial_y$ gives $X_1(R_0) = 0$. This proves the assertion. $\square$

Theorem 4. Let X_0 and X_1 be polynomial vector fields defined as in Definition 6 above. Then

$$g_0 f_1 - g_1 f_0 = \frac{R_0}{R_1} (f_{0x} + g_{0y}), \quad (13)$$

where R_1 is an integrating factor for the vector field X_1 .

Proof. By hypothesis, we have that $X_1(R_0) = 0$ (i.e., R_0 is a first integral of X_1). Therefore $R_{0x} = R_1 g_1$, $R_{0y} = -R_1 f_1$, implying that

$$g_0 f_1 - g_1 f_0 = g_0 \frac{-R_{0y}}{R_1} - f_0 \frac{R_{0x}}{R_1} = -\frac{X_0(R_0)}{R_1} = \frac{R_0}{R_1} \langle \nabla | X_0 \rangle = \frac{R_0}{R_1} (f_{0x} + g_{0y}).$$

$\square$

Corollary 3. $\frac{R_0}{R_1}$ is an inverse integrating factor for the vector field X_1 .

Proof. Since R_0 is a first integral for the vector field X_1 , then $\frac{R_1}{R_0}$ is also an integrating factor for the vector field X_1 . $\square$

Corollary 4. $\frac{R_0}{R_1}$ is a polynomial or $\langle \nabla | X_0 \rangle$ has a polynomial factor in common with R_1 or R_0 .

Proof. Since $g_0 f_1 - g_1 f_0$ is a polynomial, then $\frac{R_0}{R_1} (f_{0x} + g_{0y})$ is a polynomial. So the conclusion follows directly. $\square$

Remark 18. If R_0/R_1 is a polynomial, we can write the Equation (13) as

$$g_0 f_1 - g_1 f_0 - Y (f_{0x} + g_{0y}) = 0, \quad Y \equiv \frac{R_0}{R_1}, \quad (14)$$

and notice that it is linear in the unknown polynomials f_1 , g_1 , and Y .

3.3.2. Defining Vector Fields $\mathcal{X}_i$ Such That $\mathcal{X}_i(R) = 0$

In this subsection, we will apply the knowledge that was highlighted in Remark 18, together with the results shown in the previous subsection, to define three 2D polynomial vector fields $\mathcal{X}_i$ associated with $\mathfrak{X}_i$. First, remember that the three polynomial vector fields $\mathfrak{X}_i$ present the same Liouvillian first integral I and the same Darboux integrating factor R of $\mathfrak{X}$. Thus, there exist polynomial vector fields $\mathcal{X}_i$, which present R as their first integral.

Definition 7. Let $I \in L_S$, R , $\mathfrak{X}$, and $\mathfrak{X}_i$ defined as above. We define the polynomial vector fields $\mathcal{X}_i \equiv -\sum_{j,k} \epsilon_{ijk} \mathcal{P}_j \partial_k$ ($\mathcal{P}_i$ are coprime polynomials in $\mathbb{C}[x, y, z]$) by $\mathcal{X}_i(R) = 0$.

Remark 19. As observed in Remark 17, since any function of the first integral I multiplied by the integrating factor R is itself an integrating factor, the equation $\overline{\mathcal{X}}_i(R) = 0$ defines three equivalence classes of vector fields: $[\overline{\mathcal{X}}_i]$, $i \in \{1, 2, 3\}$.

Theorem 5. Let $I \in L_S$, R , $\mathfrak{X}$, $\mathfrak{X}_i$, and $\mathcal{X}_i$ be defined as above, and assume that the chosen representative R belongs to the Darboux-representative class. Then the vector fields $\mathcal{X}_i$ admit an integrating factor $\mathcal{R}$ such that

$$Y := \frac{R}{\mathcal{R}}$$

is an inverse integrating factor for $\mathcal{X}_i$, namely

$$\mathcal{X}_i(Y) = Y \langle \nabla | \mathcal{X}_i \rangle.$$

If for the selected representative, the possible denominator factors of $R/\mathcal{R}$ are not absorbed only through cancellations with the corresponding divergence term in the planar identity, then Y is a polynomial. This is the non-degenerate case used in ALDIF. When this polynomiality condition fails for the prescribed ansatz, the linear equations may still have solutions, but the polynomial candidate Y_c cannot represent the quotient $R/\mathcal{R}$ for that representative. In that case, ALDIF reports failure for the selected representative, equations, and degree bounds, rather than returning an uncertified result.

Proof. Apply the planar identity (13) to each coordinate plane and to the pair formed by $\mathfrak{X}_i$ and $\mathcal{X}_i$. Since $\mathcal{X}_i(R) = 0$, the quotient $R/\mathcal{R}$ is an inverse integrating factor for $\mathcal{X}_i$, as in Corollary 3. The polynomiality assertion is the corresponding non-cancellation case of Corollary 4, expressed for the selected Darboux representative. If a denominator factor of $R/\mathcal{R}$ is cancelled only by the divergence term in (13), the quotient need not be polynomial and is outside the polynomial ansatz used by ALDIF. The final sentence is part of the certifying interpretation of the algorithm: only candidates that satisfy the defining identities by direct substitution are accepted. $\square$

Theorem 6. Let the polynomials $\mathcal{P}_i$, P_i , and Y be as above. Then,

$$\langle \partial_i | \mathcal{X}_i \wedge \mathfrak{X}_i \rangle = Y \langle \nabla | \mathfrak{X}_i \rangle. \quad (15)$$

Proof. The proof follows directly from Remark 18, applied in the corresponding coordinate plane. The order of the wedge product fixes the signs used in E_1 – E_3 . $\square$

Remark 20. The equations $\langle \partial_i | \mathcal{X}_i \wedge \mathfrak{X}_i \rangle = Y \langle \nabla | \mathfrak{X}_i \rangle$ are linear in the unknown polynomials $\mathcal{P}_i$ and Y .

3.4. Constructing an Adaptive Linear Certifying Semi-Algorithm

We now construct a procedure to compute a Darboux integrating factor R of 2ODE (1). The construction uses the following property: since the vector fields $\mathfrak{X}_i$ share the integrating factor R with $\mathfrak{X}$, the Darboux polynomials occurring in R are also relevant to $\mathfrak{X}$ and to the auxiliary fields $\mathcal{X}_i$. In the non-degenerate polynomial case, this implies that the support of Y is contained in the set of Darboux polynomials shared by the vector fields $\mathfrak{X}_i$ and $\mathcal{X}_i$. We first fix the representative used for R .

