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On the Sequential (p, δ, τ)-Numerical Radius Function of Operator Sequence

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Abstract

In this article, firstly, some basic properties of the sequential (p, δ, τ)-numerical radius function are investigated. The relationships between the sequential (p, δ, τ)-numerical radius of an operator sequence and the sequential (p, δ, τ)-numerical radii of its coordinate operators are analyzed. Then, the relationships between the sequential (p, δ, τ)-numerical radius of an operator sequence and the sequential (p, δ, τ)-numerical radii of its real and imaginary parts are presented. Finally, this analysis is extended to the case in which the coordinate operators are sectorial, providing additional insight into the structural behavior of the sequential (p, δ, τ)-numerical radius function. The obtained results are generalized to some well-known famous results about the numerical radius function from the recent literature. Also, an important contribution is made to the existing literature via different and useful results.

Keywords: numerical radius; sequential (p, δ, τ)-numerical radius; sectorial operator

MSC: 47A12; 47A30; 47A63; 47B44; 15A60

1. Introduction

The concepts of numerical range and numerical radius of an operator play a very significant role in pure and applied mathematics and have been studied extensively due to their applications in engineering, quantum computing, quantum mechanics, numerical analysis, differential equations, fluid dynamics, the geometry of Banach space, etc. (see, [1–3]).

Throughout this article, $\mathcal{H}$ denotes a complex Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. Let $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{B}_\infty(\mathcal{H})$ stand for the C^*-algebra of all bounded linear operators and compact operators acting on $\mathcal{H}$, respectively. The classical numerical radius and the usual operator norm of a linear bounded operator T are, respectively, given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle| : \ x \in \mathcal{H}, \ \|x\| = 1\}$$

(see, [4]) and

$$\|T\| = \sup\{\|Tx\| : \ x \in \mathcal{H}, \ \|x\| = 1\}.$$

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In the literature there is one important formula from Yamazaki [5] for the calculation of numerical radius $\omega(T)$ of $T \in \mathfrak{B}(\mathcal{H})$:

$$\omega(T) = \sup_{t\in\mathbb{R}} \|Re(e^{it}T)\|.$$

There is another formula for the computation of generalized numerical radius:

$$\omega_N(T) = \sup_{t\in\mathbb{R}} N(Re(e^{it}T)),$$

where N is the given norm of $T \in \mathfrak{B}(H)$ [6].

Finding the exact value of $\omega(T)$ for an arbitrary $T \in \mathfrak{B}(\mathcal{H})$ is not an easy task, except in some special cases. Therefore, researchers in this field have devoted a considerable amount of time and effort to find lower and upper bounds of $\omega(T)$.

Recall that for any $T \in \mathfrak{B}(\mathcal{H})$ the numerical radius $\omega(T)$ is a norm on $\mathfrak{B}(\mathcal{H})$ and among the most basic bounds of $\omega(T)$ is the following two-sided inequality

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\| \tag{1}$$

(see, e.g., [7]). The right hand side of (1) becomes an equality when T is normal, while the left side becomes an equality when $T^2 = 0$.

Sharpening (1) has received considerable attention in the literature. We refer the reader to [8–16] as a sample of recent attempts to obtain sharper and better bounds of $\omega(T)$.

In [9], the numerical radius of accretive operators (matrices) was investigated, where several bounds were established, including both extensions of known results and newly derived inequalities. This work appears to have stimulated further interest in the study of the numerical radius of accretive and related classes of operators.

On the other hand, Bhunia et al. [10] focused on refining the inequalities in (1), with the Cartesian decomposition playing a central role in these refinements. In particular, the authors employed this decomposition to improve the first inequality in the following celebrated bound

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|, \tag{2}$$

which was originally proved in [13] as a refinement of both inequalities in (1).

Earlier, it was shown in [12] that

$$\omega(A) \leq \frac{1}{2}\big\| |A| + |A^*| \big\|,$$

which is among the sharpest refinements of the second inequality in (1). In fact, this inequality is sharper than the upper bound in (2); however, the symmetric structure of (2) makes it particularly appealing and widely cited.

Further developments were presented in [14], where various relations between an operator and its Cartesian decomposition were utilized to derive several improvements of both (1) and (2). In that work, the study of accretive–dissipative operators led to replacing the factor $\frac{1}{2}$ in (1) with $\frac{1}{\sqrt{2}}$, yielding a significant improvement for this class of operators. More recently, ref. [11] showed that this improved factor is in fact valid for all accretive operators, thereby strengthening numerous known results, particularly those in [8]. It is worth noting that the primary aim in [8] was to establish interpolated inequalities that provide a geometric perspective on several classical bounds.

Now, in alignement with our work, we proceed to the definitions of δ-numerical range and δ-numerical radius.

Definition 1 ([17]). *For the operator $T \in \mathfrak{B}(\mathcal{H})$ and $0 \leq \delta \leq \|T\|$, the δ-numerical range is defined as*

$$W_\delta(T) = cl(\{\langle Tx, x\rangle : x \in \mathcal{H},\ \|x\| = 1,\ \|Tx\| \geq \delta\}),$$

where $cl(\cdot)$ denotes the closure of a set in the complex plane.

It is readily seen that $W_\delta(T)$ is a closed subset of the closure of the classical numerical range, and that

$$W_0(T) = \bigcap_{\delta < \|T\|} W_\delta(T).$$

Furthermore, by a slight modification of a theorem from Dekker [18], it follows that $W_\delta(T)$ is connected. An interesting question is whether $W_\delta(T)$ is convex. This is indeed the case when T is normal or when the underlying Hilbert space is two-dimensional.

