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Mild Locally Small Spaces

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Abstract

We prove that a direct sum of locally small spaces, each of which is either paracompact or has its admissible open sets closed under arbitrary unions, is mild (all its subspaces are strict) and that mildness is hereditary. We show how to produce examples of non-mild spaces (answering an earlier problem) and consider finitary locally small spaces. We apply the results to paracompact locally definable spaces over o-minimal structures.

Keywords: locally small space; locally definable space; strict subspace; paracompactness; o-minimal structure

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1. Introduction

Grothendieck topologies. At the beginning of the 1960s, A. Grothendieck, using the language of category theory, introduced what we call Grothendieck topologies (see [1,2]). Grothendieck topologies, while defined in categories, can be loosely considered as generalisations of classical topologies, thereby opening new possibilities. This was important in algebraic geometry and rigid analytic geometry. In 1984, S. Bosch, U. Güntzer and R. Remmert [3] used G-topologies, which are concrete versions of Grothendieck topologies. To define a G-topology on a set X, one needs to distinguish a family of admissible open subsets of X and a family of admissible coverings for an admissible open set satisfying several conditions ([3], Definition 1 in Subsection 9.1.1). Quite often, additional conditions (G_0), (G_1), (G_2) are assumed ([3], Subsection 9.1.2, page 339).

Generalised topology of H. Delfs and M. Knebusch. The next concretisation, intended for use in semialgebraic topology, came in 1985, when H. Delfs and M. Knebusch [4] postulated two important properties: (1) admissible open sets should be closed under finite unions and (2) all finite coverings of admissible open sets should be admissible. They named such objects "generalised topological spaces". This allowed them to develop in [4,5] semialgebraic homotopy theory, which was later extended (see [6]) to a homotopy theory over o-minimal expansions of fields and even to more general contexts. (Notice that the Delfs–Knebusch "generalised topology" is completely different from Lugojan's [7] or Császár's [8] generalised topology.)

Grothendieck's tame topology and o-minimality. A. Grothendieck called for a "tame topology" to be built in his "Esquisse d'un programme" [9] in 1984 to eliminate pathologies from the usual topology. His ideas on the notion of space were found interesting even outside mathematics. For example, Cruz Morales [10] gave a history of the evolution of Grothendieck's ideas in relation to some physical and philosophical questions.

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One undisputed realisation of Grothendieck's idea of "tame topology" is o-minimality, which evolved from semialgebraic and subanalytic geometries (with prominent authors: A. Tarski, M. Coste, H. Hironaka, S. Łojasiewicz, A. Gabrielov, to name only a few).

Semialgebraic and subanalytic geometry inspired L. van den Dries to undertake some general model-theoretic considerations [11]. Based on his ideas, A. Pillay and C. Steinhorn [12,13] introduced the concept of an o-minimal structure in 1984.

But o-minimal structures as entities of model theory do not inherently have the usual topological character, which already played a role in the case of the Delfs–Knebusch generalised topology. This is why many authors worked with the notion of a "subanalytic site" (e.g., [14]) or an "o-minimal site" (e.g., [15]), which uses a Grothendieck-type approach to topology.

A simplification: locally small spaces and smopologies. Although Grothendieck's programme had been pursued over many decades in a variety of special situations, the authors of [16] observed that no precise definition had ever been given of the notion of tame topology suggested by A. Grothendieck. Some steps in further simplifying Grothendieck topologies while setting up a language for dealing with situations more general than just o-minimality were made by A. Piękosz: first making the definition of a generalised topology simpler (reducing the number of axioms from eight to five in [17]), and then introducing locally small spaces by smopologies (in [18], continuing research from [19,20]), which are used (sometimes incognito) in semialgebraic geometry ([21] Definition 7.1.14 or [22] p. 12), in o-minimal homotopy theory [4,6], and, more generally, in model theory [17,19,23,24]. Dropping some of the requirements for a topology leads to a kind of "algebra-friendly topology". Dropping the infinite unions condition gives a unitary smopology and a small space (see [17,18,20,25,26]). Weakening further the whole space condition leads to an arbitrary smopology and a locally small space (see [18,19,25,27]). Locally small spaces can and should in many cases be treated as topological spaces with additional (for example, bornological) structure. Recall that small spaces are the underlying topological structures of definable spaces, and locally small spaces are the underlying topological structures of locally definable spaces over structures with topologies (see [18,19]), which are much more general than o-minimal structures.

Main ideas of the present paper. In a series of previous papers [25–27], many equivalences of categories were proved. Those equivalences (as category-theoretic instances of symmetry between categories) tell us much about locally small spaces and morphisms (so-called bounded continuous mappings) between them. Here we continue research on locally small spaces as such, in the context of notions that have some tameness character (such as paracompactness or finitarity). While we discuss many properties, such as tautness, being Heyting, or paracompactness, we concentrate on strict subspaces. (Some mathematicians expressed doubts, in private communications, as to whether the notion of a subspace is well-behaved enough to support a theory.) We look for sufficient conditions for a space to have only strict subspaces and find an example of a space having non-strict subspaces, giving a method to produce similar examples. Then we apply theorems about locally small spaces to the case of locally definable spaces over o-minimal structures. We believe this opens up many possibilities for building theories in, for example, algebraic topology, and for developing tame topology, realising Grothendieck's postulate in a purely topological context.

Applications. The results of this paper may catalyse the development of this branch of non-classical topology. While the main focus of this paper lies in such branches of mathematics as generalisations of o-minimality, real algebraic and analytic geometry, or algebraic topology, some specialists from other disciplines (philosophy and logic, physics, computer science) may find this paper interesting.