Consider that the rational 2ODE (1) presents a first integral $I \in L_S$ and that we are in the Darboux-representative case of Remark 14. Since any function of the first integral I multiplied by an integrating factor is also an integrating factor, if the first integral I is non-elementary, then $\mathfrak{X}$ has only one Darboux integrating factor, up to multiplication by a constant, within the representative class considered here. Indeed, multiplying by a nonconstant function of a non-elementary first integral takes the representative outside this class. Hence, in this case, there is a natural choice for the integrating factor representing the equivalence class $[\mathcal{X}_i]$. In the case where the vector field $\mathfrak{X}$ presents a non-rational elementary first integral, we use the Prelle–Singer result [18] stating that, for a plane

polynomial vector field with an elementary first integral, there exists an integrating factor which is a k th root ($k \in \mathbb{N}$) of a rational function; we then choose an algebraic integrating factor to represent the class $[\overline{\mathcal{X}}_i]$. Finally, if the 2ODE presents a rational first integral, we consider the representative R of the class with minimum degree (As in this case the integrating factor will be rational, i.e., $R = \text{num}_R / \text{den}_R$, the degree of the integrating factor refers to the positive integer δ defined by $\delta = \max\{\deg(\text{num}_R), \deg(\text{den}_R)\}$).

Definition 8. Consider that the rational 2ODE (1) presents a first integral $I \in L_S$. Then

- (i) If I is non-elementary, the integrating factor R will be written as $R = e^{A/B} \prod_j p_j^{n_j}$, where p_j , A and B are polynomials in $\mathbb{C}[x, y, z]$, p_j are irreducible, A and B are coprime polynomials and B is a Darboux polynomial (possibly not irreducible).
- (ii) If I is elementary non-rational, the integrating factor R will be written as $R = \prod_j p_j^{n_j}$ (p_j in $\mathbb{C}[x, y, z]$ are irreducible polynomials and n_j are rational numbers).
- (iii) If I is rational, the integrating factor R will be written as $R = \prod_j p_j^{n_j}$ (p_j in $\mathbb{C}[x, y, z]$ are irreducible polynomials and n_j are integers such that $\delta \equiv \max\{\deg(\text{num}_R), \deg(\text{den}_R)\}$ is minimal).

Theorem 7. Consider that the rational 2ODE (1) presents a first integral $I \in L_S$ in the Darboux-representative class. Let the polynomial vector fields $\mathfrak{X}_i$, $\mathcal{X}_i$, the Darboux integrating factors R and $\mathcal{R}$ be defined as above, and suppose that

$$R = e^{A/B} \prod_j p_j^{n_j}$$

is written with $A, B \in \mathbb{C}[x, y, z]$ coprime and with the p_j irreducible. Assume also the non-degeneracy condition that no pole in the logarithmic derivatives of R is removed by an accidental cancellation outside the common irreducible factors explicitly present in $B \prod_j p_j$. Equivalently, for each irreducible factor q relevant to the logarithmic derivative of the chosen representative, the valuation of the numerator after applying $\mathcal{X}_i$ does not cancel the negative valuation forced by q unless q is invariant under $\mathcal{X}_i$. If $Y \equiv R/\mathcal{R}$ is polynomial, then:

- (a) Every irreducible factor of $B \prod_j p_j$ that is needed in the logarithmic derivatives of R is a Darboux polynomial of $\mathcal{X}_i$.
- (b) The polynomial Y has support only on these factors; in particular,

$$Y = \prod_u q_u^{k_u}, \quad k_u \in \mathbb{N},$$

where each q_u is an irreducible Darboux polynomial occurring among the factors of $B \prod_j p_j$.

Proof. We give the proof for the non-elementary case; the elementary and rational cases are obtained by deleting the exponential part or by taking the corresponding rational representative. Let

$$D := B \prod_j p_j$$

and let v_q denote the exponent of an irreducible polynomial q in a rational function. Since $\mathcal{X}_i(R) = 0$, we have

$$0 = \frac{\mathcal{X}_i(R)}{R} = \mathcal{X}_i\left(\frac{A}{B}\right) + \sum_j n_j \frac{\mathcal{X}_i(p_j)}{p_j}. \quad (16)$$

All possible poles in this logarithmic derivative are supported on the irreducible factors of D . Fix such a factor q . If $q = p_j$ for some j , the term $n_j \mathcal{X}_i(p_j)/p_j$ has negative q -valuation unless $q \mid \mathcal{X}_i(q)$. Since the full logarithmic derivative in (16) is identically zero, this negative

valuation must be canceled. In the non-degenerate case, such cancellation is allowed only among the explicit contributions supported on the same irreducible factor, hence the disappearance of the pole forces $q \mid \mathcal{X}_i(q)$. If q divides B , the term $\mathcal{X}_i(A/B)$ has possible negative q -valuation. Because A and B are coprime, the same valuation argument and the non-degeneracy hypothesis imply that the pole can disappear only when $q \mid \mathcal{X}_i(q)$. Thus every irreducible factor of D that is effectively present in the logarithmic derivatives is invariant under $\mathcal{X}_i$, i.e., is a Darboux polynomial of $\mathcal{X}_i$. This proves (a).

For (b), use $\partial_l(R) = \mathcal{R}\mathcal{P}_l$ and write

$$\frac{\partial_l R}{R} = \partial_l \left(\frac{A}{B} \right) + \sum_j n_j \frac{\partial_l p_j}{p_j} = \frac{H_l}{D_l}, \quad l = 1, 2, 3,$$

where the irreducible factors of each denominator D_l belong to the factor set of D . Hence

$$Y = \frac{R}{\mathcal{R}} = \frac{\mathcal{P}_l D_l}{H_l}$$

whenever the corresponding component is nonzero. Since Y is assumed polynomial and the representation is non-degenerate, no irreducible factor outside the factor set of D can occur in Y : any such factor would have to be introduced by a cancellation between H_l and $\mathcal{P}_l$ not explained by the denominator of the logarithmic derivative of R . Such a case is outside the structural statement. Therefore, the support of Y is contained in the support of D . By (a), these factors are Darboux polynomials of $\mathcal{X}_i$, and we can write $Y = \prod_u q_u^{k_u}$ with $k_u \in \mathbb{N}$. $\square$

Remark 21. The non-degeneracy hypothesis in Theorem 7 is a structural assumption on the selected representative. In the implementation, candidates that fall outside this structural statement are not accepted on formal grounds; a returned answer is accepted only after the final symbolic certification of the integrating factor and first integral. Thus factoring the residual coefficient space should be read as a detection mechanism inside the stated non-degenerate polynomial ansatz, not as an unconditional theorem that every relevant Darboux factor must appear in every residual parametrization. Accidental cancellations or irreducible parameter combinations may hide factors from a particular bounded search. In that event, ALDIF reports failure for the selected representative, equations and degree bounds, or the user changes the search regime; it does not certify a negative integrability statement.