Definition 2. *For the operator $T \in \mathfrak{B}(\mathcal{H})$ and $0 \leq \delta \leq \|T\|$, the δ-numerical radius function is defined as*

$$\omega_\delta(T) = \sup\{|\lambda| : \lambda \in W_\delta(T)\}.$$

From the definitions of the δ-numerical radius function, it follows immediately that $\omega_0(T) = \omega(T)$.

Now, let us give the following two new definitions.

Definition 3. *Let $1 \leq p < \infty$ and $\mathbb{A} = (A_n)$ be an operator sequence in $\mathfrak{B}(\mathcal{H})$ satisfying*

$$\sum_{n=1}^{\infty} \|A_n\|^p < \infty.$$

The following definitions are introduced:

1. $\|\mathbb{A}x\|_p = \left(\sum_{n=1}^{\infty} \|A_n x\|^p\right)^{\frac{1}{p}},\ x \in \mathcal{H};$
2. $\|\mathbb{A}\|_p = \sup\left\{\left(\sum_{n=1}^{\infty} \|A_n x\|^p\right)^{\frac{1}{p}} : x \in \mathcal{H},\ \|x\| = 1\right\};$
3. $\langle \mathbb{A}x, x\rangle = (\langle A_n x, x\rangle) \in \mathbb{C}^\infty(\mathbb{C}),\ x \in \mathcal{H};$
4. $|\langle \mathbb{A}x, x\rangle|_p = \left(\sum_{n=1}^{\infty} |\langle A_n x, x\rangle|^p\right)^{\frac{1}{p}},\ x \in \mathcal{H}.$

Definition 4. *Let $1 \leq p < \infty$ and $\mathbb{A} = (A_n)$ be an operator sequence in $\mathfrak{B}(\mathcal{H})$ satisfying*

$$\sum_{n=1}^{\infty} \|A_n\|^p < \infty.$$

The sequential (p, δ, τ)-numerical radius of the operator sequence $\mathbb{A} = (A_n)$ is defined by

$$\omega_{(p,\delta,\tau)}(\mathbb{A}) = \sup\left\{\left(\sum_{n=1}^{\infty} |\langle A_n x, x\rangle|^p\right)^{\frac{1}{p}} : \|x\| = 1,\ \delta \leq \|\mathbb{A}x\|_p \leq \tau\right\}$$

for $0 \leq \delta \leq \tau \leq \|\mathbb{A}\|_p$.

We note that the proposed definition recovers several well-known numerical radii as special cases:

(1) If $\mathbb{A} = (A_n)$, $A_m = 0$, $m \geq 2$ and $\delta = 0$, $\tau = \|\mathbb{A}\|_p = \|\mathbb{A}\|$, then this definition coincides with the definition of classical numerical radius (see [7]);
(2) If $A \in \mathfrak{B}(\mathcal{H})$, $\mathbb{A} = (A, A^*A, 0, 0 \ldots)$, $p = 2$, $\delta = 0$, $\tau = \|A\|_p$, then this definition coincides with the definition of Davis–Wielandt radius of the operator A (see [19,20]);
(3) If $\mathbb{A} = (A_n)$, $A_m = 0$, $m \geq 2$ and $0 \leq \delta \leq \|\mathbb{A}\|_p = \|\mathbb{A}\|$ and $\tau = \|\mathbb{A}\|_p = \|\mathbb{A}\|$, then this definition coincides with the definition of δ- numerical radius (see [21]);
(4) If $\mathbb{A} = (A_n)$, $A_m = 0$, $m \geq n \in \mathbb{N}$ and $p = 2$, $\delta = 0$, $\tau = \|\mathbb{A}\|_p$, then this definition coincides with the definition of p- numerical radius (see [22]);
(5) If $\mathbb{A} = (A_n)$, $A_m = 0$, $m \geq n \in \mathbb{N}$ and $p = 2$, $0 \leq \delta \leq \|\mathbb{A}\|_p$, $\tau = \|\mathbb{A}\|_p$, then this definition coincides with the definition of joint δ-numerical radius (see [23]).

Example 1. *Assume that*

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ A_1 : \mathbb{R}^2 \to \mathbb{R}^2,\ A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},\ A_2 : \mathbb{R}^2 \to \mathbb{R}^2,$$

and $\mathbb{A} = (A_1, A_2, \ldots)$, $p = 2$. *Then*

$$\|\mathbb{A}\|_2 = \sup_{x \in S_1(\mathbb{R}^2)} \left(\|A_1 x\|^2 + \|A_2 x\|^2 \right)^{\frac{1}{2}} = \sup_{x \in S_1(\mathbb{R}^2)} \left(x_1^2 + 4x_2^2 \right)^{\frac{1}{2}} = \sup_{x \in S_1(\mathbb{R}^2)} \left(1 + 3x_2^2 \right)^{\frac{1}{2}} = 2.$$

