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Computation of H-Basis and Syzygies via QR Decomposition

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Abstract

H-bases provide an attractive alternative to Gröbner bases for the study of polynomial ideals because they are independent of monomial orderings and therefore preserve structural properties that may be obscured in Gröbner basis computations. However, the practical use of H-bases has been limited by the lack of efficient algorithms for their construction, particularly due to the difficulty of computing syzygy modules and eliminating redundant generators. In this paper, we present a new algorithm for computing syzygies and H-bases based on numerical linear algebraic techniques. The proposed approach utilizes QR decomposition to construct syzygies directly from coefficient matrices, avoiding the need for monomial orderings and Gröbner basis computations. A recursive framework derived from QR factorization is developed to compute both syzygy modules and H-bases while simultaneously defining an associated reduction process. In contrast to existing methods based on Gröbner bases, Schreyer's theorem, or singular value decomposition, the proposed algorithm requires fewer matrix reductions and relies on computationally less demanding operations. Furthermore, its termination criterion is determined by an upper bound independent of Gröbner basis theory. Consequently, the method provides a more streamlined and conceptually simpler framework for the computation of syzygies and H-bases of polynomial ideals.

Keywords: H-basis; Gröbner bases; syzygy

1. Introduction

The theory of Gröbner bases, first introduced by Buchberger, has established itself as a cornerstone of computational algebra with extensive applications spanning robotics, cryptography, and signal processing; for a comprehensive introduction, see [1]. Despite their utility, Gröbner bases possess inherent limitations, most notably their dependence on a specific monomial ordering, which often breaks the natural symmetry of the underlying algebraic problem. Furthermore, these bases frequently suffer from numerical instability, where infinitesimal perturbations in polynomial coefficients can lead to divergent computational results. In contrast, H-bases—conceptually pioneered by Macaulay [2]—offer a robust alternative by relying exclusively on the homogeneous terms (leading forms) of polynomials. Because H-bases are independent of monomial ordering, they provide a more flexible framework for addressing application-related problems that are otherwise obscured by the rigid structure of Gröbner techniques.

The constructive theory of H-bases was first investigated by Spear [3] and subsequently developed by Möller and Mora [4], who established the theoretical framework for their computation. A general algorithm for constructing H-bases without imposing a monomial ordering was later introduced by Sauer [5]. This algorithm generalizes the classical Euclidean division with remainder to multivariate polynomial rings by replacing the

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cancellation of leading terms with an orthogonalization process based on an inner product. The resulting reduction procedure provides an ordering-independent characterization of H-bases and naturally leads to a Buchberger-type algorithm for their construction, referred to in this paper as the H-basis algorithm [5]. Further theoretical developments, together with applications of H-bases to polynomial system solving, interpolation, and numerical analysis, are discussed in detail by Möller and Sauer [6] and Sauer [5]. The algorithm proposed in the present work follows the same ordering-independent philosophy while replacing the orthogonalization process with a QR decomposition framework, yielding a more streamlined computational procedure based solely on linear algebraic techniques.

While H-bases offer significant theoretical advantages, their widespread adoption has been historically hindered by the lack of efficient general algorithms. Conventional approaches often derive H-bases as a byproduct of Gröbner basis computation (see [7]) or require the construction of syzygy modules, which is based on Gröbner basis computations [8]. The efficiency of such algorithms is frequently compromised by the presence of redundant generators within the syzygy module, leading to unnecessary and costly reductions. To overcome these challenges, this study proposes an algorithmic framework rooted in linear algebraic techniques, specifically utilizing extended row-reduced echelon forms and QR decomposition. By constructing syzygies through matrix row reductions without necessitating a monomial ordering, our method aims to reduce the number of computational steps and structural clarity, providing a conceptually simpler alternative to traditional symbolic methods.

QR decomposition is one of the most fundamental matrix factorization techniques in numerical linear algebra. Owing to its numerical stability and orthogonality-preserving properties, it has become a standard tool for solving least-squares problems, computing matrix rank, determining nullspaces, and developing efficient algorithms in scientific computing. Compared with Gaussian elimination and singular value decomposition, QR factorization often provides an attractive compromise between computational efficiency and numerical robustness, making it particularly suitable for large structured linear algebra problems(see, [9–11]). More broadly, recent advances in computational mathematics continue to emphasize the development of reliable numerical frameworks and efficient computational methodologies for diverse mathematical problems, including computational analysis and approximation theory [12,13]. These developments further highlight the growing role of modern numerical linear algebra techniques in computational mathematics.

Motivated by these properties, several decomposition-based approaches have recently been explored for syzygy computation. Previous studies on the computation of syzygies and H-bases have primarily relied on structured matrix constructions and decomposition techniques. For instance, Hashemi and Javanbakht [14] develop a degree-by-degree approach in which coefficient vectors of leading forms are assembled into matrices, and syzygy bases are obtained through row reduced echelon forms of both the matrix and its transpose. Their method depends heavily on Gröbner basis theory and Schreyer's theorem to determine termination, and it incorporates a rather involved orthogonalization process based on inner products. Similarly, Javanbakht and Sauer [15] propose a method that computes syzygies via singular value decomposition (SVD), leveraging its robustness in rank determination and orthogonal basis construction. While effective, these approaches either require multiple matrix reductions, depend on monomial orderings and Gröbner bases, or involve computationally expensive procedures such as SVD.

In contrast, the method developed in this work introduces a fundamentally different approach by utilizing QR decomposition for both syzygy and H-basis computation. Instead of multiple reduction steps or reliance on Gröbner bases, the proposed algorithm performs all essential computations through a unified and simpler matrix reduction framework. The

recursive structure based on QR decomposition enables the computation of the syzygy module while also defining a reduction process inherently tied to the QR factorization. Furthermore, the termination of the algorithm is governed by a distinct upper bound that is independent of Gröbner basis considerations, thereby avoiding additional matrix reductions. Consequently, the proposed method provides a more streamlined and conceptually simpler alternative to existing approaches by reducing the number of computational steps and relying solely on linear algebraic techniques.

2. Materials and Methods

Throughout this work, the polynomial ring in variables $x_1, x_2, \ldots, x_n$ over the field K will be $P = K[x_1, x_2, \ldots, x_n]$. The total degree of the monomial $x^\alpha = x^{\alpha_1}x^{\alpha_2}\cdots x^{\alpha_n}$ is $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and the total degree of a polynomial $f \in P$, $\deg(f)$, is the highest degree of the monomials of f. The vector space of homogeneous polynomials of degree d is P_d. The set of monomials of degree d, T_d^n, is a basis for P_d. Thus, the dimension of P_d is

$$\binom{n+d-1}{d}.$$

Definition 1. *Let $f = f^0 + f^1 + \cdots + f^d$ be a polynomial with $d = \deg(f)$, $f^i \in P_i$ with $f^d \neq 0$ and $I = \langle f_1, f_2, \ldots, f_s\rangle \subset P$ be an ideal. Then*

(i) The leading form of f is $LF(f) = f^d$.
(ii) The leading form of I is the set $LF(I) = \{LF(f) : f \in I\}$.