Axiomatics. For those who are interested in the set-theoretical axiomatic foundations: for most of the material, we follow the standard **ZFC** system, as in Saunders Mac Lane's Zermelo–Fraenkel axioms with the Axiom of Choice, [28] p. 23. When considering categories, we assume the existence of a set that serves as a universe, allowing us to speak about proper classes of sets (in particular, about categories) while formally staying in **ZFC** (A full explanation is given in the section "Axiomatic assumptions" of [29]).

2. Locally Small Spaces

Notation 1. *We use the following notation for operations on families of sets. For example, for family intersection and subtraction, we write*

$$\mathcal{U} \cap_1 \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}, \qquad \mathcal{U} \cap_1 V = \mathcal{U} \cap_1 \{V\},$$

$$\mathcal{U} \setminus_1 \mathcal{V} = \{U \setminus V : U \in \mathcal{U}, V \in \mathcal{V}\}, \qquad U \setminus_1 \mathcal{V} = \{U\} \setminus_1 \mathcal{V},$$

and for some other operations

$$int_1(\mathcal{A}) = \{int(A) : A \in \mathcal{A}\}, \quad cl_1(\mathcal{A}) = \{cl(A) : A \in \mathcal{A}\}.$$

We shall not use any special notation for images or preimages of families of sets under functions. For example, we write

$$f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}.$$

Let a family $\mathcal{A}$ of subsets of a set X be given. We set

$$ba(\mathcal{A}) = \text{the Boolean algebra generated by } \mathcal{A} \text{ in } \mathcal{P}(X),$$

$$\mathcal{A}^o = \{Y \subseteq X \mid Y \cap_1 \mathcal{A} \subseteq \mathcal{A}\}.$$

We say that elements of $\mathcal{A}^o$ are compatible with *the family $\mathcal{A}$, while the elements of $ba(\mathcal{A})$ are called the sets* constructible from $\mathcal{A}$.

Definition 1 ([18] Definitions 2.1 and 2.21)**.** *A* locally small space *is a pair $\mathcal{X} = (X, \mathcal{L}_X)$, where X is any set and $\mathcal{L}_X \subseteq \mathcal{P}(X)$ satisfies the following conditions:*

(LS1) $\emptyset \in \mathcal{L}_X$,

(LS2) *if $A, B \in \mathcal{L}_X$, then $A \cap B, A \cup B \in \mathcal{L}_X$,*

(LS3) $\forall x \in X \, \exists A_x \in \mathcal{L}_X \; x \in A_x$ *(i.e., $\bigcup \mathcal{L}_X = X$).*

Elements of $\mathcal{L}_X$ are called smops *(or* small admissible open *subsets) of X. A* small space *is a locally small space $(X, \mathcal{L}_X)$ for which $X \in \mathcal{L}_X$.*

Definition 2 ([27] Definition 3)**.** *We consider the following additional types of subsets or families in X:*

1. *We call the family $\mathcal{L}'_X = X \setminus_1 \mathcal{L}_X$ a* co-smopology *and its members* co-smops.
2. *The members of the family $Con(X) = ba(\mathcal{L}_X)$ are called* constructible *sets.*
3. *The members of the family $\mathcal{L}_X^o$ of subsets compatible with smops are called* admissible open sets, *and their complements are called* admissible closed sets. *The family of all admissible open sets $\mathcal{L}_X^o$ in X is also denoted by $AdOp(X)$, while the family of all admissible closed sets is denoted by $AdCl(X)$ or $\mathcal{L}_X^c$.*
4. *The bornology generated by the smopology $\mathcal{L}_X$ (i.e., the family of subsets of smops) has members known as the* small sets *$Sm(X)$ in the space.*
5. *A subset of a smop that is constructible is called a* small constructible set*. The family $sCon(X)$ contains all small constructible sets.*

6. *A subset of X is* locally constructible *if all its traces on smops are constructible. The family $LCon(X)$ contains all locally constructible sets of the space.*

Definition 3 (cf. [18] Definition 2.9)**.** *For $(X, \mathcal{L}_X)$ a locally small space, the topology $\tau(\mathcal{L}_X)$, generated by $\mathcal{L}_X$ in $\mathcal{P}(X)$, is named the* original topology *and its members are called* (weakly) open sets *in $(X, \mathcal{L}_X)$. The closure, the interior and the exterior operations in the original topology are denoted by $\overline{\cdot}$, $int(\cdot)$, $ext(\cdot)$, respectively.*

While other topologies of a locally small space were studied in [27], we shall use only the original/generated topology.

Example 1 ($\mathbb{R}_{fin}$)**.** *On the set of real numbers $\mathbb{R}$, consider as smops all finite sets and the whole space. Then the generated/original topology is discrete. The closure of a non-trivial smop is the same smop, which is not admissible closed (because it is not a co-smop).*

3. Heyting and Taut Spaces

Inspired by Delfs and Knebusch ([4], Def. 2 in I.7), we use the following two notions.

Definition 4 ([18] Definition 2.59)**.** *A locally small space $(X, \mathcal{L}_X)$ is as follows:*

1. taut *if the closure in the original topology of each smop is small (so contained in another smop), $cl_1\mathcal{L}_X \subseteq Sm(X)$,*
2. strongly taut *if the closure in the original topology of a smop is small and admissible closed, $cl_1\mathcal{L}_X \subseteq Sm(X) \cap AdCl(X)$.*

On the other hand, the following notion is inspired by the theory of spectral spaces from [30].

Definition 5 ([27], Definition 21)**.** *A locally small space is called* pre-semi-Heyting *if any of the following equivalent conditions is satisfied:*

(a) *the interior in the original topology of any admissible closed set is an admissible open set, i.e., $int_1 AdCl(X) \subseteq AdOp(X)$,*
(b) *the closure in the original topology of any admissible open set is an admissible closed set, i.e., $cl_1 AdOp(X) \subseteq AdCl(X)$.*

An example of a locally small space that is pre-semi-Heyting but not taut is given below.

