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A Partitioned Maclaurin Symmetric Mean-Based Framework for Multi-Attribute Decision-Making Under Linear Diophantine Fuzzy Sets

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Abstract

Linear Diophantine fuzzy sets (LDFSs) enlarge the admissible representation space for uncertain information through membership and non-membership degrees together with reference parameters. Existing LDFS aggregation studies have mainly considered weighted, ordered, power-based, and parametric operators, while partitioned aggregation operators that model possible interrelationships within criterion groups remain limited. Maclaurin symmetric mean (MSM) operators capture interactions among criteria, and reducible weighted MSM operators have been developed to preserve idempotency and reducibility properties but have not been extended to a partitioned MSM operator. This study develops an MSM-based aggregation framework for LDFSs, including the linear Diophantine fuzzy MSM (LDFMSM), the linear Diophantine fuzzy partitioned MSM (LDFPMSM), a reducible weighted partitioned MSM (RWPMSM) and its extension to LDFSs, and the linear Diophantine fuzzy reducible weighted partitioned MSM (LDFRWPMSM) operators. LDFMSM performs symmetric k-subset aggregation over the complete criterion set, LDFPMSM restricts joint aggregation terms to criteria within the same predefined partition, and LDFRWPMSM incorporates criterion weights while preserving idempotency and reducibility properties. Theoretical properties and special cases are established. The framework is demonstrated through numerical examples and an illustrative multi-attribute decision-making (MADM) application for an AI-driven precision agriculture platform selection. Sensitivity analyses examine the effects of the interaction parameter, criterion weights and partition structure and reveal systematic changes in the ranking results. Comparative analysis shows that the LDFPMSM can distinguish alternatives that remain tied under globally symmetric aggregation, while the LDFRWPMSM identifies a different best alternative from the benchmark operators, reflecting its joint treatment of criterion importance and within-partition interactions.

Keywords: linear Diophantine fuzzy sets; Maclaurin symmetric mean; partitioned aggregation; reducible weighted operators; MADM

Academic Editor: Jian-Qiang Wang

Received: 25 July 2026
Revised: 29 August 2026
Accepted: 31 August 2026
Published: 2 September 2026

1. Introduction

Information aggregation is essential in technical systems that combine heterogeneous measurements, signals, or assessments into a single output. Typical applications include decision making, multi-sensor monitoring and control, fault diagnosis, early-warning systems, and wearable health-monitoring systems [1–3]. In such applications, the aggregation rule determines how the available inputs contribute to the final assessment, diagnosis, or control action. Within the context of decision making, group multi-attribute decision-making (MADM) problems aim to evaluate, rank, and select the most appropriate alternative(s) from a finite set based on multiple and possibly conflicting attributes assessed by a set of decision makers. Because these contexts commonly involve subjective judgments, incomplete information, and various forms of uncertainty, evaluations are frequently represented using fuzzy sets.

Within fuzzy MADM, the representation of uncertainty and the aggregation mechanism are particularly important because they determine, respectively, how imprecise evaluations are modeled and how attribute information is combined into overall assessments. The selected representation framework should be consistent with the type of uncertainty present in the input data since it can affect both the decision process and its outcome [4]. Likewise, aggregation functions play a significant role because they determine how attribute evaluations and, where relevant, their interrelationships are incorporated into an overall assessment [5].

Fuzzy set theory, introduced by Zadeh [6], represents uncertain information through membership degrees defined on the unit interval. Several extensions have subsequently been developed to increase representational flexibility. Intuitionistic fuzzy sets (IFSs) [7] represent membership and non-membership degrees subject to the condition that their sum does not exceed one and the remaining part is interpreted as the hesitancy degree. Pythagorean fuzzy sets (PyFSs) [8], Fermatean fuzzy sets (FFSs) [9], and q-rung orthopair fuzzy sets (q-ROFSs) [10] each further enlarge the feasible assessment space by extending this framework by imposing squared, cubic and q-th power constraints, respectively, and thus providing greater flexibility in modeling uncertainty. Picture fuzzy sets (PFSs) proposed by Cuong [11] represent membership, hesitancy, and non-membership degrees, while the remaining assessment space may be interpreted as a refusal degree. Spherical fuzzy sets (SFSs) [12] and t-spherical fuzzy sets [13] respectively generalize this structure by replacing the constraint with a squared and t-th power condition. Other related paradigms, including neutrosophic sets [14], complex intuitionistic fuzzy sets [15], and interval-valued fuzzy sets, have been introduced.

Riaz and Hashmi [16] developed the linear Diophantine fuzzy set (LDFS), in which membership and non-membership degrees are associated with reference parameters. LDFSs are constrained such that the following constraints are satisfied: the membership degree and non-membership degree, along with their parameters, lie within the unit interval; the sum of the reference parameters lies within the unit interval; and the sum of membership degree and non-membership degree weighted by their respective reference parameter lies within the unit interval. LDFS allows for a greater representation space compared to IFSs and PyFSs [16]. Compared to the existing fuzzy sets, LDFS offers additional flexibility through reference parameters and makes it suitable for modeling uncertain information structures.

In classical MADM settings, aggregation is generally conducted at two stages: first, to combine the evaluation information provided by multiple experts for each alternative under each criterion; and second, to aggregate criterion values to determine the overall performance of each alternative. Aggregation operators therefore serve as essential information-fusion mechanisms. In many decision problems, however, criteria may not contribute independently to the final assessment. Their interrelationships may reflect correlation, substitutivity or complementarity, and preferential dependence [17]. Correlation concerns whether criterion evaluations tend to vary together, whereas substitutivity and complementarity indicate whether criteria provide overlapping contributions or yield an additional contribution when jointly considered. Preferential dependence arises when preferences regarding one criterion or criterion group depend on the performance levels of other criteria [17]. Chen et al. [18] further distinguished homogeneous (homo) and heterogeneous (hete) interrelationships. Under a homo structure, all criteria participate in a common interaction framework. Under a hete structure, only some criteria or criterion groups are included in the same interaction mechanism [5,18]. This distinction concerns the aggregation structure imposed on the criterion set and does not constitute a claim that criteria from different dimensions are statistically or causally independent. Rather, it reflects a structural modeling judgment whose criteria, which are conceptually related or may exhibit interrelationships relevant to the decision context, should be jointly aggregated.

Aggregation operators (AOs) account for such interrelationships through different mechanisms. The Choquet integral employs a nonadditive fuzzy measure to represent the importance of both individual criteria and criterion coalitions [19]. Accordingly, the specified fuzzy measure can encode interaction patterns at the subset level, including homogeneous or heterogeneous structures [5,17]. Power aggregation operators adjust the effective influence of each input according to the support it receives from the remaining inputs, thereby reflecting mutual agreement among evaluations [5,20]. Bonferroni mean operators generate pairwise cross-product terms over the input set and are commonly associated with homogeneous interaction structures [5,21,22]. Extended Bonferroni mean structures were introduced by [18,23] to address heterogeneous interrelationships, and Chen et al. [18] subsequently generalized the EBM through the generalized extended Bonferroni mean. Heronian mean operators also use pairwise joint aggregation; however, unlike conventional Bonferroni mean operators, they account for self-relations and reduce the redundant treatment of the same criterion pair [5,24]. Further, Maclaurin symmetric mean (MSM) operators extend joint aggregation beyond pairs by forming symmetric terms over subsets of k criteria [5,25]. Classical MSM therefore adopts a global homogeneous joint-aggregation structure in which all eligible k-criterion subsets across the complete input set are treated symmetrically. It does not identify the type, direction, or strength of relationships among criteria.

A variety of weighted, ordered, prioritized, and parametric aggregation operators has been developed under LDFSs and related Diophantine fuzzy extensions. These include Einstein- [26], Aczel–Alsina- [27], Dombi- [28], Hamacher- [29–31], and Frank-based operators [32], as well as support-based power aggregation [33], Choquet-integral aggregation [34], Bonferroni mean [35,36], and Muirhead mean [37] operators. These studies demonstrate that interaction-aware aggregation is not absent from the LDFS-related literature. However, no explicit MSM framework was identified under LDFSs in the reviewed literature. Moreover, no partitioned and weighted partitioned aggregation framework was identified under LDFSs or related Diophantine fuzzy extensions.

The distinction between homo and hete interrelationships is relevant when a criterion set contains conceptually distinguishable dimensions. For example, in the evaluation of electric vehicle alternatives, battery capacity, driving range, and charging time may form a battery-performance dimension, whereas safety, cabin comfort, and design quality may form a vehicle-quality dimension. Criteria within the same dimension may reasonably be jointly considered, while forming the same joint aggregation terms across these dimensions may not be appropriate.

Partitioned aggregation operators extend this logic by restricting joint aggregation to predefined groups of criteria. The partitioned Bonferroni mean was introduced to aggregate pairwise terms within predefined groups that reflect the heterogeneous interrelationship structure [23]. Partitioned Maclaurin symmetric mean (PMSM) operators, developed by [38], further allow symmetric k-subset aggregation within such groups. PMSM-based and weighted PMSM-based operators have subsequently been developed in several fuzzy environments, including intuitionistic fuzzy [39], q-rung orthopair fuzzy [38], probabilistic hesitant fuzzy [40], and complex Fermatean fuzzy [41] sets. These studies show that partitioned joint aggregation is an established modeling principle for decision problems involving conceptually distinguishable criterion groups.

However, the reviewed LDFS and related Diophantine fuzzy literature does not provide an MSM, PMSM or weighted MSM framework. Although weighted MSM operators have been introduced in other fuzzy environments, formulations derived from weighted averaging and geometric operators may fail to preserve idempotency and reducibility properties [42–44]. To address these limitations, Shi and Xiao [42] introduced the reducible weighted MSM operator (RWMSM). However, it has not been extended to the partitioned MSM structure.

To address this gap, this study develops aggregation operators under the LDFSs, including the linear Diophantine fuzzy MSM (LDFMSM) and the linear Diophantine fuzzy partitioned MSM (LDFPMSM). In addition, the reducible weighted partitioned MSM (RWPMSM) operator is developed to preserve the idempotency and reducibility properties in the aggregation operation under a weighted setting and is subsequently extended to LDFSs as the linear Diophantine fuzzy reducible weighted partitioned MSM (LDFRWPMSM) operator.

The LDFMSM operator provides the baseline for the unpartitioned criterion set with equal weights and aggregates all k-criterion subsets across the complete criterion set. The LDFPMSM operator extends this structure by restricting joint aggregation to criteria within the same predefined partition. The LDFRWPMSM operator incorporates criterion weighting into the same partitioned structure while preserving the idempotency and reducibility. The LDFRWPMSM operator reduces to the LDFPMSM operator under equal weights of criteria; moreover, it reduces to LDFMSM when all criteria belong to a single partition. These reducibility relations ensure consistency among the weighted partitioned, unweighted partitioned and unweighted-unpartitioned settings.

The main contributions of this study are summarized as follows.

LDFMSM is developed for unpartitioned symmetric k-subset aggregation.

LDFPMSM introduces a partition-restricted symmetric k-subset aggregation structure in which criteria are jointly aggregated only within predefined partitions, and cross-partition combinations are excluded.

The RWPMSM operator is developed to incorporate criteria weighting into the partitioned MSM framework while preserving idempotency and reducibility. It is further extended to LDFSs through the LDFRWPMSM operator, which retains these properties.

The mathematical properties, special cases, and reducibility relations of the proposed operators are established.

The proposed framework is illustrated through an AI-driven precision agriculture platform selection problem and examined through sensitivity and comparative analyses.

The remainder of this paper is organized as follows. Section 2 reviews aggregation-operator studies under LDFSs and related extensions. Section 3 presents the preliminaries on LDFSs, MSM, PMSM, and RWMSM formulations. Section 4 develops the proposed operators and investigates their properties. Section 5 provides numerical illustrations under different aggregation settings. Section 6 presents the illustrative MADM application. Sections 7 and 8 report the sensitivity and comparative analyses, respectively. Section 9 discusses the practical implications and constraints of the proposed framework. Finally, the last section presents the conclusions, limitations, and directions for future research.

https://doi.org/10.3390/sym18091477

2. Literature Review

LDFSs were introduced by Riaz et al. in 2019 [16] to provide a more flexible framework for representing uncertain information. Compared to classical fuzzy models, LDFSs enlarge the domain of representation and allow independent control of membership and non-membership degrees, which increases their ability to handle information-rich decision environments. Due to these advantages, LDFSs have been increasingly applied in MADM problems.

A literature search was conducted in the Scopus database on 17 August 2026 using the term "linear Diophantine fuzzy set" in title, abstract and keyword fields. A total of 210 publications were initially identified. Publications including the terms "Diophantine equation", "Diophantine approximation", "Diophantine system", "linear Diophantine equation" and related expressions were excluded. 183 publications remained, including 173 journal articles, 3 book chapters, 3 conference papers, 3 reviews and 1 conference review. Figure 1 shows the number of publications on LDFSs by year. Publication activity was limited during the first years but increased substantially after 2020. Records indexed during the incomplete 2026 period were excluded from Figure 1. Since 2019, various extensions of LDFSs have been introduced, including linear Diophantine fuzzy soft rough sets [45], q-rung LDFSs [46], complex LDFSs [47], and circular LDFSs [48].

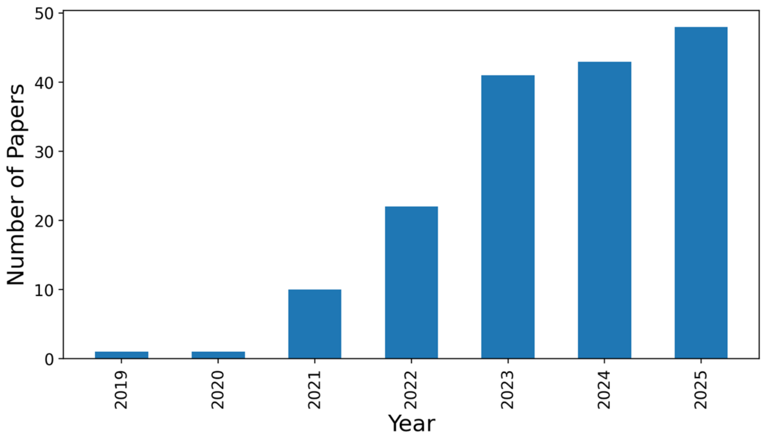

Figure 1. Number of publications on LDFSs by year.

Figure 2 presents the keyword occurrence analysis of the reviewed publications using the author keywords and index keywords. Lexical normalization was performed, and only the keywords with a minimum occurrence frequency of 5 were retained to avoid overcrowding the visualization and compromising its readability. The frequent occurrence of terms such as "aggregation operators", "mathematical operators", "decision making", "decision support systems", "MCDM", "entropy measure", "score function" and "optimization" indicates the central role of aggregation mechanisms in the LDFS literature.

Among the 183 retained publications, 66 studies explicitly develop or extend one or more aggregation operators under LDFSs or its related Diophantine fuzzy extensions. According to this review, no study has extended MSM to the LDFS environment; no partitioned aggregation operator, irrespective of the underlying aggregation family, was identified under LDFSs or its extensions. Similarly, no reducible weighted partitioned MSM formulation or its extension to LDFSs was identified. Table 1 presents selected studies with the operator families most relevant to the structural comparison, including their aggregation operations, parametrization, input weighting mechanisms, treatment of interrelationships, and capability to accommodate partitioned input structures.

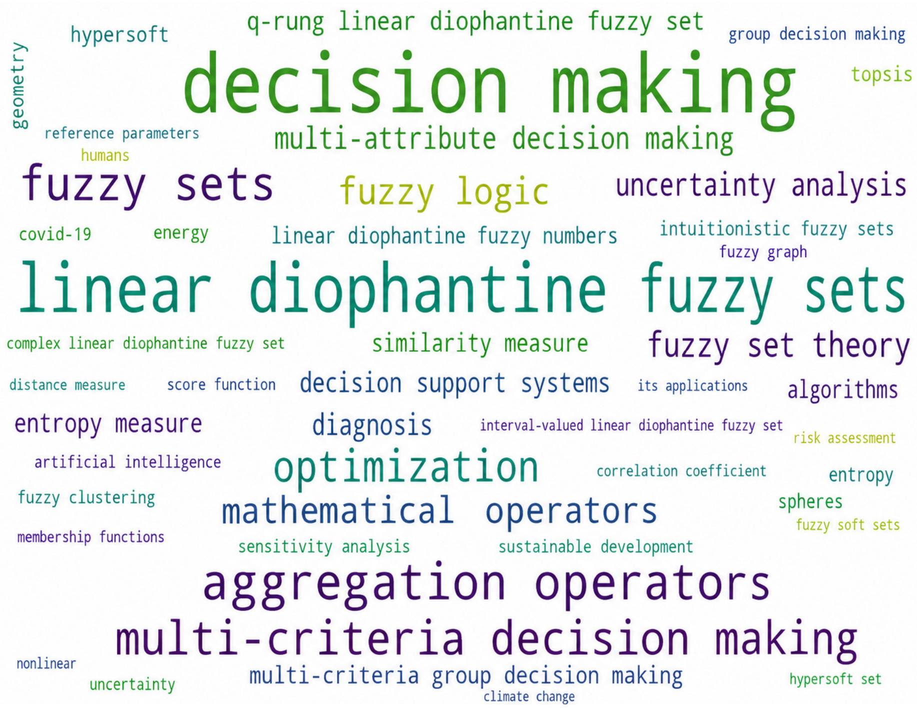

Figure 2. Keyword occurrence analysis.

Table 1. Selected aggregation-operator studies in linear Diophantine fuzzy and related fuzzy set extensions.

Study	Fuzzy Set Environment	Proposed AO(s)	Main Aggregation Operations	Parametrized	Input Weighting	Interaction Modeling	Partitioned Structure
[16]	LDFS	LDFWA, LDFWG	Weighted averaging; weighted geometric aggregation	×	Exogenous input weights	×	×
[26]	LDFS	LDFEWA, LDFEWG, LDFEOWA, LDFEOWG	Einstein-based averaging and geometric and ordered aggregation	×	Exogenous input weights in LDFEWA, LDFEWG; position weights in LDFEOWA, LDFEOWG	×	×
[27]	LDFS	LDFAAWA, LDFAAWG	Aczel–Alsina-based averaging and geometric aggregation	✓	Exogenous input weights	×	×
[28]	Sq-LDFS	Sq-LDFDWG, Sq-LDFDOWG	Dombi-based geometric and ordered aggregation	✓	Exogenous input weights in Sq-LDFDWG; position weights in Sq-LDFDOWG	×	×
[29]	LDFS	LDFHPWA, LDFHPWG,	Hamacher-based prioritized averaging and geometric aggregation	✓	Priority-induced weights	×	×
[30]	N-LDFS	N-LDFHWG, N-LDFHOWG, N-LDFHHWG	Hamacher-based geometric, ordered, and hybrid aggregation	✓	Exogenous input weights in N-LDFHWG; position weights in N-LDFHOWG; hybrid input-position weighting in N-LDFHHWG	×	×

Table 1. *Cont.*

Study	Fuzzy Set Environment	Proposed AO(s)	Main Aggregation Operations	Parametrized	Input Weighting	Interaction Modeling	Partitioned Structure
[32]	IV-LDFS	IV-LDFFWA, IV-LDFFWG	Frank-based averaging and geometric aggregation	✓	Exogenous input weights	×	×
[33]	q-RLDFS	q-LDFSSPWA, q-LDFSSPWG	Schweizer–Sklar power aggregation	✓	Exogenous input weights; support-based endogenous weights	Support-based pairwise coupling through the power aggregation	×
[34]	LDFS	LDFCIA, GLDFCIA	Choquet integral (CI) and generalized CI aggregation	× for LDFCIA, ✓ for GLDFCIA	Exogenous singleton fuzzy densities, extended to subset fuzzy measures through a λ-Sugeno fuzzy measure	Fuzzy-measure-based modeling of interrelationships among criteria	×
[35]	LDF z-number set	LDFZNBM	Bonferroni mean-based aggregation	✓	×	Pairwise cross-product aggregation	×
[49]	Interval-valued LDFS	IV-LDFPMM, IV-LDFWPMM	Power Muirhead mean aggregation and weighted power Muirhead mean	✓	Support-based endogenous weights in IV-LDFPMM and IV-LDFWPMM, Exogenous input weights in IV-LDFWPMM	Vector-parametrized multi-input aggregation	×
[37]	Fractional Diophantine fuzzy set	FDFMM, FDFWMM, FDFDMM, and FDFDWMM	Muirhead mean, weighted Muirhead mean, dual Muirhead mean, and dual weighted Muirhead mean operators.	✓	× for FDFMM and FDFDMM, ✓ for FDFWMM and FDFDWMM	Permutation-based symmetric multi-input aggregation	×
This study	LDFS	LDFMSM	Maclaurin symmetric mean	✓	×	Symmetric k-subset aggregation	×
This study	LDFS	LDFPMSM	Partitioned Maclaurin symmetric mean	✓	×	Partition-restricted symmetric k-subset aggregation	✓
This study	LDFS	LDFRWPMSM	Reducible weighted partitioned Maclaurin symmetric mean	✓	✓	Partition-restricted symmetric k-subset aggregation	✓

"✓" indicates the presence of the corresponding characteristic and "×" indicates its absence.

A substantial part of the existing LDFS literature focuses on weighted, ordered, and parametric aggregation mechanisms. The linear Diophantine fuzzy weighted averaging (LDFWA) and linear Diophantine fuzzy weighted geometric (LDFWG) operators introduced by Riaz and Hashmi [16] incorporate input weights through averaging and geometric aggregation, respectively. Iampan et al. [26] developed Einstein weighted averaging and geometric operators for LDFSs and extended them by their ordered weighted variants, which employ position weights after the inputs are ranked. Similarly, Aczel–Alsina, Dombi, Frank, and Hamacher operators introduce adjustable aggregation behavior through their respective parametric operational laws [27–30,32]. Prioritized operators (applied based on the Hamacher operator in [29]) further incorporate precedence relations through priority-induced weights. However, these approaches neither explicitly model structural interrelationships among inputs nor support aggregation within predefined input partitions.

Several studies extend the aggregation process by incorporating relationships among inputs through different mechanisms. Power aggregation operators, such as the Schweizer–Sklar power operators developed for q-RLDFSs in [33], derive effective weights from pairwise support information. Inputs receiving greater support from the remaining inputs exert a stronger influence on the final aggregated result. Thus, the power mechanism reflects mutual agreement among assessments through support-based weighting. The Choquet integral (CI) operators proposed by Riaz et al. [34] use singleton fuzzy densities that are extended to subset fuzzy measures through a λ-Sugeno fuzzy measure. This mechanism enables fuzzy-measure-based modeling of interrelationships among inputs.

Bonferroni mean, Maclaurin symmetric mean, and Muirhead mean operators provide another group of aggregation in which multiple inputs are directly combined within common product terms. The Bonferroni mean operator developed for LDF z-number sets in [35] aggregates ordered pairs of inputs through cross-product terms. In contrast, MSM operators aggregate all k-input subsets symmetrically, where the parameter k determines the number of inputs jointly included in each product term. Muirhead mean operators (introduced for fractional Diophantine fuzzy sets in [37]) provide a more general permutation-based aggregation in which a parameter vector determines the exponent pattern across inputs. Depending on the selected parameter vector, Muirhead mean operators can recover Bonferroni-type and MSM-type structures as special cases. Although these operators incorporate joint effects among multiple inputs, they are defined over the complete input set and do not impose a partition restriction on admissible joint terms.