Definition 2. *A tuple $(q_1, q_2, \ldots, q_s) \in P^s$ is called a syzygy of the polynomials $(p_1, p_2, \ldots, p_s)$ if*

$$\sum_{j=1}^{s} q_j p_j = 0.$$

The syzygy module of $p_1, p_2, \ldots, p_s$, $Syz(p_1, p_2, \ldots, p_s)$, is the set of all syzygies of $p_1, p_2, \ldots, p_s$. It is well-known that $Syz(p_1, p_2, \ldots, p_s)$ as an P-module is finitely generated.

Definition 3. *The set $\{f_1, f_2, \ldots, f_s\} \subset P$ is called an H-basis for the ideal $I = \langle f_1, f_2, \ldots, f_s\rangle$ if one of the following equivalent conditions holds:*

1. *If $f \in I$, then $LF(f) \in \langle LF(f_1), \ldots, LF(f_s)\rangle$.*
2. *Any $f \in I$ can be written as*
$$f = \sum_{i=1}^{s} h_i f_i$$
where $h_i \in P_{\leq \deg(f)-\deg(f_i)}$ for $i = 1, 2, \ldots, s$.
3. *If $(h_1, h_2, \ldots, h_s) \in Syz(LF(f_1), LF(f_2), \ldots, LF(f_s))$, then there exist $g_1, g_2, \ldots, g_s \in P$ such that*
$$\sum_{i=1}^{s} h_i f_i = \sum_{i=1}^{s} g_i f_i$$
where $\deg(g_i f_i) \geq \deg(\sum_{i=1}^{s} h_i f_i)$ for $i = 1, 2, \ldots, s$.

To determine an H-basis, at first we need to understand the reduction procedure. The Algorithm 1 gives how the reduction procedure works.

Algorithm 1 Reduction procedure algorithm.

Input: $F = \{f_1, f_2, \ldots, f_s\}, g \in P$
Output: q_{f_i}'s and r where $g = \sum q_{f_i} f_i + r$

$r = 0, q_{f_1} = 0, q_{f_2} = 0, \ldots, q_{f_s} = 0$
$d = \deg(g)$
while $g \neq 0$ **do**
  $e = d$
  $LF(g) = \sum q_{e,f_i} LF(f_i) + r_e$
  $g = g - \sum q_{e,f_i} f_i - r_e$
  $r = r + r_e$
  $q_{f_i} = q_{f_i} + q_{e,f_i}$
  $d = \deg(g)$
end while
return q_{f_i}'s, r

In the above algorithm, q_{e,f_i} represents the polynomial which ensures the degree of the multiplication of q_{e,f_i} and $LF(f_i)$ is e.

This procedure is denoted by $g \xrightarrow{F} r$. Definition 3 implies that $\{f_1, f_2, \ldots, f_s\}$ is an H-basis for $I = \langle f_1, f_2, \ldots, f_s \rangle$ if and only if for each vector $(h_1, \ldots, h_s) \in Syz(LF(f_1), LF(f_2), \ldots, LF(f_s))$,

$$\sum h_i f_i \xrightarrow{F} 0.$$

The Algorithm 2 is just a modified Buchberger's Algorithm.

Algorithm 2 H-basis algorithm.

Input: $F = \{f_1, f_2, \ldots, f_s\} \subset P$
Output: An H-basis of $\langle F \rangle$

$G =$ a basis for $Syz(LF(f_1), \ldots, LF(f_s))$
while $G \neq \emptyset$ **do**
  Select $(h_1, h_2, \ldots, h_s) \in G$
  $G = G - (h_1, h_2, \ldots, h_s)$
  Let $\sum h_i f_i \xrightarrow{F} h$
  if $h \neq 0$ **then**
    $F = F \cup \{h\}$
    $G =$ A basis of $Syz(LF(F))$
    $i = i + 1$
  end if
end while
return F

3. Results

3.1. Syzygies of Certain Degree via QR Decomposition

As can be seen in Algorithm 2, constructing an H-basis for a given ideal requires computing a syzygy basis of the leading forms of its generators and applying a reduction procedure to polynomials. Since the leading forms are homogeneous, we develop an algorithm based on QR decomposition to compute a syzygy basis for a set of homogeneous polynomials. Let $F = \{f_1, \ldots, f_s\}$ be a set of homogeneous polynomials and let $I = \langle F \rangle$.

The set of homogeneous polynomials of degree d in the ideal I is a subspace of the vector space P_d of the homogeneous polynomials of degree d. Furthermore,

$$M_d(F) = \{x^{\alpha_j} f_i \mid |x^{\alpha_j}| = d - \deg(f_i), \ 1 \leq i \leq s\}$$

is a spanning set of this subspace.

Since each polynomial $x^{\alpha_j} f_i$ can be written as a linear combination of monomials of degree d, we can obtain the coordinate vector of $x^{\alpha_j} f_i$ with respect to any given monomial order in T_d^n. Writing these coordinate vectors as a column, $M_d(F)$ can also be interpreted as a matrix. We use $M_d(F)$ to represent both the set of polynomials and their matrix representation. For convenience, the lexicographic order of monomials will be used in this study. For a reason that will become clear later, the columns of the matrix $M_d(F)$ will be arranged as follows:

The column $x^{\alpha_{j_1}} f_{i_1}$ is left to the column $x^{\alpha_{j_2}} f_{i_2}$ if $x^{\alpha_{j_1}} > x^{\alpha_{j_2}}$ in lexicographic order or, if $x^{\alpha_{j_1}} = x^{\alpha_{j_2}}$, then $i_1 < i_2$.

Definition 4. *Let A be an $m \times n$ matrix and $rank(A) = r < n$. Full QR decomposition of A is*

$$A = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

where $A = Q_1 R_1$ is the QR decomposition, Q_1 is an $m \times r$, R_1 is an $r \times n$ and Q_2 is an $m \times (m-r)$ matrices.

In the Full QR Decomposition process, the main idea is to use a full-rank matrix. Now, we will find the Full QR Decomposition of $M_d(F)$ using the following algorithm. Before giving the algorithm, let

$$v = \binom{n+d-1}{d}$$

which is the number of rows of $M_d(F)$, and let

$$w = \sum_{i=1}^{s} \binom{n+d-\deg(f_i)-1}{d-\deg(f_i)}$$

which is the number of columns of $M_d(F)$. Moreover, let $m_1, m_2, \ldots, m_w$ represent columns of $M_d(F)$ and $u_i = e_{p-i}$, the $(p-i)^{th}$ unit vector in $\mathbb{R}^v$. Notice that each monomial in degree d is a vector in $\mathbb{R}^v$.

Let R be the matrix $r \times w$ whose entries are given by

$$r_{ij} = \begin{cases} 0, & \text{if } \ j < i; \\ \langle m_j, q_i \rangle, & \text{if } \ j \geq i. \end{cases}$$

Then

$$M_d(F) = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}$$

is the Full QR decomposition of $M_d(F)$.

Notice that the columns of Q_1 are an orthonormal basis for the vector space $M_d(F)$, and Q_2 is an orthonormal basis for its complement.

Algorithm 3 is not intended to introduce a new QR factorization algorithm. Rather, it presents a modified version of the standard Full QR algorithm in order to explicitly describe the computational steps and the intermediate quantities required by the subsequent algorithms. In particular, the ordered list monlist is used in Section 3.2 and Theorem 1, while the matrices Q_1 and Q_2 are required in Section 3.3. Accordingly, in practical implementations, any standard full QR decomposition routine (e.g., a Householder-based QR implementation) may be employed to compute the required QR factors.