Example 2 (a half-plane with $\mathbb{Z}$-many teeth)**.** *Let H be the lower open half-plane $\mathbb{R} \times (-\infty, 0)$ of $\mathbb{R}^2$ and T_k be open vertical bars $(k, k + \frac{1}{2}) \times \mathbb{R}$ for $k \in \mathbb{Z}$, all with natural topologies. Define the smopology $\mathcal{L}_{T(\mathbb{Z})}$ on the union $T(\mathbb{Z})$ of those sets as the family of finite unions of open subsets of individual bars together with an open subset of H. Since any open set in $T(\mathbb{Z})$ is now admissible open and any closed set is admissible closed, the space $T(\mathbb{Z})$ is pre-semi-Heyting. The closure $cl_{T(\mathbb{Z})}H$ of H in $T(\mathbb{Z})$ is not contained in finitely many of the pieces mentioned above, so it is not small.*

The following proposition gives the first comparison between taut, strongly taut, and pre-semi-Heyting.

Proposition 1. *The following are true for locally small spaces.*

(1) *Each small space is taut.*
(2) *There exist small but not strongly taut spaces.*
(3) *For small spaces, strong tautness is equivalent to being pre-semi-Heyting.*

(4) *Each strongly taut locally small space is pre-semi-Heyting.*
(5) *There exist pre-semi-Heyting locally small spaces that are not taut.*
(6) *A locally small space is strongly taut if and only if it is taut and pre-semi-Heyting.*

Proof. (1) Every set is small in a small space.
(2) See Example 1. In $\mathbb{R}_{fin}$, the original topology is discrete and the non-trivial smops are not co-smops, so they are not admissible closed.
(3) In small spaces, the smops are exactly the admissible open sets. Moreover, all sets are small. The claim follows.
(4) Assume $Y = \overline{V}$ is the closure of an admissible open subset V in X. The intersection $Y \cap L$ of Y with a smop L is the closure of a smop $V \cap L$ in L. By the strong tautness of X, the set $Y \cap L$ is small admissible closed in L (i.e., a co-smop in L), and that is why Y is admissible closed.
(5) See Example 2.
(6) By (4), the definitions and the fact that each smop is admissible open. □

Recall one basic property of pre-semi-Heyting spaces.

Proposition 2 ([27] Proposition 8)**.** *Assume that X is a pre-semi-Heyting locally small space. Then for each $V \in AdOp(X)$, the open subspace $(V, \mathcal{L}_X \cap_1 V)$ is pre-semi-Heyting.*

We shall also use stronger properties of being pre-Heyting and Heyting.

Definition 6 ([18] Definition 3)**.** *A locally small space $(X, \mathcal{L}_X)$ is T_0 if the family $\mathcal{L}_X$ separates points. This means: for $x, y \in X$ the following condition is satisfied:*

$$\text{if } x \in A \iff y \in A \text{ for each } A \in \mathcal{L}_X, \text{ then } x = y.$$

Definition 7 ([27], Definition 21)**.** *A locally small space will be called* pre-Heyting *if any of the following equivalent conditions is satisfied:*

(a) *the interior in the original topology of any locally constructible set is an admissible open set, i.e., we have $int_1 LCon(X) \subseteq AdOp(X)$.*
(b) *the closure in the original topology of any locally constructible set is an admissible closed set, i.e., we have $cl_1 LCon(X) \subseteq AdCl(X)$.*

A locally small space will be called Heyting *if it is T_0 and pre-Heyting.*

We recall the characterisation of pre-Heyting spaces from [27].

Proposition 3 ([27], Proposition 9)**.** *For a locally small space $X = (X, \mathcal{L}_X)$, the following conditions are equivalent:*

(1) *X is pre-Heyting,*
(2) *for each $B \in \mathcal{L}_X \setminus_1 \mathcal{L}_X$, $\overline{B} \in AdCl(X)$,*
(3) *for each $A \in LCon(X)$, the subspace $(A, \mathcal{L}_X \cap_1 A)$ is pre-Heyting,*
(4) *for each $A \in LCon(X)$, the subspace $(A, \mathcal{L}_X \cap_1 A)$ is pre-semi-Heyting,*
(5) *for each $F \in AdCl(X)$, the subspace $(F, \mathcal{L}_X \cap_1 F)$ is pre-semi-Heyting.*

4. Paracompactness

Definition 8 ([25] Definition 3)**.** *A family of sets in a locally small space is as follows:*

1. essentially finite *if some finite subfamily covers the same union,*
2. locally finite *if each smop has a non-empty intersection with only finitely many members of the family,*

3. locally essentially finite *if it is essentially finite after restriction to any smop of the space.*

Corollary 1 (cf. [19] Proposition 2.1.6)**.** *The union of a locally essentially finite family of admissible open sets is again admissible open.*

Definition 9 ([18], Proposition 2.53)**.** *A locally small space $(X, \mathcal{L}_X)$ is called* paracompact *if it satisfies one of the equivalent conditions:*

(a) any locally essentially finite covering of X by smops admits a refinement that is a locally finite covering of X by smops,
(b) there exists a locally finite covering of X by smops.

The following example is inspired by [4] Example I.3.15.

Example 3 (a topological triangle in a square)**.** *Let $X = (0,1) \times (0,1)$ and $T = X \cap \{\langle x, y\rangle : x < y\}$. We declare that a smop is any subset of X that is open in the usual topology and contained in some $T \cup X_n$, where $X_n = (\frac{1}{n}, 1) \times (0,1)$, $n \geq 2$. The closure of T is not contained in any $T \cup X_n$, so it is not small. The space is neither paracompact nor taut.*

Proposition 4. *(1) Each small space is paracompact.*
(2) Each paracompact space is taut.
(3) There exist paracompact (even small) spaces that are not strongly taut.
(4) For paracompact locally small spaces, strong tautness is equivalent to being pre-semi-Heyting.