In what follows, we use the non-degenerate form of Theorem 7 to construct an adaptive linear certifying semi-algorithm (ALDIF):

Remark 22. Algorithm 2 is an adaptive certifying semi-algorithm. For fixed degrees d_1, d_2, d_3 , and d_Y it consists of linear algebra, factorization, and Darboux-polynomial tests, followed by symbolic verification. The adaptive part lies in Steps 6–9: factors revealed by the residual linear spaces are tested, retained when they are Darboux polynomials, and inserted into the next candidate for Y . This adaptation can make the search much smaller, but it does not affect correctness. Correctness comes from the final certification: every returned integrating factor is verified by direct substitution. If the required Darboux factors do not occur within the chosen degree bounds, the degrees or candidate choices must be changed. Since no general a priori upper bound for these degrees is currently known, termination is not guaranteed in complete generality. In practical use, the search failure returned by Algorithm 2 is therefore a bounded-regime failure: it refers to the chosen representative, selected equation(s) E_i , prescribed degrees and adaptive refinements. It should not be interpreted as proof that the rational 2ODE has no Liouvillian first integral or no Darboux integrating factor. A provable failure example would require independent knowledge that a Liouvillian first integral exists outside

the Darboux-representative or bounded-search regime considered here; no such example is claimed in this paper.

Remark 23. The key points for the efficiency of the algorithm are:

- The equations $E_i (\langle \partial_i | \mathcal{X}_i \wedge \mathfrak{X}_i \rangle = Y \langle \nabla | \mathfrak{X}_i \rangle)$ are linear in the coefficients of the unknown polynomials $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, Y$.
- There are three of them.
- Y is a polynomial formed by Darboux polynomials present in the integrating factor R .
- Y is an inverse integrating factor for the vector fields $\mathcal{X}_i$.

Example 3. Consider the following 2ODE:

$$\begin{aligned} z' = & z \left(x^7 y z^2 - x^6 z^3 - 2 x^5 y^2 z + x^5 y z^2 + 2 x^4 y z^2 - x^4 z^3 - x^5 z - 2 x^4 y^2 + x^3 y^3 \right. \\ & - 2 x^3 y^2 z + x^3 y^2 + 4 x^3 y z - x^2 y^2 z + 2 x^2 y z^2 + x^3 y - x^3 z - 2 x^2 y z - 2 x^2 z^2 \\ & + x y^3 + x z^2 - y^2 z + y x \left. \right) / \left(x^5 y z^2 - x^4 z^3 + x^4 y^2 - 2 x^3 y^2 z - 2 x^3 y z \right. \\ & \left. + 2 x^2 y z^2 - x^3 z + x^2 z^2 + x y^3 - y^2 z + y x \right) x. \end{aligned} \quad (17)$$

- The procedure NLS finds the symmetry (and therefore the vector fields $\mathfrak{X}_i$) in 0.03 s:

$$\begin{aligned} P_1 = & -z(x^5 y z^2 - x^4 z^3 - 2 x^4 y^2 - 2 x^3 y^2 z + 4 x^3 y z + 2 x^2 y z^2 - x^3 z - 2 x^2 z^2 + x y^3 - y^2 z + y x), \\ P_2 = & -(x^6 y z^2 - x^5 z^3 - 2 x^4 y^2 z + 2 x^3 y z^2 - x^4 z + x^2 y^3 + x^2 y^2 - x y^2 z + x^2 y - 2 x y z + z^2) x, \\ P_3 = & (x^5 y z^2 - x^4 z^3 + x^4 y^2 - 2 x^3 y^2 z - 2 x^3 y z + 2 x^2 y z^2 - x^3 z + x^2 z^2 + x y^3 - y^2 z + y x) x. \end{aligned} \quad (18)$$

- The procedure ALDIF:
 - Steps 1 and 2: $d_1 = 7, d_2 = 7, d_3 = 7$ and $d_Y = 8$.
 - Steps 3, 4, 5, and 6: (the chosen equation was E_3) 405 unknown coefficients were reduced to only 4 (in 0.14 s). The candidates $\mathcal{P}_{1c}, \mathcal{P}_{2c},$ and Y_c , depending on these 4 coefficients still to be determined, can be expressed by:

$$\begin{aligned} \mathcal{P}_{1c} = & -\frac{1}{2} \left(7 x^4 y^2 z - 12 x^3 y z^2 + x^3 z^2 - 3 x^2 y^3 + 5 x^2 z^3 + 4 x y^2 z - x y z - y z^2 \right) a_{29} \\ & + \frac{1}{4} \left(14 x^3 y^3 z + 7 x^3 y^2 z - 22 x^2 y^2 z^2 - 10 x^2 y z^2 - 6 x y^4 + 8 x y z^3 + x^2 z^2 + 5 x z^3 + 6 y^3 z \right. \\ & \left. - 2 y^2 z - 2 y z \right) c_{120} + \left(2 x^3 y z^2 - 2 x^2 z^3 + 3 x^2 y^2 - 2 x y^2 z - 4 x y z + 2 y z^2 - x z + z^2 \right) c_{86} \\ & - \frac{1}{2} \left(14 x^4 y z^2 + 14 x^3 y z^3 - 16 x^3 z^3 - 14 x^2 y^3 z - 8 x^2 z^4 + 15 x^2 y^2 z + 2 x y^2 z^2 - 2 x^2 y z \right. \\ & \left. + 6 y^4 - 10 x^2 z - 7 x z^2 - 12 y^3 - z^3 + 6 y^2 + 6 y \right) c_{94}, \\ \mathcal{P}_{2c} = & -\frac{1}{2} x \left(2 x^4 y z + x^4 z - 2 x^3 z^2 - 4 x^2 y^2 - x^2 y + 6 x y z - 2 z^2 \right) a_{29} + \frac{1}{4} \left(6 x^4 y^2 z \right. \\ & \left. + 4 x^4 y z - 6 x^3 y z^2 + x^4 z - 2 x^3 z^2 - 10 x^2 y^3 - 4 x^2 y^2 + 14 x y^2 z - 2 x^2 y + 4 x y z - 4 y z^2 \right. \\ & \left. - 2 z^2 \right) c_{120} + x^2 \left(2 x^3 y z - 2 x^2 z^2 - 2 x y^2 + 2 x y + 2 y z - x - 2 z \right) c_{86} + \frac{1}{2} \left(2 x^5 z^2 \right. \\ & \left. - 6 x^4 z^3 + 6 x^3 y z^2 - 8 x^3 y z + 10 x^2 y z^2 - x^3 z - 6 x^2 z^2 - 10 x y^3 - 4 x z^3 + 18 x y^2 \right. \\ & \left. + 4 y^2 z + 2 x y - 6 y z - 2 x - 4 z \right) c_{94}, \end{aligned} \quad (19)$$

$$\begin{aligned}
Y_c = & -\frac{1}{2}x(xy-z)^2(x^2z-y)a_{29} + \frac{1}{4}(2x^2yz+x^2z-2y^2)(xy-z)^2c_{120} \\
& + x(xy-z)^2c_{86} - \frac{1}{2}\left(2x^5yz^2+2x^4yz^3-2x^4z^3-2x^3y^3z-2x^3z^4+x^3y^2z\right. \\
& \left.-2x^3yz+2xy^4+2xyz^3-2x^3z-4xy^3-xz^3-2y^3z+2y^2x+4y^2z+2xy\right)c_{94}.
\end{aligned}$$

- Step 7: the Darboux polynomials found are $\{x, xy-z, x^2z-y\}$; the factor $xy-z$ enters Y with multiplicity two (without spending any measurable computational time).
- Steps 8, 9, and 10:

$$P_1 = 7x^4y^2z - 12x^3yz^2 + x^3z^2 - 3x^2y^3 + 5x^2z^3 + 4xy^2z - xyz - yz^2,$$

$$P_2 = x(2x^4yz + x^4z - 2x^3z^2 - 4x^2y^2 - x^2y + 6xyz - 2z^2),$$

$$Y = (x^2z-y)(xy-z)^2x.$$

- Steps 11 and 12: Since $R = 1/Y$ is an integrating factor for the vector field $\mathcal{X}_3 = P_2\partial_x - P_1\partial_y$, we can determine the first integral of $\mathcal{X}_3$ (i.e., the integrating factor R of the vector field $\mathfrak{X}$) with simple quadratures:

$$R = \frac{e^{\frac{x}{xy-z}}}{(x^2z-y)(xy-z)^2x}. \quad (20)$$

Finally, with the integrating factor R , we can obtain the Liouvillian first integral of the vector field $\mathfrak{X}$:

$$I = \frac{e^{\frac{x}{xy-z}}}{(x^2z-y)} + Ei\left(1, -\frac{x}{xy-z}\right). \quad (21)$$

Remark 24. The example illustrates several practical aspects of ALDIF.

1. The linear reduction can be substantial: in Example 3, the equation E_3 reduces 405 undetermined coefficients to 4.
2. After this reduction, factoring the coefficients that remain in Y_c produces the Darboux polynomials needed to construct Y in the tested examples.
3. The choice among E_1 , E_2 , and E_3 affects the size of the remaining linear space. In Example 3, using E_1 reduces 405 coefficients to 14, whereas E_3 reduces them to 4.
4. For elementary first integrals, the final quadrature stage can often be replaced by the usual linear determination of Darboux multiplicities in the Prelle–Singer framework.
5. For non-elementary Liouvillian first integrals, determining the multiplicities of the factors of Y is useful because it avoids a direct search through many possible exponential-factor combinations.
6. The rational 2ODE in Example 3 has a Liouvillian first integral that was not obtained by Maple's `dsolve` in our tests.

These observations should be read together with Remark 22: the method is linear for fixed candidate degrees, but no general degree bound is presently available.

Algorithm 2: ALDIF.

Input: the polynomial vector field $\mathfrak{X}$ associated with the rational 2ODE, the polynomial components P_1, P_2, P_3 obtained from NLS, degree bounds d_1, d_2, d_3, d_Y , and a choice of one or more equations among E_1, E_2, E_3 .

Output: either Darboux polynomials sufficient to construct an integrating factor R and a first integral I , both certified by direct substitution, or failure for the selected representative, prescribed degree bounds and candidate choices.

1: Construct the operators $\mathfrak{X}, \mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3$.

2: Construct four polynomials $\mathcal{P}_{1c}, \mathcal{P}_{2c}, \mathcal{P}_{3c}, Y_c$ of degrees d_1, d_2, d_3 , and d_Y , respectively, with undetermined coefficients.

3: Substitute them in one or more of the equations in (15):

$$E_1 := \mathcal{P}_2 P_3 - P_2 \mathcal{P}_3 = Y \langle \nabla | \mathfrak{X}_1 \rangle,$$

$$E_2 := \mathcal{P}_1 P_3 - P_1 \mathcal{P}_3 = Y \langle \nabla | \mathfrak{X}_2 \rangle,$$

$$E_3 := \mathcal{P}_1 P_2 - P_1 \mathcal{P}_2 = Y \langle \nabla | \mathfrak{X}_3 \rangle.$$

4: Collect each chosen equation E_i in the variables (x, y, z) , obtaining a linear system S_{E_i} for the coefficients of the polynomial candidates.

5: Solve S_{E_i} . The solution space may contain remaining free coefficients; these parameters describe the finite-dimensional family left by the linear reduction.

6: Substitute the solution in Y_c , collect it with respect to the remaining coefficients, and factor the polynomial coefficients that appear.