Partitioned aggregation structures have been developed in the broader fuzzy decision-making literature. The partitioned Bonferroni mean (PBM) was introduced to restrict pairwise joint aggregation to predefined groups of inputs [50]. The partitioned Maclaurin symmetric mean (PMSM) extends this principle by enabling symmetric k-subset aggregation within predefined partitions [38,39]. PMSM and weighted PMSM (WPMSM) operators have subsequently been formulated in several fuzzy environments, including IFSs [39], q-ROFSs [38], complex FFSs [41], probabilistic hesitant fuzzy sets [40] and complex hesitant FSs [51].

The reviewed studies demonstrate that input relationships and joint effects have been incorporated through several aggregation mechanisms. However, the reviewed LDFS literature does not provide a partition-based aggregation operator that enables a k-subset joint aggregation framework. This distinction is relevant in MADM problems where some criteria belong to the same conceptual dimension and should be jointly considered, whereas criteria belonging to different dimensions should not appear in the same joint aggregation terms. Moreover, no MSM or weighted MSM formulation was identified among the reviewed LDFS and related Diophantine fuzzy studies.

To address this limitation, this study develops the LDFMSM, LDFPMSM, and LDFRWPMSM operators under the LDFS environment. The LDFMSM operator establishes the unpartitioned MSM aggregation for LDFSs and provides global symmetric k-subset aggregation across the complete criterion set. It serves as the baseline formulation and as a reducibility reference for the proposed partitioned operators. Building on this formulation, the LDFPMSM operator introduces a k-subset joint aggregation for a partitioned attribute set with equal weights. Finally, the LDFRWPMSM operator incorporates criterion weighting into the same partitioned structure by preserving idempotency and reducibility properties.

3. Preliminaries

In this section, we present the fundamental concepts and properties of LDFS sets, the MSM aggregation operator, partitioned MSM, and the weighted partitioned MSM aggregation operators.

3.1. Linear Diophantine Fuzzy Sets

Definition 1 ([16]). *Let $\mathcal{Z}$ be a finite universe of discourse. A Linear Diophantine Fuzzy (LDF) set $\mathcal{D}$ on $\mathcal{Z}$ is an object of the form:*

$$\mathcal{D} = \{(\langle x, \langle \mu_{\mathcal{D}}(x), v_{\mathcal{D}}(x)\rangle, \langle \alpha_{\mathcal{D}}(x), \beta_{\mathcal{D}}(x)\rangle\rangle)|x \in \mathcal{Z}\} \tag{1}$$

where $\mu_{\mathcal{D}}(x)$, $v_{\mathcal{D}}(x)$, $\alpha_{\mathcal{D}}(x)$, $\beta_{\mathcal{D}}(x)$ denote the membership degree, the non-membership degree and the corresponding reference parameters, respectively. These degrees satisfy the following conditions: (i) $\mu_{\mathcal{D}}$: $\mathcal{Z} \to [0,\ 1]$, $v_{\mathcal{D}}$: $\mathcal{Z} \to [0,\ 1]$, (ii) $\alpha_{\mathcal{D}}(x), \beta_{\mathcal{D}}(x) \in [0,\ 1]$, (iii) $0 \le \alpha_{\mathcal{D}}(x)\mu_{\mathcal{D}}(x) + \beta_{\mathcal{D}}(x)v_{\mathcal{D}}(x) \le 1$, (iv) $0 \le \alpha_{\mathcal{D}}(x) + \beta_{\mathcal{D}}(x) \le 1$.

The hesitation term is given by $\gamma(x)\pi_{\mathcal{D}}(x) = 1 - (\alpha_{\mathcal{D}}(x)\mu_{\mathcal{D}}(x) + \beta_{\mathcal{D}}(x)v_{\mathcal{D}}(x))$, where $\gamma(x)$ denotes the reference parameter of the hesitancy degree $\pi_{\mathcal{D}}(x)$ of $\mathcal{D}$.

Definition 2 ([16]). *An LDFS of the form $\mathcal{D}^1 = \{(\langle x, \langle 1,0\rangle, \langle 1,0\rangle\rangle)|x \in \mathcal{Z}\}$ is called an absolute LDFS and $\mathcal{D}^0 = \{(\langle x, \langle 0,1\rangle, \langle 0,1\rangle\rangle)|x \in \mathcal{Z}\}$ is an empty or null LDFS.*

Definition 3 ([16]). *Let $\phi = (\langle \mu_\phi, v_\phi\rangle, \langle \alpha_\phi, \beta_\phi\rangle)$ be a linear Diophantine fuzzy number (LDFN), then the score function is a mapping $\mathfrak{S}(\phi)$: $LDFN(\mathcal{Z}) \to [-1,\ 1]$ which is defined as:*

$$\mathfrak{S}(\phi) = \frac{1}{2}\left[(\mu_\phi - v_\phi) + (\alpha_\phi - \beta_\phi)\right] \tag{2}$$

Definition 4 ([16]). *Let $\phi = (\langle \mu_\phi, v_\phi\rangle, \langle \alpha_\phi, \beta_\phi\rangle)$ be an LDFN, then the accuracy function is a mapping $\mathcal{H}(\phi)$: $LDFN(\mathcal{Z}) \to [0,\ 1]$ and is given by:*

$$\mathcal{H}(\phi) = \frac{1}{2}\left[\left(\frac{\mu_\phi + v_\phi}{2}\right) + (\alpha_\phi + \beta_\phi)\right] \tag{3}$$

Definition 5 ([16]). *Let $\phi_1 = (\langle \mu_{\phi_1}, v_{\phi_1}\rangle, \langle \alpha_{\phi_1}, \beta_{\phi_1}\rangle)$ and $\phi_2 = (\langle \mu_{\phi_2}, v_{\phi_2}\rangle, \langle \alpha_{\phi_2}, \beta_{\phi_2}\rangle)$ be two LDFNs. These two LDFNs can be compared by using the score function and accuracy function given in Definition 3 and Definition 4, respectively, as:*

If $\mathfrak{S}(\phi_1) < \mathfrak{S}(\phi_2)$, then $\phi_1 < \phi_2$,
If $\mathfrak{S}(\phi_1) > \mathfrak{S}(\phi_2)$, then $\phi_1 > \phi_2$,
If $\mathfrak{S}(\phi_1) = \mathfrak{S}(\phi_2)$, then,
If $\mathcal{H}(\phi_1) < \mathcal{H}(\phi_2)$, then $\phi_1 < \phi_2$,
If $\mathcal{H}(\phi_1) > \mathcal{H}(\phi_2)$, then $\phi_1 > \phi_2$,
If $\mathcal{H}(\phi_1) = \mathcal{H}(\phi_2)$, then $\phi_1 \approx \phi_2$.

Definition 6 ([16]). *Let $\phi = (\langle \mu_\phi, v_\phi\rangle, \langle \alpha_\phi, \beta_\phi\rangle)$ be an LDFN, then the expectation score function (ESF) $\mathcal{S}(\phi)$ is a mapping on $LDFN(\mathcal{Z}) \to [0,\ 1]$ and is defined by:*

$$\mathcal{S}(\phi) = \frac{1}{2}\left[\left(\frac{\mu_\phi - v_\phi + 1}{2}\right) + \left(\frac{\alpha_\phi - \beta_\phi + 1}{2}\right)\right] \tag{4}$$

Definition 7 ([16]). *Let $\phi_1 = (\langle \mu_{\phi_1}, v_{\phi_1} \rangle, \langle \alpha_{\phi_1}, \beta_{\phi_1} \rangle)$ and $\phi_2 = (\langle \mu_{\phi_2}, v_{\phi_2} \rangle, \langle \alpha_{\phi_2}, \beta_{\phi_2} \rangle)$ be two LDFNs on $\mathcal{Z}$, and let $\varrho > 0$ be a scalar. The arithmetic operations between these LDFNs are defined by Equations (5)–(9):*

$$\phi_1 \oplus \phi_2 = (\langle \mu_{\phi_1} + \mu_{\phi_2} - \mu_{\phi_1}\mu_{\phi_2}, v_{\phi_1}v_{\phi_2} \rangle, \langle \alpha_{\phi_1} + \alpha_{\phi_2} - \alpha_{\phi_1}\alpha_{\phi_2}, \beta_{\phi_1}\beta_{\phi_2} \rangle) \quad (5)$$

$$\phi_1 \otimes \phi_2 = (\langle \mu_{\phi_1}\mu_{\phi_2}, v_{\phi_1} + v_{\phi_2} - v_{\phi_1}v_{\phi_2} \rangle, \langle \alpha_{\phi_1}\alpha_{\phi_2}, \beta_{\phi_1} + \beta_{\phi_2} - \beta_{\phi_1}\beta_{\phi_2} \rangle) \quad (6)$$

$$\varrho\phi_1 = \left(\langle 1 - (1 - \mu_{\phi_1})^{\varrho}, v_{\phi_1}^{\varrho} \rangle, \langle 1 - (1 - \alpha_{\phi_1})^{\varrho}, \beta_{\phi_1}^{\varrho} \rangle \right) \quad (7)$$

$$\phi_1^{\varrho} = \left(\langle \mu_{\phi_1}^{\varrho}, 1 - (1 - v_{\phi_1})^{\varrho} \rangle, \langle \alpha_{\phi_1}^{\varrho}, 1 - (1 - \beta_{\phi_1})^{\varrho} \rangle \right) \quad (8)$$

$$\phi_1^{c} = (\langle v_{\phi_1}, \mu_{\phi_1} \rangle, \langle \beta_{\phi_1}, \alpha_{\phi_1} \rangle) \quad (9)$$

where the superscript c denotes the complement of LDFN.

Definition 8 ([52]). *Let $\phi_1 = (\langle \mu_{\phi_1}, v_{\phi_1} \rangle, \langle \alpha_{\phi_1}, \beta_{\phi_1} \rangle)$ and $\phi_2 = (\langle \mu_{\phi_2}, v_{\phi_2} \rangle, \langle \alpha_{\phi_2}, \beta_{\phi_2} \rangle)$ be two LDFNs on $\mathcal{Z}$. The Hamming distance d_H and Euclidean distance d_E between these LDFNs are given by Equations (10) and (11):*

$$d_H(\phi_1, \phi_2) = \frac{1}{4}\left[|\mu_{\phi_1} - \mu_{\phi_2}| + |v_{\phi_1} - v_{\phi_2}| + |\alpha_{\phi_1} - \alpha_{\phi_2}| + |\beta_{\phi_1} - \beta_{\phi_2}| \right] \quad (10)$$

$$d_E(\phi_1, \phi_2) = \left\{ \frac{1}{4}\left[(\mu_{\phi_1} - \mu_{\phi_2})^2 + (v_{\phi_1} - v_{\phi_2})^2 + (\alpha_{\phi_1} - \alpha_{\phi_2})^2 + (\beta_{\phi_1} - \beta_{\phi_2})^2 \right] \right\}^{\frac{1}{2}} \quad (11)$$

The graphical representation of LDFSs can be illustrated as in Figure 3 for an LDFN with constant reference parameters of $\alpha = 0.7$ and $\beta = 0.3$. The horizontal axes represent the membership degree μ and the non-membership degree ν, both defined on the interval $[0,\ 1]$. The surface corresponds to the linear combination $z = \alpha\mu + \beta\nu$, which represents the weighted contribution of membership and non-membership degrees under the LDF constraint $0 \le \alpha\mu + \beta\nu \le 1$.

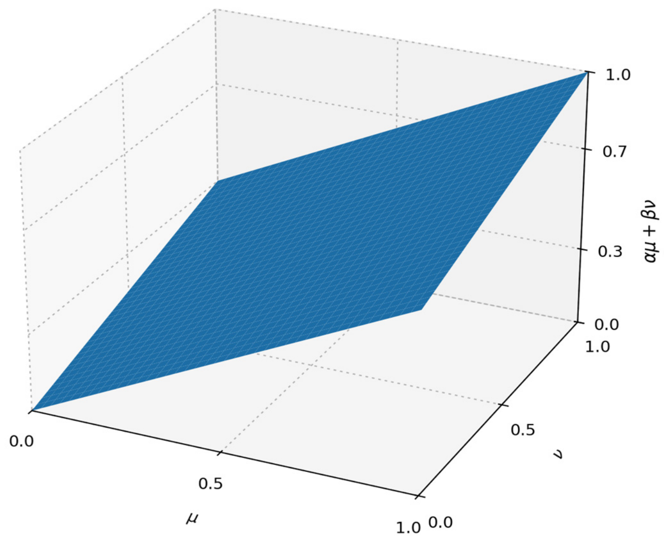

Figure 3. Linear Diophantine fuzzy representation surface for $\alpha = 0.7$ and $\beta = 0.3$.

3.2. MSM Aggregation Operator

The Maclaurin symmetric mean (MSM) is introduced as an aggregation technique characterized by its ability to model the interrelationships among multiple input arguments [53]. The formal definition of the MSM is given in Definition 9.

Definition 9 ([53]). *Let $\theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be a collection of nonnegative real numbers and let $k = 1, 2, \ldots, n$. Then the $MSM^{(k)}$ operation is defined as:*

$$MSM^{(k)}(\theta_1, \theta_2, \ldots, \theta_n) = \left(\frac{\sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} \theta_{ij}}{C_n^k} \right)^{1/k} \tag{12}$$

where k is the interaction parameter, n is the number of attributes, $(i_1, i_2, \ldots, i_k)$ traverse all k-tuple combinations of (1, 2, . . ., n) and C_n^k is the binomial coefficient expressed by $C_n^k = \frac{n!}{k!(n-k)!}$. The admissible range for k is $1 \le k \le n$.

The MSM operator provides the following properties [25]:
$MSM^{(k)}(0, 0, \ldots, 0) = 0$,
$MSM^{(k)}(a, a, \ldots, a) = a$,
$MSM^{(k)}(\theta_1, \theta_2, \ldots, \theta_n) \le MSM^{(k)}(\eta_1, \eta_2, \ldots, \eta_n)$, *if* $\theta_i \le \eta_i$ *for all* i,
$\min_i \{\theta_i\} \le MSM^{(k)}(\theta_1, \theta_2, \ldots, \theta_n) \le \max_i \{\theta_i\}$.

3.3. PMSM Aggregation Operator

In some decision-making problems, criteria can be partitioned into predefined groups according to the conceptual structure of the problem. Suppose that $C = \{X_1, X_2, \ldots, X_n\}$ is a collection of attributes and $\phi = (\phi_1, \ldots, \phi_i, \ldots, \phi_n)$, ($i$ = 1, 2, . . ., n) represents the evaluation values associated with these attributes. In practical situations, the attribute set C can be partitioned into s mutually disjoint groups $P_1, P_2, \ldots, P_s$ such that $P_i \cap P_j = \emptyset$, $(i \ne j)$ and $\bigcup_{r=1}^{s} P_r = C$. While the classical MSM operator forms symmetric joint terms over k-input subsets of the complete criterion set, the partitioned Maclaurin symmetric mean (PMSM) operator applies this aggregation separately within each partition.

Definition 10 ([39]). *Let $x_1, x_2, \ldots, x_n$ be a collection of non-negative real numbers. Suppose that the attribute set is divided into s partitions $P_1, P_2, \ldots, P_s$. Then the PMSM operator is defined by Equation (13):*

$$PMSM^{(k)}(x_1, x_2, \ldots, x_n) = \frac{1}{s} \sum_{r=1}^{s} \left(\frac{\sum_{1 \le i_1 < \cdots < i_k \le h_r} \prod_{j=1}^{k} x_{ij}}{C_{h_r}^k} \right)^{1/k} \tag{13}$$

where s denotes the number of partitions, k is the interaction parameter satisfying $k = 1, 2, \ldots, n$, h_r is the number of attributes in partition P_r, $(i_1, i_2, \ldots, i_k)$ traverse all k-tuple combinations of (1, 2, . . ., h_r), and $C_{h_r}^k = \frac{h_r!}{k!(h_r-k)!}$ denotes the binomial coefficient.

In the partitioned formulations in this study, a single common interaction parameter k is used across all partitions. Accordingly, its admissible range is restricted to $1 \le k \le h_{min}$, where $h_{min} = \min_{1 \le r \le s} h_r$, so that an admissible k-subset exists in every partition. This common feasible range is used throughout the proposed partitioned operators and the numerical analysis.

3.4. Reducible Weighted MSM Operator

The MSM and PMSM operators assign equal importance to all inputs. To accommodate unequal input importance, several weighted extensions of the MSM operator have been developed in different fuzzy environments. However, classical weighted MSM operators have two structural limitations as pointed out by [42–44], namely the loss of idempotency and the failure to reduce to the MSM under equal input weights. To address these limitations, Shi and Xiao [42] proposed the reducible weighted MSM (RWMSM) operator, which shows both the idempotency and reducibility properties, as in Definition 11.

Definition 11 ([42])**.** *Let x_i $(i = 1, 2, \ldots, n)$ be a collection of non-negative real numbers representing the evaluations for the attribute set and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be the associated weight vector of these attributes such that $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$. The attributes may exhibit interrelationships. Then, the RWMSM operator is given by Equation (14):*

$$RWMSM^{(k)}(x_1, x_2, \ldots, x_n) = \left(\frac{\sum_{1 \le i_1 < \cdots < i_k \le n} \left(\prod_{j=1}^{k} \omega_{ij} \right) \left(\prod_{j=1}^{k} x_{ij} \right)}{\sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} \omega_{ij}} \right)^{1/k} \tag{14}$$

where k is the interaction parameter satisfying $k = 1, 2, \ldots, n$, n denotes the number of attributes, $(i_1, i_2, \ldots, i_k)$ traverse all k-tuple combinations of (1, 2, …, n). For each k-subset, $(i_1, i_2, \ldots, i_k)$, the term $\prod_{j=1}^{k} \omega_{ij}$ represents the joint weight associated with the corresponding subset, and the term $\sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} \omega_{ij}$ gives the sum of the joint weights over all admissible k-subsets. The term $\frac{\prod_{j=1}^{k} \omega_{ij}}{\sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} \omega_{ij}}$ denotes the normalized relative joint weight of the k-subset $(i_1, i_2, \ldots, i_k)$.

4. The Proposed LDFMSM, LDFPMSM and LDFRWPMSM Aggregation Operators

4.1. LDFMSM Aggregation Operator

In this section, we propose the linear Diophantine fuzzy Maclaurin symmetric mean (LDFMSM) aggregation operator and discuss its properties. The operator is intended for unpartitioned settings in which all *k*-criterion subsets are considered in symmetric joint aggregation and the inputs are assigned equal importance.

Definition 12. *Let $\phi_i = \left(\langle \mu_{\phi_i}, v_{\phi_i} \rangle, \langle \alpha_{\phi_i}, \beta_{\phi_i} \rangle \right)$, (i = 1, 2, …, n) be a collection of LDFNs and let k = 1, 2, …, n. Then we define an LDFMSM aggregation operation as in Equation (15):*

$$LDFMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) = \left(\frac{\bigoplus_{1 \le i_1 < \cdots < i_k \le n} \left(\bigotimes_{j=1}^{k} \phi_{i_j} \right)}{C_n^k} \right)^{1/k} \tag{15}$$

where k is the interaction parameter, n is the number of attributes, $(i_1, i_2, \ldots, i_k)$ traverse all k-tuple combinations of (1, 2, …, n) and C_n^k is the binomial coefficient expressed by $C_n^k = \frac{n!}{k!(n-k)!}$. The admissible range for k is $1 \le k \le n$.

Using the LDFS arithmetic operations given in Definition 7, we can derive Theorem 1.

Theorem 1. *Let $\phi_i = \left(\langle \mu_{\phi_i}, v_{\phi_i} \rangle, \langle \alpha_{\phi_i}, \beta_{\phi_i} \rangle \right)$, (i = 1, 2, …, n) be a collection of LDFNs and k = 1, 2, …, n. Then the aggregated value obtained by the LDFMSM operator $\phi^* = \left(\langle \mu^*, v^* \rangle, \langle \alpha^*, \beta^* \rangle \right)$ is also an LDFN, and*

$$LDFMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) = \left(\begin{array}{l} \langle \left(1 - \left(\prod_{1 \le i_1 < \ldots < i_k \le n} \left(1 - \prod_{j=1}^{k} \mu_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}, 1 - \left(1 - \left(\prod_{1 \le i_1 < \ldots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - v_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \rangle, \\ \langle \left(1 - \left(\prod_{1 \le i_1 < \ldots < i_k \le n} \left(1 - \prod_{j=1}^{k} \alpha_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}, 1 - \left(1 - \left(\prod_{1 \le i_1 < \ldots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - \beta_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \rangle \end{array} \right) \tag{16}$$

Proof. By the operational laws of LDFS in Definition 7, we have

$$\underset{j=1}{\overset{k}{\otimes}} \phi_{i_j} = \left(\langle \prod_{j=1}^{k} \mu_{\phi_{i_j}}, 1 - \prod_{j=1}^{k} \left(1 - v_{\phi_{i_j}}\right) \rangle, \langle \prod_{j=1}^{k} \alpha_{\phi_{i_j}}, 1 - \prod_{j=1}^{k} \left(1 - \beta_{\phi_{i_j}}\right) \rangle \right)$$

and

$$\underset{1 \le i_1 < \cdots < i_k \le n}{\oplus} \left(\underset{j=1}{\overset{k}{\otimes}} \phi_{i_j} \right) = \left(\begin{array}{l} \langle 1 - \prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \mu_{\phi_{i_j}}\right), \prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - v_{\phi_{i_j}}\right)\right) \rangle, \\ \langle 1 - \prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \alpha_{\phi_{i_j}}\right), \prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - \beta_{\phi_{i_j}}\right)\right) \rangle \end{array} \right)$$

then we obtain

$$\frac{1}{C_n^k} \left(\underset{1 \le i_1 < \cdots < i_k \le n}{\oplus} \left(\underset{j=1}{\overset{k}{\otimes}} \phi_{i_j} \right) \right) = \left(\begin{array}{l} \langle 1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \mu_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^k}}, \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - v_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}} \rangle, \\ \langle 1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \alpha_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^k}}, \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - \beta_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}} \rangle \end{array} \right)$$

Therefore,

$$LDFMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) = \begin{pmatrix} \left\langle \begin{array}{c} \left(1-\left(\prod\limits_{1\le i_1<\cdots<i_k\le n}\left(1-\prod\limits_{j=1}^{k}\mu_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}, \\ 1-\left(1-\left(\prod\limits_{1\le i_1<\cdots<i_k\le n}\left(1-\prod\limits_{j=1}^{k}\left(1-v_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \end{array} \right\rangle, \\ \left\langle \left(1-\left(\prod\limits_{1\le i_1<\cdots<i_k\le n}\left(1-\prod\limits_{j=1}^{k}\alpha_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}, 1-\left(1-\left(\prod\limits_{1\le i_1<\cdots<i_k\le n}\left(1-\prod\limits_{j=1}^{k}\left(1-\beta_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \right\rangle \end{pmatrix}$$

To verify that the aggregated value in Equation (16) is an LDFN, assume that $1 \le k \le n$, and define for $\mathbf{x} = (x_1, \ldots, x_n) \in [0, 1]^n$,

$$F_k(\mathbf{x}) = \left[1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^{k}x_{i_j}\right)\right)^{1/C_n^k}\right]^{1/k}$$

Then, Equation (16) can be expressed as

$$LDFMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) = (\langle F_k(\boldsymbol{\mu}), 1-F_k(\mathbf{1}-\boldsymbol{v})\rangle, \langle F_k(\boldsymbol{\alpha}), 1-F_k(\mathbf{1}-\boldsymbol{\beta})\rangle)$$

where

$$\boldsymbol{\mu} = (\mu_{\phi_1}, \ldots, \mu_{\phi_n}), \boldsymbol{v} = (v_{\phi_1}, \ldots, v_{\phi_n}), \boldsymbol{\alpha} = (\alpha_{\phi_1}, \ldots, \alpha_{\phi_n}), \boldsymbol{\beta} = (\beta_{\phi_1}, \ldots, \beta_{\phi_n})$$

and $\mathbf{1} = (1, \ldots, 1)$.