Proof. (1) is clear. For (2), each closure of a smop L is contained in a finite union of smops since it is disjoint from smops disjoint from L. (3) was shown in Proposition 1 (2). (4) Pre-semi-Heyting implies strongly taut: The closure of a smop is small by paracompactness, see (2), and is admissible closed by the definition of a pre-semi-Heyting space. Strongly taut implies pre-semi-Heyting: Proposition 1 (4). □

Problem 1. *Is there a paracompact pre-semi-Heyting space that is not pre-Heyting?*

Example 4 ($\mathbb{R}_{fin}$ again)**.** *The space $(\mathbb{R}, \mathrm{Fin}(\mathbb{R}) \cup \{\mathbb{R}\})$ from Example 1 is a small discrete paracompact space but with infinitely many connected components.*

5. Direct Sums

Definition 10. *The* direct sum $\oplus_{i\in I}\mathcal{X}_i$ *of a family of locally small spaces $\mathcal{X}_i = (X_i, \mathcal{L}_i), i \in I$, is a locally small space whose underlying set $X = \sqcup_{i\in I} X_i$ is the disjoint union of the underlying sets X_i and whose smopology $\mathcal{L}$ consists of finite unions of elements of $\mathcal{L}_i$'s.*

Corollary 2. *All sets X_i, $i \in I$, as well as their unions, are admissible clopen in $\oplus_{i\in I}\mathcal{X}_i$ from the previous definition.*

Proposition 5. *The following statements hold:*

(1) Direct sums of non-empty small spaces are small if and only if they are finite.
(2) Infinite direct sums of non-empty locally small spaces are locally small but not small.
(3) Direct sums of taut spaces are taut.
(4) Direct sums of strongly taut spaces are strongly taut.
(5) Direct sums of pre-semi-Heyting locally small spaces are pre-semi-Heyting.
(6) Direct sums of pre-Heyting locally small spaces are pre-Heyting.
(7) Direct sums of Heyting locally small spaces are Heyting.
(8) Direct sums of paracompact locally small spaces are paracompact.

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Proof. (1) If each X_i is a smop and I is finite, then X is a smop. See also (2).

(2) If I is infinite and all X_i are non-empty, then X is not a smop.

(3) The closure $cl(L)$ of each smop L in X is a finite union of some family of small sets $cl(L \cap X_i)$, $i \in I_0$ finite subset of I. Hence, $cl(L)$ is also small.

(4) We now check admissible closedness. Since each $cl(L \cap X_i)$ is admissible closed in the respective X_i, the set $cl(L)$ is admissible closed in X.

(5) Take $V \in AdOp(X)$. Then $cl(V)$ is a union of $cl(V \cap X_i)$ taken in the respective X_i. Each $cl(V \cap X_i)$ is admissible closed (the complement of an admissible open set) in X_i; hence the whole $cl(V)$ is admissible closed (the complement of an admissible open set) in X.

(6) Assume $A \in LCon(X)$. Then each $A_i = A \cap X_i$ is locally constructible in the respective X_i. Hence, all $cl(A_i)$ are admissible closed in the respective X_i. Their union $cl(A)$ is admissible closed in X, similarly to (5).

(7) If all X_i are T_0, then the direct sum X is T_0.

(8) Take a locally finite covering $\mathcal{W}_i$ of X_i by smops for each X_i. Then their union $\mathcal{W} = \bigcup \mathcal{W}_i$ is a locally finite covering of X by smops. Indeed, each smop L of X has non-empty traces on only finitely many X_i, so also on finitely many members of $\mathcal{W}$.

□

The following two examples show that if the underlying set X has finitely many connected components in the original topology, then the space need not be the direct sum of its components.

Example 5 (($\mathbb{R} \setminus \{e\})_{\mathbb{Q}}$)**.** *We declare the smops to be restrictions to $\mathbb{R} \setminus \{e\}$ of finite unions of intervals in $\mathbb{R}$ with ends in $\mathbb{Q} \cup \{-\infty, +\infty\}$. Topologically, this space has two connected components. But it is not the direct sum of its connected components. For example, $(e, +\infty)$ is not a smop.*

Example 6 ($(-\infty, e)_{\mathbb{Q}} \oplus (e, +\infty)_{\mathbb{Q}}$)**.** *Here the smops are the finite unions of restrictions to $(-\infty, e)$ or to $(e, +\infty)$ of intervals with ends in $\mathbb{Q} \cup \{-\infty, +\infty\}$. For example, $(e, +\infty)$ is a smop.*

The next example shows how the topological decomposition into a direct sum may differ from the decomposition into a direct sum of locally small spaces.

Example 7. *Consider the locally small space $\mathbb{N}_{fin} = (\mathbb{N}, \{\mathbb{N}\} \cup \mathrm{Fin}(\mathbb{N}))$ where the smops are exactly the finite sets or the whole space. Then the generated topology is discrete. All non-empty proper subsets of $\mathbb{N}$ are topologically clopen but not admissible clopen. The space does not decompose into a direct sum of finitely or infinitely many non-empty proper subspaces since these subspaces would be admissible clopen. Moreover, the space does not decompose into a direct sum of infinitely many non-empty subspaces since $\mathbb{N}$ is a smop.*

6. Strict Subspaces and Mild Locally Small Spaces

Definition 11 ([17] Definition 2.2.41)**.** *For any locally small space $(X, \mathcal{L}_X)$ and a subset $Y \subseteq X$, we induce a locally small space on Y taking the smopology $\mathcal{L}_Y = \mathcal{L}_X \cap_1 Y$. Then $(Y, \mathcal{L}_Y)$ is called a* subspace *of $(X, \mathcal{L}_X)$. If moreover the admissible open subsets of Y are exactly the traces of admissible open subsets of X on Y (i.e., if $AdOp(Y) = AdOp(X) \cap_1 Y$), then Y is called a* strict subspace *of X. In other words: we say, for short, that a* subset $Y \subseteq X$ *is* strict *if $\mathcal{L}_X^o \cap_1 Y = (\mathcal{L}_X \cap_1 Y)^o$ in $\mathcal{P}(Y)$.*

Naturally, the inclusion $\mathcal{L}_X^o \cap_1 Y \subseteq (\mathcal{L}_X \cap_1 Y)^o$ is obvious.