Algorithm 3 Full QR algorithm.

Input: M, vectors u_i
Output: Q_1, Q_2, index, monlist
$r = 0$
index $= \emptyset$
monlist $= \emptyset$
for $i = 1$ to w **do**
    $\tilde{q}_i = m_i - \sum_{j=1}^{i-1} \langle q_j, m_i \rangle q_j$
    if $\tilde{q}_i \neq 0$ **then**
        $r = r + 1$
        $q_r = \frac{\tilde{q}_i}{||\tilde{q}_i||}$
    else
        index $=$ index $\cup \{i\}$
    end if
end for
$Q_1 = [q_1, q_2, \ldots, q_r]$
if $r = v$ **then**
    return $Q_1, \emptyset$, index, $\emptyset$
end if
$i = 1$
$k = 1$
while $k \leq v - r$ **do**
    $\tilde{p}_k = u_i - \sum_{j=1}^{r} \langle u_i, q_j \rangle q_j - \sum_{j=1}^{k-1} \langle u_i, p_j \rangle p_j$
    if $\tilde{p}_k \neq 0$ **then**
        $p_k = \frac{\tilde{p}_k}{\|\tilde{p}_k\|}$
        $k = k + 1$
        $i = i + 1$
    else
        monlist $=$ Append(monlist,x^{u_i})
        $i = i + 1$
    end if
end while
$Q_2 = [\, p_1, p_2, \ldots, p_{v-r} \,]$
return Q_1, Q_2, index, monlist

Definition 5. *Let* $A = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ *be the Full QR decomposition of* A. R_1 *can be written as*

$$\begin{bmatrix} R_{11} & R_{12} \end{bmatrix}$$

where R_{11} *is* $r \times r$ *upper triangular matrix by interchanging the columns of* R_1. *Hence*

$$A = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} P^T$$

where P *is an* $n \times n$ *permutation matrix. This form is called the Pivoted QR decomposition of* A.

Using the above definition, we can obtain the Pivoted QR decomposition of $M_d(F)$ as follows: If $M_d(F) = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}$ is the Full QR decomposition of M_d, we can arrange R as

$$\begin{bmatrix} R_1 & R_2 \end{bmatrix}$$

where R_1 is $r \times r$ upper triangular matrix. Hence,

$$M_d(F) = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} P^T$$

where P is $w \times w$ permutation matrix.

Lemma 1. *If*

$$M_d(F) = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} P^T$$

is the Pivoted QR decomposition of $M_d(F)$, then the columns of

$$N = P \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix}$$

form a basis for the nullspace of $M_d(F)$.

Proof. Since $P^T P = I$,

$$\begin{aligned} M_d(F) P \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} &= \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} P^T P \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} \\ &= \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} \\ &= \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} -R_1 R_1^{-1} R_2 + R_2 I \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} -R_2 + R_2 \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \end{aligned}$$

□

By the construction of the matrix N, each column of N represents a linearly dependent relation of a column $x^\alpha f_k$ of $M_d(F)$ with respect to the columns left to it. This linear dependence expresses a certain syzygy

$$p = \sum_{i=1}^{s} h_i f_i = 0. \tag{1}$$

So, $p \in szy_d(f_1, \ldots, f_s)$. Therefore, the columns of N form a basis for

$$szy_d(f_1, \ldots, f_s) = \{(h_1, \ldots, h_s) \in Syz(f_1, \ldots, f_s) : \deg(h_i f_i) = d\}$$

for some d. Thus, for every d, we can find $szy_d(f_1, \ldots, f_s)$ with QR-Decomposition. Evidently, a generating set for the module $Syz(f_1, \ldots, f_s)$ can be obtained by taking the union of these bases for a suitable selection of d. Therefore, the primary challenge within this framework is to determine an appropriate upper limit for the degree d to guarantee that a set of generators for $Syz(f_1, \ldots, f_s)$ has been identified. The following section will clarify how to identify this upper limit.

3.2. Finding Upper Limit

Before we define extended syzygies, we seek details about the total number of monomials associated with a certain degree in a monomial ideal.

Let $\{x^{\alpha(1)}, x^{\alpha(2)}, \ldots, x^{\alpha(t)}\}$ be the set of minimal generating monomials of a monomial ideal J. We aim to find the number of monomials of degree e of the ideal J. We define A_i as the set of all monomial multiples $x^\beta x^{\alpha(i)}$ such that $|x^\beta x^{\alpha(i)}| = e$ for $1 \leq i \leq t$. Notice that

$$|A_i| = \binom{n+e-|x^{\alpha(i)}|-1}{e-|x^{\alpha(i)}|}$$

The number of monomials of degree e of the ideal J is

$$|A_1 \cup A_2 \cup \cdots \cup A_t|.$$

If we apply the Inclusion-Exclusion Principle,

$$|\bigcup_{i=1}^{t} A_i| = \sum_{i=1}^{t}(-1)^{k+1}(\sum_{1\leq i_i<\cdots<i_k\leq t} |A_{i_i} \cap A_{i_2} \cap \cdots \cap A_{i_k}|) \quad (2)$$

Notice that if $x^\gamma = LCM(x^{\alpha(i_1)}, x^{\alpha(i_2)}, \ldots, x^{\alpha(i_k)})$ and $|x^\gamma| = f$, then

$$|A_{i_i} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \binom{n+e-f-1}{e-f}$$

Let us define the following weight order on $M_d(F)$. $x^\alpha f_i > x^\beta f_j$ if $i > j$ or $i = j$ and $x^\alpha < x^\beta$ with respect to lexicographic order. Notice that this is the reverse of the order we define for the columns of the matrix form of $M_d(F)$. Hence according to the order described above, the leading monomial of the syzygy p given in Equation (1) is $x^{\alpha_j} f_k$. We write this as $LM(p) = x^{\alpha_j} f_k$.

Notice that $x^\beta p$ is also a syzygy in $szy_e(f_1, \ldots, f_s)$ if $|x^\beta| = e - d$. Furthermore $x^{\alpha+\beta} f_j$ is the leading monomial of this syzygy. We will refer to syzygy p as a basic syzygy and $x^\beta p$ as an extended syzygy. Notice that some monomial products of two different basic syzygies can be produce two extended syzygies with the same leading monomial. Hence we cannot say that all extended syzygies are linearly independent. To determine the least number of linearly independent extended syzygies, we can categorize the basic syzygies into groups based on the polynomial that is being multiplied.

Definition 6. *For* $1 \leq k \leq s$,

$$J_k^d = \{x^\alpha \mid LM(p) = x^\alpha f_k \text{ for some basis element } p \text{ of } szy_d(f_1, \ldots, f_s)\}.$$

Similarly,

$$J_k^{d,e} = \{x^\beta x^\alpha \mid LM(x^\alpha p) = x^\beta x^\alpha f_k \text{ for some basis element } p \text{ of } szy_d(f_1, \ldots, f_s)\}.$$

We select an extended syzygy for each $x^\beta x^\alpha \in J_k^{d,e}$, $1 \leq k \leq s$. By organizing these syzygies into rows of a matrix, starting with the syzygies that have the highest leading monomial and moving towards those with the lowest leading monomial, we can ensure that the matrix is presented in row echelon form. Consequently, the systematic arrangement of syzygies results in a matrix that conforms to the row echelon form. Hence these extended syzygies will be linearly independent. Therefore we may conclude that we have at least

$$|J_1^{d,e}| + |J_2^{d,e}| + \cdots + |J_s^{d,e}|$$

linearly independent syzygies of degree e.