For any $\mathbf{x} \in [0, 1]^n$, each term satisfies $\left(1-\prod_{j=1}^{k}x_{i_j}\right) \in [0,1]$. Therefore, $F_k(\mathbf{x}) \in [0,1]$. Hence, $0 \le \mu^*, v^*, \alpha^*, \beta^* \le 1$.

Moreover, $F_k(\mathbf{x})$ is nondecreasing in each component of $\mathbf{x}$. Since every ϕ_i satisfies $\alpha_{\phi_i} + \beta_{\phi_i} \le 1$, we have $\alpha_{\phi_i} \le 1 - \beta_{\phi_i}$, $i = 1, 2, \ldots, n$. Therefore, $F_k(\boldsymbol{\alpha}) \le F_k(\mathbf{1}-\boldsymbol{\beta})$.

Using the definitions in Equation (16), it follows that

$$\alpha^* = F_k(\boldsymbol{\alpha}) \le F_k(\mathbf{1}-\boldsymbol{\beta}) = 1-\beta^*$$

Thus,

$$\alpha^* + \beta^* \le 1$$

Since $\mu^*, v^* \in [0,1]$,

$$0 \le \alpha^*\mu^* + \beta^*v^* \le \alpha^* + \beta^* \le 1$$

Hence, the aggregated LDFN obtained by LDFMSM satisfies all conditions in Definition 1 and is therefore an LDFN. Thus, the proof of Theorem 1 is complete. □

It can be proved that LDFMSM has the following properties:

Property 1 (Idempotency)**.** *If all ϕ_i (i = 1, 2, . . ., n) are equal, i.e., $\phi_i = \phi$ for all i, then*

$$LDFMSM(\phi_1,\ \phi_2,\ \ldots,\ \phi_n) = \phi$$

Proof. Since $\phi = \left(\langle \mu_\phi, v_\phi\rangle, \langle \alpha_\phi, \beta_\phi\rangle\right)$, based on Theorem 1, we have

$$LDFMSM(\phi,\ \phi,\ \ldots,\ \phi) = \left(\begin{array}{c} \left\langle \left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^{k}\mu_\phi\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}},\ 1-\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^{k}(1-v_\phi)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right\rangle, \\ \left\langle \left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^{k}\alpha_\phi\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}},\ 1-\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^{k}(1-\beta_\phi)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right\rangle \end{array}\right)$$

$$= \left(\begin{array}{c} \left\langle \left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(1-(\mu_\phi)^k\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}},\ 1-\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(1-(1-v_\phi)^k\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right\rangle, \\ \left\langle \left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(1-(\alpha_\phi)^k\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}},\ 1-\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(1-(1-\beta_\phi)^k\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right\rangle \end{array}\right)$$

$$= \left(\begin{array}{c} \left\langle \left(1-\left(\left(1-(\mu_\phi)^k\right)^{C_n^k}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}},\ 1-\left(1-\left(\left(1-(1-v_\phi)^k\right)^{C_n^k}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right\rangle, \\ \left\langle \left(1-\left(\left(1-(\alpha_\phi)^k\right)^{C_n^k}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}},\ 1-\left(1-\left(\left(1-(1-\beta_\phi)^k\right)^{C_n^k}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right\rangle \end{array}\right)$$

$$= \left(\begin{array}{c} \left\langle \left(1-\left(1-(\mu_\phi)^k\right)\right)^{\frac{1}{k}},\ 1-\left(1-\left(1-(1-v_\phi)^k\right)\right)^{\frac{1}{k}}\right\rangle, \\ \left\langle \left(1-\left(1-(\alpha_\phi)^k\right)\right)^{\frac{1}{k}},\ 1-\left(1-\left(1-(1-\beta_\phi)^k\right)\right)^{\frac{1}{k}}\right\rangle \end{array}\right) = \left(\langle \mu_\phi, v_\phi\rangle, \langle \alpha_\phi, \beta_\phi\rangle\right) = \phi$$

which completes the proof of Property 1. □

Property 2 (Commutativity)**.** *If $\phi_i = \left(\langle \mu_{\phi_i}, v_{\phi_i}\rangle, \langle \alpha_{\phi_i},\ \beta_{\phi_i}\rangle\right)$, (i = 1, 2, . . ., n) is a collection of LDFNs and $\phi_i' = \left(\langle \mu_{\phi_i}',\ v_{\phi_i}'\rangle, \langle \alpha_{\phi_i}', \beta_{\phi_i}'\rangle\right)$, (i = 1, 2, . . ., n) is any permutation of ϕ_i, then*

$$LDFMSM(\phi_1, \phi_2,\ \ldots,\ \phi_n) = LDFMSM\left(\phi_1', \phi_2',\ \ldots,\ \phi_n'\right)$$

Proof. Since $\left(\phi_1',\ \phi_2',\ \ldots,\ \phi_n'\right)$ is any permutation of $(\phi_1,\ \phi_2,\ \ldots,\ \phi_n)$, we have, based on Definition 9,

$$LDFMSM^{(k)}(\phi_1, \phi_2,\ \ldots,\ \phi_n) = \left(\frac{\bigoplus_{1\le i_1<\cdots<i_k\le n}\left(\bigotimes_{j=1}^{k}\phi_{i_j}\right)}{C_n^k}\right)^{1/k}$$

$$= \left(\frac{\bigoplus_{1\le i_1<\cdots<i_k\le n}\left(\bigotimes_{j=1}^{k}\phi_{i_j}'\right)}{C_n^k}\right)^{1/k} = LDFMSM^{(k)}\left(\phi_1',\ \phi_2',\ \ldots,\ \phi_n'\right)$$

□

Property 3 (Monotonicity). *Let* $\phi_i = \left(\langle \mu_{\phi_i}, v_{\phi_i}\rangle, \langle \alpha_{\phi_i}, \beta_{\phi_i}\rangle\right)$ *and* $\phi'_i = \left(\langle \mu'_{\phi_i}, v'_{\phi_i}\rangle, \langle \alpha'_{\phi_i}, \beta'_{\phi_i}\rangle\right)$ *be two collections of LDFNs, if* $\mu_{\phi_i} \geq \mu'_{\phi_i}$, $v_{\phi_i} \leq v'_{\phi_i}$, $\alpha_{\phi_i} \geq \alpha'_{\phi_i}$ *and* $\beta_{\phi_i} \leq \beta'_{\phi_i}$ *for all i, then*

$$LDFMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) \geq LDFMSM^{(k)}(\phi'_1, \phi'_2, \ldots, \phi'_n)$$

Proof. Since $k \geq 1$, $\mu_{\phi_i} \geq \mu'_{\phi_i}$ and $v_{\phi_i} \leq v'_{\phi_i}$ for all i, then we have $\mu_{\phi_{i_j}} \geq \mu'_{\phi_{i_j}} \geq 0$, $v_{\phi_{i_j}} \leq v'_{\phi_{i_j}} \leq 1$. Then for all i and j ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, k$), we obtain

$$\begin{aligned}&\prod_{j=1}^{k} \mu_{\phi_{i_j}} \geq \prod_{j=1}^{k} \mu'_{\phi_{i_j}} \Rightarrow 1 - \prod_{j=1}^{k} \mu_{\phi_{i_j}} \leq 1 - \prod_{j=1}^{k} \mu'_{\phi_{i_j}} \Rightarrow \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(1 - \prod_{j=1}^{k} \mu_{\phi_{i_j}}\right) \leq \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(1 - \prod_{j=1}^{k} \mu'_{\phi_{i_j}}\right) \\ &\Rightarrow \left(1 - \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(1 - \prod_{j=1}^{k} \mu_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \geq \left(1 - \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(1 - \prod_{j=1}^{k} \mu'_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\end{aligned} \tag{17}$$

Similarly, we have

$$\begin{aligned}\left(1 - v_{\phi_{i_j}}\right) \geq \left(1 - v'_{\phi_{i_j}}\right) \quad &\Rightarrow 1 - \prod_{j=1}^{k}\left(1 - v_{\phi_{i_j}}\right) \leq 1 - \prod_{j=1}^{k}\left(1 - v'_{\phi_{i_j}}\right) \Rightarrow \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n}\left(1 - \prod_{j=1}^{k}\left(1 - v_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}} \\ &\leq \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n}\left(1 - \prod_{j=1}^{k}\left(1 - v'_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}} \Rightarrow \left(1 - \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n}\left(1 - \prod_{j=1}^{k}\left(1 - v_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \\ &\geq \left(1 - \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n}\left(1 - \prod_{j=1}^{k}\left(1 - v'_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\end{aligned} \tag{18}$$

Again, $\alpha_{\phi_i} \geq \alpha'_{\phi_i}$ and $\beta_{\phi_i} \leq \beta'_{\phi_i}$ for all i, we obtain

$$\begin{aligned}&\prod_{j=1}^{k} \alpha_{\phi_{i_j}} \geq \prod_{j=1}^{k} \alpha'_{\phi_{i_j}} \Rightarrow 1 - \prod_{j=1}^{k} \alpha_{\phi_{i_j}} \leq 1 - \prod_{j=1}^{k} \alpha'_{\phi_{i_j}} \Rightarrow \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(1 - \prod_{j=1}^{k} \alpha_{\phi_{i_j}}\right) \leq \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(1 - \prod_{j=1}^{k} \alpha'_{\phi_{i_j}}\right) \\ &\Rightarrow \left(1 - \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(1 - \prod_{j=1}^{k} \alpha_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \geq \left(1 - \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(1 - \prod_{j=1}^{k} \alpha'_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\end{aligned} \tag{19}$$

and similarly,

$$\begin{aligned}\left(1 - \beta_{\phi_{i_j}}\right) \quad &\geq \left(1 - \beta'_{\phi_{i_j}}\right) \Rightarrow 1 - \prod_{j=1}^{k}\left(1 - \beta_{\phi_{i_j}}\right) \leq 1 - \prod_{j=1}^{k}\left(1 - \beta'_{\phi_{i_j}}\right) \Rightarrow \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n}\left(1 - \prod_{j=1}^{k}\left(1 - \beta_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}} \\ &\leq \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n}\left(1 - \prod_{j=1}^{k}\left(1 - \beta'_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}} \Rightarrow \left(1 - \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n}\left(1 - \prod_{j=1}^{k}\left(1 - \beta_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \\ &\geq \left(1 - \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n}\left(1 - \prod_{j=1}^{k}\left(1 - \beta'_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\end{aligned} \tag{20}$$

By Equations (17)–(20), it follows that for all i, $\mu_{\phi_i} \geq \mu'_{\phi_i}$, $v_{\phi_i} \leq v'_{\phi_i}$, $\alpha_{\phi_i} \geq \alpha'_{\phi_i}$ and $\beta_{\phi_i} \leq \beta'_{\phi_i}$.

According to Definition 3, since $\mu_{\phi_i} \geq \mu'_{\phi_i}$, $-v_{\phi_i} \geq -v'_{\phi_i}$, $\alpha_{\phi_i} \geq \alpha'_{\phi_i}$ and $-\beta_{\phi_i} \geq -\beta'_{\phi_i}$, it follows that $\mathfrak{S}(\overline{\phi}) \geq \mathfrak{S}\left(\overline{\phi}'\right)$. Therefore, by Definition 5, we obtain

$$LDFMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) \geq LDFMSM^{(k)}(\phi'_1, \phi'_2, \ldots, \phi'_n).$$

Thus, the proof of Property 3 is complete. □

Property 4 (Boundedness). *Let* $\phi_i = \left(\langle \mu_{\phi_i}, v_{\phi_i}\rangle, \langle \alpha_{\phi_i}, \beta_{\phi_i}\rangle\right)$, *(i = 1, 2, . . ., n) be a collection of LDFNs and k = 1, 2, . . ., n, and let*

$$\phi^- = \left(\langle \min_i \mu_{\phi_i}, \max_i v_{\phi_i}\rangle, \langle \min_i \alpha_{\phi_i}, \max_i \beta_{\phi_i}\rangle\right)$$

$$\phi^+ = \left(\langle \max_i \mu_{\phi_i}, \min_i v_{\phi_i}\rangle, \langle \max_i \alpha_{\phi_i}, \min_i \beta_{\phi_i}\rangle\right)$$

then

$$\phi^- \leq LDFMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) \leq \phi^+$$

Proof. Based on Property 1 and Property 3, and $\phi^- \leq \phi_i \leq \phi^+$ we obtain

$$LDFMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) \geq LDFMSM^{(k)}(\phi^-, \phi^-, \ldots, \phi^-) = \phi^-$$

and

$$LDFMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) \leq LDFMSM^{(k)}(\phi^+, \phi^+, \ldots, \phi^+) = \phi^+$$

□

Computational Examination of Parameter Monotonicity of LDFMSM

To examine the behavior of the LDFMSM operator with respect to interaction parameter k, an exhaustive finite-grid search for counterexamples was conducted. Exhaustive evaluation over discretized continuous domains has been employed in aggregation function studies. Tang and Chen [54] discretized the unit interval and exhaustively evaluated the grid-defined input configurations to obtain empirical numerical results. Following this general principle, the present study discretizes the admissible input domain and systematically searches for any configuration for which increasing the interaction order k to $k+1$ produces a larger LDFMSM aggregation value, which would constitute a counterexample to the hypothesized non-increasing behavior with respect to k.

Let $G = \{0, 0.1, \ldots, 1\}$ denote the eleven-level finite grid. Using the closed-form expression of the LDFMSM operator, the four components can be represented through a common scalar kernel $F_{n,k}(\mathbf{x})$, as follows:

$$F_{n,k}(\mathbf{x}) = \left[1 - \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(1 - \prod_{j=1}^{k} x_{i_j}\right)\right)^{1/C_n^k}\right]^{1/k}$$

and

$$LDFMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) = \left(\langle F_{n,k}(\boldsymbol{\mu}), 1 - F_{n,k}(\mathbf{1} - \boldsymbol{v})\rangle, \langle F_{n,k}(\boldsymbol{\alpha}), 1 - F_{n,k}(\mathbf{1} - \boldsymbol{\beta})\rangle\right)$$

where

$$\boldsymbol{\mu} = (\mu_{\phi_1}, \ldots, \mu_{\phi_n}), \boldsymbol{v} = (v_{\phi_1}, \ldots, v_{\phi_n}), \boldsymbol{\alpha} = (\alpha_{\phi_1}, \ldots, \alpha_{\phi_n}), \boldsymbol{\beta} = (\beta_{\phi_1}, \ldots, \beta_{\phi_n})$$

and $\mathbf{1} = (1, \ldots, 1)$.

Therefore, parameter monotonicity can be examined by evaluating the consecutive-order condition $F_{n,k+1}(\mathbf{x}) \leq F_{n,k}(\mathbf{x})$, for every examined $\mathbf{x} \in G^{\boldsymbol{n}}$ and $k = 1, \ldots, n-1$. The counterexample search is based on the margin $\Delta_{n,k}(\mathbf{x}) = F_{n,k}(\mathbf{x}) - F_{n,k+1}(\mathbf{x})$, where a negative value indicates a violation.

The search was conducted for n = 2, …, 8. Since LDFMSM is commutative, only non-decreasing vectors were evaluated. The complete finite grid contains 11^n configurations for each n. However, configurations differing only in the order of their components yield identical aggregation values. Therefore, the search considered only $\binom{11+n-1}{n}$ permutation-distinct configurations. As a complementary admissibility check, all 11^4 candidate $(\mu, v, \alpha, \beta) \in G^{\mathbf{4}}$ were generated and screened according to the LDFN conditions. This screening yielded 7986 valid finite-grid LDFNs. Because each of the four LDFMSM components is obtained by applying the same scalar kernel separately to the corresponding component vectors, every component vector arising from these valid LDFNs is already included in the complete scalar grid $G^{\boldsymbol{n}}$ considered in the exhaustive search.

In total, 75,570 distinct scalar configurations were evaluated, representing 235,794,757 ordered grid configurations and yielding 478,687 direct consecutive-k comparisons. Using a numerical tolerance of 10^{-10}, no violation of the condition $F_{n,k+1}(\mathbf{x}) \leq F_{n,k}(\mathbf{x})$ was identified. The numerical calculations were performed using the publicly available software tools Python 3.13.9, NumPy (v2.3.5) and the Spyder development environment (v6.1.0), and the pseudocode is provided in the Supplementary Material. Since the input space is continuous, the observed non-increasing behavior constitutes finite grid evidence for the examined discretized domain. It is not general analytical evidence of parameter monotonicity over the continuous input space.

Now, we can discuss some special cases of the LDFMSM operator with respect to the parameter k.

Case 1. *If k = 1, we obtain:*

$$
\begin{aligned}
&LDFMSM^{(1)}(\phi_1, \phi_2, \ldots, \phi_n) \\
&= \left(\begin{array}{l}
\left\langle \left(1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \prod_{j=1}^{1} \mu_{\phi_{i_j}} \right) \right)^{\frac{1}{C_n^1}} \right)^{\frac{1}{1}}, 1 - \left(1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \prod_{j=1}^{1} \left(1 - v_{\phi_{i_j}} \right) \right) \right)^{\frac{1}{C_n^1}} \right)^{\frac{1}{1}} \right\rangle, \\
\left\langle \left(1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \prod_{j=1}^{1} \alpha_{\phi_{i_j}} \right) \right)^{\frac{1}{C_n^1}} \right)^{\frac{1}{1}}, 1 - \left(1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \prod_{j=1}^{1} \left(1 - \beta_{\phi_{i_j}} \right) \right) \right)^{\frac{1}{C_n^1}} \right)^{\frac{1}{1}} \right\rangle
\end{array} \right) \\
&= \left(\begin{array}{l}
\left\langle 1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \mu_{\phi_{i_1}} \right) \right)^{\frac{1}{n}}, 1 - \left(1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \left(1 - v_{\phi_{i_1}} \right) \right) \right)^{\frac{1}{n}} \right) \right\rangle, \\
\left\langle 1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \alpha_{\phi_{i_1}} \right) \right)^{\frac{1}{n}}, 1 - \left(1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \left(1 - \beta_{\phi_{i_1}} \right) \right) \right)^{\frac{1}{n}} \right) \right\rangle
\end{array} \right) \\
&= \left(\begin{array}{l}
\left\langle 1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \mu_{\phi_{i_1}} \right) \right)^{\frac{1}{n}}, 1 - \left(1 - \left(\prod_{1 \leq i_1 \leq n} v_{\phi_{i_1}} \right)^{\frac{1}{n}} \right) \right\rangle, \\
\left\langle 1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \alpha_{\phi_{i_1}} \right) \right)^{\frac{1}{n}}, 1 - \left(1 - \left(\prod_{1 \leq i_1 \leq n} \beta_{\phi_{i_1}} \right)^{\frac{1}{n}} \right) \right\rangle
\end{array} \right)
\end{aligned}
$$

Let $i_1 = i$, then

$$LDFMSM^{(1)}(\phi_1, \phi_2, \ldots, \phi_n) = \begin{pmatrix} \langle 1-\left(\prod_{i=1}^{n}\left(1-\mu_{\phi_i}\right)\right)^{\frac{1}{n}}, \left(\prod_{i=1}^{n} v_{\phi_i}\right)^{\frac{1}{n}}\rangle, \\ \langle 1-\left(\prod_{i=1}^{n}\left(1-\alpha_{\phi_i}\right)\right)^{\frac{1}{n}}, \left(\prod_{i=1}^{n} \beta_{\phi_i}\right)^{\frac{1}{n}}\rangle \end{pmatrix}$$

which is a simplification of linear Diophantine fuzzy weighted averaging (LDFWA) aggregation operator proposed by Riaz et al. in [55], where the weight is $\frac{1}{n}$ for all i.

Case 2. *If k = 2, we obtain:*

$$LDFMSM^{(2)}(\phi_1, \phi_2, \ldots, \phi_n) = \begin{pmatrix} \langle \left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^{2}\mu_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}, 1-\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^{2}\left(1-v_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}\rangle, \\ \langle \left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^{2}\alpha_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}, 1-\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^{2}\left(1-\beta_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle \left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\mu_{\phi_{i_1}}\mu_{\phi_{i_2}}\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}, 1-\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\left(1-v_{\phi_{i_1}}\right)\left(1-v_{\phi_{i_2}}\right)\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}\rangle, \\ \langle \left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\alpha_{\phi_{i_1}}\alpha_{\phi_{i_2}}\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}, 1-\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\left(1-\beta_{\phi_{i_1}}\right)\left(1-\beta_{\phi_{i_2}}\right)\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle \left(1-\left(\left(\prod_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n}\left(1-\mu_{\phi_{i_1}}\mu_{\phi_{i_2}}\right)\right)^{2}\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}, 1-\left(1-\left(\left(\prod_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n}\left(1-\left(1-v_{\phi_{i_1}}\right)\left(1-v_{\phi_{i_2}}\right)\right)\right)^{2}\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}\rangle, \\ \langle \left(1-\left(\left(\prod_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n}\left(1-\alpha_{\phi_{i_1}}\alpha_{\phi_{i_2}}\right)\right)^{2}\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}, 1-\left(1-\left(\left(\prod_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n}\left(1-\left(1-\beta_{\phi_{i_1}}\right)\left(1-\beta_{\phi_{i_2}}\right)\right)\right)^{2}\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle \left(1-\left(\prod_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n}\left(1-\mu_{\phi_{i_1}}\mu_{\phi_{i_2}}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}, 1-\left(1-\left(\prod_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n}\left(1-\left(1-v_{\phi_{i_1}}\right)\left(1-v_{\phi_{i_2}}\right)\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\rangle, \\ \langle \left(1-\left(\prod_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n}\left(1-\alpha_{\phi_{i_1}}\alpha_{\phi_{i_2}}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}, 1-\left(1-\left(\prod_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n}\left(1-\left(1-\beta_{\phi_{i_1}}\right)\left(1-\beta_{\phi_{i_2}}\right)\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\rangle \end{pmatrix}$$

In this case, the LDFMSM operator reduces to the linear Diophantine fuzzy Bonferroni mean (LDFBM) operator with the Bonferroni parameters p = q = 1 given in [36].