Proposition 6 ([27] Lemma 1)**.** *Each admissible open, admissible closed, small, or constructibly dense (see [27] Definition 4(4)) set in a locally small space forms a strict subspace.*

Definition 12 ([18] Definition 2.40)**.** *A mapping $f : \mathcal{X} \to \mathcal{Y}$ between locally small spaces is called* bounded *if the image of a smop is contained in a smop, while it is called* continuous *if the preimage of a smop is admissible open.*

Problem 2. *Does the preimage of a strict subspace under a bounded continuous mapping always form a strict subspace of the domain?*

We introduce the following notion:

Definition 13. *A locally small space is* mild *if all its subsets are strict (i.e., form strict subspaces).*

We call such spaces mild because no pathology of the subspace operation occurs. Notice that Problem 2 is interesting only when the domain is not known to be mild. From the literature we have only the following consequence of [27] Lemma 1, quoted above as Proposition 6:

Corollary 3. *All small spaces are mild.*

In particular, a locally small space is mild if its smopology is a topology.

Problem 3. *Characterise mild locally small spaces.*

We shall give a partial answer to this problem.

Lemma 1. *(1) Mildness is a hereditary property.*
(2) Paracompactness is a hereditary property.

Proof. (1) Take a subspace $\mathcal{Z}$ of a subspace $\mathcal{Y}$ of the space $\mathcal{X}$. Then $\mathcal{Z}$ is a subspace of $\mathcal{X}$. Since $\mathcal{X}$ is mild, both $\mathcal{Y}$ and $\mathcal{Z}$ are strict in $\mathcal{X}$. Hence, $\mathcal{Z}$ is strict in $\mathcal{Y}$.
(2) Local finiteness of a family of sets is preserved under taking the intersection with any set. □

Example 8 (a semialgebraic triangle in a square)**.** *Let $X = (0,1) \times (0,1)$ and $T = X \cap \{\langle x, y\rangle : x < y\}$. We declare that a smop is any semialgebraic open subset of X contained in some $T \cup X_n$, where $X_n = (\frac{1}{n}, 1) \times (0,1)$, $n \geq 2$. The space is neither paracompact nor taut (the topology of our space $\mathcal{X}$ is the same as in Example 3).*

Proposition 7. *The space from Example 8 is not mild.*

Proof. Consider the sets $W = \{\langle \frac{1}{n}, \frac{1}{n}\rangle : n \geq 2\}$ and $Y = W \cup \{\langle \frac{1}{n}, \frac{1}{n} + e^{-n}\rangle : n \geq 2\}$. Then $Y \setminus W$ is contained in one smop T of the space. Notice that W is admissible open in the induced space $\mathcal{Y}$ on Y. Indeed, each smop in $\mathcal{Y}$ has a finite trace on W (since the trace is contained in some X_n) and each finite subset of W is a smop of $\mathcal{Y}$ (small discs easily avoid $Y \setminus W$).

If W were equal to $V \cap Y$ for some $V \in AdOp(X)$, then V would have to contain a non-empty semialgebraic region $T \cap V$ (since it is contained in the smop T) whose projection on the first axis contains some interval with the left limit 0, having the origin in its closure. This region would have to be disjoint from $Y \setminus W$. The semialgebraic function of height of this region (defined and positive for all x sufficiently close to 0 by o-minimality) near the diagonal

$$h(x) = \sup\{\alpha : \forall \beta \in (0, \alpha)\ \langle x, x+\beta \rangle \in T \cap V\}$$

would have to satisfy $h(\frac{1}{m}) \leq e^{-m}$ for $m \geq 2$. But semialgebraic functions are polynomially bounded, so such a semialgebraic h cannot exist. A contradiction; hence Y is not strict in $\mathcal{X}$ and the space $\mathcal{X}$ is not mild. □

This answers Question 2.1.25 in [19] asking if any subset of a locally small space forms a strict subspace. One can easily generalise the above example not only to polynomially bounded o-minimal expansions of the real field, but also to exponentially bounded ones, using a sequence $\varepsilon(m) = \frac{1}{\exp_m(m)}$ where $\exp_m(x) = \exp(\exp_{m-1}(x))$, $\exp_1(x) = \exp(x)$.

Proposition 8. *Direct sums of mild spaces are mild spaces.*

Proof. If $\mathcal{X} = \bigoplus_{i \in I} \mathcal{X}_i$ is a decomposition of a space $\mathcal{X}$ as a direct sum, then a subset $V \subseteq X$ is admissible open iff all its traces $V \cap X_i$ are admissible open in the respective $\mathcal{X}_i$'s. Consider any subset $Y \subseteq X$ and an admissible open set W in the subspace induced on Y. Since for each i we have $W \cap X_i = Y \cap M_i \in AdOp(Y \cap X_i)$ for some admissible open $M_i \subseteq X_i$, we obtain $W = \bigcup_{i \in I} M_i \cap Y$ with $\bigcup_{i \in I} M_i$ admissible open in $\mathcal{X}$. Hence, Y forms a strict subspace of $\mathcal{X}$. □