Notice that each $|J_k^{d,e}|$ can be computed using Equation (2).

This provides insight into the dimension of the nullspace associated with the matrix $M_e(F)$. We shall now proceed to examine the rank of this matrix.

For a polynomial $f \in k[x_1, \ldots, x_n]$, the leading monomial of f, denoted by $LM(f)$, is the largest monomial with respect to a given monomial order, which is lex order in our case.

Definition 7. *For a given set $F = \{f_1, \ldots, f_s\}$ of homogeneous polynomials and the ideal $I = \langle F \rangle$, we define*

$$LM(F) = \{LM(f) \mid f \in I\}$$

and for a non-negative integer d,

$$LM_d(F) = \{LM(f) \mid f \in I \text{ and } \deg(f) = d\}.$$

The subsequent theorem demonstrates how the set $LM_d(F)$ can be derived as a result of the Full QR Algorithm.

Theorem 1. *Let $monlist = \{x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_s}\}$ be the ordered list obtained in Full QR decomposition of $M_d(F)$. Then*

$$LM_d(F) = T_d^n \backslash monlist$$

Proof. Let $x^\alpha \notin$ monlist. Then

$$x^{\alpha_{j+1}} < x^\alpha < x^{\alpha_j}$$

with respect to the lexicographic order. Therefore,

$$x^\alpha \in span(q_1, \ldots, q_r, x^{\alpha_1}, \ldots, x^{\alpha_j}).$$

This implies there is a polynomial

$$f = x^\alpha + a_1 x^{\alpha_1} + \cdots + a_j x^{\alpha_j} \in M_d(F).$$

Hence, $LM(f) = x^\alpha \in LM_d(F)$.

Conversely, suppose that $x^\alpha \in LM_d(F)$. Therefore, there is a homogeneous polynomial f such that $\deg(f) = d$ and

$$f = x^\alpha + \sum_{i=1}^{s} a_{\beta_i} x^{\beta_i}$$

where each $x^{\beta_i} < x^\alpha$. Hence

$$x^\alpha \in span(q_1, \ldots, q_r, x^{\beta_1}, \ldots, x^{\beta_s}).$$

If some $x^{\beta_i} \notin$ monlist, then by the above argument it can be written as a linear combination of $q_1, \ldots, q_r$ and monomials in monlist which are smaller than x^{β_i}. Hence, we may assume that each $x^{\beta_i} \in$ monlist. Therefore,

$$\alpha - \sum_{j=1}^{r} \langle \alpha, q_j \rangle q_j - \sum_{j=1}^{s} \langle \alpha, \beta_j \rangle \beta_j = 0$$

and $x^\alpha \notin$ monlist. □

This theorem clearly indicates that the rank of the matrix $M_d(F)$ is $|LM_d(F)|$. A minimal generating set for the monomial ideal $\langle LM_d(F) \rangle$ can be easily obtained as a consequence of the theorem. Let $LM_{d,e}(F)$ be the set of monomials in degree $e > d$ in $LM_d(F)$. The number of elements in $LM_{d,e}(F)$, denoted by $|LM_{d,e}(F)|$, can be found by Equation (2). Notice that $LM_{d,e}(F) \subseteq LM_e(F)$. Therefore

$$|LM_{d,e}(F)| \leq |LM_e(F)|.$$

The Rank-Nullity Theorem is widely recognized for stating that the sum of a matrix's rank and its nullity is always equal to the total number of its columns. So

$$|LM_{d,e}(F)| + \sum_{i=1}^{s} |J_i^{d,e}| \leq |LM_e(F)| + \sum_{i=1}^{s} |J_i^e| = \sum_{i=1}^{s} \binom{n+e-\deg(f_i)-1}{e-\deg(f_i)}.$$

If for some d

$$|LM_{d,e}(F)| + \sum_{i=1}^{s} |J_i^{d,e}| = \sum_{i=1}^{s} \binom{n+e-\deg(f_i)-1}{e-\deg(f_i)},$$

then that means Full QR decomposition of $M_e(F)$ for $e > d$ will not produce a new syzygy. Hence at the step d, a set of generators for $Syz(f_1, \ldots, f_s)$ has been found.

3.3. Recursive QR Decomposition

The results achieved up to this point establish a method for determining syzygy: acquiring the QR decompositions of the $M_d(F)$ matrices progressively by degree, starting from $d = \min\{\deg(f_i)\}$ and then obtaining the syzygy basis through nullspace. An upper bound check is executed at each degree, and these operations are continued until the upper bound check is satisfied.

Now, we give a recursive method for computing the Full QR decomposition of $M_{d+1}(F)$. That means instead of computing the Full QR decomposition of the $M_{d+1}(F)$ from scratch, we will compute it using the QR decomposition of the $M_d(F)$.

Notice that $M_{d+1}(F)$ can be written as

$$M_{d+1}(F) = \begin{bmatrix} M_d(F) & \bar{A} \\ 0 & \bar{B} \end{bmatrix}.$$

First of all, to find all linearly independent extended syzygies of degree $d+1$, $syz_{d,d+1}(F)$, we can write all extended syzygies of degree $d+1$ as rows of a matrix as described above. This matrix will not be in row echelon form but very close to it. Applying a few row reduction operations, we can obtain a matrix in row reduced echelon form. Hence, we can find all linearly independent extended syzygies of degree $d+1$. Since the column corresponding to the leading monomial of each extended syzygy of degree $d+1$ can be expressed as a linear combination of those columns of $M_{d+1}(F)$ that do not correspond to leading monomials, the columns associated with the leading monomials are redundant and may therefore be removed from $M_{d+1}(F)$.

Furthermore, if $M_d(F) = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ is the Full QR decomposition, we can substitute Q_1 for $M_d(F)$ in the matrix of $M_{d+1}(F)$ since Q_1 is an orthonormal basis for the column space of $M_d(F)$. So, it will be enough to find the Full QR decomposition of

$$M_{d+1}(F) = \begin{bmatrix} Q_1 & A \\ 0 & B \end{bmatrix} \tag{3}$$

where A and B are obtained from the matrices $\bar{A}$ and $\bar{B}$ by dropping some columns, respectively.

Theorem 2. *Let $M_d(F), Q_1, Q_2, R_1, M_{d+1}(F), A$ and B be matrices as defined above, and r be the number of columns of Q_1. If*

$$\begin{bmatrix} Q_2^T A \\ B \end{bmatrix} = \begin{bmatrix} \bar{Q}_1 & \bar{Q}_2 \end{bmatrix} \begin{bmatrix} \bar{R}_1 \\ 0 \end{bmatrix}$$

is the Full QR decomposition and

$$\begin{bmatrix} \bar{Q}_1 & \bar{Q}_2 \end{bmatrix} = \begin{bmatrix} S_1 & | & S_2 \\ T_1 & | & T_2 \end{bmatrix},$$

then

$$\begin{bmatrix} Q_1 & A \\ 0 & B \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2S_1 & Q_2S_2 \\ 0 & T_1 & T_2 \end{bmatrix} \begin{bmatrix} I_r & Q_1^TA \\ 0 & \bar{R}_1 \\ 0 & 0 \end{bmatrix}$$

is the Full QR decomposition of $\begin{bmatrix} Q_1 & A \\ 0 & B \end{bmatrix}$.