On the other hand, for $\delta \leq \|\mathbb{A}x\| \leq \tau \leq 2$ *and* $x \in \mathbb{R}^2$ *from the last relation*

$$\delta^2 \leq \|A_1 x\|^2 + \|A_2 x\|^2 \leq \tau^2,$$

i.e.,

$$\delta^2 \leq x_1^2 + 4x_2^2 \leq \tau^2,$$

it is obtained that

$$\frac{\delta^2 - 1}{3} \leq x_2^2 \leq \frac{\tau^2 - 1}{3}.$$

Therefore,

$$\begin{aligned}
\omega^2_{(2,\delta,\tau)}(\mathbb{A}) &= \sup\left\{ \langle A_1 x, x\rangle^2 + \langle A_2 x, x\rangle^2 :\ x \in S_1(\mathbb{R}^2),\ \delta \leq \|\mathbb{A}x\| \leq \tau \right\} \\
&= \sup\left\{ x_1^4 + 4x_2^4 :\ x \in S_1(\mathbb{R}^2),\ \delta \leq \|\mathbb{A}x\| \leq \tau \right\} \\
&= \sup\left\{ 1 + 7x_2^2 \left(x_2^2 - \frac{2}{7} \right) :\ x \in S_1(\mathbb{R}^2),\ \delta \leq \|\mathbb{A}x\| \leq \tau \right\} \\
&= 1 + 7\left(\frac{\tau^2 - 1}{3} \right)\left(\frac{\tau^2 - 1}{3} - \frac{2}{7} \right) \\
&= \frac{7\tau^4 - 20\tau^2 + 16}{3} \\
&= \frac{7}{3}\left(\tau^2 - \frac{10}{7} \right)^2 + \frac{4}{7}.
\end{aligned}$$

Throughout this paper, we consider operator sequences $\mathbb{A} = (A_n) \subset \mathfrak{B}(\mathcal{H})$ and adopt the following conventions for the parameters δ and τ:

(1) If $\delta < 0$ and $0 \leq \tau \leq \|\mathbb{A}\|_p$, then $\omega_{(p,\delta,\tau)}(\mathbb{A}) = \omega_{(p,0,\tau)}(\mathbb{A})$.
(2) If $0 \leq \delta \leq \|\mathbb{A}\|_p$ and $\tau > \|\mathbb{A}\|_p$, then $\omega_{(p,\delta,\tau)}(\mathbb{A}) = \omega_{(p,\delta,\|\mathbb{A}\|_p)}(\mathbb{A})$.
(3) If $\delta < 0$ and $\tau > \|\mathbb{A}\|_p$, then $\omega_{(p,\delta,\tau)}(\mathbb{A}) = \omega_{(p,0,\|\mathbb{A}\|_p)}(\mathbb{A})$.

Also for any operator sequences $\mathbb{A} = (A_n) \subset \mathfrak{B}(\mathcal{H})$, $1 \leq p < \infty$ and $t, s \in \mathbb{R}$, $-\infty < t \leq \|\mathbb{A}\|_p$, $0 \leq s < \infty$ it will be defined

$$\Delta_{(p,t,s)}(\mathbb{A}) = \{x \in S_1(\mathcal{H}) : t \leq \|\mathbb{A}x\| \leq s\},$$

where $S_1(\mathcal{H})$ denotes the unit sphere of the Hilbert space $\mathcal{H}$.

Definition 5. *For any fixed* $-\frac{\pi}{2} < \gamma \leq \alpha < \frac{\pi}{2}$,

(i) $S(\gamma, \alpha) = \{re^{i\varphi} : r \geq 0,\ \gamma \leq \varphi \leq \alpha\} \subset \mathbb{C}$ *is a sector with a vertex at the origin and angles* γ *and* α.

(ii) *The operator* $T : \mathcal{H} \to \mathcal{H}$, $T \in \mathfrak{B}(\mathcal{H})$ *in any Hilbert space* $\mathcal{H}$ *is said to be* (γ, α)*-sectorial if* $W(T) \subseteq S(\gamma, \alpha)$.

(iii) *The class of all* (γ, α)*-sectorial operators on Hilbert space* $\mathcal{H}$ *is denoted by* $Sec_{(\gamma,\alpha)}(\mathcal{H})$.

It is clear that, if $A \in Sec_\alpha(\mathcal{H})$, $0 \leq \alpha \leq \frac{\pi}{2}$, then A is accretive. Remember that an operator $A \in \mathfrak{B}(\mathcal{H})$ is called accretive if $Re\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. It is called dissipative if $Im\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$.

This work is organized as follows: In Section 2, fundamental properties of the sequential (p, δ, τ)-numerical radius function are investigated. The relationships between the sequential (p, δ, τ)-numerical radius of an operator sequence and the sequential (p, δ, τ)-numerical radii of its coordinate operators are then analyzed. In addition, the relationships between the sequential (p, δ, τ)-numerical radius of an operator sequence and the sequential (p, δ, τ)-numerical radii of its real and imaginary parts are presented. Section 3 extends the study to the case in which the coordinate operators are sectorial.

The main goal of this work is to develop and generalize several well-known classical inequalities related to the numerical radius, as studied in the recent literature (see, e.g., [7,8,13,19–26]), with particular emphasis on obtaining refined lower and upper bounds for the numerical radius.

In this direction, we introduce a sequential (p, δ, τ)-numerical radius framework for operator sequences, which provides a unified setting for studying numerical radius-type quantities and allows the derivation of both new inequalities and extensions of several known results.

Moreover, this approach provides a unified perspective on operator sequences and generates improved spectral estimates for the numerical radius.