Lemma 2. *Each paracompact locally small space is mild.*

Proof. Let Y be an arbitrary set in the paracompact space $(X, \mathcal{L}_X)$. Consider a locally finite covering $\{V_i\}_{i \in I}$ of X by smops. To prove the non-trivial inclusion (each admissible open set in Y is a trace of an admissible open set in X), consider an admissible open subset W of Y. Then each $W_i = W \cap V_i$ is a smop in Y, so $W_i = M_i \cap Y$ for some smop M_i in X. We may assume $M_i \subseteq V_i$. Then $\{M_i\}_{i \in I}$ is also a locally finite family and its union $M = \bigcup_i M_i$ is an admissible open set (Corollary 1) satisfying $M \cap Y = W$. This proves Y is strict and $(X, \mathcal{L}_X)$ is mild. □

Example 9 (Khalimsky plane, cf. [25] Example 2)**.** *Declare the locally small space called the Khalimsky digital plane to be $(\mathbb{Z}^2, \mathcal{L}_{\mathbb{Z}^2})$ with the smopology $\mathcal{L}_{\mathbb{Z}^2}$ consisting of those finite sets which, together with an even point $(2m, 2n)$, contain the set $\{2m-1, 2m, 2m+1\} \times \{2n-1, 2n, 2n+1\}$; together with an even-odd point $(2m, 2n+1)$, contain the set $\{2m-1, 2m, 2m+1\} \times \{2n+1\}$ and similarly for an odd-even point. Then $AdOp(\mathbb{Z}^2, \mathcal{L}_{\mathbb{Z}^2})$ is the family of arbitrary unions of smops, so it is a topology equal to the original topology. Notice that $LCon(\mathbb{Z}^2, \mathcal{L}_{\mathbb{Z}^2}) = \mathcal{P}(\mathbb{Z}^2)$. The Khalimsky plane is topologically connected, paracompact, and Heyting. By Lemma 2, the Khalimsky plane is mild.*

Proposition 9. *Assume that the family of all admissible open sets in a locally small space is closed under arbitrary unions. Then the space is mild.*

Proof. For a set Y in the space $\mathcal{X} = (X, \mathcal{L}_X)$ and for $W \in AdOp(Y)$, we can express W as a union of intersections of some smops $V_i (i \in I)$ in $\mathcal{X}$ with Y. As $AdOp(X)$ is closed under arbitrary unions, the union $V = \bigcup_i V_i$ is admissible open and $W = V \cap Y$. □

Proposition 9 helps to see that the spaces from Examples 2 and 3 are mild but not paracompact. Comparing Examples 3 and 8, we can also see that mildness is not a topological property.

Theorem 1. *The class of mild locally small spaces is closed under subspaces and direct sums, and contains all spaces that are either paracompact or have the admissible open sets closed under arbitrary unions.*

Proof. Follows from Lemma 1, Lemma 2, Proposition 9 and Proposition 8. □

The space from Example 1 is paracompact, but its admissible open sets are not closed under arbitrary unions, while Examples 2 and 3 are of the second kind but not paracompact; hence the two assumptions on spaces appearing in Theorem 1 are independent. We do not know if there exists a mild space that is not the direct sum of a paracompact space and a space whose admissible open sets are closed under arbitrary unions.

7. Finitary Locally Small Spaces

Definition 14 (admissible connectedness, cf. [18] Definition 2.56, [6] Remark 16)**.** *A space is* admissibly connected *if it cannot be decomposed into two non-empty admissible open sets. Equivalently: a space $\mathcal{X}$ is admissibly connected iff it is not a direct sum $\mathcal{X}_1 \oplus \mathcal{X}_2$ of its non-empty open subspaces.*

Definition 15 (cf. [17] Definition 2.2.89)**.** *An* admissible component *of a locally small space is a maximal admissibly connected set in this space.*

Proposition 10 (finitely many admissible components)**.** *A space has only finitely many admissible components if and only if it decomposes into finitely many admissible open sets that are admissibly connected.*

Proof. Each admissible component is contained in every admissible clopen set non-trivially intersecting the component. If the decomposition exists, then the summands are admissible clopen and are the admissible components. If the space has only a finite number of admissible components, then an easy induction on the number of admissible components shows, using Proposition 6, that those components must be admissible open (so also admissible closed). □

The following proposition may be of independent interest.

Proposition 11. *In a paracompact locally small space $(X, \mathcal{L}_X)$ with a finite number of* admissible *components, there exists a locally essentially finite non-decreasing countable covering of X by smops $\{X_n\}_{n\in\mathbb{N}}$ such that $bd(X_k) \cap bd(X_l) = \emptyset$ for $k \neq l$ and with the property that each smop $L \in \mathcal{L}_X$ is contained in some X_m.*

Proof. Choose a locally finite covering $\mathcal{W}$ of X by smops and let V_1 be a finite union of members of $\mathcal{W}$ that intersects all admissible components of X. Denote by V_{n+1} the union of all smops in the covering $\mathcal{W}$ that intersect V_n, getting a non-decreasing family $\{V_n\}_{n\in\mathbb{N}}$ of smops. Since the union $\bigcup_{n\in\mathbb{N}} V_n$ is both admissible open and admissible closed, by Corollary 1, and intersects all admissible components, it is the whole space X. Since each V_k is a finite union of smops, it is a smop, and each smop is a subset of some member V_m of the mentioned locally essentially finite covering of X. Put $X_n = V_{2n+1}$. Clearly, $\{X_n\}_{n\in\mathbb{N}}$ is locally essentially finite. Consider the sets $bd(X_k)$ and $bd(X_l)$ for $k < l$. Then $bd(X_k)$, a subset of $cl(X_k)$, is contained in V_{2k+2} and $bd(X_l)$ is disjoint from V_{2k+3}, so $bd(X_k) \cap bd(X_l) = \emptyset$. □

Notice that a topologically connected set is always admissibly connected. Hence, the connected components are always contained in the admissible components.