7: Test the irreducible factors obtained in the preceding step. Keep those that pass the relevant Darboux-polynomial tests for the associated construction; in particular, factors retained for the final integrating factor are tested for the vector field $\mathfrak{X}$, and the corresponding $\mathcal{X}_i$ identities are checked in the certification step.

8: Refine the candidate Y_c using the Darboux polynomials found. For instance, if a Darboux polynomial p_1 of degree d_1 is found, reconstruct the candidate as $Y_c = Y_p p_1$, where the degree of Y_p is $d_Y - d_1$.

9: Repeat the linear solve, factorization, and Darboux-test loop until no new Darboux polynomials are obtained inside the prescribed degree bounds.

10: If the required polynomial data have been found and Y is polynomial, substitute the complete solution into the polynomial candidates, obtaining $\mathcal{P}_j, \mathcal{P}_k, Y$. Otherwise, report failure for the present representative and bounds, or increase the degree bounds or change the candidate-selection strategy and restart the search.

11: From $\mathcal{P}_j, \mathcal{P}_k, Y$ determine an integrating factor R by quadratures, whenever the resulting compatibility conditions are satisfied.

12: Compute a first integral I of the rational 2ODE by quadratures and certify the output by checking $\mathfrak{X}(I) = 0$ and the corresponding integrating-factor identity.

4. Computational Results and Discussion

4.1. Performance of the Algorithms

This section reports computational tests of the procedures *NLS* and *ALDIF*, including a compact summary of the linear reductions obtained in the certified examples. The aim is not to present an optimized software benchmark, but to document the computational scale of the proposed method and to show how the linear reduction behaves on nontrivial rational 2ODEs. The implementation used in these experiments is a Maple 17 pre-implementation run on a notebook with an Intel i5 processor and 4 GB of RAM.

For each example, we report the most expensive symbolic stages: computation of the nonlocal symmetry, coefficient reduction by the linear systems S_{E_i} , Darboux-polynomial detection, reconstruction of Y , and construction of the Liouvillian first integral. The com-

parison is made with the S-function method of [14]. This comparison is diagnostic rather than exhaustive: in these tests, the S-function method is considered unsuccessful on a given example if it does not return the relevant symmetry within 30 s or if memory use reaches 300 MB. When the symmetry is obtained but the associated first order equation is not solved by Maple's `dsolve`, this is indicated separately.

All certified answers returned by NLS-ALDIF are checked symbolically: the computed first integral is substituted in $\mathfrak{X}(I) = 0$, and the corresponding integrating factor is checked against the defining Darboux/integrating-factor identities. Therefore, the tables measure search cost; they do not replace the final symbolic verification.

4.2. Some Difficult 2ODEs

We report five fully certified rational 2ODEs with Liouvillian first integrals. The certified examples are divided into two groups (please see Table 1):

- The first group contains four 2ODEs, three of them with non-elementary Liouvillian first integrals. These equations were selected as challenging cases in our Maple 17 tests: the standard routines used under the limits stated above did not return the desired solutions, point-symmetry searches did not provide a direct reduction, and the corresponding Darboux data include factors of relatively high degree.
- The second certified group contains one comparison 2ODE for which the S-function method can compute the S-function with moderate time and memory use.

The Supplementary Maple Files S1–S5 are open reproducibility records for the Darboux-polynomial search. They document the selected equations, candidate degree regimes, residual linear spaces, retained factors and final symbolic certifications used in the five examples. In a few examples, the worksheets use structural information already visible in the associated fields to choose an equivalent candidate representation. For instance, in Supplementary Maple Files S2 the candidate for $Y = R/\mathcal{R}$ is written with the denominator $2x^4z^2 - 2y^3 - 3$; this is the non-polynomial quotient allowed by Corollary 4 and by the discussion following Theorem 5, and the final candidate is certified by direct substitution. In Supplementary Maple Files S4, the shaped ansatz $N_1 = z \text{Polg}([y, z], 2)$ and $P_1 = y \text{Polg}([y, z], 1)$ reflects the visible factors of the corresponding associated-field components N and P . In the elementary/Darbouxian cases Supplementary Maple Files S4 and S5, once the Darboux polynomials have been identified, the remaining multiplicities and integrating factor may also be obtained by the classical Prolle–Singer linear step. In all cases the retained factors are tested as Darboux polynomials and the final pair (R, I) is certified by the identities $\mathfrak{X}(I) = 0$ and the integrating-factor relation.

The five examples also show different roles of the adaptive stage. In 2ODE-1 and 2ODE-2, a single selected associated equation already produces a strong coefficient reduction and the retained Darboux factors determine the certified integrating factor. In 2ODE-3, two different choices of associated equations are reported because they lead to different residual dimensions; this illustrates that the choice of E_i is part of the practical search strategy rather than a cosmetic implementation detail. In 2ODE-4, the shaped ansatz uses visible factors in the associated fields to reduce the candidate space before the final certification. Finally, 2ODE-5 is included as a comparison example where the S-function computation is feasible, showing that ALDIF is not intended only as a replacement for existing routines but as an additional certifying route when the Darboux search is the bottleneck.

Table 1. Reproducibility data for the certified runs. The Supplementary Maple Files S1–S5 contain the input equations, the degree/search regimes, the monomial candidates used in the linear systems, the retained Darboux-polynomial factors and the final symbolic certification commands. The table records the selected equations when stated in the text, the linear reductions, the Darboux factors retained after testing, and the symbolic identities used for final certification.

Example	Selected Equation (s)	Reproducibility Record	Unknowns → Residual	Darboux Factors Retained	Final Check
2ODE-1	E_3	Supplementary Maple File run S1	469 → 14	$x, y, z, x^2y^2 + z^2$	$\mathfrak{X}(I) = 0$, IF identity
2ODE-2	reported run	Supplementary Maple File run S2	1235 → 1	$x, y, z, x^4z^2 - y^3 - 1$	$\mathfrak{X}(I) = 0$, IF identity
2ODE-3	two reported reductions	Supplementary Maple File run S3	1575 → 4; 3083 → 121	$2xz - y, xy^4z^4 + \frac{2}{2}$	$\mathfrak{X}(I) = 0$, IF identity
2ODE-4	reported run	Supplementary Maple File run S4	203 → 11	$x, y, z, -z^4 - y^3 + x^2$	$\mathfrak{X}(I) = 0$, IF identity
2ODE-5	reported run	Supplementary Maple File run S5	126 → 26	$x - z, x^2 - yz$	$\mathfrak{X}(I) = 0$, IF identity

4.2.1. First Set

Remark 25. The following Tables 2 and 4–7 describe the memory and CPU time expenditures of the most costly routines (computationally speaking) of the procedures NLS and ALDIF when applied to the 2ODEs that follow.