To be more precise, we prove several fundamental properties of the sequential (p, δ, τ)-numerical radius function for operator sequences. One of our main results is to establish precise relationships between the sequential (p, δ, τ)-numerical radius of an operator sequence and those of its coordinate operators. This generalizes the well-known inequality due to Drnovšek and Müller [24,25], which states that for $\mathbb{A} = (A_1, \ldots, A_n, 0, 0, \ldots) \in \mathfrak{B}(\mathcal{H})$,

$$\frac{1}{2\sqrt{n}}\|\mathbb{A}\| \leq \omega(\mathbb{A}).$$

Another main contribution of this work is to establish relationships between the sequential (p, δ, τ)-numerical radius of an operator sequence and the sequential (p, δ, τ)-numerical radii of its real and imaginary parts. This generalizes the well-known inequality due to Kittaneh [13], which states that for $A \in \mathfrak{B}(\mathcal{H})$,

$$\frac{1}{4}\|A^*A + AA^*\| \leq \omega^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|.$$

Furthermore, we extend our analysis to the case of sectorial coordinate operators, obtaining new structural insights and sharper estimates. The results obtained in this section generalize the inequalities due to Bedrani et al. [9], which state that for $A \in Sec_{\alpha}(\mathcal{H})$, $0 < \alpha < \frac{\pi}{2}$,

$$\omega(A) \leq \sec(\alpha)\omega(ReA) \quad \text{and} \quad \omega(A) \leq \csc(\alpha)\omega(ImA).$$

Moreover, our results improve the inequalities due to Sammour et al. [8], which state that for $A \in Sec_{\alpha}(\mathcal{H})$, $0 \leq \alpha < \dfrac{\pi}{2}$,

$$\|A\| \leq \sqrt{1+\sin^2(\alpha)}\,\omega(A),$$

and which state that for $A, B \in \mathfrak{B}(\mathcal{H})$ satisfying

$$W(A), W(B) \subset \left\{re^{i\varphi} : r \geq 0,\ 0 < \gamma < \alpha < \frac{\pi}{2}\right\},$$

$$\omega(AB) \leq (\cos^2\gamma + \sin^2\alpha)\omega(A)\omega(B).$$

The main advantage of the sequential (p, δ, τ)-numerical radius framework is that it gives a unified way to study different numerical radius-type quantities for operator sequences. It also allows us to recover many known results as special cases and to obtain new inequalities and improved estimates.

As a consequence, the results presented in this paper both generalize and significantly improve upon previously known results, while also contributing new and useful techniques to the study of numerical radius-type functions.

Note that each operator $T \in \mathfrak{B}(\mathcal{H})$ can be expressed in the Cartesian decomposition form as $T = ReT + iImT$, where $ReT = \dfrac{T+T^*}{2}$ and $ImT = \dfrac{T-T^*}{2i}$. Here, T^* denotes the adjoint of T. Throughout this study we denote by $|T| = (T^*T)^{1/2}$ the absolute value of an operator $T \in \mathfrak{B}(\mathcal{H})$.

2. Sequential (p, δ, τ)-Numerical Radius

In this section, some important properties of the sequential (p, δ, τ)-numerical radius function will be investigated.

Theorem 1. *Let $\mathbb{A} = (A_n)$, $\mathbb{B} = (B_n)$ be two operator sequences in $\mathfrak{B}(\mathcal{H})$.*

(1) If $\delta \leq \|\lambda\mathbb{A}\|_p \leq \tau$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$, then

$$\omega_{(p,\delta,\tau)}(\lambda\mathbb{A}) = |\lambda|\omega_{\left(p,\frac{\delta}{|\lambda|},\frac{\tau}{|\lambda|}\right)}(\mathbb{A}).$$

(2) If $\delta \leq \|\mathbb{A}+\mathbb{B}\|_p \leq \tau$, then

$$\omega_{(p,\delta,\tau)}(\mathbb{A}+\mathbb{B}) \leq \omega_{\left(p,\delta-\|\mathbb{B}\|_p,\tau+\|\mathbb{B}\|_p\right)}(\mathbb{A}) + \omega_{\left(p,\delta-\|\mathbb{A}\|_p,\tau+\|\mathbb{A}\|_p\right)}(\mathbb{B}).$$

(3) If $\mathbb{A} = (A_n)$ is a normal operator sequence in $\mathfrak{B}(\mathcal{H})$, then for any $\delta \leq \|\mathbb{A}\|_p \leq \tau$,

$$\omega_{(p,\delta,\tau)}(\mathbb{A}) = \omega_{(p,\delta,\tau)}(\mathbb{A}^*).$$

(4) If $\mathbb{A} = (A_n)$ is a hyponormal operator sequence in $\mathfrak{B}(\mathcal{H})$, then for any $0 \leq \delta \leq \|\mathbb{A}^*\|_f$

$$\omega_{(p,\delta,\tau)}(\mathbb{A}^*) \leq \omega_{(p,\delta,\tau)}(\mathbb{A}).$$

Proof.

(1) For any $x \in \mathcal{H}$, it is clear that

$$|\langle \lambda \mathbb{A}x, x\rangle|_p = |\lambda||\langle \mathbb{A}x, x\rangle|_p \tag{3}$$

On the other hand, for $x \in \Delta_{(p,\delta,\lambda)}(\lambda\mathbb{A})$, $\lambda \in \mathbb{C}\setminus\{0\}$, i.e.,

$$\delta \leq \|\lambda \mathbb{A}x\|_p \leq \tau$$

it is obtained that

$$\frac{\delta}{|\lambda|} \leq \|\mathbb{A}x\|_p \leq \frac{\tau}{|\lambda|}.$$

This means that

$$\Delta_{(p,\delta,\tau)}(\lambda\mathbb{A}) = \Delta_{\left(p,\frac{\delta}{|\lambda|},\frac{\tau}{|\lambda|}\right)}(\mathbb{A}). \tag{4}$$

Therefore, from (3) and (4) it is found that

$$\omega_{(p,\delta,\tau)}(\lambda\mathbb{A}) = |\lambda|\omega_{\left(p,\frac{\delta}{|\lambda|},\frac{\tau}{|\lambda|}\right)}(\mathbb{A}),$$

for $\delta \leq \|\lambda\mathbb{A}\|_p \leq \tau$.