Corollary 4. *If a paracompact locally small space has a finite number of topological connected components, then it satisfies the assumptions of Proposition 11. Hence, its conclusion holds.*

Definition 16 (finitary space)**.** *A locally small space is* finitary *if all its smops have finitely many admissible components.*

The notions of being finitary and having finitely many topological connected components are independent, since we have the following examples.

Example 10 (finitary, infinitely many c.c.)**.** *Take the ordered field of real algebraic numbers $\overline{\mathbb{Q}}^r$, the real closure of $\mathbb{Q}$, and treat it as an o-minimal structure. This means that the smops are the finite unions of open intervals in $\overline{\mathbb{Q}}^r$. Then each smop has a finite number of admissible components. On the other hand, there exist countably many topological connected components of $\overline{\mathbb{Q}}^r$, all being singletons.*

Example 11 (finitely many c.c., non-finitary)**.** *Consider the real line with the usual topology as the smopology $(\mathbb{R}, \tau_{nat})$. Then all weakly open sets are smops. Topologically connected components are admissible components. The space is both topologically and admissibly connected. Any open set with infinitely many topological connected components is a witness that the space is not finitary.*

Finitary locally small spaces do not have to be mild, see Example 8.

Proposition 12. *In a finitary paracompact locally small space, all admissible components are admissible clopen, and they form a locally finite family. Hence, the space is the direct sum of those admissible components.*

Proof. Take a locally finite covering $\mathcal{W}$ of the space by smops. Each $W \in \mathcal{W}$ decomposes into a finite family of relatively admissible clopen admissibly connected sets W_i, $i \leq k(W)$. The new family $\mathcal{W}^* = \{W_i : i \leq k(W), W \in \mathcal{W}\}$ is still locally finite. Since the union of two admissibly connected sets that non-trivially intersect is admissibly connected, admissible components of the space are unions of members of $\mathcal{W}^*$, and they form a locally finite family. Consequently, admissible components are admissible clopen (Corollary 1). Since the smops of the space are exactly the finite unions of the smops of admissible components, the space is the direct sum of those admissible components. □

8. Mild Locally Definable Spaces

Now we apply our knowledge of locally small spaces in a context closer to model theory and o-minimal geometry and topology.

Definition 17 ([18] Definition 3.1)**.** *Assume M is any non-empty set. A family $\mathcal{F}$ of functions $h : U \to M$, where $U \in \mathcal{L}^o_X$, which is closed under:*

(a) *restrictions of functions to admissible open subsets $V \subseteq U$,*
(b) *gluings of compatible families of functions defined on members of locally essentially finite families of admissible open sets*

is called a function sheaf over *M on a locally small space $\mathcal{X} = (X, \mathcal{L}_X)$. By $\mathrm{FSh}(\mathcal{X}, M)$ we denote the family of all function sheaves on a space $\mathcal{X}$ over a set M.*

Notation 2 ([18] Definition 3.2)**.** *For given function sheaves $\mathcal{F} \in \mathrm{FSh}(\mathcal{X}, M)$, $\mathcal{G} \in \mathrm{FSh}(\mathcal{Y}, M)$ and a mapping $f : X \to Y$, we consider the following operations*

1. *the image $f_*\mathcal{F} = \{h : V \to M | V \in \mathcal{L}^o_Y,\ h \circ f \in \mathcal{F}\}$ of $\mathcal{F}$ by f,*
2. *the preimage $f^{-1}\mathcal{G} = \mathcal{G} \circ f = \{h \circ f | h \in \mathcal{G}\}$ of $\mathcal{G}$ by f.*

Definition 18 ([18] Definition 3.3)**.** *The following notions will describe locally definable spaces.*

1. *A* locally small space over M *is a pair* $(\mathcal{X}, \mathcal{O}_X)$*, where* $\mathcal{X}$ *is a locally small space and* $\mathcal{O}_X$ *is a function sheaf over* M *on* X*. A* morphism $f : (\mathcal{X}, \mathcal{O}_X) \to (\mathcal{Y}, \mathcal{O}_Y)$ *of locally small spaces over* M *is a bounded continuous mapping* $f : \mathcal{X} \to \mathcal{Y}$ *such that* $\mathcal{O}_Y \subseteq f_* \mathcal{O}_X$ *(equivalently:* $f^{-1}\mathcal{O}_Y \subseteq \mathcal{O}_X$*).*
   We obtain the category $\mathbf{LSS}(M)$ *of locally small spaces over* M *and their morphisms.*
2. *An object of* $\mathbf{LSS}(M)$ *is said to be* small, (topologically) connected, *or* paracompact *if its underlying locally small space is such.*

Corollary 5. *An isomorphism* $f : (\mathcal{X}, \mathcal{O}_X) \to (\mathcal{Y}, \mathcal{O}_Y)$ *is a bijection* $f : X \to Y$ *such that* $f(\mathcal{L}_X) = \mathcal{L}_Y$ *and* $\mathcal{O}_Y = f_* \mathcal{O}_X$ *(equivalently:* $f^{-1}\mathcal{O}_Y = \mathcal{O}_X$*).*

Definition 19 ([18] Definitions 3.4 and 3.5)**.** *Moreover, we use the following notions:*