2ODE-1 (Table 2):

$$\begin{aligned}
 z' = & \left(2x^5y^4z^2 - x^5y^4 - 2x^5y^3z + 4x^3y^2z^4 - x^4y^4 - 4x^4y^2z^2 + 2x^5yz - 2x^3y^3z \right. \\
 & \left. - 2x^3y^2z^2 - 2x^3yz^3 + 2xz^6 + x^4y^2 + 2xyz^3 - xz^4 - x^2z^2 + z^4 \right) / \\
 & \left(-2x \left(x^4y^5 + 2x^2y^3z^2 - x^2y^2z + yz^4 - 2xyz^2 + x^2z - z^3 \right) \right)
 \end{aligned} \quad (22)$$

Table 2. Time and memory use—2ODE (22).

Algorithm	Task	Memory (MB)	Time (s)
NLS	$\{P_3, P_2, P_1\}$	1	0.047
ALDIF (part 1)	Coeff. reduction	2	0.078
ALDIF (part 2)	Compute DPs	0	0.032
ALDIF (part 3)	Compute Y	1	0.390
NLS–ALDIF	Find a LFI I	4	0.547

Procedure NLS:

- (1) NLS computes $P_2 = 2x(x^4y^4z - x^4y^3 + 2x^2y^2z^3 - 2x^3y^2z + x^4y - x^2yz^2 + z^5)$ and $P_1 = -x^5y^4 - x^4y^4 - 2x^3y^3z - 2x^3y^2z^2 + x^4y^2 + 2xyz^3 - xz^4 - x^2z^2 + z^4$.

Procedure ALDIF:

- (2) Reduction of undetermined coefficients (part 1): 469 → 14
 (3) DPs found (part 2): $\{p_1 = x, p_2 = y, p_3 = z, p_4 = x^2y^2 + z^2\}$.

- (4) Exponents found (part 3): $\{n_1 = 1, n_2 = 0, n_3 = 0, n_4 = 2\}$. So, $\mathcal{P}_3 = -2xz(2x^2y^2 + 2z^2 + x)$, $\mathcal{P}_2 = -2x^3y(2x^2y^2 + 2z^2 + x)$, $\mathcal{P}_1 = -5x^4y^4 - 6x^2y^2z^2 - x^3y^2 - z^4 + xz^2$ and $Y = x(x^2y^2 + z^2)^2$.

Remark 26. The other parts have a very small algorithmic cost compared to those shown in Table 3: Since Y is an inverse integrating factor for $\mathcal{X}_i$, we can find R with simple quadratures: $\mathcal{R}_z = \frac{\mathcal{P}_3}{Y}$, $\mathcal{R}_y = \frac{\mathcal{P}_2}{Y}$, $\mathcal{R}_x = \frac{\mathcal{P}_1}{Y}$ and so

$$R = \frac{e^{\frac{x}{x^2y^2+z^2}}}{(x^2y^2+z^2)x}. \quad (23)$$

Therefore, since $I_z = R \mathcal{P}_3$, $I_y = R \mathcal{P}_2$, $I_x = R \mathcal{P}_1$, we have

$$I = e^{\frac{x}{x^2y^2+z^2}} (-2yz + x) + Ei\left(1, -\frac{x}{x^2y^2+z^2}\right). \quad (24)$$

2ODE-2 (Table 4):

$$\begin{aligned} z' = & \left(-x^9z^5 - 4x^9z^4 + x^8yz^4 + 3x^6y^2z^3 + 2x^5y^3z^3 + 4x^5y^3z^2 - 2x^4y^4z^2 + 2x^5z^3 \right. \\ & - 3x^2y^5z - xy^6z + 4x^5z^2 - 6x^4yz^2 + y^7 - 3x^2y^2z + xy^3z \\ & \left. + 2y^4 - xz + y \right) / \left(2x^5z(x^5z^2 - xy^3 - x + y) \right). \end{aligned} \quad (25)$$

Table 3. Summary of the computational tests. The column “Main reduction” reports the reduction in the number of undetermined coefficients in the linear stage. For each row, the final pair (R, I) is displayed and checked symbolically.

Example	Main Reduction	Total Time (s)	Memory (MB)	Comparison Status	Certified
2ODE-1	469 $\rightarrow$ 14	0.547	4	difficult set	yes
2ODE-2	1235 $\rightarrow$ 1	0.735	8	difficult set	yes
2ODE-3	1575 $\rightarrow$ 4; 3083 $\rightarrow$ 121	2.015	5	difficult set	yes
2ODE-4	203 $\rightarrow$ 11	0.079	1	difficult set	yes
2ODE-5	126 $\rightarrow$ 26	0.140	1	comparison set	yes

Table 4. Time and memory use—2ODE (25).

Algorithm	Task	Memory (MB)	Time (s)
NLS	$\{P_3, P_2, P_1\}$	1	0.047
ALDIF (part 1)	Coeff. reduction	6	0.422
ALDIF (part 2)	Compute DPs	0	0.141
ALDIF (part 3)	Compute Y	1	0.125
NLS–ALDIF	Find a LFI I	8	0.735

Procedure NLS:

- (1) NLS_{2I} computes $P_2 = (x^8z^4 - 3x^5y^2z^2 - 2x^4y^3z^2 - 2x^4z^2 + 3xy^5 + y^6 + 3y^2x - y^3 + 1)x$ and $P_1 = 4x^9z^4 - x^8yz^4 - 4x^5y^3z^2 + 2x^4y^4z^2 - 4x^5z^2 + 6x^4z^2y - y^7 - 2y^4 - y$.

Procedure ALDIF:

- (2) Reduction of undetermined coefficients (part 1): 1235 $\rightarrow$ 1

- (3) DPs found (part 2): $\{p_1 = x, p_2 = y, p_3 = z, p_4 = x^4z^2 - y^3 - 1\}$.
- (4) Exponents found (part 3): $\{n_1 = 1, n_2 = 0, n_3 = 0, n_4 = 2\}$. So, $\mathcal{P}_3 = 2x^5z(2x^4z^2 - 2y^3 - 3)$, $\mathcal{P}_2 = -3xy^2(2x^4z^2 - 2y^3 - 3)$, $\mathcal{P}_1 = 10x^8z^4 - 12x^4y^3z^2 - 16x^4z^2 + 2y^6 + 4y^3 + 2$.