Proof. Notice that

$$\begin{bmatrix} Q_1 & A \\ 0 & B \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I_r & Q_1^TA \\ 0 & Q_2^TA \\ 0 & B \end{bmatrix} \tag{4}$$

On the other hand,

$$\begin{bmatrix} Q_2^TA \\ B \end{bmatrix} = \begin{bmatrix} S_1 & | & S_2 \\ T_1 & | & T_2 \end{bmatrix} \begin{bmatrix} \bar{R}_1 \\ 0 \end{bmatrix}$$

implies

$$\begin{bmatrix} I_r & Q_1^TA \\ 0 & Q_2^TA \\ 0 & B \end{bmatrix} = \begin{bmatrix} I_r & 0 & 0 \\ 0 & S_1 & S_2 \\ 0 & T_1 & T_2 \end{bmatrix} \begin{bmatrix} I_r & Q_1^TA \\ 0 & \bar{R} \\ 0 & 0 \end{bmatrix}.$$

Hence,

$$\begin{aligned} \begin{bmatrix} Q_1 & A \\ 0 & B \end{bmatrix} &= \begin{bmatrix} Q_1 & Q_2 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I_r & 0 & 0 \\ 0 & S_1 & S_2 \\ 0 & T_1 & T_2 \end{bmatrix} \begin{bmatrix} I_r & Q_1^TA \\ 0 & \bar{R} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} Q_1 & Q_2S_1 & Q_2S_2 \\ 0 & T_1 & T_2 \end{bmatrix} \begin{bmatrix} I_r & Q_1^TA \\ 0 & \bar{R} \\ 0 & 0 \end{bmatrix} \end{aligned}$$

□

Corollary 1. *Let*

$$\begin{bmatrix} Q_2^TA \\ B \end{bmatrix} = \begin{bmatrix} \bar{Q}_1 & | & \bar{Q}_2 \end{bmatrix} \begin{bmatrix} \bar{R}_{11} & \bar{R}_{12} \\ 0 & 0 \end{bmatrix} P^T$$

be the Pivoted QR decomposition and

$$\begin{bmatrix} Q_1^TA \\ B \end{bmatrix} = \begin{bmatrix} C & D \\ \bar{R_{11}} & \bar{R_{12}} \end{bmatrix} P^T.$$

Then

$$-\begin{bmatrix} I_r & | & P \end{bmatrix} = \begin{bmatrix} D - C\bar{R}_{11}^{-1}\bar{R}_{12} \\ \bar{R}_{11}^{-1}\bar{R}_{12} \\ I \end{bmatrix}$$

gives a nullspace for the matrix

$$\begin{bmatrix} Q_1 & A \\ 0 & B \end{bmatrix}.$$

Proof.

$$\begin{bmatrix} I_r & Q_1^TA \\ 0 & \bar{R} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_r & C & : & D \\ 0 & \bar{R}_{11} & : & \bar{R}_{12} \\ 0 & 0 & : & 0 \end{bmatrix} \begin{bmatrix} I_r & | & P^T \end{bmatrix}.$$

The nullspace can be obtained as

$$-\begin{bmatrix} I_r & | & P \end{bmatrix}\begin{bmatrix} I_r & C \\ 0 & \bar{R}_{11} \end{bmatrix}^{-1}\begin{bmatrix} D \\ \bar{R}_{12} \end{bmatrix} = -\begin{bmatrix} I_r & | & P \end{bmatrix}\begin{bmatrix} I_r & -C\bar{R}_{11}^{-1} \\ 0 & \bar{R}_{11}^{-1} \end{bmatrix}\begin{bmatrix} D \\ \bar{R}_{12} \end{bmatrix}$$

$$= -\begin{bmatrix} I_r & | & P \end{bmatrix}\begin{bmatrix} D - C\bar{R}_{11}^{-1}\bar{R}_{12} \\ \bar{R}_{11}^{-1}\bar{R}_{12} \end{bmatrix}$$

□

Lemma 2. *Let* $K = \begin{bmatrix} L & M \end{bmatrix}$ *be an* $m \times s$ *matrix where* $L = AB$ *is an* $m \times r$ *matrix. If* $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_s \end{bmatrix}$ *is in the nullspace of* K*, then*

$$\begin{bmatrix} B\begin{bmatrix} v_1 \\ \vdots \\ v_r \end{bmatrix} \\ v_{r+1} \\ \vdots \\ v_s \end{bmatrix}$$

is in the nullspace of $\begin{bmatrix} A & M \end{bmatrix}$.

Proof. Let $K = \begin{bmatrix} L & M \end{bmatrix}$ and $L = AB$. Columns of K can be written as

$$\begin{bmatrix} L_1 & L_2 & \dots & L_r & | & M_1 & M_2 & \dots & M_{s-r} \end{bmatrix}$$

Let $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_s \end{bmatrix}$ be a column vector such that $Kv = 0$. So,

$$Kv = v_1L_1 + v_2L_2 + \cdots + v_rL_r + v_{r+1}M_1 + \cdots + v_sM_{s-r} = 0. \quad (5)$$

Since $L = AB$,

$$L_i = b_{1i}A_1 + b_{2i}A_2 + \cdots + b_{ri}A_r.$$

Replacing these equations into Equation (5),

$$\begin{aligned} Kv &= v_1(b_{11}A_1 + b_{21}A_2 + \cdots + b_{r1}A_r) + v_2(b_{12}A_1 + b_{22}A_2 + \cdots + b_{r2}A_r) + \cdots + \\ &\quad v_r(b_{1r}A_1 + b_{2r}A_2 + \cdots + b_{rr}A_r) + v_{r+1}M_1 + \cdots + v_sM_{s-r} \\ &= (v_1b_{11} + v_2b_{12} + \cdots + v_rb_{1r})A_1 + (v_1b_{21} + v_2b_{22} + \cdots + v_rb_{2r})A_2 + \cdots + \\ &\quad (v_1b_{r1} + v_2b_{r2} + \cdots + v_rb_{rr})A_r + v_{r+1}M_1 + \cdots + v_sM_{s-r} \\ &= 0 \end{aligned}$$

implies that

$$\begin{bmatrix} A & M \end{bmatrix} \begin{bmatrix} B \begin{bmatrix} v_1 \\ \vdots \\ v_r \end{bmatrix} \\ v_{r+1} \\ \vdots \\ v_s \end{bmatrix} = 0$$

□

Corollary 2. *Let* $M_{d+1}(F) = \begin{bmatrix} Q_1 & A \\ 0 & B \end{bmatrix}$ *where* $Q_1 = M_d(F)(R^{(d)})^{-1}$, *and let* $\begin{bmatrix} \bar{Q}_1 & \bar{Q}_2 \end{bmatrix} \begin{bmatrix} \bar{R}_1 & \bar{R}_2 \\ 0 & 0 \end{bmatrix} P^T$ *be the pivoted QR decomposition of* $M_{d+1}(F)$. *If* $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_s \end{bmatrix}$ *is in the nullspace of* M_{d+1}, *then*

$$\begin{bmatrix} (R^{(d)})^{-1} \begin{bmatrix} v_1 \\ \vdots \\ v_r \end{bmatrix} \\ P \begin{bmatrix} v_{r+1} \\ \vdots \\ v_s \end{bmatrix} \end{bmatrix}$$

is in the nullspace of $\begin{bmatrix} M_d(F) & A \\ 0 & B \end{bmatrix}$.