(2) For any $x \in \mathcal{H}$, it is easy to see that

$$|\langle (\mathbb{A}+\mathbb{B})x, x\rangle|_p \leq |\langle \mathbb{A}x, x\rangle|_p + |\langle \mathbb{B}x, x\rangle|_p. \tag{5}$$

On the other hand, if $x \in S_1(\mathcal{H})$ and $\delta \leq \|(\mathbb{A}+\mathbb{B})x\|_p \leq \tau$, then

$$\delta \leq \|(\mathbb{A}+\mathbb{B})x\|_p \leq \|\mathbb{A}x\|_p + \|\mathbb{B}\|_p$$

and

$$\delta \leq \|(\mathbb{A}+\mathbb{B})x\|_p \leq \|\mathbb{A}\|_p + \|\mathbb{B}x\|_p.$$

Hence,

$$\delta - \|\mathbb{B}\|_p \leq \|\mathbb{A}x\|_p \text{ and } \delta - \|\mathbb{A}\|_p \leq \|\mathbb{B}x\|_p. \tag{6}$$

Also, if $x \in S_1(\mathcal{H})$ and $\|(\mathbb{A}+\mathbb{B})x\|_p \leq \tau$, then

$$\begin{aligned} \|\mathbb{A}x\|_p &\leq \|(\mathbb{A}+\mathbb{B})x\|_p + \|\mathbb{B}\|_p \leq \tau + \|\mathbb{B}\|_p, \\ \|\mathbb{B}x\|_p &\leq \|(\mathbb{A}+\mathbb{B})x\|_p + \|\mathbb{A}\|_p \leq \tau + \|\mathbb{A}\|_p. \end{aligned} \tag{7}$$

Therefore, from (6) and (7) it is obtained that for $x \in S_1(\mathcal{H})$

$$\delta - \|\mathbb{B}\|_p \leq \|\mathbb{A}x\|_p \leq \tau + \|\mathbb{B}\|_p \text{ and } \delta - \|\mathbb{A}\|_p \leq \|\mathbb{B}x\|_p \leq \tau + \|\mathbb{A}\|_p.$$

Consequently,

$$\Delta_{(p,\delta,\tau)}(\mathbb{A}+\mathbb{B}) \subset \Delta_{(p,\delta-\|\mathbb{B}\|_p,\tau+\|\mathbb{B}\|_p)}(\mathbb{A}) \cap \Delta_{(p,\delta-\|\mathbb{A}\|_p,\tau+\|\mathbb{A}\|_p)}(\mathbb{B}). \tag{8}$$

Finally, from (5) and (8) it is obtained that

$$\omega_{(p,\delta,\tau)}(\mathbb{A}+\mathbb{B}) \leq \omega_{\left(p,\delta-\|\mathbb{B}\|_p,\tau+\|\mathbb{B}\|_p\right)}(\mathbb{A}) + \omega_{\left(p,\delta-\|\mathbb{A}\|_p,\tau+\|\mathbb{A}\|_p\right)}(\mathbb{B}).$$

(3) For any $x \in \mathcal{H}$

$$|\langle \mathbb{A}x, x\rangle|_p = |\langle \mathbb{A}^*x, x\rangle|_p. \tag{9}$$

As well as for any $x \in S_1(\mathcal{H})$

$$\delta \leq \|\mathbb{A}x\|_p = \left(\sum_{n=1}^{\infty} \|A_n x\|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} \|A_n^* x\|^p\right)^{\frac{1}{p}} = \|\mathbb{A}^* x\|_p \leq \tau.$$

From these relations it is established that

$$\omega_{(p,\delta,\tau)}(\mathbb{A}) = \omega_{(p,\delta,\tau)}(\mathbb{A}^*).$$

(4) From the equality (9) and

$$\delta \leq \|\mathbb{A}^* x\|_p = \left(\sum_{n=1}^{\infty} \|A_n^* x\|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} \|A_n x\|^p\right)^{\frac{1}{p}} = \|\mathbb{A}x\|_p,\ x \in S_1(\mathcal{H}),$$

it is obtained that

$$\omega_{(p,\delta,\tau)}(\mathbb{A}^*) \leq \omega_{(p,\delta,\tau)}(\mathbb{A}).$$

□

Remark 1. *Let* $\mathbb{A} = (A_n),\ A_m = 0,\ m > n,\ p = 2.$

(1) If $\delta = 0$ *and* $\tau \geq \|\mathbb{A}\|_2$, *then Theorem 1 coincides with well-known result found in [26].*

(2) If $0 \leq \delta \leq \|\mathbb{A}\|_2$ *and* $\tau \geq \|\mathbb{A}\|_2$, *then Theorem 1 is a generalization which has a corresponding result in [23].*

Now, we provide a useful auxiliary lemma.