Case 3. *If k = n, we obtain:*

$$LDFMSM^{(n)}(\phi_1,\ \phi_2,\ \ldots,\ \phi_n) = \begin{pmatrix} \langle \left(1-\left(\prod_{1\le i_1<\cdots<i_n\le n}\left(1-\prod_{j=1}^{n}\mu_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^n}}\right)^{\frac{1}{n}}, 1-\left(1-\left(\prod_{1\le i_1<\cdots<i_n\le n}\left(1-\prod_{j=1}^{n}\left(1-v_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^n}}\right)^{\frac{1}{n}} \rangle, \\ \langle \left(1-\left(\prod_{1\le i_1<\cdots<i_n\le n}\left(1-\prod_{j=1}^{n}\alpha_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^n}}\right)^{\frac{1}{n}}, 1-\left(1-\left(\prod_{1\le i_1<\cdots<i_n\le n}\left(1-\prod_{j=1}^{n}\left(1-\beta_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_n^n}}\right)^{\frac{1}{n}} \rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle \left(1-\left(1-\prod_{j=1}^{n}\mu_{\phi_{i_j}}\right)^{\frac{1}{C_n^n}}\right)^{\frac{1}{n}}, 1-\left(1-\left(1-\prod_{j=1}^{n}\left(1-v_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^n}}\right)^{\frac{1}{n}} \rangle, \\ \langle \left(1-\left(1-\prod_{j=1}^{n}\alpha_{\phi_{i_j}}\right)^{\frac{1}{C_n^n}}\right)^{\frac{1}{n}}, 1-\left(1-\left(1-\prod_{j=1}^{n}\left(1-\beta_{\phi_{i_j}}\right)\right)^{\frac{1}{C_n^n}}\right)^{\frac{1}{n}} \rangle \end{pmatrix}$$

let $i_j = i$:

$$LDFMSM^{(n)}(\phi_1, \phi_2,\ \ldots,\ \phi_n) = \left(\langle \left(\prod_{i=1}^{n}\mu_{\phi_i}\right)^{\frac{1}{n}}, 1-\left(\prod_{i=1}^{n}\left(1-v_{\phi_i}\right)\right)^{\frac{1}{n}} \rangle, \langle \left(\prod_{i=1}^{n}\alpha_{\phi_i}\right)^{\frac{1}{n}}, 1 - \left(\prod_{i=1}^{n}\left(1-\beta_{\phi_i}\right)\right)^{\frac{1}{n}} \rangle \right)$$

which is a simplification of the linear Diophantine fuzzy weighted geometric (LDFWG) aggregation operator proposed by Riaz et al. in [55], where the weight is $\frac{1}{n}$ for all i.

4.2. LDFPMSM Aggregation Operator

In this section, we propose the Linear Diophantine Fuzzy Partitioned Maclaurin Symmetric Mean (LDFPMSM) aggregation operator and discuss its properties. The operator is intended for partitioned settings in which k-subset symmetric joint aggregation is performed only among criteria within the same predefined partition, while the criteria are assigned equal importance.

The classical PMSM operator in Section 3.3 assigns the coefficient $\frac{1}{s}$ to every partition, which gives equal importance to partitions irrespective of their cardinalities. When the partition sizes are unequal, such a structure may assign greater implicit importance to criteria belonging to smaller partitions. Therefore, the proposed LDFPMSM operator assigns each partition a weighting coefficient proportional to its cardinality. The LDFPMSM operator is defined in Definition 13.

Definition 13. *Let $\phi_i = \left(\langle\mu_{\phi_i}, v_{\phi_i}\rangle, \langle\alpha_{\phi_i}, \beta_{\phi_i}\rangle\right)$, ($i = 1, 2, \ldots, n$) be a collection of LDFNs. Suppose that the attribute set is partitioned into s subsets $P_1, P_2,\ \ldots,\ P_s$, and let $k = 1,\ 2,\ \ldots,\ h_{min}$, where $h_{min} = \min_{1\le r\le s} h_r$. Then the LDFPMSM operator is defined by Equation (21):*

$$LDFPMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) = \underset{r=1}{\overset{s}{\oplus}} \frac{h_r}{n}\left(\frac{\underset{1\le i_1<\cdots<i_k\le h_r}{\oplus}\left(\underset{j=1}{\overset{k}{\otimes}} \phi_{i_j}\right)}{C_{h_r}^k}\right)^{1/k} \tag{21}$$

where s denotes the number of partitions, h_r is the number of criteria in partition P_r, and k is the interaction parameter. Here, $(i_1, i_2, \ldots, i_k)$ traverse all k-tuple combinations of $(1, 2, \ldots, h_r)$ in each partition P_r, and $C_{h_r}^k$ is the binomial coefficient expressed by $C_{h_r}^k = \frac{h_r\,!}{k!(h_r-k)!}$. The coefficient $\frac{h_r}{n}$ represents the proportion of the criterion set in partition P_r in the complete criterion set and satisfies $\sum_{r=1}^{s} \frac{h_r}{n} = 1$.

Using the LDFS arithmetic operations given in Definition 7, we can derive Theorem 2.

Theorem 2. *Let $\phi_i = (\langle \mu_{\phi_i}, v_{\phi_i}\rangle, \langle \alpha_{\phi_i}, \beta_{\phi_i}\rangle)$, (i = 1, 2, ..., n) be a collection of LDFNs and let $1 \le k \le h_{min}$ and $h_{min} = \underset{1\le r\le s}{min}\, h_r$. Then the aggregated value obtained by the LDFPMSM operator $\phi^* = (\langle \mu^*, v^*\rangle, \langle \alpha^*, \beta^*\rangle)$ is also an LDFN, and*

$$LDFPMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) = \left(\begin{array}{c} \left\langle \begin{array}{c} \left(1 - \left(\prod_{r=1}^{s}\left(1 - \left(1 - \left(\prod_{1\le i_1<\cdots<i_k\le h_r}\left(1 - \prod_{j=1}^{k} \mu_{\phi_{i_j}}\right)\right)^{\frac{1}{C_{h_r}^k}}\right)^{\frac{1}{k}}\right)^{\frac{h_r}{n}}\right)\right), \\ \left(\prod_{r=1}^{s}\left(1 - \left(1 - \left(\prod_{1\le i_1<\cdots<i_k\le h_r}\left(1 - \prod_{j=1}^{k}\left(1 - v_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_{h_r}^k}}\right)^{\frac{1}{k}}\right)^{\frac{h_r}{n}}\right) \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \left(1 - \left(\prod_{r=1}^{s}\left(1 - \left(1 - \left(\prod_{1\le i_1<\cdots<i_k\le h_r}\left(1 - \prod_{j=1}^{k} \alpha_{\phi_{i_j}}\right)\right)^{\frac{1}{C_{h_r}^k}}\right)^{\frac{1}{k}}\right)^{\frac{h_r}{n}}\right)\right), \\ \left(\prod_{r=1}^{s}\left(1 - \left(1 - \left(\prod_{1\le i_1<\cdots<i_k\le h_r}\left(1 - \prod_{j=1}^{k}\left(1 - \beta_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_{h_r}^k}}\right)^{\frac{1}{k}}\right)^{\frac{h_r}{n}}\right) \end{array} \right\rangle \end{array}\right) \tag{22}$$

Proof. The proof of Theorem 2 is provided in Supplementary Material. □

It can be proved that LDFPMSM has the following properties:

Property 5 (Idempotency)**.** *If all ϕ_i (i = 1, 2, . . ., n) are equal, i.e., $\phi_i = \phi$ for all i, then*

$$LDFPMSM(\phi_1, \phi_2, \ldots, \phi_n) = \phi$$

Proof. The proof of Property 5 is provided in Supplementary Material. □

Property 6 (Monotonicity)**.** *Let $\phi_i = (\langle \mu_{\phi_i}, v_{\phi_i} \rangle, \langle \alpha_{\phi_i}, \beta_{\phi_i} \rangle)$ and $\phi'_i = \left(\langle \mu'_{\phi_i}, v'_{\phi_i} \rangle, \langle \alpha'_{\phi_i}, \beta'_{\phi_i} \rangle \right)$ be two collections of LDFNs. For a fixed partition structure P_r, $(i \in P_r)$ and fixed k, if $\mu_{\phi_i} \geq \mu'_{\phi_i}$, $v_{\phi_i} \leq v'_{\phi_i}$, $\alpha_{\phi_i} \geq \alpha'_{\phi_i}$ and $\beta_{\phi_i} \leq \beta'_{\phi_i}$ for all i, then*

$$LDFPMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) \geq LDFPMSM^{(k)}(\phi'_1, \phi'_2, \ldots, \phi'_n)$$

Proof. The proof is similar to that of Property 3 and is therefore omitted. □

Property 7 (Boundedness)**.** *Let $\phi_i = (\langle \mu_{\phi_i}, v_{\phi_i} \rangle, \langle \alpha_{\phi_i}, \beta_{\phi_i} \rangle)$, (i = 1, 2, . . ., n) be a collection of LDFNs and $1 \leq k \leq h_{min}$, and let*

$$\phi^- = \left(\langle \min_i \mu_{\phi_i}, \max_i v_{\phi_i} \rangle, \langle \min_i \alpha_{\phi_i}, \max_i \beta_{\phi_i} \rangle \right)$$

$$\phi^+ = \left(\langle \max_i \mu_{\phi_i}, \min_i v_{\phi_i} \rangle, \langle \max_i \alpha_{\phi_i}, \min_i \beta_{\phi_i} \rangle \right)$$

then

$$\phi^- \leq LDFPMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) \leq \phi^+$$

Proof. The proof is similar to Property 4 since the LDFPMSM operator also satisfies idempotency and monotonicity properties. Hence, it is omitted. □

Computational Examination of Parameter Monotonicity of LDFPMSM

To examine the behavior of the LDFPMSM operator with respect to interaction parameter k, an exhaustive finite-grid search for counterexamples was conducted. Since the LDFPMSM operator depends on the predefined partition structure, the computational examination was performed over both finite-grid input configurations and admissible partition cardinality profiles. Let $P = \{P_1, \ldots, P_s\}$ denote the partitions of the n inputs, where $h_r = |P_r|$ and $\sum_{r=1}^{s} h_r = n$, and let $h_{min} = \min\limits_{1 \leq r \leq s} h_r$, $1 \leq k \leq h_{min}$. The LDFPMSM aggregation can be represented through the common scalar kernel as follows:

$$F_{P,k}(\boldsymbol{x}) = 1 - \prod_{r=1}^{s} \left(1 - \left[1 - \left(\prod_{1 \leq i_1 < \cdots < i_k \leq h_r} \left(1 - \prod_{j=1}^{k} x_{i_j} \right) \right)^{1/C_{h_r}^k} \right]^{1/k} \right)^{\frac{h_r}{n}}$$

and

$$LDFPMSM_P^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) = (\langle F_{P,k}(\boldsymbol{\mu}), 1 - F_{P,k}(\mathbf{1} - \boldsymbol{v}) \rangle, \langle F_{P,k}(\boldsymbol{\alpha}), 1 - F_{P,k}(\mathbf{1} - \boldsymbol{\beta}) \rangle)$$

where

$$\boldsymbol{\mu} = (\mu_{\phi_1}, \ldots, \mu_{\phi_n}), \boldsymbol{v} = (v_{\phi_1}, \ldots, v_{\phi_n}), \boldsymbol{\alpha} = (\alpha_{\phi_1}, \ldots, \alpha_{\phi_n}), \boldsymbol{\beta} = (\beta_{\phi_1}, \ldots, \beta_{\phi_n})$$

and $\mathbf{1} = (1, \ldots, 1)$.

Therefore, for a fixed partition structure, parameter monotonicity can be examined by evaluating the consecutive-order condition $F_{P,k+1}(\mathbf{x}) \leq F_{P,k}(\mathbf{x})$, for every examined $\mathbf{x} \in G^{\boldsymbol{n}}$ and $k = 1, \ldots, h_{min} - 1$. The counterexample search is based on the margin $\Delta_{P,k}(\mathbf{x}) = F_{P,k}(\mathbf{x}) - F_{P,k+1}(\mathbf{x})$, where a negative value indicates a violation.

The finite input domain was defined using the five-level $= \{0,\ 0.25,\ 0.50,\ 0.75,\ 1\}$, and the search was conducted for $n = 2, \ldots, 8$. For each n, all ordered input configurations of 5^n in the complete finite grid $G_P^{\boldsymbol{n}}$ were evaluated for each admissible partition cardinality profile. The five-level discretization was selected to enable exhaustive enumeration of the finite input domain across all admissible partition cardinality profiles while maintaining computational feasibility. All integer partition cardinality profiles satisfying $h_r \geq 2$ for every partition were examined, and the partition structures with $h_{min} = 1$ were excluded since no consecutive-k comparison can be performed.

Across $n = 2, \ldots, 8$, a total of 21 admissible partition cardinality profiles were identified. For each profile, one canonical partition was explicitly evaluated. Although different labeled partition structures sharing the same cardinality profile may yield different aggregation values for a fixed input vector, each such structure can be obtained from the corresponding canonical partition through a permutation of the input indices. Since all ordered input vectors in $G_P^{\boldsymbol{n}}$ were exhaustively enumerated, the corresponding permuted input configurations are also included in the search. Therefore, separate evaluation of every labeled partition structure is not required. As a complementary admissibility check, all 5^4 candidates $(\mu,\ v,\ \alpha,\ \beta) \in G^{\boldsymbol{4}}$ were generated and screened according to the LDFN conditions. This screening yielded 375 valid finite-grid LDFNs. Because each of the four LDFPMSM components is obtained by applying the same scalar kernel separately to the corresponding component vectors, every component vector arising from these valid LDFNs is already included in the complete scalar grid $G_P^{\boldsymbol{n}}$ considered in the exhaustive search.

In total, 3,117,025 grid-vector/partition-profile configurations were explicitly evaluated across the 21 admissible partition cardinality profiles. These configurations represent 292,630,775 input-configuration/labeled-partition pairs when all labeled partition structures corresponding to the examined cardinality profiles are taken into account. The search yielded 7,190,275 direct consecutive-k comparisons, and 347,547,775 corresponding comparisons across all labeled partition structures. Using a numerical tolerance of 10^{-10}, no violation of the condition $F_{P,k+1}(\mathbf{x}) \leq F_{P,k}(\mathbf{x})$ was identified. The numerical calculations were performed using the publicly available software tools Python 3.13.9, NumPy (v2.3.5) and the Spyder development environment (v6.1.0), and the pseudocode is provided in the Supplementary Material. Since the input space is continuous, the observed non-increasing behavior constitutes finite grid evidence for the examined discretized domain. It is not general analytical evidence of parameter monotonicity over the continuous input space.

Now, we can discuss some special cases of the LDFPMSM operator with respect to the parameter k.

Case 1. *If $k = 1$, then the LDFPMSM operator reduces to the linear Diophantine fuzzy weighted averaging operator with equal weights, which coincides with the LDFMSM operator with $k = 1$, as follows:*

$$LDFPMSM^{(1)}(\phi_1,\ \phi_2,\ \ldots,\ \phi_n) = \begin{pmatrix} \left\langle \begin{matrix} \left(1-\left(\prod_{r=1}^{s}\left(\left(\prod_{i=1}^{h_r}\left(1-\mu_{\phi_i}\right)\right)^{\frac{1}{h_r}}\right)^{\frac{h_r}{n}}\right)\right), \\ \left(\prod_{r=1}^{s}\left(\left(\prod_{i=1}^{h_r}v_{\phi_i}\right)^{\frac{1}{h_r}}\right)^{\frac{h_r}{n}}\right) \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} \left(1-\left(\prod_{r=1}^{s}\left(\left(\prod_{i=1}^{h_r}\left(1-\alpha_{\phi_i}\right)\right)^{\frac{1}{h_r}}\right)^{\frac{h_r}{n}}\right)\right), \\ \left(\prod_{r=1}^{s}\left(\left(\prod_{i=1}^{h_r}\beta_{\phi_i}\right)^{\frac{1}{h_r}}\right)^{\frac{h_r}{n}}\right) \end{matrix} \right\rangle \end{pmatrix}$$

and since $\left(\frac{1}{h_r}\cdot\frac{h_r}{n}\right)=\frac{1}{n}$ *and mutually disjoint partitions constituting of* n *criteria, we obtain*

$$\begin{aligned} LDFPMSM^{(1)}(\phi_1,\ \phi_2,\ \ldots,\ \phi_n) &= \begin{pmatrix} \left\langle 1-\left(\prod_{i=1}^{n}\left(1-\mu_{\phi_i}\right)\right)^{\frac{1}{n}},\ \left(\prod_{i=1}^{n}v_{\phi_i}\right)^{\frac{1}{n}} \right\rangle, \\ \left\langle 1-\left(\prod_{i=1}^{n}\left(1-\alpha_{\phi_i}\right)\right)^{\frac{1}{n}},\ \left(\prod_{i=1}^{n}\beta_{\phi_i}\right)^{\frac{1}{n}} \right\rangle \end{pmatrix} \\ &= LDFMSM^{(1)}(\phi_1,\ \phi_2,\ \ldots,\ \phi_n) \end{aligned}$$

The detailed derivation is given in Supplementary Material.

Case 2. *If* k *= 2, provided that* $h_{min} \geq 2$*, we obtain:*

$$\begin{aligned} & LDFPMSM^{(2)}(\phi_1,\phi_2,\ \ldots,\ \phi_n) \\ &= \begin{pmatrix} \left\langle \begin{matrix} \left(1-\left(\prod_{r=1}^{s}\left(1-\left(1-\left(\prod_{\substack{i_1,i_2=1 \\ i_1\neq i_2}}^{h_r}\left(1-\mu_{\phi_{i_1}}\mu_{\phi_{i_2}}\right)\right)^{\frac{1}{h_r(h_r-1)}}\right)^{\frac{1}{2}}\right)^{\frac{h_r}{n}}\right)\right), \\ \left(\prod_{r=1}^{s}\left(1-\left(1-\left(\prod_{\substack{i_1,i_2=1 \\ i_1\neq i_2}}^{h_r}\left(1-\left(1-v_{\phi_{i_1}}\right)\left(1-v_{\phi_{i_2}}\right)\right)\right)^{\frac{1}{h_r(h_r-1)}}\right)^{\frac{1}{2}}\right)^{\frac{h_r}{n}}\right) \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} \left(1-\left(\prod_{r=1}^{s}\left(1-\left(1-\left(\prod_{\substack{i_1,i_2=1 \\ i_1\neq i_2}}^{h_r}\left(1-\alpha_{\phi_{i_1}}\alpha_{\phi_{i_2}}\right)\right)^{\frac{1}{h_r(h_r-1)}}\right)^{\frac{1}{2}}\right)^{\frac{h_r}{n}}\right)\right), \\ \left(\prod_{r=1}^{s}\left(1-\left(1-\left(\prod_{\substack{i_1,i_2=1 \\ i_1\neq i_2}}^{h_r}\left(1-\left(1-\beta_{\phi_{i_1}}\right)\left(1-\beta_{\phi_{i_2}}\right)\right)\right)^{\frac{1}{h_r(h_r-1)}}\right)^{\frac{1}{2}}\right)^{\frac{h_r}{n}}\right) \end{matrix} \right\rangle \end{pmatrix} \end{aligned}$$

which exhibits a Bonferroni-type pairwise interaction structure with (p = q = 1) within each partition as in [39,56], while employing a partition-size proportional weighting.

For the sake of brevity, the derivation is presented in Supplementary Material.

Remark 1 (Reducibility of LDFPMSM to LDFMSM)**.** *When s = 1, there exists a single partition which contains all the criteria, thus* $h_1 = n$ *and* $\frac{h_1}{n} = 1$*. Then, the LDFPMSM operator reduces to the LDFMSM operator, which is defined in Theorem 1. Its derivation is provided in Supplementary Material.*

4.3. RWPMSM and LDFRWPMSM Aggregation Operators

To the best of our knowledge, the reducible weighted MSM operator introduced by [42] has not been incorporated into a partitioned MSM framework. Therefore, we first introduce a reducible weighted partitioned MSM operator in Definition 14 by adopting the normalized k-subset weighting principle, in which each k-subset is assigned a weight proportional to the product of the weights of its constituent criteria. To accommodate the partitioned structure and preserve criterion importance at different aggregation levels, between subsets and partitions, we introduce partition-level and within-subset-level weighting coefficients. Then we establish the idempotency, monotonicity, boundedness and reducibility properties. Subsequently, the proposed RWPMSM operator is extended to the linear Diophantine fuzzy sets as the linear Diophantine fuzzy reducible weighted partitioned MSM (LDFRWPMSM) operator in Definition 15.

Definition 14. *Let* $x_1, x_2, \ldots, x_n$ *be a collection of non-negative real numbers. Suppose that the attribute set is partitioned into s subsets* $P_1, P_2, \ldots, P_s$*. Let* ω_i *be the weight of* x_i *satisfying* $\omega_i > 0$ *and* $\sum_{i=1}^{n} \omega_i = 1$*, and let* $k = 1, 2, \ldots, h_{min}$*, where* $h_{min} = min_{1 \le r \le s} h_r$*. Then the RWPMSM operator is defined by Equation (23):*

$$RWPMSM^{(k)}(x_1, x_2, \ldots, x_n) = \sum_{r=1}^{s} \Omega_r \left[\sum_{1 \le i_1 < \cdots < i_k \le h_r} \lambda^{(r)}_{i_1, i_2, \ldots, i_k} \left(\prod_{j=1}^{k} \left(x^{(r)}_{i_j} \right)^{\theta^{(r)}_{i_j | i_1, i_2, \ldots, i_k}} \right) \right]^{\frac{1}{k}} \tag{23}$$

where s denotes the number of partitions and h_r *is the number of attributes in partition* P_r*. Within each partition* P_r*, its constituent attributes and corresponding weights are indexed as* $x^{(r)}_i$ *and* $\omega^{(r)}_i$*, respectively for i = 1, 2, …,* h_r*. The indices* $1 \le i_1 < \cdots < i_k \le h_r$ *traverse all k-tuple combinations of* h_r*. The coefficient* Ω_r*, defined by* $\Omega_r = \sum_{i=1}^{h_r} \omega^{(r)}_i$*, denotes the weight of partition* P_r *and satisfies* $\sum_{r=1}^{s} \Omega_r = 1$*. The conditional weight of criterion i,* $\overline{\omega}^{(r)}_i$*, in partition* P_r $(i \in P_r)$ *is given by* $\overline{\omega}^{(r)}_i = \frac{\omega^{(r)}_i}{\Omega_r}$ *such that* $\sum_{i=1}^{h_r} \overline{\omega}^{(r)}_i = 1$*. For each k-tuple* $(i_1, i_2, \ldots, i_k)$ *in* P_r*, the normalized joint subset weight,* $\lambda^{(r)}_{i_1, i_2, \ldots, i_k}$*, is defined as* $\lambda^{(r)}_{i_1, i_2, \ldots, i_k} = \frac{\prod_{j=1}^{k} \overline{\omega}^{(r)}_{i_j}}{\sum_{1 \le l_1 < \cdots < l_k \le h_r} \prod_{j=1}^{k} \overline{\omega}^{(r)}_{l_j}}$ *such that* $\lambda^{(r)}_{i_1, i_2, \ldots, i_k} > 0$ *and* $\sum_{1 \le i_1 < \cdots < i_k \le h_r} \lambda^{(r)}_{i_1, i_2, \ldots, i_k} = 1$*. It represents the relative contribution of the selected k-subset* $(i_1, i_2, \ldots, i_k)$ *to the aggregation within* P_r*. The coefficient* $\theta^{(r)}_{i_j | i_1, i_2, \ldots, i_k}$ *denotes the within-subset weighting coefficient of criterion* i_j *in the selected k-subset* $(i_1, i_2, \ldots, i_k)$ *and is defined as* $\theta^{(r)}_{i_j | i_1, i_2, \ldots, i_k} = \frac{k \overline{\omega}^{(r)}_{i_j}}{\sum_{l=1}^{k} \overline{\omega}^{(r)}_{i_l}}$ *for j = 1, …, k, satisfying* $\sum_{j=1}^{k} \theta^{(r)}_{i_j | i_1, i_2, \ldots, i_k} = k$*.*

It can be proved that RWPMSM has the following properties:

Property 8 (Idempotency)**.** *If all* x_i *(i = 1, 2, …, n) are equal, i.e.,* $x_i = x$ *for all i, then*

$$RWPMSM^{(k)}(x_1, x_2, \ldots, x_n) = x$$

Proof. The proof of Property 8 is provided in Supplementary Material. □

Property 9 (Monotonicity). *Let $X_i = (x_1, x_2, \ldots, x_n)$ and $X'_i = \left(x'_1, x'_2, \ldots, x'_n\right)$ be two collections of non-negative real numbers. For a fixed partition structure P_r, $(i \in P_r)$, fixed k and weight vector $\omega_i = \omega'_i$, if $x_i \geq x'_i$ for all i, then*

$$RWPMSM^{(k)}(x_1, x_2, \ldots, x_n) \geq RWPMSM^{(k)}\left(x'_1, x'_2, \ldots, x'_n\right)$$

Proof. The proof of Property 9 is provided in Supplementary Material. □

Property 10 (Boundedness). *Let $x^- = \min_i x_i$, and $x^+ = \max_i x_i$. Then*

$$x^- \leq RWPMSM^{(k)}(x_1, x_2, \ldots, x_n) \leq x^+$$

Proof. The proof of Property 10 is provided in Supplementary Material. □

Now, we can discuss some special cases and reducibility properties of the RW-PMSM operator.