Proof. By Lemma 2, $K = M_{d+1}(F)$, $A = M_d$, and $B = (R^{(d)})^{-1}$ give the desired result. □

The above lemma allows us to find the syzygies of $M_{d+1}(F)$. We can use Algorithm 4 to find the minimal basis for $Syz(F)$.

Algorithm 4 Syzygy computation algorithm.

Input: $F = (f_1, f_2, \ldots, f_s) \subseteq k[x_1, x_2, \ldots, x_n]^s$, an s-tuple of homogenous polynomials
Output: S, a minimal basis for $Syz(F)$

$S = \emptyset$
$r(e) = \sum_{i=1}^{s} \binom{n+e-\deg(f_i)-1}{e-\deg(f_i)}$
$d = \min\{\deg(f_i) : 1 \leq i \leq s\}$
repeat
  Find Full QR Decomposition of $M_d(F)$
  Find $LM_d(F)$
  Find $syz_d(F)$
  $S = S \cup syz_d(F)$
  for $k = 1$ to s **do**
    Find J_k^d
  end for
  if $|LM_{d,e}(F)| + \sum_{k=1}^{s} |J_k^{d,e}| = r(e)$ **then**
    Exit
  end if
  Obtain extended syzygies $syz_{d,d+1}(F)$
  Apply row reduction to $syz_{d,d+1}(F)$
  Construct $M_{d+1}(F)$ using Equation (3)
  for $k = 1$ to s **do**
    Find $J_k^{d,d+1}$
    Delete the columns $\{x^\alpha f_k : x^\alpha \in J_k^{d,d+1}\}$ from $M_{d+1}(F)$
  end for
  $d = d + 1$
until $|LM_{d,e}(F)| + \sum_{k=1}^{s} |J_k^{d,e}| < r(e)$
return S

Example 1. *Let* $F = \{f_1 = x_1 + x_2, f_2 = x_1x_2 + x_3^2, f_3 = x_1x_3 + x_2x_3\}$. *Then*

$$r = \binom{3+e-1-1}{e-1} + \binom{3+e-2-1}{e-2} + \binom{3+e-2-1}{e-2} = \binom{e+1}{e-1} + 2\binom{e}{e-2},$$

and

$$M_1(F) = \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

If we apply the Full QR algorithm to $M_1(F)$,

$$Q = \begin{matrix} \\ \\ \\ \\ Q_1 \end{matrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & : & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & : & 0 & \frac{1}{\sqrt{2}} \\ 0 & : & 1 & 0 \end{bmatrix}, \quad \begin{matrix} \\ \\ \\ \\ Q_2 \end{matrix}$$

$$R = \begin{bmatrix} R^{(1)} \\ - \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ - \\ 0 \\ 0 \end{bmatrix}, \text{ and } (R^{(1)})^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}}. \end{bmatrix}$$

Hence $monlist = \{x_3, x_2\}$ *and* $LM_1(F) = (monlist)^c = \{x_1\}$. *Thus, there are no syzygies. Now, consider* $d = 2$. *So,* $M_2(F) = \{x_1f_1 : x_2f_1, x_3f_1, f_2, f_3\}$, *and*

$$M_2(F) = \begin{matrix} x_1^2 \\ x_1x_2 \\ x_1x_3 \\ \\ x_2^2 \\ x_2x_3 \\ x_3^2 \end{matrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & : & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & : & 1 & 0 & 1 & 0 \\ 0 & : & 0 & 1 & 0 & 1 \\ \dots & : & \dots & \dots & \dots & \dots \\ 0 & : & 1 & 0 & 0 & 0 \\ 0 & : & 0 & 1 & 0 & 1 \\ 0 & : & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} Q_1 : A \\ 0 : B \end{bmatrix}$$

According to Theorem 2, we need to find the Full QR Decomposition of the matrix

$$\begin{bmatrix} Q_2^T A \\ B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The Full QR Decomposition gives that

$$\bar{Q} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & : & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{6}} & : & \frac{-1}{\sqrt{2}} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & : & \frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & : & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{\sqrt{3}}{2} & : & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} S_1 & : & S_2 \\ \dots & \dots & \dots \\ T_1 & : & T_2 \end{bmatrix}$$

and

$$\bar{R} = \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & : & 0 & \\ 0 & \sqrt{2} & 0 & : & \sqrt{2} & \\ 0 & 0 & \frac{2}{\sqrt{3}} & : & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & : & 0 & \\ 0 & 0 & 0 & : & 0 & \end{bmatrix} = \begin{bmatrix} \bar{R}_1 & : & \bar{R}_2 \\ 0 & : & 0 \end{bmatrix}.$$

Hence, monlist $= \{x_3^2, x_2x_3\}$ *and* $LM_2(F) = \{x_1^2, x_1x_2, x_1x_3, x_2^2\}$. *Thus,* $\langle LM_2(F)\rangle = \langle x_1, x_2^2\rangle$. *Therefore we can write Full QR Decomposition of* $M_2(F)$ *as*

$$Q = \begin{bmatrix} Q_1 & Q_2S_1 & : & Q_2S_2 \\ 0 & T_1 & : & T_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & 0 & \frac{-1}{2\sqrt{3}} & : & \frac{1}{2} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 & \frac{1}{2\sqrt{3}} & : & \frac{-1}{2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & : & 0 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & : & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & : & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & : & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} Q_1 & : Q_2 \end{bmatrix},$$

and

$$R = \begin{bmatrix} I_1 & Q_1^TA \\ 0 & \bar{R} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & : & 0 \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & : & 0 \\ 0 & 0 & \sqrt{2} & 0 & : & \sqrt{2} \\ 0 & 0 & 0 & \frac{2}{\sqrt{3}} & : & 0 \\ \cdots & \cdots & \cdots & \cdots & : & \cdots \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix} = \begin{bmatrix} R_1 & | & R_2 \end{bmatrix}$$

To find the syzygy of degree 2, *we need to use Corollary 1. Then we have*

$$Q_1^TA = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & : & 0 \end{bmatrix} = \begin{bmatrix} C & : & 0 \end{bmatrix}, R^{(2)} = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$

$$\begin{bmatrix} -D + C\bar{R}_1^{-1}\bar{R}_2 \\ -\bar{R}_1^{-1}\bar{R}_2 \\ I_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

The above vector is the syzygy of the matrix $M_2(F)$, *but we need to find the syzygy of the matrix* $\begin{bmatrix} M_1(F) & A \\ 0 & B \end{bmatrix}$. *Corollary 2 implies that the corresponding syzygy is*

$$\begin{bmatrix} x_1f_1 & : & x_2f_2 & x_3f_1 & f_2 & f_3 \end{bmatrix} \begin{bmatrix} (R^{(1)})^{-1}.0 \\ \cdots \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = f_3 - x_3f_1.$$

Hence $syz_2(F) = \{s_1 = f_3 - x_3 f_1\}$ *and* $J_3^2 = \langle 1 \rangle$.