Lemma 1 ([27]). *If* $T \in \mathfrak{B}(\mathcal{H}),\ T \geq 0$, *then for any* $x \in T$,

$$\|Tx\|^2 \leq \|T\|\langle Tx, x\rangle.$$

Theorem 2. *Let* $\mathbb{A} = (A_n)$ *be an operator sequence in* $\mathfrak{B}(\mathcal{H})$ *and* $0 \leq \delta \leq \|\mathbb{A}x\|_p \leq \tau$. *Then*

$$\omega^2_{(p,\delta,\tau)}(\mathbb{A}) \leq a\omega_{(\frac{p}{2},\delta,\tau)}(|\mathbb{A}|),$$

where $a = \sup_{n\geq 1} \| |A_n| \|$.

Proof. For any $x \in \Delta_{(p,\delta,\tau)}(\mathbb{A})$ it is clear that

$$\begin{aligned}
\delta^p &\leq \|\mathbb{A}x\|_p^p \\
&= \sum_{n=1}^{\infty} \|A_n x\|^p \\
&= \sum_{n=1}^{\infty} \langle A_n^* A_n x, x\rangle^{\frac{p}{2}} \\
&= \sum_{n=1}^{\infty} |(A_n^* A_n)^{\frac{1}{2}} x|^p \\
&= \| |\mathbb{A}|x\|^p \\
&\leq \tau^p.
\end{aligned}$$

Hence,

$$\delta \leq \| |\mathbb{A}|x\| \leq \tau,$$

this shows that

$$\Delta_{(p,\delta,\tau)}(\mathbb{A}) = \Delta_{(p,\delta,\tau)}(|\mathbb{A}|). \tag{10}$$

On the other hand, from Lemma 1, for $x \in \Delta_{(p,\delta,\tau)}(\mathbb{A})$ it is clear that

$$\begin{aligned} |\langle \mathbb{A}x, x\rangle|_p^p &= \sum_{n=1}^{\infty} |\langle A_n x, x\rangle|^p \\ &\leq \sum_{n=1}^{\infty} \|A_n x\|^p \\ &= \sum_{n=1}^{\infty} \langle A_n^* A_n x, x\rangle^{\frac{p}{2}} \\ &= \sum_{n=1}^{\infty} \left(\| \, |A_n| x\|^2 \right)^{\frac{p}{2}} \\ &\leq \sum_{n=1}^{\infty} (\| \, |A_n| \, \| \langle |A_n| x, x\rangle)^{\frac{p}{2}} \\ &\leq \sup_{n\geq 1} \| \, |A_n| \, \|^{\frac{p}{2}} \sum_{n=1}^{\infty} \langle |A_n| x, x\rangle^{\frac{p}{2}} \\ &= a^{\frac{p}{2}} |\langle |\mathbb{A}| x, x\rangle|_{\frac{p}{2}}^{\frac{p}{2}}, \end{aligned} \tag{11}$$

where $a = \sup_{n\geq 1} \| \, |A_n| \, \|$. Thus, from (10) and (11) it is obtained that

$$\omega_{(p,\delta,\tau)}^p(\mathbb{A}) \leq a^{\frac{p}{2}} \omega_{(\frac{p}{2},\delta,\tau)}^{\frac{p}{2}}(|\mathbb{A}|),$$

where $a = \sup_{n\geq 1} \| \, |A_n| \, \|$. □

Example 2. *Assume that*

$$A_n = \begin{pmatrix} 0 & a_n \\ a_n & 0 \end{pmatrix}, \; a_n \geq 0, \; n \geq 1,$$

and let $\mathbb{A} = (A_n)$, $p = 2$. *Assume also that* $0 \leq \delta \leq \|\mathbb{A}x\|_2 \leq \tau$, $\sup_{n\geq 1} |a_n| < \infty$, $\sum_{n=1}^{\infty} a_n^2 < \infty$.

In this case,

$$|A_n| = \begin{pmatrix} a_n & 0 \\ 0 & a_n \end{pmatrix}, \; n \geq 1.$$

Then, by Theorem 2 it is obtained that

$$\omega_{(2,\delta,\tau)}^2(\mathbb{A}) \leq \sup_{n\geq 1} |a_n| \omega_{(1,\delta,\tau)}(|\mathbb{A}|).$$

In a special case, if $\delta = 0$, $\tau = \sum_{n=1}^{\infty} a_n^2$, *then from the last inequality we have*

$$\omega_{(2,0,\tau)}^2(\mathbb{A}) \leq \sup_{n\geq 1} |a_n| \sum_{n=1}^{\infty} a_n^2.$$

The following two theorems describe the relationships between the sequential (p,δ,τ)-numerical radius of an operator sequence and the sequential (p,δ,τ)-numerical radii of its coordinate operators.

Theorem 3. *Let $\mathbb{A} = (A_n)$ be an operator sequence in $\mathfrak{B}(\mathcal{H})$ and $0 \leq \delta_n \leq \tau_n$, $n \geq 1$. Then*

$$\omega_{(\delta_n,\tau_n)}(A_n) \leq \omega_{(p,\delta,\tau)}(\mathbb{A}),$$

where $\delta = \left(\sum_{n=1}^{\infty} \delta_n^p\right)^{\frac{1}{p}}$ *and* $\tau = \left(\sum_{n=1}^{\infty} \tau_n^p\right)^{\frac{1}{p}}$.