The integrating factor and the LFI are given by

$$R = \frac{e^{\frac{-1}{x^4z^2 - y^3 - 1}}}{(x^4z^2 - y^3 - 1)^2 x^2}, \quad I = \frac{e^{(-x^4z^2 + y^3 + 1)^{-1}} y}{x} + Ei\left(1, -(-x^4z^2 + y^3 + 1)^{-1}\right). \quad (26)$$

2ODE-3 (Table 5):

$$\begin{aligned} z' = & \left(16x^5y^7z^{11} - 16x^4y^8z^{10} + 4x^3y^9z^9 + x^3y^8z^9 + 32x^4y^3z^7 + 8x^4y^3z^6 - 40x^3y^4z^6 \right. \\ & - 4x^3y^4z^5 + 16x^2y^5z^5 + 4x^2y^4z^5 - 2xy^6z^4 - 16x^2z^2 - 4x^2z + 16xyz + 2xy + 4xz - 4y^2 \Big) \\ & / \left(-2x^2 \left(8x^3y^8z^9 - 8x^2y^9z^8 + x^2y^8z^8 + 2xy^{10}z^7 + 16x^2y^4z^5 + 4x^2y^4z^4 - 16xy^5z^4 \right. \right. \\ & \left. \left. - 2xy^5z^3 + 4xy^4z^4 + 4y^6z^3 + 4 \right) \right). \end{aligned} \quad (27)$$

Table 5. Time and memory use for two ALDIF choices—2ODE (27). The NLS stage is common to both runs; the two ALDIF blocks compare the effect of the selected equation/candidate strategy on the residual linear space.

Run	Algorithm	Task	Memory (MB)	Time (s)
shared	NLS	$\{P_3, P_2, P_1\}$	1	0.063
one way	ALDIF (part 1)	Coeff. reduction	4	1.750
one way	ALDIF (part 2)	Compute DPs	0	0.202
one way	ALDIF (part 3)	Compute Y	0	0.000
one way	NLS–ALDIF	Find a LFI I	5	2.015
another way	ALDIF (part 1)	Coeff. reduction	17	10.500
another way	ALDIF (part 2)	Compute DPs	1	3.281
another way	ALDIF (part 3)	Compute Y	0	0.000
another way	NLS–ALDIF	Find a LFI I	19	13.844

Procedure NLS:

- (1) NLS computes $P_2 = (16x^4y^7z^{10} - 16x^3y^8z^9 + 4x^2y^9z^8 - x^2y^8z^8 + 32x^3y^3z^6 + 8x^3y^3z^5 - 32x^2y^4z^5 - 4x^2y^4z^4 + 8xy^5z^4 - 4xy^4z^4 - 4x)$ and $P_1 = 2(8x^3y^8z^9 - 8x^2y^9z^8 + x^2y^8z^8 + 2xy^{10}z^7 + 16x^2y^4z^5 + 4x^2y^4z^4 - 16xy^5z^4 - 2xy^5z^3 + 4xy^4z^4 + 4y^6z^3 + 4)x^2$.

Procedure ALDIF:

- (2) Reduction of coefficients (part 1—one way): $1575 \rightarrow 4$ and $3083 \rightarrow 121$
- (3) Reduction of coefficients (part 1—another way): $4851 \rightarrow 260$
- (4) DPs found (part 2): $\{2xz - y, xy^4z^4 + 2\}$.
- (5) Exponents found (part 3): $\{n_1 = 1, n_2 = 2\}$. So, $\mathcal{P}_3 = -2x(5x^2y^8z^8 - 2xy^9z^7 + 2x^2y^4z^4 - xy^5z^3 + 12xy^4z^4 - 4y^5z^3 + 4)$, $\mathcal{P}_2 = -8x^3y^7z^9 + 5x^2y^8z^8 - 4x^3y^3z^5 + 2x^2y^4z^4 - 16x^2y^3z^5 + 12xy^4z^4 + 4$, $\mathcal{P}_1 = -4x^2y^8z^9 + xy^9z^8 - 12xy^4z^5 + 2y^5z^4 + 2xz - y - 8z$.

The integrating factor and the LFI are given by

$$R = \frac{e^{\frac{x}{xy^4z^4+2}}}{(xy^4z^4+2)^2(2xz-y)^2}, \quad I = e^{\frac{x}{xy^4z^4+2}}(-2xz+y)^{-1} + Ei\left(1, -\frac{x}{xy^4z^4+2}\right). \quad (28)$$

2ODE-4 (Table 6):

$$z' = \frac{3y^2z^5+3y^5z+2yz^5-9x^2y^2z+2xz^4-y^4z-9y^2z^3-2x^2yz+2xy^3+2x^3+2xy^2+6xz^2}{2z(-2z^6-2y^3z^2+6x^2z^2+2y^2z^2+3z^4-3y^3+3x^2)}. \quad (29)$$

Table 6. Time and memory use—2ODE (29).

Algorithm	Task	Memory (MB)	Time (s)
NLS	$\{P_3, P_2, P_1\}$	1	0.031
ALDIF (part 1)	Coeff. reduction	0	0.032
ALDIF (part 2)	Compute DPs	0	0.016
ALDIF (part 3)	Compute Y	0	0.000
NLS–ALDIF	Find a LFI I	1	0.079

Procedure NLS:

- (1) NLS computes $P_2 = xz(x^2yz - 3xyz^2 - 2x + 6z - 1)(x - 3z)$ and $P_1 = x^3y^2z^2 - 6x^2y^2z^3 + 9xy^2z^4 - x^2y^2z^2 - 2x^2yz + 12xyz^2 - 18yz^3 + 3xyz + 3yz^2 - 4$.

Procedure ALDIF:

- (2) Reduction of coefficients (part 1): $203 \rightarrow 11$
- (3) DPs found (part 2): $\{p_1 = x, p_2 = y, p_3 = z, p_4 = -z^4 - y^3 + x^2\}$.
- (4) Exponents of R found directly (elementary first integral): $\{n_1 = 0, n_2 = 0, n_3 = 0, n_4 = 2\}$.