1. *An* open subspace *of an object* $((X, \mathcal{L}_X), \mathcal{O}_X)$ *of* $\mathbf{LSS}(M)$ *is an object* $((Y, \mathcal{L}_Y), \mathcal{O}_Y)$ *where* $Y \in \mathcal{L}_X^o$, $\mathcal{L}_Y = \mathcal{L}_X \cap_1 Y$, $\mathcal{O}_Y = i_{YX}^{-1}\mathcal{O}_X$*, where* $i_{YX} : Y \to X$ *denotes the inclusion.*
2. *An object* $(\mathcal{X}, \mathcal{O}_X)$ *of* $\mathbf{LSS}(M)$ *is the* admissible union *of a family* $\{(\mathcal{X}_i, \mathcal{O}_i)\}_{i \in I}$ *of objects in* $\mathbf{LSS}(M)$ *if: each* X_i *is an admissible open set in* $\mathcal{X}$*, the family* $\{\mathcal{X}_i\}_{i \in I}$ *is locally essentially finite in* $\mathcal{X}$, $\mathcal{L}_X \cap_1 X_i = \mathcal{L}_i$ *for each* $i \in I$*, and* $\mathcal{O}_X$ *is the smallest function sheaf containing* $\bigcup_{i \in I} \mathcal{O}_i$*. We write then*
$$(\mathcal{X}, \mathcal{O}_X) = \bigcup_{i \in I}^{a} (\mathcal{X}_i, \mathcal{O}_i).$$
   If I *is finite, such an admissible union will be called a* finite open union *of objects of* $\mathbf{LSS}(M)$*.*

O-minimal structures are well known in model theory. A typical reference to the theory of o-minimal structures is [23].

Definition 20. *An* o-minimal structure *is a structure* $\mathcal{M} = (M, \leq, \ldots)$ *in the sense of model theory such that* $\leq$ *is a dense linear order without endpoints and all definable subsets of* M *are finite unions of intervals (of any kind). Any o-minimal structure has an intrinsic order topology on* M *and the product topology on each Cartesian power* M^n, $n \in \mathbb{N}$*. They are called* Euclidean topologies*.*

Definition 21 ([18] Definition 3.13, [6] pp. 579–580)**.** *We use the following notions of spaces over structures:*

1. *An* affine definable space *over an (o-minimal) structure* $\mathcal{M} = (M, \leq, \ldots)$ *is an object* $((D, \mathcal{L}_D), \mathcal{O}_D)$ *of* $\mathbf{LSS}(M)$ *that is isomorphic to a definable set* $D \subseteq M^n$ *with the smopology* $\mathcal{L}_D$ *of definable relatively open subsets of* D *and the function sheaf of definable (in* $\mathcal{M}$*) continuous functions* $f : E \to M$, $E \in \mathcal{L}_D = \mathcal{L}_D^o$*.*
2. *A* locally definable space *over a structure* $\mathcal{M} = (M, \leq, \ldots)$ *is an object of the category* $\mathbf{LSS}(M)$ *that has a locally essentially finite covering by affine definable open subspaces (It is then the admissible union of the family of those subspaces by (b) in Definition 17).*

The concept of a definable component in a locally definable space (over an o-minimal structure) is well known.

Definition 22 (cf. [23] p. 57)**.** *A* definable set *in a locally definable space is a small subset whose intersections with (the underlying sets of) all affine definable subspaces are definable. (It is enough to check this for one locally essentially finite covering of the space by affine subspaces.) A* definably connected component *of a definable set* D *in a locally definable space is a maximal definably connected subset of* D*.*

Proposition 13. *Assume D is a definable set in a locally definable space over an o-minimal structure. Then the following are equivalent for a subset E of D:*

(a) *E is a definably connected component of D,*
(b) *E is an admissible component of D.*

Proof. Case 1: The space is affine definable.

If $D \subseteq M^n$ is definable, then the classical o-minimal arguments suffice. Indeed, each definable set decomposes into finitely many relatively clopen definable subsets which are definably connected, so are the definably connected components of D (see [23], (2.9) and (2.18) of Chapter 3). The smops of D are the relatively open definable subsets of D. This means that the definably connected components of D are the admissible components of D. The equivalence is transported to all affine definable spaces by definable homeomorphisms (which are the isomorphisms between affine definable spaces).

Case 2: The space is arbitrary locally definable.

If D is a definable set in a space $\mathcal{X}$, then, by smallness, it is contained in some finite union of affine definable subspaces $\mathcal{X}_i (i \in I_0)$ of $\mathcal{X}$. Both definably connected components and admissible components of D are those finite unions of admissible components of the traces $D \cap X_i (i \in I_0)$ which are admissibly connected. □

Theorem 2. *All paracompact locally definable spaces over o-minimal structures $(M, \leq, \ldots)$ are as follows:*

1. *mild as locally small spaces,*
2. *finitary,*
3. *Heyting (so strongly taut by paracompactness and Proposition 4).*

Proof. Those spaces are mild by Lemma 2. Moreover, by o-minimality, they are finitary since each definable set has a finite number of admissible components, see the proof of Proposition 13.

They are Heyting since they are T_0 (even every single smop is T_0), and the locally constructible sets (also called the locally definable sets) have closures that are the closed locally definable sets (i.e., admissible closed sets). Indeed, the closure of a definable open set is a definable closed set (compare [23], (3.4) in Chapter 1). Moreover, Boolean combinations of definable open sets are arbitrary definable sets. Now, by paracompactness, those properties extend from definable sets to locally definable sets. Let L be a smop contained in a finite union U of members of a locally finite covering by smops. For a locally constructible A, we have $cl(A) \cap L = cl(A \cap U) \cap L$. Then $A \cap U$ is constructible in a smop, so it is definable, and its closure is a definable closed set. This proves $cl(A) \in AdCl(X)$. □

9. Conclusions

We have given some sufficient conditions for a locally small space to be mild, including Theorem 1. We have also presented an example of a non-mild space as well as other examples of locally small spaces. We have then applied this knowledge to the setting in o-minimal geometry/topology in Theorem 2. Consequently, all paracompact locally definable spaces over o-minimal structures are "safe" in the sense that they are mild as locally small spaces, so that no unexpected phenomena involving the notion of a subspace will occur. We have also posed some problems related to locally small spaces.

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