Extended syzygies are $syz_{2,3} = \{x_3 f_3 - x_3^2 f_1, x_2 f_3 - x_2 x_3 f_1, x_1 f_3 - x_1 x_3 f_1\}$. *The coefficient matrix*

$$\begin{array}{c} \begin{matrix} x_3f_3 & x_3f_2 & x_3^2f_1 & x_2f_3 & x_2f_2 & x_2x_3f_1 & x_2^2f_1 & x_1f_3 & x_1f_2 & x_!x_3f_1 & x_1x_2f_1 & x_1^2f_1 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix} \end{array}$$

is already in row echelon form. Therefore we can drop the columns x_3f_3, x_2f_3 *and* x_1f_3 *when calculating* $M_3(F)$.

Since $\langle LM_2(F) \rangle = \langle x_1, x_2^2 \rangle$ *and* $J_3^2 = \langle 1 \rangle$,

$$\begin{aligned} \mid LM_{2,e}(F) \mid &= \binom{3+e-1-1}{e-1} + \binom{3+e-2-1}{e-2} - \binom{3+e-3-1}{e-3} \\ &= \binom{e+1}{e-1} + \binom{e}{e-2} - \binom{e-1}{e-3}, \end{aligned}$$

and

$$|J_3^{2,e}| = \binom{3+e-2-1}{e-2} = \binom{e}{e-2}$$

imply that $\mid LM_{2,e}(F) \mid + |J_3^{2,e}| < r(e)$. *So we need to continue for* $d = 3$.

If $d = 3$, $M_3(F) = \{x_1^2 f_1, x_1 x_2 f_1, x_1 x_3 f_1, x_1 f_2 : x_2^2 f_1, x_2 x_3 f_1, x_2 f_2, x_3^2 f_1, x_3 f_2\}$, *and*

$$M_3(F) = \begin{array}{c} \\ x_1^2 \\ x_1^2x_2 \\ x_1^2x_3 \\ x_1x_2^2 \\ x_1x_2x_3 \\ x_1x_3^2 \\ \dots \\ x_2^3 \\ x_2^2x_3 \\ x_2x_3^2 \\ x_3^2 \end{array} \begin{array}{c} \begin{matrix} x_2^2f_1 & x_2x_3f_1 & x_2f_2 & x_3^2f_1 & x_3f_2 \end{matrix} \\ \left[\begin{array}{ccccccc} & : & 0 & 0 & 0 & 0 & 0 \\ & : & 0 & 0 & 0 & 0 & 0 \\ Q_1 & : & 0 & 0 & 0 & 0 & 0 \\ & : & 1 & 0 & 1 & 0 & 0 \\ & : & 0 & 1 & 0 & 0 & 1 \\ & : & 0 & 0 & 0 & 1 & 0 \\ \dots & : & \dots & \dots & \dots & \dots & \\ & : & 1 & 0 & 0 & 0 & 0 \\ 0 & : & 0 & 1 & 0 & 0 & 0 \\ & : & 0 & 0 & 1 & 1 & 0 \\ & : & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} = \begin{bmatrix} Q_1 : A \\ 0 : B \end{bmatrix}.$$

Then $Q_2^T A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$. *We find the Full QR Decomposition of the matrix* $\begin{bmatrix} Q_2^T A \\ B \end{bmatrix}$ *and get*

$$\bar{Q} = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{\sqrt{2}}{\sqrt{5}} & 0 & : & 0 & \frac{-\sqrt{2}}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{6}} & : & \frac{-1}{\sqrt{2}} & 0 \\ \frac{2}{\sqrt{5}} & 0 & \frac{-1}{\sqrt{30}} & 0 & : & 0 & \frac{1}{\sqrt{6}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & : & \frac{1}{2} & 0 \\ 0 & 0 & \frac{\sqrt{5}}{\sqrt{6}} & 0 & : & 0 & \frac{1}{\sqrt{6}} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & : & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} S_1 : S_2 \\ T_1 : T_2 \end{bmatrix},$$

$$\bar{R} = \begin{bmatrix} \frac{\sqrt{5}}{2} & 0 & \frac{1}{2\sqrt{5}} & 0 & : & \frac{1}{2\sqrt{5}} \\ 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{\sqrt{6}} & : & 0 \\ 0 & 0 & \frac{\sqrt{6}}{\sqrt{5}} & 0 & : & \frac{\sqrt{6}}{\sqrt{5}} \\ 0 & 0 & 0 & \frac{2}{\sqrt{3}} & : & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix} P^T = \begin{bmatrix} \bar{R}_1 & : & \bar{R}_2 \end{bmatrix} P^T$$

where $P^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. *Hence,*

$$Q_1^T A = \begin{bmatrix} 0 & 0 & 0 & 0 & : & 0 \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} & 0 & : & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & : & 0 \\ \frac{-1}{2\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & 0 & : & \frac{\sqrt{3}}{2} \end{bmatrix} P^T = \begin{bmatrix} C & : D \end{bmatrix} P^T$$

and the following matrices

$$\bar{R_1}^{-1} R_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, D - C\bar{R_1}^{-1} R_2 = \begin{bmatrix} 0 \\ \frac{-\sqrt{2}}{\sqrt{3}} \\ 0 \\ \frac{2}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} -D + C\bar{R_1}^{-1} R_2 \\ -\bar{R_1}^{-1} R_2 \\ I_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \\ \frac{-2}{\sqrt{3}} \\ \cdots \\ 0 \\ 0 \\ -1 \\ 0 \\ \cdots \\ 1 \end{bmatrix}$$

imply that the corresponding syzygy is

$$\begin{bmatrix} (R^{(2)})^{-1} \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \\ \frac{-2}{\sqrt{3}} \end{bmatrix} \\ \cdots \\ P \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$s_2 = x_3^2 f_1 - x_2 f_2 - x_1 f_2 + x_1 x_2 f_1$. *Thus,*

$$S = \{s_1, s_2\} \text{ and } J_1^3 = \{x_3^2\}.$$

Since $| LM_{3,e}(F) | = \binom{e+1}{e-1} + \binom{e}{e-2} - \binom{e-1}{e-3}$, $|J_3^{3,e}| = \binom{e}{e-2}$, *and* $|J_1^{3,e}| = \binom{e-1}{e-3}$,

$$| LM_{3,e}(F) | + |J_3^{3,e}| + |J_1^{3,e}| = \binom{e+1}{e-1} + 2\binom{e}{e-2} = r(e)$$

which means that the algorithm terminates.

3.4. Reduction Procedure

Let $F = \{f_1, f_2, \ldots, f_s\}$ be the set of homogeneous polynomials, and $W_d(F)$ be the orthogonal complement of $M_d(F)$, yielding the direct sum decomposition

$$P_d = M_d(F) \oplus W_d(F).$$

Accordingly, every homogeneous polynomial $h \in P_d$ admits a unique orthgonal decomposition into components in $M_d(F)$ and $W_d(F)$.

Let $Q = \begin{bmatrix} Q_1 & | & Q_2 \end{bmatrix}$ denote the orthogonal matrix arising from the Full QR decomposition of $A = M_d(F)$, where $rank(A) = k$. If v is the coefficient vector of $g \in P_d$, then

$$v = \sum_{i=1}^{k} \langle v, (Q_1)_i \rangle (Q_1)_i + \sum_{i=1}^{m-k} \langle v, (Q_2)_i \rangle (Q_2)_i \tag{6}$$

where $m = \binom{n+d-1}{d}$ and $(Q_1)_i, (Q_2)_i$ are the i^{th} columns of Q_1, Q_2 respectively.