Case 1. *If k = 1, then the RWPMSM operator reduces to a weighted averaging (WA) operator as follows:*

For k = 1, $\theta^{(r)}_{i|i} = \frac{\overline{\omega}^{(r)}_i}{\overline{\omega}^{(r)}_i} = 1$ and $\lambda^{(r)}_i = \frac{\overline{\omega}^{(r)}_i}{\sum_{l \in P_r} \overline{\omega}^{(r)}_l} = \overline{\omega}^{(r)}_i$.
Therefore,

$$RWPMSM^{(1)}(x_1, x_2, \ldots, x_n) = \sum_{r=1}^{s} \Omega_r \sum_{i \in P_r} \overline{\omega}^{(r)}_i x_i$$

Since $\Omega_r \overline{\omega}^{(r)}_i = \omega_i$, we obtain

$$RWPMSM^{(1)}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \omega_i x_i = WA$$

Case 2. *If k = n and s = 1, then the RWPMSM operator reduces to a weighted geometric (WG) operator as follows:*

Since s = 1, $h_1 = n$ and $\Omega_1 = 1$. Thus, the conditional weights satisfy $\overline{\omega}^{(1)}_i = \frac{\omega_i}{\Omega_1} = \omega_i$.
For k = n, there exists only one admissible n-subset (1, 2, . . . , n). Thus, $\lambda^{(1)}_{1,\,2,\,\ldots,\,n} = 1$.
For each i = 1, . . . , n, the within subset criterion $\theta^{(1)}_{i|1,\,2,\,\ldots,\,n} = \frac{n\overline{\omega}^{(1)}_i}{\sum_{l=1}^{n} \overline{\omega}^{(1)}_l}$.
Since $\sum_{l=1}^{n} \overline{\omega}^{(1)}_l = 1$, $\theta^{(r)}_{i|1,\,2,\,\ldots,\,n} = n\omega_i$. Thus, the operator reduces to the weighted geometric (WG) operator.

$$RWPMSM^{(n)}(x_1, x_2, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i^{n\omega_i}\right)^{\frac{1}{n}} = \prod_{i=1}^{n} x_i^{\omega_i} = WG_\omega(x_1, x_2, \ldots, x_n)$$

Remark 2 (Reducibility of RWPMSM under equal weights to cardinality-proportional PMSM). *When the criteria have equal weights $\left(\omega_i = \frac{1}{n}\right)$ for all i, then RWPMSM reduces to a cardinality-proportional PMSM structure similar to the proposed LDFPMSM operator in Definition 13, as follows:*

Since $\omega_i = \frac{1}{n}$ for all i, then $\Omega_r = \frac{h_r}{n}$, $\overline{\omega}^{(r)}_i = \frac{1}{h_r}$. For every k-subset, $\theta^{(r)}_{i_j|i_1,\,i_2,\,\ldots,\,i_k} = \frac{k/h_r}{k/h_r} = 1$.
Furthermore, $\lambda^{(r)}_{i_1,\,i_2,\,\ldots,\,i_k} = \frac{(1/h_r)^k}{C^k_{h_r}(1/h_r)^k} = \frac{1}{C^k_{h_r}}$.

Thus,

$$RWPMSM^{(k)}(x_1, x_2, \ldots, x_n) = \sum_{r=1}^{s} \frac{h_r}{n} \left[\frac{1}{C_{h_r}^k} \sum_{1 \leq i_1 < \cdots < i_k \leq h_r} \left(\prod_{j=1}^{k} x_{i_j} \right) \right]^{\frac{1}{k}}$$

Remark 3 (Reducibility of RWPMSM under equal weights and equal-cardinality partitions to PMSM)**.** *Following Remark 2, if each partition has the same cardinality $h_r = \frac{n}{s}$ for every r, then $\frac{h_r}{n} = \frac{1}{s}$. Thus,*

$$RWPMSM^{(k)}(x_1, x_2, \ldots, x_n) = PMSM^{(k)}(x_1, x_2, \ldots, x_n)$$

where PMSM is the operator given in Definition 10.

Remark 4 (Reducibility of RWPMSM under a single partition and equal weights to MSM)**.** *Under a single partition (s = 1) and equally weighted criteria $\left(\omega_i = \frac{1}{n}\right)$ for all i:*

If s = 1, then $P_1 = C$, $h_1 = n$, $\Omega_1 = 1$.

If $\omega_i = \frac{1}{n}$, then $\overline{\omega}_i^{(1)} = \frac{1}{n}$. Thus, $\theta_{i_j|i_1, i_2, \ldots, i_k}^{(1)} = 1$, and $\lambda_{i_1, i_2, \ldots, i_k}^{(1)} = \frac{1}{C_n^k}$.

Thus,

$$RWPMSM^{(k)}(x_1, x_2, \ldots, x_n) = \left(\frac{1}{C_n^k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} x_{ij} \right)^{1/k} = MSM^{(k)}(x_1, x_2, \ldots, x_n)$$

The RWPMSM reduces to MSM, as defined in Definition 9.

Based on Definition 14, we extend the RWPMSM operator to LDFSs, as given in Definition 15.

Definition 15. *Let $\phi_i = (\langle \mu_{\phi_i}, v_{\phi_i} \rangle, \langle \alpha_{\phi_i}, \beta_{\phi_i} \rangle)$, (i = 1, 2, ..., n) be a collection of LDFNs, and ω_i be the weight vector of ϕ_i satisfying $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$. Suppose that the attribute set is partitioned into s subsets $P_1, P_2, \ldots, P_s$, and let $k = 1, 2, \ldots, h_{min}$, where $h_{min} = \min_{1 \leq r \leq s} h_r$. Then the LDFRWPMSM operator is defined by Equation (24):*

$$LDFRWPMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) = \underset{r=1}{\overset{s}{\oplus}} \Omega_r \left(\underset{1 \leq i_1 < \cdots < i_k \leq h_r}{\oplus} \lambda_{i_1, i_2, \ldots, i_k}^{(r)} \left(\underset{j=1}{\overset{k}{\otimes}} \phi_{i_j}^{\theta_{i_j|i_1, i_2, \ldots, i_k}^{(r)}} \right) \right)^{1/k} \tag{24}$$

where s denotes the number of partitions and h_r is the number of attributes in partition P_r. Within each partition P_r, its constituent inputs and corresponding weights are indexed as $\phi_i^{(r)}$ and $\omega_i^{(r)}$, respectively for i = 1, 2, ..., h_r. The indices $1 \leq i_1 < \cdots < i_k \leq h_r$ traverse all k-tuple combinations of h_r. The coefficient Ω_r, defined by $\Omega_r = \sum_{i=1}^{h_r} \omega_i^{(r)}$, denotes the weight of partition P_r and satisfies $\sum_{r=1}^{s} \Omega_r = 1$. The conditional weight of criterion i, $\overline{\omega}_i^{(r)}$, in partition P_r $(i \in P_r)$ is given by $\overline{\omega}_i^{(r)} = \frac{\omega_i^{(r)}}{\Omega_r}$ such that $\sum_{i=1}^{h_r} \overline{\omega}_i^{(r)} = 1$. For each k-tuple $(i_1, i_2, \ldots, i_k)$ in P_r, the normalized joint subset weight, $\lambda_{i_1, i_2, \ldots, i_k}^{(r)}$, is defined as $\lambda_{i_1, i_2, \ldots, i_k}^{(r)} = \frac{\prod_{j=1}^{k} \overline{\omega}_{i_j}^{(r)}}{\sum_{1 \leq l_1 < \cdots < l_k \leq h_r} \prod_{j=1}^{k} \overline{\omega}_{l_j}^{(r)}}$ such that $\lambda_{i_1, i_2, \ldots, i_k}^{(r)} > 0$ and $\sum_{1 \leq i_1 < \cdots < i_k \leq h_r} \lambda_{i_1, i_2, \ldots, i_k}^{(r)} = 1$. It represents the relative contribution of the selected k-subset $(i_1, i_2, \ldots, i_k)$ to the aggregation within P_r. The coefficient $\theta_{i_j|i_1, i_2, \ldots, i_k}^{(r)}$ denotes the within-subset weighting coefficient of criterion i_j in the selected *k-subset* $(i_1, i_2, \ldots, i_k)$ *and is defined as* $\theta^{(r)}_{i_j|i_1, i_2, \ldots, i_k} = \frac{k\overline{\omega}^{(r)}_{i_j}}{\sum_{l=1}^{k} \overline{\omega}^{(r)}_{i_l}}$ *for j = 1, …, k, satisfying* $\sum_{j=1}^{k} \theta^{(r)}_{i_j|i_1, i_2, \ldots, i_k} = k$.

Using the LDFS arithmetic operations in Definition 7, we derive the LDFRWPMSM operator, as given in Theorem 3.

Theorem 3. *Let* $\phi_i = (\langle \mu_{\phi_i}, v_{\phi_i}\rangle, \langle \alpha_{\phi_i}, \beta_{\phi_i}\rangle)$, *(i = 1, 2, …, n) be a collection of LDFNs, and* ω_i *be the weight vector of* ϕ_i *satisfying* $\omega_i > 0$ *and* $\sum_{i=1}^{n} \omega_i = 1$. *Suppose that the attribute set is partitioned into s subsets* $P_1, P_2, \ldots, P_s$ *and let* $k = 1, 2, \ldots, h_{min}$, *where* $h_{min} = \min_{1 \le r \le s} h_r$. *Then the aggregated value obtained by the LDFRWPMSM operator* $\phi^* = (\langle \mu^*, v^*\rangle, \langle \alpha^*, \beta^*\rangle)$ *is also an LDFN, and*

$$LDFRWPMSM^{(k)}(\phi_1, \phi_2, \ldots, \phi_n) = \left(\begin{array}{c} \left\langle \begin{array}{c} \left(1 - \left(\prod_{r=1}^{s}\left(1 - \left(1 - \prod_{1 \le i_1 < \cdots < i_k \le h_r} \left(1 - \prod_{j=1}^{k} \mu_{\phi_{i_j}}^{\theta^{(r)}_{i_j|i_1, i_2, \ldots, i_k}}\right)^{\lambda^{(r)}_{i_1, i_2, \ldots, i_k}}\right)^{\frac{1}{k}}\right)^{\Omega_r}\right)\right), \\ \left(\prod_{r=1}^{s}\left(1 - \left(1 - \prod_{1 \le i_1 < \cdots < i_k \le h_r} \left(1 - \prod_{j=1}^{k} \left(1 - v_{\phi_{i_j}}\right)^{\theta^{(r)}_{i_j|i_1, i_2, \ldots, i_k}}\right)^{\lambda^{(r)}_{i_1, i_2, \ldots, i_k}}\right)^{\frac{1}{k}}\right)^{\Omega_r}\right) \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \left(1 - \left(\prod_{r=1}^{s}\left(1 - \left(1 - \prod_{1 \le i_1 < \cdots < i_k \le h_r} \left(1 - \prod_{j=1}^{k} \alpha_{\phi_{i_j}}^{\theta^{(r)}_{i_j|i_1, i_2, \ldots, i_k}}\right)^{\lambda^{(r)}_{i_1, i_2, \ldots, i_k}}\right)^{\frac{1}{k}}\right)^{\Omega_r}\right)\right), \\ \left(\prod_{r=1}^{s}\left(1 - \left(1 - \prod_{1 \le i_1 < \cdots < i_k \le h_r} \left(1 - \prod_{j=1}^{k} \left(1 - \beta_{\phi_{i_j}}\right)^{\theta^{(r)}_{i_j|i_1, i_2, \ldots, i_k}}\right)^{\lambda^{(r)}_{i_1, i_2, \ldots, i_k}}\right)^{\frac{1}{k}}\right)^{\Omega_r}\right) \end{array} \right\rangle \end{array} \right) \tag{25}$$

where s denotes the number of partitions and h_r *is the number of attributes in partition* P_r. *Within each partition* P_r, *its constituent attributes and corresponding weights are indexed as* $\phi_i^{(r)}$ *and* $\omega_i^{(r)}$, *respectively for i = 1, 2, …,* h_r. *The indices* $1 \le i_1 < \cdots < i_k \le h_r$ *traverse all k-tuple combinations of* h_r. *The coefficient* Ω_r, *defined by* $\Omega_r = \sum_{i=1}^{h_r} \omega_i^{(r)}$, *denotes the weight of partition* P_r *and satisfies* $\sum_{r=1}^{s} \Omega_r = 1$. *The conditional weight of criterion i,* $\overline{\omega}_i^{(r)}$, *in partition* P_r $(i \in P_r)$ *is given by* $\overline{\omega}_i^{(r)} = \frac{\omega_i^{(r)}}{\Omega_r}$ *such that* $\sum_{i=1}^{h_r} \overline{\omega}_i^{(r)} = 1$. *For each k-tuple* $(i_1, i_2, \ldots, i_k)$ *in* P_r, *the normalized joint subset weight,* $\lambda^{(r)}_{i_1, i_2, \ldots, i_k}$, *is defined as* $\lambda^{(r)}_{i_1, i_2, \ldots, i_k} = \frac{\prod_{j=1}^{k} \overline{\omega}^{(r)}_{i_j}}{\sum_{1 \le l_1 < \cdots < l_k \le h_r} \prod_{j=1}^{k} \overline{\omega}^{(r)}_{l_j}}$ *such that* $\lambda^{(r)}_{i_1, i_2, \ldots, i_k} > 0$ *and* $\sum_{1 \le i_1 < \cdots < i_k \le h_r} \lambda^{(r)}_{i_1, i_2, \ldots, i_k} = 1$. *It represents the relative contribution of the selected k-subset* $(i_1, i_2, \ldots, i_k)$ *to the aggregation within* P_r. *The coefficient* $\theta^{(r)}_{i_j|i_1, i_2, \ldots, i_k}$ *denotes the within-subset weighting coefficient of criterion* i_j *in the selected k-subset* $(i_1, i_2, \ldots, i_k)$ *and is defined as* $\theta^{(r)}_{i_j|i_1, i_2, \ldots, i_k} = \frac{k\overline{\omega}^{(r)}_{i_j}}{\sum_{l=1}^{k} \overline{\omega}^{(r)}_{i_l}}$ *for j = 1, …, k, satisfying* $\sum_{j=1}^{k} \theta^{(r)}_{i_j|i_1, i_2, \ldots, i_k} = k$.

Proof. The proof of Theorem 3 is provided in Supplementary Material. □

It can be proved that LDFRWPMSM has the following properties:

Property 11 (Idempotency)**.** *If all ϕ_i (i = 1, 2, ..., n) are equal, i.e., $\phi_i = \phi$ for all i, then*

$$LDFRWPMSM(\phi_1,\ \phi_2,\ \ldots,\ \phi_n) = \phi$$

Proof. Let $\phi_i = \phi$ for all i = 1, . . ., n. For every k-subset in partition P_r, we have

$$\prod_{j=1}^{k} \mu_\phi^{\theta^{(r)}_{i_j|i_1,\ i_2,\ldots,i_k}} = \mu_\phi^{\sum_{j=1}^{k} \theta^{(r)}_{i_j|i_1,\ i_2,\ldots,i_k}} = \mu_\phi^k$$

and similarly,

$$\prod_{j=1}^{k} (1 - v_\phi)^{\theta^{(r)}_{i_j\,|\,i_1,\ i_2,\ldots,i_k}} = (1 - v_\phi)^{\sum_{j=1}^{k} \theta^{(r)}_{i_j\,|\,i_1,\ i_2,\ldots,i_k}} = (1 - v_\phi)^k$$

$$\prod_{j=1}^{k} \alpha_\phi^{\theta^{(r)}_{i_j|i_1,\ i_2,\ldots,i_k}} = \alpha_\phi^{\sum_{j=1}^{k} \theta^{(r)}_{i_j|i_1,\ i_2,\ldots,i_k}} = \alpha_\phi^k$$

$$\prod_{j=1}^{k} (1 - \beta_\phi)^{\theta^{(r)}_{i_j\,|\,i_1,\ i_2,\ldots,i_k}} = (1 - \beta_\phi)^{\sum_{j=1}^{k} \theta^{(r)}_{i_j\,|\,i_1,\ i_2,\ldots,i_k}} = (1 - \beta_\phi)^k$$

Based on Theorem 3, we have

$$LDFRWPMSM^{(k)}(\phi,\ \phi,\ \ldots,\ \phi) = \left(\begin{array}{c} \left\langle \begin{array}{c} \left(1 - \left(\prod_{r=1}^{s} \left(1 - \left(1 - \prod\limits_{1 \le i_1 < \cdots < i_k \le h_r} \left(1 - \mu_\phi^k \right)^{\lambda^{(r)}_{i_1, i_2,\ \ldots,\ i_k}} \right)^{\frac{1}{k}} \right)^{\Omega_r} \right) \right), \\ \left(\prod_{r=1}^{s} \left(1 - \left(1 - \prod\limits_{1 \le i_1 < \cdots < i_k \le h_r} \left(1 - (1 - v_\phi)^k \right)^{\lambda^{(r)}_{i_1, i_2,\ \ldots,\ i_k}} \right)^{\frac{1}{k}} \right)^{\Omega_r} \right) \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \left(1 - \left(\prod_{r=1}^{s} \left(1 - \left(1 - \prod\limits_{1 \le i_1 < \cdots < i_k \le h_r} \left(1 - \alpha_\phi^k \right)^{\lambda^{(r)}_{i_1, i_2,\ \ldots,\ i_k}} \right)^{\frac{1}{k}} \right)^{\Omega_r} \right) \right), \\ \left(\prod_{r=1}^{s} \left(1 - \left(1 - \prod\limits_{1 \le i_1 < \cdots < i_k \le h_r} \left(1 - (1 - \beta_\phi)^k \right)^{\lambda^{(r)}_{i_1, i_2,\ \ldots,\ i_k}} \right)^{\frac{1}{k}} \right)^{\Omega_r} \right) \end{array} \right\rangle \end{array} \right)$$

Since for every partition, it satisfies $\sum_{1\le i_1<\cdots<i_k\le h_r}\lambda^{(r)}_{i_1,i_2,\ldots,i_k}=1$

$$
\begin{aligned}
&LDFRWPMSM^{(k)}(\phi,\ \phi,\ \ldots,\ \phi)\\
&=\left(\begin{array}{c}
\left\langle \begin{array}{c}\left(1-\left(\prod_{r=1}^{s}\left(1-\left(1-\left(1-\mu_{\phi}^{k}\right)^{\sum_{1\le i_1<\cdots<i_k\le h_r}\lambda^{(r)}_{i_1,i_2,\ldots,i_k}}\right)^{\frac{1}{k}}\right)^{\Omega_r}\right)\right),\\
\left(\prod_{r=1}^{s}\left(1-\left(1-\left(1-(1-v_{\phi})^{k}\right)^{\sum_{1\le i_1<\cdots<i_k\le h_r}\lambda^{(r)}_{i_1,i_2,\ldots,i_k}}\right)^{\frac{1}{k}}\right)^{\Omega_r}\right)\end{array}\right\rangle,\\
\left\langle \begin{array}{c}\left(1-\left(\prod_{r=1}^{s}\left(1-\left(1-\left(1-\alpha_{\phi}^{k}\right)^{\sum_{1\le i_1<\cdots<i_k\le h_r}\lambda^{(r)}_{i_1,i_2,\ldots,i_k}}\right)^{\frac{1}{k}}\right)^{\Omega_r}\right)\right),\\
\left(\prod_{r=1}^{s}\left(1-\left(1-\left(1-(1-\beta_{\phi})^{k}\right)^{\sum_{1\le i_1<\cdots<i_k\le h_r}\lambda^{(r)}_{i_1,i_2,\ldots,i_k}}\right)^{\frac{1}{k}}\right)^{\Omega_r}\right)\end{array}\right\rangle
\end{array}\right)\\
&=\left(\begin{array}{c}
\left\langle \begin{array}{c}\left(1-\left(\prod_{r=1}^{s}\left(1-\left(1-\left(1-\mu_{\phi}^{k}\right)\right)^{\frac{1}{k}}\right)^{\Omega_r}\right)\right),\\
\left(\prod_{r=1}^{s}\left(1-\left(1-\left(1-(1-v_{\phi})^{k}\right)\right)^{\frac{1}{k}}\right)^{\Omega_r}\right)\end{array}\right\rangle,\\
\left\langle \begin{array}{c}\left(1-\left(\prod_{r=1}^{s}\left(1-\left(1-\left(1-\alpha_{\phi}^{k}\right)\right)^{\frac{1}{k}}\right)^{\Omega_r}\right)\right),\\
\left(\prod_{r=1}^{s}\left(1-\left(1-\left(1-(1-\beta_{\phi})^{k}\right)\right)^{\frac{1}{k}}\right)^{\Omega_r}\right)\end{array}\right\rangle
\end{array}\right)\\
&=\left(\begin{array}{c}
\left\langle \begin{array}{c}\left(1-\left(\prod_{r=1}^{s}\left(1-\left(\mu_{\phi}^{k}\right)^{\frac{1}{k}}\right)^{\Omega_r}\right)\right),\\
\left(\prod_{r=1}^{s}\left(1-\left((1-v_{\phi})^{k}\right)^{\frac{1}{k}}\right)^{\Omega_r}\right)\end{array}\right\rangle,\\
\left\langle \begin{array}{c}\left(1-\left(\prod_{r=1}^{s}\left(1-\left(\alpha_{\phi}^{k}\right)^{\frac{1}{k}}\right)^{\Omega_r}\right)\right),\\
\left(\prod_{r=1}^{s}\left(1-\left((1-\beta_{\phi})^{k}\right)^{\frac{1}{k}}\right)^{\Omega_r}\right)\end{array}\right\rangle
\end{array}\right)
\end{aligned}
$$

Since $0\le\mu_{\phi},v_{\phi},\ \alpha_{\phi},\beta_{\phi}\le 1$, we have

$$\left(\mu_\phi^k\right)^{\frac{1}{k}} = \mu_\phi, \left(\left(1 - v_\phi\right)^k\right)^{\frac{1}{k}} = 1 - v_\phi, \left(\alpha_\phi^k\right)^{\frac{1}{k}} = \alpha_\phi, \left(\left(1 - \beta_\phi\right)^k\right)^{\frac{1}{k}} = 1 - \beta_\phi.$$

Thus,

$$LDFRWPMSM^{(k)}(\phi,\ \phi,\ \ldots,\ \phi) = \begin{pmatrix} \left\langle \begin{matrix} \left(1 - \left(\prod_{r=1}^{s}\left(1 - \mu_\phi\right)^{\Omega_r}\right)\right), \\ \left(\prod_{r=1}^{s}(v_\phi)^{\Omega_r}\right) \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} \left(1 - \left(\prod_{r=1}^{s}(1 - \alpha_\phi)^{\Omega_r}\right)\right), \\ \left(\prod_{r=1}^{s}(\beta_\phi)^{\Omega_r}\right) \end{matrix} \right\rangle \end{pmatrix}$$