Proof. It is clear that for $n \geq 1$ and $x \in \mathcal{H}$,

$$|\langle A_n x, x\rangle|^p \leq \sum_{n=1}^{\infty} |\langle A_n x, x\rangle|^p = |\langle \mathbb{A}x, x\rangle|_p^p. \tag{12}$$

On the other hand, since for any $x \in \Delta_{(\delta_n,\tau_n)}(A_n)$, $n \geq 1$

$$\sum_{n=1}^{\infty} \delta_n^p \leq \sum_{n=1}^{\infty} \|A_n x\|^p \leq \sum_{n=1}^{\infty} \tau_n^p,$$

then

$$\Delta_{(\delta_n,\tau_n)}(A_n) \subset \Delta_{(\delta,\tau)}(\mathbb{A}), \tag{13}$$

where $\delta = \left(\sum_{n=1}^{\infty} \delta_n^p\right)^{\frac{1}{p}}$ and $\tau = \left(\sum_{n=1}^{\infty} \tau_n^p\right)^{\frac{1}{p}}$.

Thus, from (12) and (13) it is obtained that for $n \geq 1$

$$\omega_{(\delta_n,\tau_n)}(A_n) \leq \omega_{(p,\delta,\tau)}(\mathbb{A}),$$

where $\delta = \left(\sum_{n=1}^{\infty} \delta_n^p\right)^{\frac{1}{p}}$ and $\tau = \left(\sum_{n=1}^{\infty} \tau_n^p\right)^{\frac{1}{p}}$. □

From Theorem 3, we obtain the next results.

Corollary 1. *Let $\mathbb{A} = (A_n)$, $A_m = 0$, $m > n$, $p = 2$, and $\delta = \left(\sum_{m=1}^{n} \delta_m^p\right)^{\frac{1}{p}} = 0$ and $\tau = \left(\sum_{m=1}^{n} \tau_m^p\right)^{\frac{1}{p}} \geq \|\mathbb{A}\|_p$. Then, for any $m \geq 1$*

$$\frac{1}{m} \sum_{n=1}^{m} \omega_{(\delta_n,\tau_n)}(A_n) \leq \omega_{(p,\delta,\tau)}(\mathbb{A}).$$

Corollary 2. *Let $\mathbb{A} = (A_1, \ldots, A_n, 0, 0, \ldots)$. Then*

$$\frac{1}{\sqrt{n}} \left(\sum_{m=1}^{n} \omega^2(A_n)\right)^{\frac{1}{p}} \leq \omega(\mathbb{A}).$$

On the other hand, by using the following well-known inequality

$$\frac{1}{2}\|A_m\| \leq \omega(A_m),\ 1 \leq m \leq n$$

it is obtained that

$$\frac{1}{2\sqrt{n}} \left(\sum_{m=1}^{n} \|A_m\|^2\right)^{\frac{1}{2}} \leq \omega(\mathbb{A}).$$

Also, it is clear that

$$\|\mathbb{A}\|_2 \leq \left(\sum_{m=1}^{n} \|A_m\|^2\right)^{\frac{1}{2}}.$$

From this inequality and the previous relation it is established that

$$\frac{1}{2\sqrt{n}}\|\mathbb{A}\| \leq \omega(\mathbb{A})$$

which was obtained in [24,25].

Theorem 4. *Let $\mathbb{A} = (A_n)$ be an operator sequence in $\mathfrak{B}(\mathcal{H})$ and $\delta \leq \tau$. Then*

$$\omega^p_{(p,\delta,\tau)}(\mathbb{A}) \leq \sum_{n=1}^{\infty} \omega^p_{(\delta_n,\tau)}(A_n),$$

where $\delta_n = \delta - \alpha_n$, $\alpha_n = \left(\sum\limits_{\substack{m=1 \\ m\neq n}}^{\infty} \|A_m\|^p\right)^{\frac{1}{p}}$, $n \geq 1$.

Proof. For any $x \in \Delta_{(p,\delta,\tau)}(\mathbb{A})$

$$\delta \leq \|\mathbb{A}x\|_p = \left(\sum_{n=1}^{\infty} \|A_n x\|^p\right)^{\frac{1}{p}} \leq \left(\|A_n x\|^p + \alpha_n^p\right)^{\frac{1}{p}} \leq \|A_n x\| + \alpha_n,$$

where $\alpha_n = \left(\sum\limits_{\substack{m=1 \\ m\neq n}}^{\infty} \|A_m\|^p\right)^{\frac{1}{p}}$, $n \geq 1$. Hence,

$$\delta_n = \delta - \alpha_n \leq \|A_n x\|,\ x \in \mathcal{H},\ n \geq 1. \tag{14}$$

Also, for any $n \geq 1$ from the last estimate

$$\|A_n x\| \leq \left(\sum_{n=1}^{\infty} \|A_n x\|^p\right)^{\frac{1}{p}} \leq \tau,\ x \in \mathcal{H}$$

and previous relation (14) it is established that

$$\Delta_{(p,\delta,\tau)}(\mathbb{A}) \subset \bigcap_{n=1}^{\infty} \Delta_{(\delta_n,\tau)}(A_n). \tag{15}$$

On the other hand, for $x \in \mathcal{H}$ it is clear that

$$|\langle \mathbb{A}x, x\rangle|^p_p = \sum_{n=1}^{\infty} |\langle A_n x, x\rangle|^p. \tag{16}$$

Thus, from (15) and (16) it is obtained that

$$\omega^p_{(p,\delta,\tau)}(\mathbb{A}) \leq \sum_{n=1}^{\infty} \omega^p_{(\delta_n,\tau)}(A_n),$$

where $\delta_n = \delta - \alpha_n$, $\alpha_n = \left(\sum\limits_{\substack{m=1 \\ m \neq n}}^{\infty} \|A_m\|^p \right)^{\frac{1}{p}}$, $n \geq 1$. □

The following corollary follows from Theorem 4.