The integrating factor and the LFI are given by

$$R = \frac{1}{(-z^4 - y^3 + x^2)^2}, \quad I = \frac{e^{\frac{2x^2+y^2+3z^2}{-z^4-y^3+x^2}}}{(-z^4 - y^3 + x^2)}. \quad (30)$$

4.2.2. Certified Comparison Example

We conclude the benchmark section with a fully verified comparison example for which the S-function method can compute the S-function with moderate time and memory use.

2ODE-5 (Table 7):

$$z' = \frac{2x^4z-2x^3z^2-2x^2yz^2+2xyz^3-2x^2yz-2xz^3+2y^2z^2+yz^3+z^4-2x^3+4x^2z-2xz^2}{-(2x^3y+2x^3z-x^2z^2-2xy^2z-2xyz^2+yz^3+x^2y-2xyz+yz^2)}. \quad (31)$$

Table 7. Time and memory use—2ODE (31).

Algorithm	Task	Memory (MB)	Time (s)
NLS	$\{P_3, P_2, P_1\}$	1	0.047
ALDIF (part 1)	Coeff. reduction	0	0.015
ALDIF (part 2)	Compute DPs	0	0.078
ALDIF (part 3)	Compute Y	0	0.000
NLS–ALDIF	Find a LFI I	1	0.140

Procedure NLS:

- (1) NLS computes $P_2 = (2x^3 - 2xyz + xz - z^2)(x - z)$ and $P_1 = -2x^2yz - x^2z^2 + 2y^2z^2 + yz^3 - 2x^3 + 4x^2z - 2xz^2$.

Procedure ALDIF:

- (2) Reduction of coefficients: $126 \rightarrow 26$
 (3) DPs found (part 2): $\{x - z, x^2 - yz\}$.
 (4) Exponents found (part 3): $\{n_1 = 1, n_2 = 2\}$. So, $\mathcal{P}_3 = -2x^3y - 2x^3z + x^2z^2 + 2xy^2z + 2xyz^2 - yz^3 - 2x^3 - 2x^2y + 2x^2z + 6xyz - 4yz^2$, $\mathcal{P}_1 = 2x^2yz + x^2z^2 - 2y^2z^2 - yz^3 + 6x^3 - 10x^2z - 2xyz + 4xz^2 + 2yz^2$, $\mathcal{P}_2 = -2(x - z)(x^3 - xyz + xz - z^2)$.

The integrating factor and the LFI are given by

$$R = \frac{e^{\frac{2xy+z^2}{x-z}}}{(x-z)^2(x^2-zy)^2}, \quad I = \frac{2xy+z^2}{x-z} - \ln(x^2-zy). \quad (32)$$

5. Conclusions

The examples above support the practical usefulness of the associated-field construction as a computational method. In the five certified tests, the computation of the nonlocal symmetry by NLS was fast, and the subsequent ALDIF stage reduced the Darboux-factor search to linear systems of moderate size. In the first set of examples, the comparison with the S-function method is favorable because the direct use of the associated first order equation is ineffective in the tested computer-algebra environment.

The comparison reported here is intentionally diagnostic rather than a broad benchmark against a common corpus of equations already solvable by standard routines. The examples were chosen because they lie in a region where classical point-symmetry reductions, direct Darboux searches, or the direct solution of the associated first order equations do not provide the desired certified first integral under the stated Maple limits. Consequently, the Tables 2 and 4–7 should not be read as claiming universal speed superiority over existing symbolic methods; they document that, for these reproducible cases, the associated-field and residual-factor strategy reaches a certified answer where the direct routes tested in the same environment did not.

From a computational point of view, the main advantage of the method is that the expensive nonlinear Darboux search is replaced, for fixed degree bounds, by linear algebra plus factorization and Darboux tests. The examples also show that the choice among E_1, E_2, E_3 matters: different equations can leave residual linear spaces of different dimensions. This suggests that practical implementations should try the three equations, or combine information obtained from them, before increasing the degree bounds.

The method has clear limitations. ALDIF is not yet a complete algorithm in the unrestricted class considered here, because no general upper bound is known for the degrees of $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$, and Y . Moreover, the present implementation is a Maple pre-implementation, not an optimized package. The reported timings should therefore be read as evidence that the approach is computationally feasible on the tested examples, not as definitive performance data. For reproducibility, the monomial bases, degree bounds, and linear systems used in the reported runs are included in the Supplementary Maple Worksheets.

A larger test library, including documented bounded-regime failures and comparisons on current computer-algebra systems, would be valuable for assessing success rates and practical parameter choices. We regard this as a natural next step for a packaged implementation of NLS–ALDIF. In the present manuscript, the five examples are intended to be certified and reproducible illustrations of distinct search mechanisms, rather than

statistical evidence for the behavior of the semi-algorithm on the full class of rational second order ODEs.

These points lead to natural developments. One is to establish degree bounds for the non-degenerate case in which Y is polynomial; such bounds would turn ALDIF into a finite algorithm in that class. Another is to study the quotient-rigidity problem described in Remarks 15 and 16: if the Darboux-relative quotient between a planar Darboux integrating factor and the common integrating factor R cannot contain an essential passive-coordinate obstruction, then the Darboux-representative hypothesis could be replaced by a stronger structural theorem. A third is to study the dimensions of the linear solution spaces S_{E_i} and relate them to intrinsic data of the vector fields $\mathfrak{X}_i$, such as their Darboux polynomials and exponential factors. A fourth is to design reproducible selection strategies for the adaptive stage, so that candidate factors are refined according to explicitly stated rules and the success rate can be estimated experimentally.

Finally, the associated-field viewpoint may be useful beyond the examples treated here, but such extensions remain open. Possible directions include higher order ODEs and, more speculatively, restricted classes of partial differential equations. Before pursuing these extensions, the most immediate computational step is to turn NLS–ALDIF into a documented computer-algebra package, with automatic certification of the returned integrating factor and first integral.

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Abbreviations

The following abbreviations are used in this manuscript:

2ODE	second-order ordinary differential equation
ALDIF	adaptive linear Darboux integrating factor procedure
DP	Darboux polynomial
LFI	Liouvillian first integral
NLS	nonlocal symmetry procedure

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