Since $\sum_{r=1}^{s}\Omega_r = 1$, we obtain

$$LDFRWPMSM^{(k)}(\phi,\ \phi,\ \ldots,\ \phi) = \begin{pmatrix} \left\langle \begin{matrix} \left(1 - \left(\left(1 - \mu_\phi\right)^{\sum_{r=1}^{s}\Omega_r}\right)\right), \\ \left((v_\phi)^{\sum_{r=1}^{s}\Omega_r}\right) \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} \left(1 - \left((1 - \alpha_\phi)^{\sum_{r=1}^{s}\Omega_r}\right)\right), \\ \left((\beta_\phi)^{\sum_{r=1}^{s}\Omega_r}\right) \end{matrix} \right\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \begin{matrix} \left(1 - \left(1 - \mu_\phi\right)\right), \\ (v_\phi) \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} (1 - (1 - \alpha_\phi)), \\ (\beta_\phi) \end{matrix} \right\rangle \end{pmatrix} = (\langle \mu_\phi, v_\phi \rangle, \langle \alpha_\phi, \beta_\phi \rangle) = \phi$$

Hence,

$$LDFRWPMSM^{(k)}(\phi,\ \phi,\ \ldots,\ \phi) = \phi,$$

which completes the proof of the idempotency property of the LDFRWPMSM operator. □

Property 12 (Monotonicity). *Let $\phi_i = (\langle \mu_{\phi_i}, v_{\phi_i} \rangle, \langle \alpha_{\phi_i}, \beta_{\phi_i} \rangle)$ and $\phi_i' = \left(\langle \mu_{\phi_i}', v_{\phi_i}' \rangle, \langle \alpha_{\phi_i}', \beta_{\phi_i}' \rangle\right)$ be two collections of LDFNs. For a fixed partition structure P_r, $(i \in P_r)$, fixed k and weight vector $\omega_i = \omega_i'$, if $\mu_{\phi_i} \geq \mu_{\phi_i}'$, $v_{\phi_i} \leq v_{\phi_i}'$, $\alpha_{\phi_i} \geq \alpha_{\phi_i}'$ and $\beta_{\phi_i} \leq \beta_{\phi_i}'$ for all i, then*

$$LDFRWPMSM^{(k)}(\phi_1,\ \phi_2,\ \ldots,\ \phi_n) \geq LDFRWPMSM^{(k)}(\phi_1',\ \phi_2',\ \ldots,\ \phi_n')$$

Proof. The proof of Property 12 is provided in Supplementary Material. □

Property 13 (Boundedness). *Let $\phi_i = (\langle \mu_{\phi_i}, v_{\phi_i} \rangle, \langle \alpha_{\phi_i}, \beta_{\phi_i} \rangle)$, (i = 1, 2, …, n) be a collection of LDFNs and $1 \leq k \leq h_{min}$, and let*

$$\phi^- = \left(\langle \min_i \mu_{\phi_i}, \max_i v_{\phi_i} \rangle, \langle \min_i \alpha_{\phi_i}, \max_i \beta_{\phi_i} \rangle\right)$$

$$\phi^+ = \left(\langle \max_i \mu_{\phi_i}, \min_i v_{\phi_i} \rangle, \langle \max_i \alpha_{\phi_i}, \min_i \beta_{\phi_i} \rangle\right)$$

then

$$\phi^- \leq LDFRWPMSM^{(k)}(\phi_1,\ \phi_2,\ \ldots,\ \phi_n) \leq \phi^+$$

Proof. The proof is similar to that of Property 4 and Property 10 and is therefore omitted. □

Now, we can discuss some special cases and reducibility properties of the LDFRWPMSM operator. Since the weighting structure of LDFRWPMSM is inherited from the RWPMSM operator, the derivations of special cases and reducibility relations are similar to RWPMSM. Therefore, the derivations are omitted for brevity.

Case 1. *If k = 1, then the LDFRWPMSM operator reduces to the linear Diophantine fuzzy weighted averaging (LDFWA) operator in [55].*
Case 2. *If k = n and s = 1, then the LDFRWPMSM operator reduces to the LDFWG operator in [55].*

Remark 5 (Reducibility of LDFRWPMSM under equal weights to LDFPMSM)**.** *When the criteria have equal weights $\left(\omega_i = \frac{1}{n}\right)$ for all i, then LDFRWPMSM reduces to the LDFPMSM operator in Definition 13.*

Remark 6 (Reducibility of LDFRWPMSM under a single partition and equal weights to LDFMSM)**.** *Under a single partition (s = 1) and equally weighted criteria $\left(\omega_i = \frac{1}{n}\right)$ for all i, LDFRWPMSM reduces to LDFMSM in Definition 12.*

5. A Numerical Example for the Proposed Aggregation Operators

This section illustrates the proposed LDFMSM, LDFPMSM, and LDFRWPMSM operators through numerical examples representing unpartitioned, partitioned, and weighted-partitioned aggregation settings. In each example, the operator is selected according to the specified aggregation structure. For simplicity, all criteria are assumed to be benefit type; therefore, normalization is not required.

Example 1. *Suppose that the evaluations of an alternative with respect to four attributes C_1, C_2, C_3, C_4 are expressed by the following linear Diophantine fuzzy numbers: $\phi_c = \{(\langle 0.82,\ 0.12\rangle,\ \langle 0.60,\ 0.20\rangle),\ (\langle 0.71,\ 0.18\rangle,\ \langle 0.50,\ 0.25\rangle),\ (\langle 0.56,\ 0.32\rangle,\ \langle 0.42,\ 0.30\rangle),\ (\langle 0.38,\ 0.49\rangle,\ \langle 0.28,\ 0.36\rangle)\}$. All criteria are treated as equally important, and the criteria are assumed to exhibit interrelationships. Accordingly, all k-subsets are included in the symmetric joint aggregation, and the LDFMSM operator in Equation (16) is applied with k = 2.*

$$\begin{aligned}\phi_1 \otimes \phi_2 &= (\langle 0.82 \times 0.71,\ 0.12 + 0.18 - 0.12 \times 0.18\rangle, \langle 0.60 \times 0.50,\ 0.20 + 0.25 - 0.20 \times 0.25\rangle) \\ &= (\langle 0.582, 0.278\rangle, \langle 0.300, 0.400\rangle)\end{aligned}$$

$$\begin{aligned}\phi_1 \otimes \phi_3 &= (\langle 0.82 \times 0.56,\ 0.12 + 0.32 - 0.12 \times 0.32\rangle, \langle 0.60 \times 0.42,\ 0.20 + 0.30 - 0.20 \times 0.30\rangle) \\ &= (\langle 0.459, 0.402\rangle, \langle 0.252, 0.440\rangle)\end{aligned}$$

$$\begin{aligned}\phi_1 \otimes \phi_4 &= (\langle 0.82 \times 0.38,\ 0.12 + 0.49 - 0.12 \times 0.49\rangle, \langle 0.60 \times 0.28,\ 0.20 + 0.36 - 0.20 \times 0.36\rangle) \\ &= (\langle 0.312, 0.551\rangle, \langle 0.168, 0.488\rangle)\end{aligned}$$

$$\begin{aligned}\phi_2 \otimes \phi_3 &= (\langle 0.71 \times 0.56,\ 0.18 + 0.32 - 0.18 \times 0.32\rangle, \langle 0.50 \times 0.42,\ 0.25 + 0.30 - 0.25 \times 0.30\rangle) \\ &= (\langle 0.398, 0.442\rangle, \langle 0.210, 0.475\rangle)\end{aligned}$$

$$\begin{aligned}\phi_2 \otimes \phi_4 &= (\langle 0.71 \times 0.38,\ 0.18 + 0.49 - 0.18 \times 0.49\rangle, \langle 0.50 \times 0.28,\ 0.25 + 0.36 - 0.25 \times 0.36\rangle) \\ &= (\langle 0.270, 0.582\rangle, \langle 0.140, 0.520\rangle)\end{aligned}$$

$$\begin{aligned}\phi_3 \otimes \phi_4 &= (\langle 0.56 \times 0.38,\ 0.32 + 0.49 - 0.32 \times 0.49\rangle, \langle 0.42 \times 0.28,\ 0.30 + 0.36 - 0.30 \times 0.36\rangle) \\ &= (\langle 0.213, 0.653\rangle, \langle 0.118, 0.552\rangle)\end{aligned}$$

Using Equation (16), the aggregated result obtained by the LDFMSM operator is:

$$\mu^{agg} = \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le 4}\left(1 - \prod_{j=1}^{2} \mu_{\phi_{i_j}}\right)\right)^{\frac{1}{C_4^2}}\right)^{\frac{1}{2}}$$

$$= \Big(1 - \big((1 - (0.820 \times 0.710)) \times (1 - (0.820 \times 0.560)) \times (1 - (0.820 \times 0.380))$$
$$\times (1 - (0.710 \times 0.560)) \times (1 - (0.710 \times 0.380))$$
$$\times (1 - (0.560 \times 0.380))\big)^{\frac{1}{C_4^2}}\Big)^{\frac{1}{2}} = 0.621$$

$$v^{agg} = 1 - \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le 4}\left(1 - \prod_{j=1}^{2}\left(1 - v_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_4^2}}\right)^{\frac{1}{2}}$$

$$= 1 - \Big(1 - \big((1 - (1 - 0.120)(1 - 0.180)) \times (1 - (1 - 0.120)(1 - 0.320))$$
$$\times (1 - (1 - 0.120)(1 - 0.490)) \times (1 - (1 - 0.180)(1 - 0.320))$$
$$\times (1 - (1 - 0.180)(1 - 0.490)) \times (1 - (1 - 0.320)(1 - 0.490))\big)^{\frac{1}{C_4^2}}\Big)^{\frac{1}{2}}$$
$$= 0.270$$

and similarly calculated for α^{agg} and β^{agg}.

Thus,

$$LDFMSM^{(2)}(\phi_1, \phi_2, \ldots, \phi_4) = \left(\frac{\bigoplus_{1 \le i_1 < i_2 \le 4}(\phi_{i_1} \otimes \phi_{i_2})}{C_4^2}\right)^{1/2} = (\langle 0.621, 0.270\rangle, \langle 0.448, 0.277\rangle)$$

Example 2. *This example uses the same data set as Example 1. Here, the attributes are organized into two predefined partitions, $P_1 = \{C_1, C_2\}$ and $P_2 = \{C_3, C_4\}$, reflecting the assumption that C_1 and C_2 may be jointly considered, as may C_3 and C_4. The evaluations of the alternative are aggregated using the LDFPMSM operator in Equation (22) with k = 2, as follows:*

$$\phi_1 \otimes \phi_2 = (\langle 0.82 \times 0.71,\ 0.12 + 0.18 - 0.12 \times 0.18\rangle, \langle 0.60 \times 0.50,\ 0.20 + 0.25 - 0.20 \times 0.25\rangle)$$
$$= (\langle 0.582, 0.278\rangle, \langle 0.300, 0.400\rangle)$$

$$\phi_3 \otimes \phi_4 = (\langle 0.56 \times 0.38,\ 0.32 + 0.49 - 0.32 \times 0.49\rangle, \langle 0.42 \times 0.28,\ 0.30 + 0.36 - 0.30 \times 0.36\rangle)$$
$$= (\langle 0.213, 0.653\rangle, \langle 0.118, 0.552\rangle)$$

Using Equation (22), the aggregated result obtained by the LDFPMSM operator is ($h_1 = 2$, $h_2 = 2$):

$$\mu^{agg} = \left(1 - \left(\prod_{r=1}^{2}\left(1 - \left(1 - \left(\prod_{1\le i_1<\cdots<i_k\le h_r}\left(1 - \prod_{j=1}^{2}\mu_{\phi_{i_j}}\right)\right)^{\frac{1}{C_{h_r}^2}}\right)^{\frac{1}{2}}\right)\right)\right)^{\frac{1}{2}}$$

$$= \left(1 - \left(\left(1 - \left(1 - (1 - (0.820 \times 0.710))^{\frac{1}{C_2^2}}\right)^{\frac{1}{2}}\right) \times \left(1 - \left(1 - (1 - (0.560 \times 0.380))^{\frac{1}{C_2^2}}\right)^{\frac{1}{2}}\right)\right)\right)^{\frac{1}{2}} = 0.643$$

$$v^{agg} = \left(\left(\prod_{r=1}^{2}\left(1 - \left(1 - \left(\prod_{1\le i_1<\cdots<i_k\le h_r}\left(1 - \prod_{j=1}^{2}\left(1 - v_{\phi_{i_j}}\right)\right)\right)^{\frac{1}{C_{h_r}^2}}\right)^{\frac{1}{2}}\right)\right)\right)^{\frac{1}{2}}$$

$$= \left(\left(\left(1 - \left(1 - ((1 - (1 - 0.120)(1 - 0.180)))^{\frac{1}{C_2^2}}\right)^{\frac{1}{2}}\right) \times \left(1 - \left(1 - ((1 - (1 - 0.320)(1 - 0.490)))^{\frac{1}{C_2^2}}\right)^{\frac{1}{2}}\right)\right)\right)^{\frac{1}{2}} = 0.249$$

and similarly calculated for α^{agg} and β^{agg}.
Thus,

$$LDFPMSM^{(2)}(\phi_1, \phi_2, \ldots, \phi_4) = \frac{1}{2}\bigoplus_{r=1}^{2}\left(\frac{\bigoplus_{1\le i_1<\cdots<i_k\le h_r}\left(\bigotimes_{j=1}^{2}\phi_{i_j}\right)}{C_{h_r}^2}\right)^{1/2} = (\langle 0.643, 0.249\rangle, \langle 0.455, 0.273\rangle)$$

Example 3. *This example uses the same evaluations and partitioned structure as Example 2, with $P_1 = \{C_1, C_2\}$ and $P_2 = \{C_3, C_4\}$, and k = 2. However, the criteria are assigned unequal input weights given by $\omega_c = (0.4, 0.3, 0.2, 0.1)$ satisfying $\sum_{c=1}^{4}\omega_c = 1$. The evaluations are then aggregated using the LDFRWPMSM operator in Equation (25), as follows:*

The coefficients $\overline{\omega}_i^{(r)}$, $\theta_{i_j|i_1, i_2, \ldots, i_k}^{(r)}$, $\lambda_{i_1, i_2, \ldots, i_k}^{(r)}$ are given for each k-subset as in Table 2.

Table 2. Weighting coefficients of LDFRWPMSM for Example 3.

	$\overline{\omega}_i^{(r)}$		$\theta_{i_j\mid i_1,i_2,\ldots,i_k}^{(r)}$		$\lambda_{i_1,i_2,\ldots,i_k}^{(r)}$
$\overline{\omega}_1^{(1)}$	0.571	$\theta_{1\|(1,2)}^{(1)}$	1.143	$\lambda_{\{1,2\}}^{(1)}$	1
$\overline{\omega}_2^{(1)}$	0.429	$\theta_{2\|(1,2)}^{(1)}$	0.857	$\lambda_{\{3,4\}}^{(2)}$	1
$\overline{\omega}_3^{(2)}$	0.667	$\theta_{3\|(3,4)}^{(2)}$	1.333		
$\overline{\omega}_4^{(2)}$	0.333	$\theta_{4\|(3,4)}^{(2)}$	0.667		

https://doi.org/10.3390/sym18091477

As an example:

$$\overline{\omega}_1^{(1)} = \frac{\omega_1^{(1)}}{\Omega_1} = \frac{0.4}{(0.4+0.3)} = 0.571$$

$$\theta_{1|\{1,\,2\}}^{(1)} = \frac{k\overline{\omega}_{1_{(1,\,2)}}^{(1)}}{\sum_{l=1}^{k}\overline{\omega}_{1_l}^{(1)}} = \frac{2\times 0.571}{(0.571+0.429)} = 1.143$$

and since there is a single k-subset $\{1,\ 2\}$ in P_1:

$$\lambda_{\{1,2\}}^{(1)} = \frac{\prod_{j=1}^{k}\overline{\omega}_{i_j}^{(1)}}{\sum_{1\le l_1<l_2\le h_1}\prod_{j=1}^{k}\overline{\omega}_{l_j}^{(1)}} = \frac{0.571{\cdot}0.429}{0.571{\cdot}0.429} = 1$$

$$\begin{aligned}\mu^{agg} &= \left(1-\left(\prod_{r=1}^{2}\left(1-\left(1-\prod_{1\le i_1<\ldots<i_k\le h_r}\left(1-\prod_{j=1}^{2}\mu_{\phi_{i_j}}^{\theta_{i_j|i_1,i_2,\ldots,i_k}^{(r)}}\right)^{\lambda_{i_1,i_2,\ldots,i_k}^{(r)}}\right)^{\frac{1}{2}}\right)^{\Omega_r}\right)\right)\\ &= \left(1-\left(\left(1-\left(1-\left(1-0.820^{1.143}\times 0.710^{0.857}\right)^{1}\right)^{\frac{1}{2}}\right)^{0.700}\times\left(1-\left(1-\left(1-0.560^{1.333}\times 0.380^{0.667}\right)^{1}\right)^{\frac{1}{2}}\right)^{0.300}\right)\right)\\ &= 0.709\end{aligned}$$

$$\begin{aligned}v^{agg} &= \left(\prod_{r=1}^{2}\left(1-\left(1-\prod_{1\le i_1<\ldots<i_k\le h_r}\left(1-\prod_{j=1}^{2}\left(1-v_{\phi_{i_j}}\right)^{\theta_{i_j|i_1,i_2,\ldots,i_k}^{(r)}}\right)^{\lambda_{i_1,i_2,\ldots,i_k}^{(r)}}\right)^{\frac{1}{2}}\right)^{\Omega_r}\right)\\ &= \left(\left(1-\left(1-\prod_{1\le i_1<\cdots<i_k\le h_r}\left(1-(1-0.120)^{1.143}\times(1-0.180)^{0.857}\right)^{1}\right)^{\frac{1}{2}}\right)^{0.700}\right.\\ &\quad\left.\times\left(1-\left(1-\prod_{1\le i_1<\cdots<i_k\le h_r}\left(1-(1-0.320)^{1.133}\times(1-0.490)^{0.667}\right)^{1}\right)^{\frac{1}{2}}\right)^{0.300}\right) = 0.195\end{aligned}$$

and similarly calculated for α^{agg} and β^{agg}.

Thus,

$$LDFRWPMSM^{(2)}(\phi_1,\ \phi_2,\ \ldots,\ \phi_4) = \bigoplus_{r=1}^{2}\Omega_r\left(\bigoplus_{1\le i_1<\cdots<i_k\le h_r}\lambda_{i_1,\,i_2,\,\ldots,\,i_k}^{(r)}\left(\bigotimes_{j=1}^{k}\phi_{i_j}^{\theta_{i_j|i_1,\,i_2,\,\ldots,\,i_k}^{(r)}}\right)\right)^{1/2} = (\langle 0.709,\ 0.195\rangle, \langle 0.505,\ 0.248\rangle)$$

Table 3 compares the aggregated LDFNs obtained under three aggregation settings given by Numerical Examples 1–3. The differing results reflect the aggregation structure and weighting scheme specified for each setting.

Table 3. Comparison of the aggregated LDFNs obtained by the proposed AOs.

Aggregation Operator	Aggregation Setting	Aggregated LDFN
LDFMSM	Unpartitioned criterion set with equal importance weights; global symmetric k-subset aggregation	$(\langle 0.621, 0.270\rangle, \langle 0.448, 0.277\rangle)$
LDFPMSM	Predefined partitioned criteria with equal importance weights; within-partition symmetric k-subset aggregation	$(\langle 0.643, 0.249\rangle, \langle 0.455, 0.273\rangle)$
LDFRWPMSM	Predefined partitioned criteria with different importance weighting; within-partition symmetric k-subset aggregation	$(\langle 0.709,\ 0.195\rangle, \langle 0.505,\ 0.248\rangle)$

6. An Illustrative MADM Example: Selection of an AI-Driven Precision Agriculture Platform

6.1. Problem Definition

AI-driven precision agriculture platforms are digital decision-support systems that integrate IoT sensor technologies, remote-sensing data, farm-management information, and AI-enabled analytics to support precision farming, real-time monitoring, and data-driven decision-making [57]. These systems collect and process field-level information on soil, crop, weather, and environmental conditions to support applications such as irrigation management, fertilization, crop monitoring, disease detection, and yield forecasting [58]. By enabling site-specific management, these systems can support variable-rate input application, targeted interventions, and more efficient field operations. Such capabilities can improve resource-use efficiency, agricultural productivity, and environmental sustainability [59,60].

The selection of an AI-driven precision agriculture platform requires the joint consideration of economic, technical, agricultural, and environmental dimensions. This study evaluates alternative platforms using six criteria identified from the literature on AI-driven precision agriculture.

Implementation Cost (C_1)

Implementation cost refers to the initial investment required to adopt the platform, including sensors, communication infrastructure, software, automation hardware, and system-integration expenses. High initial investment costs for smart sensors, IoT infrastructure, and advanced analytics can constrain adoption, particularly for small-scale farms [58]. Economic conditions and the availability of technological infrastructure are also important factors affecting digital agriculture adoption [61]. This criterion therefore reflects the financial feasibility of deploying the platform.

Operational Cost (C_2)

Operational cost represents the recurring expenses associated with operating and maintaining the platform, including connectivity, cloud-based data services, sensor maintenance, calibration, power consumption, and technical support. IoT-based precision agriculture systems may require continued investment in sensor maintenance, calibration, power supply, connectivity, and data-management infrastructure [58]. Cost effectiveness, power requirements, and wireless communication are also identified as relevant barriers to the adoption of IoT-based precision agriculture practices [62]. This criterion evaluates the long-term economic viability of the platform.

Crop Yield Improvement Capability (C_3)

Crop-yield improvement capability reflects the platform's ability to support productivity-enhancing farm management decisions. Artificial intelligence of things (AIoT)-based systems can support site-specific irrigation, nutrient management, crop monitoring, and timely disease-management interventions through sensor networks and predictive analytics [63]. Digital agricultural technologies, including variable-rate technologies and farm-management information systems, have also been associated with yield improvements in specific agricultural applications [64]. This criterion evaluates the expected contribution of the platform to agricultural productivity rather than a fixed yield increase.

Data Analytics Accuracy (C_4)

Data analytics accuracy reflects the platform's ability to convert heterogeneous agricultural data into reliable predictions, classifications, and management recommendations. In AI-driven precision agriculture, analytical outputs inform decisions on irrigation scheduling, nutrient application, crop monitoring, and disease or stress management [57,63]. Therefore, inaccurate or unreliable analytical results may lead to inappropriate timing, targeting, or intensity of farm-management interventions. This criterion assesses not only the predictive or classification performance of the platform, but also the reliability of its analytical outputs under different field conditions. In particular, data quality, the representativeness of training data, model robustness, scalability, and real-time applicability affect whether analytical results can be translated into dependable on-farm decisions [65,66]. Accordingly, this criterion evaluates the extent to which the platform can provide reliable and actionable decision support for precision agriculture applications.

IoT Sensor and Equipment Integration (C_5)

IoT sensor and equipment integration reflects the platform's ability to connect and use data from heterogeneous sources, including soil sensors, weather stations, unmanned aerial vehicles (UAVs), satellite imagery, and farm-management systems [57]. Integrating such data sources can support real-time monitoring and decision-making, but may require compatible data formats, interoperability protocols, and data-fusion mechanisms [57]. Connectivity limitations in rural settings and the practical complexity of integrating multiple platforms can further constrain deployment [58]. This criterion evaluates the platform's ability to integrate diverse agricultural data sources and equipment effectively.