Substituting into Equation (6) yields, for any $h \in P_d$,

$$h = \sum_{i=1}^{s} g_i f_i + r^d,$$

where $g_i \in P_{d-\deg(f_i)}$ and $r^d \in W_d(F)$.

Now let $p \in P$ with $\deg(p) = d$ and $\{h_1, h_2, \ldots, h_s\} \subseteq P$, $f_i = LF(h_i)$ $1 \leq i \leq s$. Then

$$\mathrm{LF}(p) = \sum_{i=1}^{s} g_i f_i + r^d,$$

with $g_i \in P_{d-\deg(f_i)}$ and $r^d \in W_d(F)$. Applying the same procedure to

$$p - \sum_{i=1}^{s} g_i h_i - r^d$$

and iterating, after at most d steps we obtain

$$p = \sum_{i=1}^{s} q_i h_i + r, \tag{7}$$

where $\deg(q_i h_i) \leq d$ and $r \in \bigcup_{k=0}^{d} W_k(F)$.

We refer to r as the remainder of p modulo H and write $p \xrightarrow{H} r$. Algorithm 5 gives the reduction procedure by using QR decomposition.

Algorithm 5 Reduction QR algorithm.

Input: $p \in P$, generating set $H = \{h_1, \ldots, h_s\}$
Output: Decomposition $p = \sum_{i=1}^{s} q_i h_i + r$ satisfying Equation (7)
1: $F = \{f_1 = LF(h_1), \ldots, f_s = LF(h_s)\}$
2: $r = 0, q_i = 0, 1 \leq i \leq s$
3: **while** $f \neq 0$ **do**
4: $g = LF(p), d = \deg(g)$
5: Construct $M_d(F)$
6: **if** $M_d(F) \neq 0$ **then**
7: Find Full QR decomposition of $M_d(F)$
8: Compute the expression $g = \sum_{i=1}^{s} h_i f_i + r^d$
9: $r = r + r^d$
10: $p = p - \sum_{i=1}^{s} g_i h_i - r^d$
11: $q_i = q_i + g_i, 1 \leq i \leq s$
12: **else**
13: $r = r + g$
14: **end if**
15: **end while**
16: **return** $p = \sum_{i=1}^{s} q_i h_i + r$

3.5. H-Basis Computation

The results established in the previous sections provide all the necessary ingredients for constructing an H-basis using only QR decomposition. Recall that Algorithm 2 requires two essential operations: the computation of a basis for the syzygy module of the leading forms (Step 1) and the reduction of the corresponding polynomial combination (Step 5). In the proposed framework, these two steps are completely replaced by Algorithm 4 and Algorithm 5, respectively. Consequently, the entire H-basis computation can be performed without Gröbner basis techniques, monomial orderings, singular value decomposition, or additional symbolic elimination procedures. Every stage of the algorithm is reduced to QR-based linear algebra computations.

More precisely, Algorithm 4 constructs a minimal generating set for the syzygy module of the leading forms degree by degree through recursive QR decompositions. Once a nontrivial syzygy is obtained, the associated polynomial combination is reduced by Algorithm 5, whose reduction procedure is also formulated entirely in terms of QR decomposition. Therefore, the proposed H-basis algorithm is based on a single computational framework, where both syzygy construction and polynomial reduction are carried out by successive QR factorizations.

An important computational advantage of this approach arises from the recursive construction of the QR decompositions. During the computation of the syzygy module, the QR decompositions of the matrices $M_d(F)$ are already available for every degree processed by Algorithm 4. Consequently, when Algorithm 5 performs reductions whose leading degree does not exceed the highest degree previously reached during the syzygy computation, these QR decompositions do not need to be recomputed. Instead, the previously computed orthogonal bases are reused directly. As a result, the reduction algorithm avoids redundant matrix factorizations and requires additional QR computations only for degrees that have not yet been processed. This substantially decreases the computational cost of repeated reductions occurring throughout the H-basis construction.

A second source of efficiency appears whenever a nonzero remainder is produced during the reduction process and is appended to the current generating set. According to Algorithm 2, the syzygy module of the enlarged set must then be recomputed. In the proposed framework, however, this recomputation does not require restarting the orthogonalization procedure from the beginning. Since the orthonormal basis associated

with all previously existing generators has already been constructed, the QR decomposition is updated incrementally. The orthogonalization step is performed only for the newly added polynomial, while the orthogonal basis corresponding to the original generators is preserved unchanged. Hence, extending the generating set requires only a local update of the QR decomposition rather than a complete reconstruction of the orthogonal basis.

These observations demonstrate that the proposed method is not merely a replacement of symbolic computations by numerical linear algebra, but an incremental QR-based framework in which previously computed information is systematically reused throughout the H-basis construction. Consequently, the algorithm significantly reduces redundant orthogonalization and matrix factorization steps while preserving numerical stability and computational efficiency.

4. Discussion

The results presented in this work demonstrate that the computation of H-bases can be reformulated as a sequence of linear algebra problems without relying on Gröbner basis techniques. By expressing homogeneous polynomial spaces through coefficient matrices and exploiting QR decomposition, both syzygy computation and the associated reduction process are unified within a single computational framework. This perspective establishes a direct connection between computational algebra and numerical linear algebra while preserving the algebraic properties required for H-basis construction.

Compared with existing approaches, the proposed method substantially simplifies the computation of syzygy modules. Earlier algorithms either derive syzygies through Gröbner basis computations and Schreyer's theorem or employ singular value decomposition to determine nullspaces. In contrast, the present approach computes syzygies directly from QR decompositions of coefficient matrices. Since QR decomposition generally requires fewer computational resources than singular value decomposition and avoids Gröbner basis calculations entirely, the proposed algorithm offers a conceptually straightforward framework based primarily on standard linear algebra techniques.

A further advantage of the proposed framework is its recursive structure. Instead of recomputing the decomposition of the coefficient matrix at every degree, the recursive QR algorithm updates the decomposition using information obtained at the previous degree. Together with the elimination of redundant columns arising from extended syzygies, this considerably decreases the amount of matrix reduction required during successive iterations. Consequently, the algorithm becomes increasingly efficient as the degree increases.

Another distinguishing feature is the termination criterion. Existing H-basis algorithms often rely on Gröbner basis computations or auxiliary algebraic results to determine when all necessary syzygies have been generated. Here, termination follows from an upper bound obtained by combining the dimensions of the leading monomial space and the independently generated extended syzygies. This criterion is derived entirely within the proposed framework and therefore removes one of the principal dependencies on Gröbner basis theory.

Although the present work is primarily theoretical, the proposed algorithm is well suited for implementation in computer algebra systems. Since QR decomposition is supported by highly optimized numerical linear algebra libraries, the method has the potential to provide practical advantages for large polynomial systems. Future work will focus on implementing the algorithm, performing computational comparisons with existing Gröbner basis-, RREF-, and SVD-based methods, and investigating its performance on applications arising in computational algebra, coding theory, robotics, and polynomial optimization.

Overall, the proposed QR-based framework provides a conceptually simple alternative for computing syzygy modules and H-bases. It also highlights the close relationship

between homogeneous polynomial ideals and orthogonal matrix factorizations, suggesting new directions for the interaction between symbolic computation and numerical linear algebra.

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