

Symmetry in Mathematical Theory and Simulation Methods for Backward Problems

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Backward problems, also referred to as inverse problems, aim to recover an unknown cause—an initial state, a boundary condition, a source term, a material parameter, or a hidden internal structure—from indirect observations of its effects. In this sense, they are the mirror image of the corresponding forward problems, which propagate a known cause to its measurable consequences. This reversal of the cause–effect direction is precisely what places backward problems within the scope of *Symmetry*. Unlike forward problems, backward problems are typically ill-posed in the sense of Hadamard [1]: a solution may not exist, may not be unique, or may depend discontinuously on the data, so that small measurement errors can be amplified without bound. The mathematical theory of such problems, from the foundational work of Tikhonov to the modern functional-analytic treatments, is by now well established [2–7]. Overcoming their intrinsic instability nevertheless still requires both a rigorous understanding of the underlying operators and carefully designed simulation methods.

On the computational side, a rich collection of stabilization strategies has been developed, including the quasi-reversibility and quasi-boundary methods [8–10], mollification [11], and a variety of parameter-choice rules such as the discrepancy principle, generalized cross-validation, and the L-curve criterion [12–14]. Symmetry considerations offer powerful tools that complement these strategies. Lie-group and symmetry-operator structures reveal invariants and evolutionary operators of the governing equations [15–18]; in particular, group-preserving and Lie-group shooting schemes exploit the cone structure of the augmented dynamics to integrate parabolic problems backward in time without a priori regularization [19–22]. Geometric symmetry of the domain reduces the dimensionality of a problem and exposes its singular points; symmetric transformations of series and special functions yield closed-form expressions that serve as analytical benchmarks; and structural symmetry of a network can act as a diagnostic quantity in its own right. These ideas are increasingly combined with meshless collocation and fundamental-solution methods [23–26] and, most recently, with physics-informed neural networks [27]. The applications are correspondingly broad, ranging from inverse heat conduction [28–30], mechanics, and acoustics to medical imaging, nondestructive testing, geophysics, finance, and the operation of modern energy systems.

This Special Issue collects six papers that, taken together, illustrate this spectrum—from abstract operator theory, through inverse and fractional models with their numerical treatment, to combinatorial identities and a data-driven application in electricity markets.

At the theoretical end, Ludkowski [31] develops the noncommutative integration of the diffusion partial differential equation and of several of its generalizations. Since many approaches to both forward and backward evolution problems rest on evolutionary operators obtained as solutions of the underlying equation, extending these operators to a



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noncommutative setting enlarges the class of diffusion-type problems that can be treated analytically and clarifies the algebraic symmetry behind them.

Two contributions address inverse and fractional models of direct engineering relevance. Koleva and Vulkov [32] study inverse interface problems for parabolic equations with cylindrical symmetry, in which unknown boundary conditions must be identified from point observations. In the two-layer configuration, the degeneracy of the equation at the axis of symmetry makes the problem particularly delicate; the authors exploit the symmetry to construct well-posed reformulations and efficient numerical procedures for recovering the missing boundary data. Fractional-order models add a further layer of difficulty, because the memory kernel of the Caputo derivative couples every time level to the entire history of the solution [33–35]; backward and inverse problems for such equations have accordingly attracted intense interest [36–38]. Against this background, Vivas-Cortez et al. [39] turn to the time-fractional generalization of the KdV–Burgers equation [40] and propose a rational non-polynomial spline scheme whose construction draws on symmetric functions. The resulting method attains high accuracy for a class of time-fractional problems that frequently appear as the forward model in backward reconstruction.

Two additional papers strengthen the analytical toolkit on which such reconstructions rely. Chen and Chu [41] examine four classes of multiple symmetric sums over cyclic products of binomial coefficients and, by combining the generating-function approach with a recursive construction, express them analytically as coefficients of rational functions together with explicit recurrence relations. Li and Chu [42] apply the coefficient extraction method to a symmetric transformation of hypergeometric series and obtain a systematic family of infinite series of convergence rate $-1/4$ involving harmonic numbers. Closed-form identities of this kind, rooted in the classical theory of special functions and hypergeometric summation [43–45], are indispensable for building exact solutions, verifying numerical schemes, and accelerating series-based algorithms in inverse computations.

Finally, Zicmane et al. [46] carry the theme into power engineering. The reconstruction of nodal prices and generation patterns in network-constrained electricity markets [47] is itself an inverse analysis problem, closely related to power-system state estimation [48] and complicated by congestion-induced non-uniqueness and limited observability. The authors introduce node symmetry analysis as an early indicator of the growth of locational marginal prices under high renewable penetration, showing that structural symmetry of the network carries predictive information that conventional market data alone do not reveal.

I hope that this Special Issue will serve as a useful reference for mathematicians, computational scientists, and engineers working on backward problems, and that it will stimulate further exploration of the ways in which symmetry—algebraic, geometric, combinatorial, and structural—can be harnessed to make ill-posed problems tractable.

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