Environmental Sustainability Support (C_6)

Environmental sustainability support reflects the platform's ability to promote efficient resource use and reduce the environmental effects of farming operations. Digital agricultural technologies, particularly variable-rate technologies, have been associated with reductions in fertilizer, pesticide, and water use in specific crop-production settings [64]. However, environmental sustainability is not inherent to precision agriculture technologies, and the resulting benefits depend on the application and implementation context [67]. This criterion therefore evaluates the platform's potential to support environmentally sustainable farm management.

6.2. Numerical Example

Assume that the owner of a large-scale farm seeks to improve agricultural productivity while controlling implementation and operational costs and supporting environmentally sustainable farming practices. To support this objective, an AI-driven precision agriculture platform is selected for data-driven farm management. Let the set of alternatives be denoted by $\mathcal{H} = \{\mathcal{H}_1, \ldots, \mathcal{H}_5\}$, ($a = 1, \ldots, 5$). The alternatives are evaluated with respect to the criterion set $C = \{C_1, \ldots, C_6\}$, ($i = 1, \ldots, 6$). The criteria are defined as follows:

C_1 = implementation cost,
C_2 = operational cost,
C_3 = crop yield improvement capability,
C_4 = data analytics accuracy,
C_5 = IoT sensor and equipment integration,
C_6 = environmental sustainability support.

Among these criteria, C_1 and C_2 are cost criteria, whereas the remaining criteria are benefit criteria. The criteria are organized into two predefined partitions (s = 2) to reflect the interrelationships among the criteria. A single decision maker evaluates the alternatives using the LDFS linguistic scale presented in Table 4. Since the application is illustrative, the criterion weights are prespecified and are given by $\omega = (0.10,\ 0.12,\ 0.24,\ 0.24,\ 0.13,\ 0.17)^T$, ω_i denotes the importance weight of criterion C_i, (i = 1, . . ., 6), and $\sum_{i=1}^{6} \omega_i = 1$. The weight profile assigns the highest importance to crop yield improvement capability (C_3) and data analytics accuracy (C_4), followed by environmental sustainability support (C_6). The partitions are defined ex ante according to their conceptual role in the decision problem. Partition 1, $P_1 = \{C_1, C_2, C_3\}$, groups implementation cost, operational cost, and the crop-yield improvement capability to capture the economic-productivity trade-off, whereas Partition 2, $P_2 = \{C_4, C_5, C_6\}$, combines data analytic accuracy, IoT integration, and environmental sustainability support as interrelated dimensions of technology-enabled operational capabilities of the platform. The numerical calculations reported in Sections 6–8 were implemented using Microsoft Excel for Mac (v16.112.1) without intermediate rounding. For reporting purposes only, rounding was applied to the aggregated LDFNs and expectation score values.

Table 4. Linguistic LDF scale.

Linguistic Terms	LDFNs
Very High (VH)	(<0.95, 0.05>, <0.80, 0.20>)
High (H)	(<0.80, 0.20>, <0.70, 0.30>)
Medium High (MH)	(<0.65, 0.35>, <0.60, 0.40>)
Medium (M)	(<0.50, 0.50>, <0.50, 0.50>)
Medium Low (ML)	(<0.35, 0.65>, <0.40, 0.60>)
Low (L)	(<0.20, 0.80>, <0.30, 0.70>)
Very Low (VL)	(<0.05, 0.95>, <0.20, 0.80>)

The proposed MADM approach is as follows:

Step 1. Construct the LDF decision matrix. The decision maker evaluates each alternative under each criterion by using the linguistic LDF scale as shown in Table 5. The corresponding LDFNs are obtained, and the decision matrix is constructed as $\mathfrak{D} = [d_{ai}]_{5\times 6}$, ($a$ = 1, . . ., 5; i = 1, . . ., 6), where d_{ai} denotes the LDFN evaluation of alternative $\mathcal{H}_a$ with respect to criterion C_i. The obtained decision matrix is given in Table 6.

Table 5. Linguistic evaluations of each alternative with respect to criteria.

	C_1	C_2	C_3	C_4	C_5	C_6
$\mathcal{H}_1$	L	L	H	L	M	L
$\mathcal{H}_2$	M	L	L	H	L	H
$\mathcal{H}_3$	H	MH	MH	MH	ML	M
$\mathcal{H}_4$	MH	M	MH	M	MH	M
$\mathcal{H}_5$	H	M	L	MH	ML	H

Table 6. Linear Diophantine fuzzy decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6
$\mathcal{H}_1$	(<0.2, 0.8>, <0.3, 0.7>)	(<0.2, 0.8>, <0.3, 0.7>)	(<0.8, 0.2>, <0.7, 0.3>)	(<0.2, 0.8>, <0.3, 0.7>)	(<0.5, 0.5>, <0.5, 0.5>)	(<0.2, 0.8>, <0.3, 0.7>)
$\mathcal{H}_2$	(<0.5, 0.5>, <0.5, 0.5>)	(<0.2, 0.8>, <0.3, 0.7>)	(<0.2, 0.8>, <0.3, 0.7>)	(<0.8, 0.2>, <0.7, 0.3>)	(<0.2, 0.8>, <0.3, 0.7>)	(<0.8, 0.2>, <0.7, 0.3>)
$\mathcal{H}_3$	(<0.8, 0.2>, <0.7, 0.3>)	(<0.65, 0.35>, <0.6, 0.4>)	(<0.65, 0.35>, <0.6, 0.4>)	(<0.65, 0.35>, <0.6, 0.4>)	(<0.35, 0.65>, <0.4, 0.6>)	(<0.5, 0.5>, <0.5, 0.5>)
$\mathcal{H}_4$	(<0.65, 0.35>, <0.6, 0.4>)	(<0.5, 0.5>, <0.5, 0.5>)	(<0.65, 0.35>, <0.6, 0.4>)	(<0.5, 0.5>, <0.5, 0.5>)	(<0.65, 0.35>, <0.6, 0.4>)	(<0.5, 0.5>, <0.5, 0.5>)
$\mathcal{H}_5$	(<0.8, 0.2>, <0.7, 0.3>)	(<0.5, 0.5>, <0.5, 0.5>)	(<0.2, 0.8>, <0.3, 0.7>)	(<0.65, 0.35>, <0.6, 0.4>)	(<0.35, 0.65>, <0.4, 0.6>)	(<0.8, 0.2>, <0.7, 0.3>)

Step 2. Normalize the cost criteria. Since C_1 and C_2 are cost criteria, the corresponding LDFNs are normalized by using the complement operator defined in Definition 7. The normalized decision matrix is presented in Table 7.

Table 7. Normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6
$\mathcal{H}_1$	(<0.8, 0.2>, <0.7, 0.3>)	(<0.8, 0.2>, <0.7, 0.3>)	(<0.8, 0.2>, <0.7, 0.3>)	(<0.2, 0.8>, <0.3, 0.7>)	(<0.5, 0.5>, <0.5, 0.5>)	(<0.2, 0.8>, <0.3, 0.7>)
$\mathcal{H}_2$	(<0.5, 0.5>, <0.5, 0.5>)	(<0.8, 0.2>, <0.7, 0.3>)	(<0.2, 0.8>, <0.3, 0.7>)	(<0.8, 0.2>, <0.7, 0.3>)	(<0.2, 0.8>, <0.3, 0.7>)	(<0.8, 0.2>, <0.7, 0.3>)
$\mathcal{H}_3$	(<0.2, 0.8>, <0.3, 0.7>)	(<0.35, 0.65>, <0.4, 0.6>)	(<0.65, 0.35>, <0.6, 0.4>)	(<0.65, 0.35>, <0.6, 0.4>)	(<0.35, 0.65>, <0.4, 0.6>)	(<0.5, 0.5>, <0.5, 0.5>)
$\mathcal{H}_4$	(<0.35, 0.65>, <0.4, 0.6>)	(<0.5, 0.5>, <0.5, 0.5>)	(<0.65, 0.35>, <0.6, 0.4>)	(<0.5, 0.5>, <0.5, 0.5>)	(<0.65, 0.35>, <0.6, 0.4>)	(<0.5, 0.5>, <0.5, 0.5>)
$\mathcal{H}_5$	(<0.2, 0.8>, <0.3, 0.7>)	(<0.5, 0.5>, <0.5, 0.5>)	(<0.2, 0.8>, <0.3, 0.7>)	(<0.65, 0.35>, <0.6, 0.4>)	(<0.35, 0.65>, <0.4, 0.6>)	(<0.8, 0.2>, <0.7, 0.3>)

Step 3. Aggregate the normalized evaluations. The normalized LDFNs of each alternative are aggregated by using the proposed aggregation operators LDFMSM, LDFPMSM and LDFRWPMSM.

Step 3.1. Aggregation using the LDFMSM operator. The normalized LDFNs are aggregated by applying the LDFMSM operator defined in Equation (16) to obtain the aggregated assessment for each alternative. The number of criteria is $n = 6$, and the aggregation parameter is set as $k = 2$.

Step 3.2. Aggregation using the LDFPMSM operator. The normalized LDFNs are aggregated by applying the LDFPMSM operator defined in Equation (22) to obtain the aggregated assessment for each alternative. The criterion set is partitioned as $P_1 = \{C_1, C_2, C_3\}, P_2 = \{C_4, C_5, C_6\}$, with $h_1 = 3$, $h_2 = 3$ and $s = 2$. By setting $k = 2$, the aggregated LDFNs of the alternatives are obtained.

Step 3.3. Aggregation using the LDFRWPMSM operator. The normalized LDFNs are aggregated by applying the LDFRWPMSM operator defined in Equation (25) to obtain the aggregated assessment for each alternative. The same partition structure is used, and the normalized criterion weights are given by $w = (0.10,\ 0.12,\ 0.24,\ 0.24,\ 0.13,\ 0.17)^T$. The partition weights are calculated as $\Omega_1 = 0.46$ and $\Omega_2 = 0.54$. The aggregation parameter is set as $k = 2$.

The aggregated results obtained in Steps 3.1–3.3 are given in Table 8.

Table 8. Aggregated LDFNs obtained under the proposed aggregation operators.

	LDFMSM	LDFPMSM	LDFRWPMSM
$\mathcal{H}_1$	(<0.570, 0.430>, <0.539, 0.461>)	(<0.621, 0.379>, <0.562, 0.438>)	(<0.592, 0.408>, <0.540, 0.460>)
$\mathcal{H}_2$	(<0.570, 0.430>, <0.539, 0.461>)	(<0.548, 0.452>, <0.529, 0.471>)	(<0.562, 0.438>, <0.535, 0.465>)
$\mathcal{H}_3$	(<0.451, 0.549>, <0.467, 0.533>)	(<0.441, 0.559>, <0.463, 0.537>)	(<0.504, 0.496>, <0.506, 0.494>)
$\mathcal{H}_4$	(<0.526, 0.474>, <0.517, 0.483>)	(<0.523, 0.477>, <0.516, 0.484>)	(<0.540, 0.460>, <0.527, 0.473>)
$\mathcal{H}_5$	(<0.453, 0.547>, <0.468, 0.532>)	(<0.463, 0.537>, <0.472, 0.528>)	(<0.488, 0.512>, <0.487, 0.513>)

Step 4. Calculate expectation score values. The aggregated LDFNs obtained in Steps 3.1–3.3 are transformed into crisp values by using the expectation score function defined in Equation (4). The obtained values are shown in Table 9 for the three aggregation methods.

Table 9. Expectation score values of the aggregated LDFNs.

	LDFMSM	LDFPMSM	LDFRWPMSM
$\mathcal{H}_1$	0.5545	0.5919	0.5664
$\mathcal{H}_2$	0.5545	0.5382	0.5484
$\mathcal{H}_3$	0.4591	0.4520	0.5050
$\mathcal{H}_4$	0.5217	0.5190	0.5337
$\mathcal{H}_5$	0.4602	0.4673	0.4876

Step 5. Ranking of the alternatives. The alternatives are then ranked according to their score values in Step 4 in descending order. The corresponding ranking results are shown in Table 10.

Table 10. Ranking of alternatives obtained under the proposed aggregation operators.

	Ranking of Alternatives
LDFMSM	$\mathcal{H}_1 = \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_5 \succ \mathcal{H}_3$
LDFPMSM	$\mathcal{H}_1 \succ \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_5 \succ \mathcal{H}_3$
LDFRWPMSM	$\mathcal{H}_1 \succ \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_5$

Table 10 shows that the three proposed operators produce consistent rankings, with minor differences revealing the effects of partitioning and criterion weighting. All operators identify $\mathcal{H}_1$ as the best and rank $\mathcal{H}_4$ as the third-best alternative. The LDFMSM operator ranks $\mathcal{H}_1$ and $\mathcal{H}_2$ in the first place, and incorporating the partition structure by the LDFPMSM operator resolves this tie in favor of $\mathcal{H}_1$. This result can be explained by examining the normalized linguistic evaluations. Both alternatives contain the same set of evaluations consisting of three "H", one "M", and two "L" assessments; however, their distributions across the partitions differ: $\mathcal{H}_1 = (H, H, H|L, M, L)$ and $\mathcal{H}_2 = (M, H, L|H, L, H)$. Since the LDFMSM operator performs a symmetric aggregation over the entire criterion set, it is invariant to the permutations of the same collection of evaluations to criteria. Therefore, it assigns the same score to both alternatives. In contrast, the LDFPMSM operator restricts joint aggregation to criteria belonging to the same partition. Therefore, differences in how the same evaluation multiset is distributed across the predefined partitions can affect the aggregation result. The three "H" values in the first partition of $\mathcal{H}_1$ create a strong within-partition joint contribution, which results in a higher aggregated score for $\mathcal{H}_1$ than for $\mathcal{H}_2$.

Further, incorporating the criteria weights through the LDFRWPMSM operator preserves the relative order of the best three alternatives while reversing the order of the fourth and fifth alternatives. Under LDFPMSM, $\mathcal{H}_5$ slightly outranks $\mathcal{H}_3$ with scores of 0.4673 and 0.4520, respectively, whereas under LDFRWPMSM, $\mathcal{H}_3$ outranks $\mathcal{H}_5$ with scores of 0.5050 and 0.4876, respectively. After the normalization, the two alternatives differ only on C_2, C_3 and C_6. In particular, $\mathcal{H}_3$ substantially outperforms $\mathcal{H}_5$ on the criterion C_3 which has a relatively high weight of $w_3 = 0.24$ (MH vs. L). By comparison, $\mathcal{H}_5$ outperforms $\mathcal{H}_3$ on C_2 and C_6, which have lower weights of $w_2 = 0.12$ and $w_6 = 0.17$, respectively. Since C_3 has the highest weight among these criteria and receives a stronger contribution within the weighted subsets of P_1, the advantage of $\mathcal{H}_3$ on C_3 is amplified sufficiently to offset the advantages of $\mathcal{H}_5$ on C_2 and C_6, resulting in the observed change in ranking orders.

7. Sensitivity Analysis

7.1. Sensitivity Analysis with Respect to Criteria Weights

A one-at-a-time (OAT) sensitivity analysis was conducted for the LDFRWPMSM operator to examine the effect of changes in criterion importance on the alternative rankings. Each criterion weight was varied independently from −40% to +40% of its baseline value in 5% increments. For each perturbation, the remaining criterion weights were proportionally rescaled relative to their baseline values so that their mutual weight ratios were preserved and the normalization condition remained satisfied. When a change in the ranking orders occurred, the corresponding crossover threshold was approximated by linear interpolation. As shown in Figure 4, the ranking remained unchanged over the observed weight range for C_1 and C_2, indicating that the decision outcome is relatively stable with respect to variations in these two criteria.

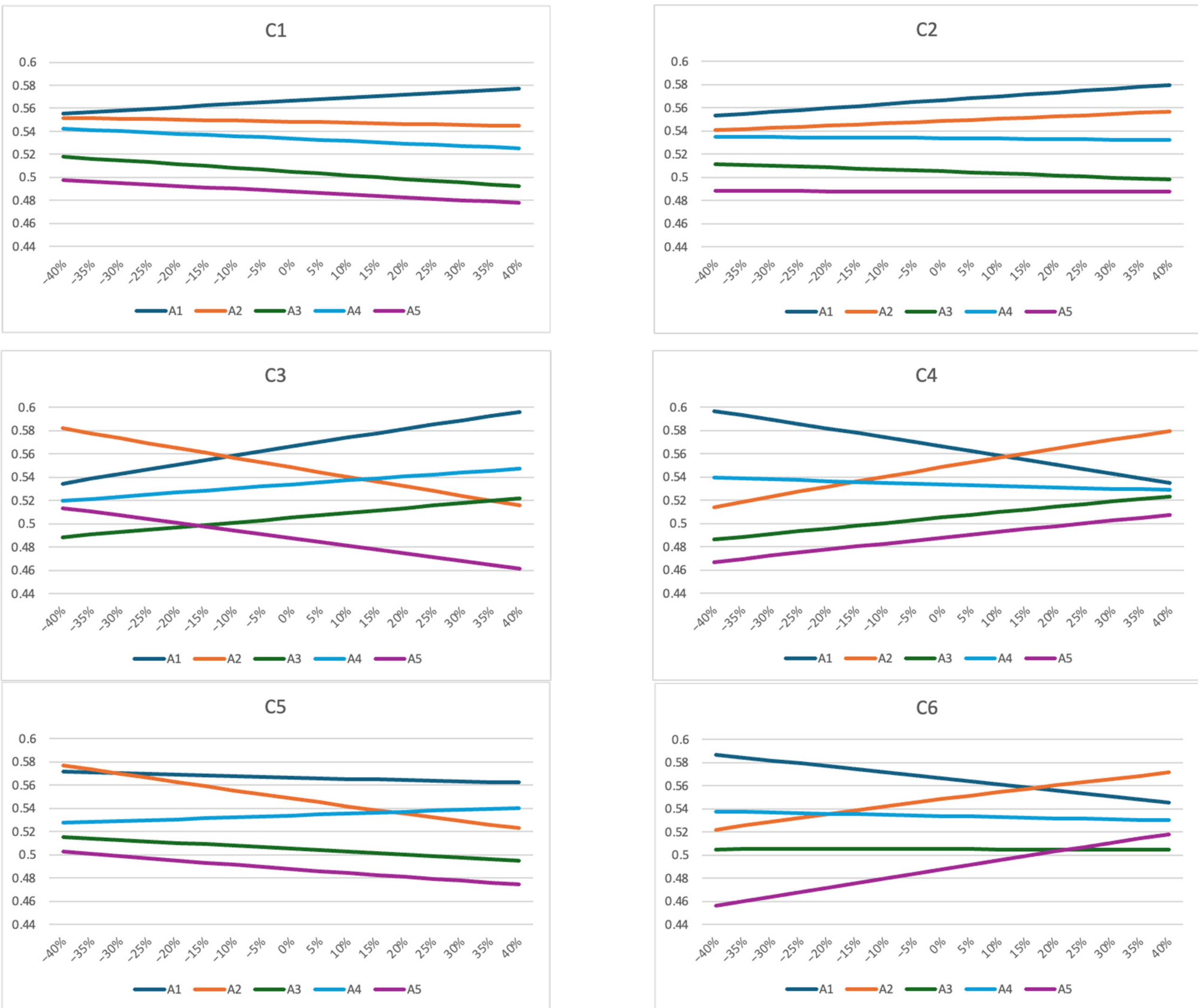

Figure 4. Alternative scores under one-at-a-time analysis of each criterion.

In contrast, changes in C_3, C_4, C_5, and C_6 resulted in observable ranking shifts. In particular, the best alternative changed at approximate weight variations of −11.2%, +11.3%, −30.7%, +16.2% for C_3, C_4, C_5 and C_6 respectively. Additional changes in the relative order of lower-ranked alternatives occurred under the variation of criteria C_3 and C_6. Overall, although the variations in C_3, C_4, C_5, and C_6 produced ranking changes, the best alternative was restricted to $\mathcal{H}_1$ and $\mathcal{H}_2$ throughout the entire variation range. Overall, the analysis indicates localized sensitivity to criterion weights; variations in four of the six criterion weights produced ranking changes, while the leading position remained confined to two alternatives.

7.2. Sensitivity Analysis with Respect to Partition Structure

To examine the effect of partition composition on the alternative rankings, all possible equal-cardinality partition configurations were considered. With six criteria divided into two partitions, a total of ten unique bipartitions can be formed. To isolate the effect of partition composition, the LDFPMSM operator was applied while the partition sizes and interaction parameter k were held fixed, and the results are presented in Table 11. Across ten partition configurations, six distinct ranking profiles were obtained. In five configurations, $\mathcal{H}_1$ and $\mathcal{H}_2$ were tied as the best alternatives, $\mathcal{H}_1$ was uniquely ranked first in two configurations, and $\mathcal{H}_2$ in three configurations. $\mathcal{H}_4$ was consistently ranked third across all configurations, while the relative order of the two lowest-ranked alternatives, $\mathcal{H}_3$ and $\mathcal{H}_5$, varied across the configurations. These results demonstrate that partition composition can affect the ranking result. At the same time, the occurrence of similar ranking profiles indicates that the overall decision remains relatively stable under changes in partition composition.

Table 11. Ranking of alternatives under different partition compositions.

Partition Composition	Ranking of Alternatives
$P^{(0)} = \{\{C_1, C_2, C_3\}, \{C_4, C_5, C_6\}\}$	$\mathcal{H}_1 \succ \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_5 \succ \mathcal{H}_3$
$P^{(1)} = \{\{C_1, C_2, C_4\}, \{C_3, C_5, C_6\}\}$	$\mathcal{H}_2 \succ \mathcal{H}_1 \succ \mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_5$
$P^{(2)} = \{\{C_1, C_2, C_5\}, \{C_3, C_4, C_6\}\}$	$\mathcal{H}_1 \succ \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_5$
$P^{(3)} = \{\{C_1, C_2, C_6\}, \{C_3, C_4, C_5\}\}$	$\mathcal{H}_2 \succ \mathcal{H}_1 \succ \mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_5$
$P^{(4)} = \{\{C_1, C_3, C_4\}, \{C_2, C_5, C_6\}\}$	$\mathcal{H}_1 = \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_5$
$P^{(5)} = \{\{C_1, C_3, C_5\}, \{C_2, C_4, C_6\}\}$	$\mathcal{H}_2 \succ \mathcal{H}_1 \succ \mathcal{H}_4 \succ \mathcal{H}_5 \succ \mathcal{H}_3$
$P^{(6)} = \{\{C_1, C_3, C_6\}, \{C_2, C_4, C_5\}\}$	$\mathcal{H}_1 = \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_5$
$P^{(7)} = \{\{C_1, C_4, C_5\}, \{C_2, C_3, C_6\}\}$	$\mathcal{H}_1 = \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_5$
$P^{(8)} = \{\{C_1, C_4, C_6\}, \{C_2, C_3, C_5\}\}$	$\mathcal{H}_1 = \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_5 \succ \mathcal{H}_3$
$P^{(9)} = \{\{C_1, C_5, C_6\}, \{C_2, C_3, C_4\}\}$	$\mathcal{H}_1 = \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_5$

Partitioned aggregation operators may employ different weighting approaches for assigning importance to partitions. For example, the PMSM operator in Definition 10 assigns equal weights to each partition. However, when partition cardinalities are unequal, this may create an imbalance in the partition-level contribution associated with criteria from different partitions, since a criterion in a smaller partition receives a greater implicit weight than one in a larger partition. To address this issue, the proposed LDFPMSM operator in Definition 13 assigns partition weights in proportion to their cardinalities.

To examine the effect of partition weighting on the decision outcome, an unequal-cardinality partition structure was considered. Using the LDFPMSM operator with $k = 2$, the aggregation was performed under two weighting schemes: equal weighting of partitions, $\left(\frac{1}{s}\right)$, and cardinality-proportional weighting, $\left(\frac{h_r}{n}\right)$, adopted in the proposed LDFPMSM operator. For this analysis, the partition structure was specified as $P = \{\{C_1, C_2\}, \{C_3, C_4, C_5, C_6\}\}$

with $h_1 = 2$, $h_2 = 4$, $n = 6$ and $s = 2$. Under equal weighting, a criterion in the first partition receives an implicit weight of 0.25 $\left(\frac{0.5}{2}\right)$, whereas a criterion in the second partition receives 0.125 $\left(\frac{0.5}{4}\right)$. The same partitioned aggregation was then evaluated under both partition-weighting schemes. The expectation score vector for equal weighting is obtained as: $\boldsymbol{\mathcal{S}}^E = (0.62432,\ 0.56028,\ 0.42963,\ 0.50201,\ 0.42962)$, whereas the corresponding vector under cardinality-proportional weighting as $\boldsymbol{\mathcal{S}}^S = (0.56842,\ 0.54134,\ 0.46571,\ 0.52308,\ 0.45335)$. As expected, the cardinality-proportional weighting reduces the relative contribution of the smaller partition and increases that of the larger partition compared with equal partition weighting. Accordingly, alternatives whose stronger evaluations are concentrated in the smaller partition lose part of the advantage created by equal partition weighting. Consequently, the expectation scores of $\mathcal{H}_1$ and $\mathcal{H}_2$ decrease by approximately 8.95% and 3.38%, respectively, whereas those of $\mathcal{H}_3$, $\mathcal{H}_4$ and $\mathcal{H}_5$ increase. These results illustrate that equal partition weighting can affect the ranking of the alternatives when partition cardinalities are unequal, whereas cardinality-proportional weighting avoids the implicit over-weighting of criteria belonging to smaller partitions.

7.3. Sensitivity Analysis with Respect to the Interaction Parameter k

To investigate the influence of the interaction parameter k on the aggregation results, a sensitivity analysis was conducted by considering different values of k (i.e., $k = 1, 2, 3$), which satisfy the admissible range since $h_{min} = 3$ for the considered partition structure. The expectation score values of the alternatives obtained under the proposed LDFMSM, LDFPMSM, and LDFRWPMSM operators are presented in Supplementary Material and illustrated in Figure 5.

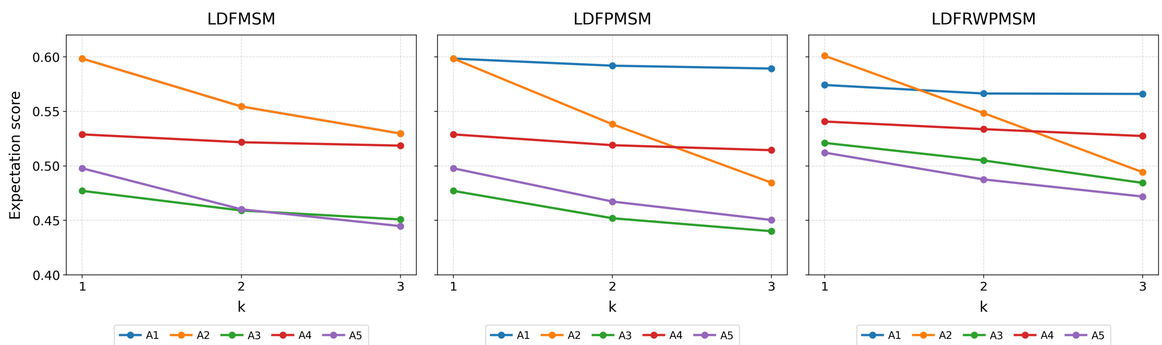

Figure 5. Expectation score values of alternatives for different values of k under the proposed operators.

For the present numerical example, the expectation score values obtained by LDFMSM, LDFPMSM and LDFRWPMSM operators decrease for all alternatives as the interaction parameter k varies from 1 to 3. This decreasing pattern is an empirical observation for the considered numerical instance.

However, the magnitude of the decrease as k increases differs across alternatives. This behavior can be explained by the composition of evaluations within the predefined partitions. Although $\mathcal{H}_1$ and $\mathcal{H}_2$ contain the same set of normalized evaluations, these evaluations are distributed differently across the two partitions. For $\mathcal{H}_2$, the normalized evaluations are ((M, H, L), (H, L, H)), indicating a relatively dispersed performance pattern, in which high and low evaluations occur within the same partition. In contrast, $\mathcal{H}_1$ has the normalized evaluation set ((H, H, H), (L, M, L)), with a more coherent and stronger performance profile in the first partition. As k increases, the higher-order joint aggregation increasingly combines the evaluations within each partition jointly. In the present example, the dispersed configuration containing both high and low evaluations within the same partition is associated with a larger decrease in the expectation score as k increases. This larger decrease is sufficient to alter the relative ranking of $\mathcal{H}_2$ under the LDFPMSM and LDFRWPMSM operators, whereas $\mathcal{H}_1$, whose high evaluations are more concentrated within the first partition, exhibits a smaller decline. These results indicate that, with the other conditions fixed, the effect of k depends on the composition of the evaluation sets within the predefined partitions, where increasing k gives higher emphasis to higher-order joint performance and therefore affects alternatives differently depending on their performance configuration.

8. Comparative Analysis

The comparative behavior of the proposed operators is examined using selected aggregation operators developed under LDFSs and their extensions. The same normalized decision matrix is processed using each operator, and the expectation score function in Equation (4) is used to obtain crisp scores for the alternatives. The resulting scores and ranking orders are presented in Table 12. The comparison is limited to operators that can be implemented using the available normalized LDFN decision matrix and, where applicable, the specified criterion weights. Choquet-integral operators are not included because they require a fuzzy measure over criterion coalitions, which introduces additional preference information beyond the decision matrix and criterion weights. Bonferroni and Muirhead operators reviewed in this study are defined under different Diophantine fuzzy representations and are therefore not directly comparable with the single-valued LDFN setting considered in this paper.

Table 12. Expectation score values and ranking results obtained by comparative and proposed AOs.

AOs	$\mathcal{S}(\mathcal{H}_1)$	$\mathcal{S}(\mathcal{H}_2)$	$\mathcal{S}(\mathcal{H}_3)$	$\mathcal{S}(\mathcal{H}_4)$	$\mathcal{S}(\mathcal{H}_5)$	Ranking of Alternatives
LDF AA Average ($\rho = 2$) [27]	0.6472	0.6472	0.5042	0.5391	0.5503	$\mathcal{H}_1 = \mathcal{H}_2 \succ \mathcal{H}_5 \succ \mathcal{H}_4 \succ \mathcal{H}_3$
LDF AA Geometric ($\rho = 2$) [27]	0.4134	0.4134	0.4076	0.5015	0.3780	$\mathcal{H}_4 \succ \mathcal{H}_1 = \mathcal{H}_2 \succ \mathcal{H}_3 \succ \mathcal{H}_5$
LDF Einstein Average [26]	0.5833	0.5833	0.4703	0.5264	0.4845	$\mathcal{H}_1 = \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_5 \succ \mathcal{H}_3$
LDF Einstein Geometric [26]	0.5016	0.5016	0.4410	0.5157	0.4287	$\mathcal{H}_4 \succ \mathcal{H}_1 = \mathcal{H}_2 \succ \mathcal{H}_3 \succ \mathcal{H}_5$
q-LDF SS Power Average $\rho = -5$, $q = 1$ [33]	0.7161	0.7161	0.5513	0.5639	0.6516	$\mathcal{H}_1 = \mathcal{H}_2 \succ \mathcal{H}_5 \succ \mathcal{H}_4 \succ \mathcal{H}_3$
q-LDF SS Power Geometric $\rho = -5$, $q = 1$ [33]	0.3135	0.3135	0.3372	0.4737	0.3042	$\mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_1 = \mathcal{H}_2 \succ \mathcal{H}_5$
LDFMSM ($k = 2$) (proposed)	0.5545	0.5545	0.4591	0.5217	0.4602	$\mathcal{H}_1 = \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_5 \succ \mathcal{H}_3$
LDFPMSM ($k = 2$) (proposed)	0.5919	0.5382	0.452	0.519	0.4673	$\mathcal{H}_1 \succ \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_5 \succ \mathcal{H}_3$
LDFAAWA ($\rho = 2$) [27]	0.6317	0.6519	0.5422	0.5492	0.5629	$\mathcal{H}_2 \succ \mathcal{H}_1 \succ \mathcal{H}_5 \succ \mathcal{H}_4 \succ \mathcal{H}_3$
LDFAAWG ($\rho = 2$) [27]	0.3859	0.404	0.4522	0.5173	0.3834	$\mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_2 \succ \mathcal{H}_1 \succ \mathcal{H}_5$
LDFEWA [26]	0.5567	0.5848	0.5154	0.5386	0.4987	$\mathcal{H}_2 \succ \mathcal{H}_1 \succ \mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_5$
LDFEWG [26]	0.4687	0.4963	0.4886	0.5297	0.4396	$\mathcal{H}_4 \succ \mathcal{H}_2 \succ \mathcal{H}_3 \succ \mathcal{H}_1 \succ \mathcal{H}_5$
q-LDFSSPWA ($\rho = -5$, $q = 1$) [33]	0.7118	0.7192	0.5764	0.5703	0.6560	$\mathcal{H}_2 \succ \mathcal{H}_1 \succ \mathcal{H}_5 \succ \mathcal{H}_3 \succ \mathcal{H}_4$
q-LDFSSPWG ($\rho = -5$, $q = 1$) [33]	0.3011	0.3076	0.3655	0.4924	0.3039	$\mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_2 \succ \mathcal{H}_5 \succ \mathcal{H}_1$
LDFRWPMSM ($k = 2$) (proposed)	0.5664	0.5484	0.505	0.5337	0.4876	$\mathcal{H}_1 \succ \mathcal{H}_2 \succ \mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_5$

The first comparison considers operators that do not incorporate criteria weights. LDFMSM and LDFPMSM are compared with the LDF Aczel–Alsina (AA) average and geometric operators, LDF Einstein average and geometric operators, and q-rung LDF Schweizer–Sklar (SS) power average and geometric operators, and the ranking results are illustrated in Figure 6. The results reveal a clear distinction among the unweighted aggregation operators and the proposed LDFPMSM operator. All selected unweighted benchmark operators and the proposed LDFMSM operator assign identical scores to $\mathcal{H}_1$

and $\mathcal{H}_2$. This follows from the fact that the two alternatives contain the same overall set of normalized evaluations, although the same evaluations are distributed differently across the predefined partitions. Since these operators aggregate the complete criterion set symmetrically without incorporating criterion weights or partition structure, they are invariant to the distribution of evaluations and therefore cannot distinguish between these two alternatives. In contrast, LDFPMSM restricts the joint aggregation to the predefined partitions and therefore makes the distribution of evaluations across partitions decision-relevant and consequently ranks $\mathcal{H}_1$ above $\mathcal{H}_2$.

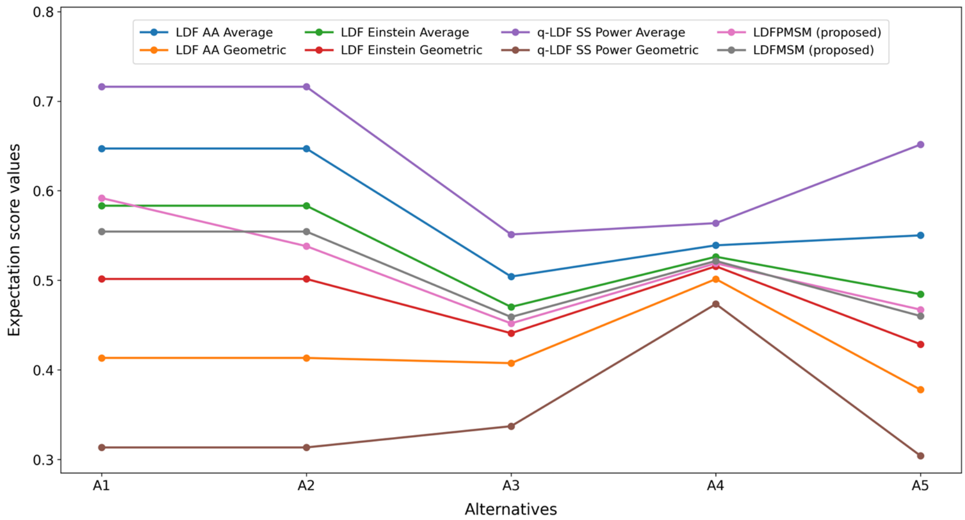

Figure 6. Comparison of expectation score values of unweighted aggregation operators.

A second pattern concerns the distinction between the averaging- and geometric-type aggregation operators. The unweighted LDF AA average and LDF Einstein average operators both rank $\mathcal{H}_1$ and $\mathcal{H}_2$ as the first alternatives, whereas their geometric counterparts evaluated $\mathcal{H}_4$, with the normalized evaluation ((ML, M, MH), (M, MH, M)), as the best alternative. This difference is consistent with a greater sensitivity of geometric-type aggregation to low evaluations, whereas averaging-type aggregations allow stronger evaluations to compensate more for weak evaluations. Geometric-type aggregations reduce such compensation and favor the alternatives with a more balanced performance.

The second comparison considers operators that incorporate criteria weights. The proposed LDFRWPMSM operator is compared with the LDFAAWA, LDFAAWG, LDFEWA, LDFEWG, q-LDFSSPWA and q-LDFSSPWG operators, and the ranking results are illustrated in Figure 7. The results reveal a clearer differentiation among averaging and geometric type operators. All the weighted averaging operators identify $\mathcal{H}_2$ as the best alternative, which can be attributed to the compensatory nature for low evaluations of averaging aggregation. Under every considered geometric type operator, $\mathcal{H}_4$ is selected as the best alternative, reflecting the greater sensitivity to low evaluations; therefore, they favor the alternative with a more balanced performance profile. In contrast to these operators, the proposed LDFRWPMSM operator identifies $\mathcal{H}_1$ as the best alternative, followed by $\mathcal{H}_2$ and $\mathcal{H}_4$. This difference arises from the aggregation structure of LDFRWPMSM, which simultaneously incorporates criterion importance, the predefined partition structure, and higher-order joint interactions among criteria within each partition. Consequently, its ranking cannot be interpreted solely in terms of the compensatory behavior that characterizes averaging and geometric type operators.

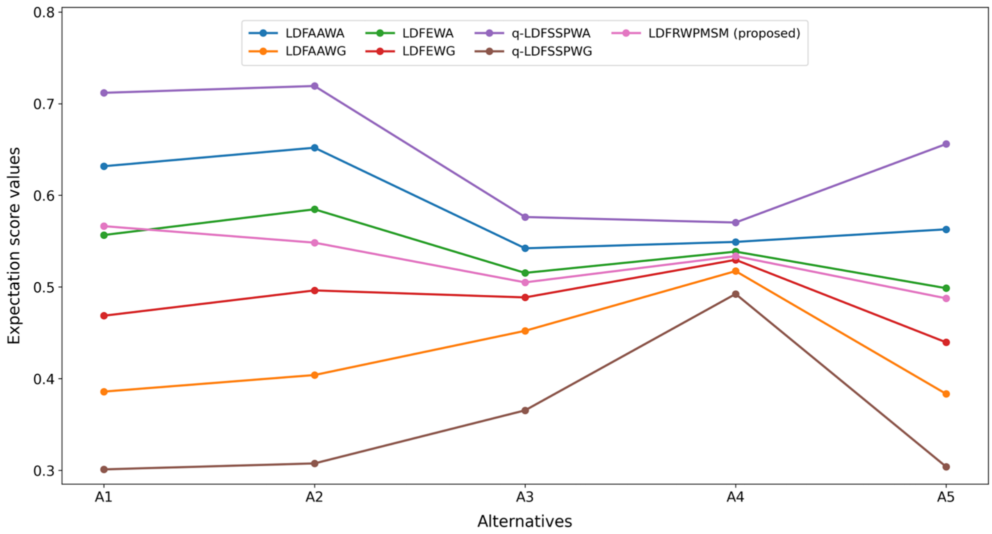

Figure 7. Comparison of expectation score values of weighted aggregation operators.

The relative ranking of $\mathcal{H}_3$, $\mathcal{H}_4$, and $\mathcal{H}_5$ further illustrates the differences among the aggregation operators. The weighted averaging operators do not produce a common ordering for these three alternatives: LDFAAWA, LDFEWA and q-LDFSSPWA give $\mathcal{H}_5 \succ \mathcal{H}_4 \succ \mathcal{H}_3$, $\mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_5$, and $\mathcal{H}_5 \succ \mathcal{H}_3 \succ \mathcal{H}_4$, respectively. In contrast, all the weighted geometric operators produce the common order of $\mathcal{H}_4 \succ \mathcal{H}_3 \succ \mathcal{H}_5$, which reflects their preference for more balanced profiles. LDFRWPMSM operator also yields the same order, but through a different aggregation mechanism: $\mathcal{H}_4$ has a relatively balanced evaluation configuration within both partitions, which strengthens its contribution through k-subset interaction effect. The outperformance of $\mathcal{H}_3$ over $\mathcal{H}_5$ is supported by its better performance on C_3, whose contribution is incorporated through the weighted within-partition interactions.

9. Discussion

Beyond precision agriculture, the proposed framework may be useful in technology-selection and investment decisions where criteria can be organized into conceptually distinct groups. Examples include smart-grid and energy-storage technology selection, healthcare IT and cybersecurity vendor assessment, industrial equipment selection, and sustainable supplier or digital-platform evaluation. In such problems, cost and lifecycle considerations, technical performance and interoperability requirements, and sustainability, risk, or compliance criteria may be assessed within distinct groups.

In this study, we propose three LDF aggregation operators that are suitable for different decision settings. LDFMSM is appropriate when the criteria are equally important and their interactions can be considered over the complete criterion set. It captures k-order joint effects through symmetric-subset aggregation. However, this symmetric aggregation makes the operator insensitive to the criterion-wise allocation of an identical set of evaluations. Accordingly, alternatives with the same evaluation multiset may receive identical scores, as observed in the comparative analysis for LDFMSM and all selected unweighted benchmark operators. The LDFPMSM operator addresses this limitation by restricting the interaction terms to predefined criterion groups, thereby allowing differences in the allocation of evaluations across partitions to affect the aggregation result while retaining symmetry within each partition. In the sensitivity analysis with respect to partition structure, the analysis of ten admissible equal-cardinality bipartitions yielded six different ranking profiles, indicating that partition structure can affect the decision outcome. To construct the weighted partitioned MSM operator for LDFSs, the RWPMSM operator was first introduced as a preliminary operator and subsequently extended to the LDFS environment as the LDFRWPMSM operator. The LDFRWPMSM operator is suitable for the decision problems in which both the partition structure and unequal criterion importance are relevant. It further incorporates criterion weights while preserving the symmetric k-order within-partition structure. All three proposed LDF aggregation operators, LDFMSM, LDFPMSM and LDFRWPMSM, satisfy both idempotency and reducibility properties, which ensure consistency across the proposed operator framework. The LDFPMSM reduces to LDFMSM under a single partition, while the LDFRWPMSM operator reduces to LDFPMSM under equal criterion weights and, under a single partition with equal criterion weights, it further reduces to the LDFMSM operator. Since the OAT sensitivity analysis showed that variations in four of the six criterion weights can alter the ranking, weight sensitivity should be assessed when the specification of criterion weights is uncertain.

Computational burden is an important consideration in interaction-aware aggregation operators because multiple combinations of input arguments may need to be evaluated jointly. Therefore, accelerated computation has been investigated for Bonferroni mean aggregation [68]. Chen et al. [69] reported that the computational burden increases significantly for MSM- and Muirhead mean-based operators, particularly as the number of aggregated inputs increases. For the LDFMSM the computationally heavy step is the evaluation of all admissible k-subsets, and the exact number of joint-subset evaluations is $\begin{pmatrix} n \\ k \end{pmatrix}$. For LDFMSM operation, the time complexity with respect to the number of criteria n for fixed k is $O\left(n^k\right)$. When $k \geq 2$, partitioning the criteria set reduces the number of joint evaluations for a given set of n criteria. For LDFPMSM and LDFRWPMSM operations, the number of subset evaluations becomes $\sum_{r=1}^{s} \begin{pmatrix} h_r \\ k \end{pmatrix}$, and the corresponding complexity is $O\left(\sum_{r=1}^{s} {h_r}^k\right)$. LDFRWPMSM operator evaluates the same number of joint subsets, but additional weighting operations increase the constant computational cost without changing the asymptotic order.

10. Conclusions

This study developed an MSM-based aggregation framework for linear Diophantine fuzzy sets. The LDFMSM operator was proposed for global symmetric k-subset aggregation as a baseline formulation for problems in which all criteria are considered under a common aggregation structure and have equal importance, and LDFPMSM extended this setting by incorporating predefined criterion partitions. To incorporate criterion importance, the RWPMSM operator was first formulated and subsequently extended to the LDFS as the LDFRWPMSM operator. The proposed operators satisfy idempotency, monotonicity and boundedness properties and provide reducibility among unweighted, partitioned and weighted-partitioned aggregation settings.

The sensitivity analyses demonstrated that criterion weights, partition structure and interaction order k can change the ranking results. In particular, holding all other conditions fixed, the effect of interaction order k depends on how the evaluations are distributed within the predefined partitions. The ten admissible equal-cardinality bipartition configurations produced six different ranking profiles, while variations in four of the six criterion weights produced ranking changes in the OAT analysis, including changes in the leading alternative. The comparative analysis further showed that LDFPMSM can differentiate alternatives that are assigned identical scores by LDFMSM and all selected unweighted benchmark operators employing global symmetric aggregation, while LDFRWPMSM produces different ranking behavior from the weighted benchmark operators by simultaneously considering criterion importance, partition structure and within-partition interactions.

The proposed framework has some limitations. Criterion partitions and weights are specified externally, and, as the sensitivity analyses demonstrated, their specification can have a significant effect on the aggregation result. In addition, the use of the proposed MSM-based operators may become computationally demanding as the number of criteria increases. Future research may investigate expert-supported and data-driven procedures for constructing criterion partitions from influence, similarity, or preference information. Such procedures may combine relationship-elicitation methods with clustering or network-based partitioning techniques. The proposed framework may also be extended to other fuzzy sets, group decision-making settings, and large-scale decision problems. For large-scale applications, the computational burden may be alleviated by parallel evaluation of independent k-subset terms and by precomputing recurring weighting coefficients.

Supplementary Materials: The following supporting information can be downloaded at https://www.mdpi.com/article/10.3390/sym18091477/s1. Algorithm S1: Exhaustive finite-grid counterexample search for parameter monotonicity of LDFMSM; Algorithm S2: Exhaustive finite-grid counterexample search for parameter monotonicity of LDFPMSM; Table S1: Expectation score values obtained by proposed aggregation operators under different values of interaction parameter k.

Author Contributions: Conceptualization, A.Y.G. and C.K.; methodology, A.Y.G.; software, A.Y.G.; validation, A.Y.G.; formal analysis, A.Y.G.; investigation, A.Y.G.; writing—original draft preparation, A.Y.G.; writing—review and editing, A.Y.G. and C.K.; visualization, A.Y.G.; supervision, C.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article/Supplementary Material. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

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