Corollary 3. *Let $\mathbb{A} = (A_1, \ldots, A_n, 0, 0, \ldots)$ be an operator sequence in $\mathfrak{B}(\mathcal{H})$and $\delta \leq \tau$. Then*

$$\omega_{(p,\delta,\tau)}(\mathbb{A}) \leq \left(\sum_{m=1}^{n} \|A_m\|^p \right)^{\frac{1}{p}}.$$

Also, from this, if $n = 1$ (i.e., $\mathbb{A} = A \in \mathfrak{B}(\mathcal{H})$), $\delta = 0$, $\tau > \|A\|$, then we obtain the well-known inequality

$$\omega(A) \leq \|A\|.$$

Theorem 5. *Let $\mathbb{A} = (A_n)$ be an operator sequence in $\mathfrak{B}(\mathcal{H})$ and $0 \leq \delta \leq \tau \leq \|\mathbb{A}\|_p$. Then*

$$\omega^p_{(p,\delta,\tau)}(\mathbb{A}) \leq \left(\sum_{n=1}^{\infty} \| \, |A_n| \, \|^p \right)^{\frac{1}{2}} \omega^{\frac{p}{2}}_{(p,\delta,\tau)}(|\mathbb{A}|).$$

Proof. By using Lemma 1, for any $x \in S_1(\mathcal{H})$

$$\begin{aligned} |\langle \mathbb{A}x, x \rangle|_p^p &= \sum_{n=1}^{\infty} |\langle A_n x, x \rangle|^p \\ &\leq \sum_{n=1}^{\infty} \|A_n x\|^p \\ &= \sum_{n=1}^{\infty} \langle A_n^* A_n x, x \rangle^{\frac{p}{2}} \\ &= \sum_{n=1}^{\infty} \| \, |A_n| x\|^p \\ &\leq \sum_{n=1}^{\infty} \| \, |A_n| \, \|^{\frac{p}{2}} \langle |A_n| x, x \rangle^{\frac{p}{2}} \\ &\leq \left(\sum_{n=1}^{\infty} \| \, |A_n| \, \|^p \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |\langle A_n x, x \rangle|^p \right)^{\frac{1}{2}} \\ &= \left(\sum_{n=1}^{\infty} \| \, |A_n| \, \|^p \right)^{\frac{1}{2}} |\langle |\mathbb{A}| x, x \rangle|_p^{\frac{p}{2}}. \end{aligned} \tag{17}$$

On the other hand, for $x \in S_1(\mathcal{H})$, since

$$\delta^p \leq \|\mathbb{A}x\|_p^p = \sum_{n=1}^{\infty} \|A_n x\|^p = \sum_{n=1}^{\infty} \| \, |A_n| x\|^p = \| \, |\mathbb{A}| \, x\|_p^p \leq \tau^p,$$

then

$$\Delta_{(p,\delta,\tau)}(\mathbb{A}) = \Delta_{(p,\delta,\tau)}(|\mathbb{A}|). \tag{18}$$

Thus, from (17) and (18) it is obtained that

$$\omega^p_{(p,\delta,\tau)}(\mathbb{A}) \leq \left(\sum_{n=1}^{\infty} \| \, |A_n| \, \|^p \right)^{\frac{1}{2}} \omega^{\frac{p}{2}}_{(p,\delta,\tau)}(|\mathbb{A}|).$$

□

Now, let us present a result that establishes the relationships between the sequential (p,δ,τ)-numerical radius of an operator sequence and the sequential (p,δ,τ)-numerical radii of its real and imaginary parts.

Theorem 6. *Let $\mathbb{A}=(A_n)$ be an operator sequence in $\mathfrak{B}(\mathcal{H})$, $2\leq p<\infty$, and $0\leq\delta\leq\tau\leq\|\mathbb{A}\|_p$. Then*

$$\omega^p_{(p,\delta,\tau)}(\mathbb{A})\leq\omega^{\frac{p}{2}}_{\left(\frac{p}{2},\delta^2-\alpha,\tau^2+\alpha\right)}\left(Re^2\mathbb{A}+Im^2\mathbb{A}\right),$$

where $\alpha=\left(\sum\limits_{n=1}^{\infty}\alpha_n^{\frac{p}{2}}\right)^{\frac{2}{p}}$, $\alpha_n=\|ReA_nImA_n-ImA_nReA_n\|$, $n\geq1$.

Proof. For any $x\in S_1(\mathcal{H})$ and $n\geq1$

$$\begin{aligned}|\langle A_nx,x\rangle|^p&=(|\langle ReA_nx,x\rangle|^p+|\langle ImA_nx,x\rangle|^p)^{\frac{p}{2}}\\&\leq(\|ReA_nx\|^p+\|ImA_nx\|^p)^{\frac{p}{2}}\\&=|\langle\left(Re^2A_n+Im^2A_n\right)x,x\rangle|^{\frac{p}{2}}.\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty}|\langle A_nx,x\rangle|^p\leq\sum_{n=1}^{\infty}|\langle\left(Re^2A_n+Im^2A_n\right)x,x\rangle|^{\frac{p}{2}}.$$

Therefore,

$$|\langle\mathbb{A}x,x\rangle|_p^p\leq|\langle\left(Re^2\mathbb{A}+Im^2\mathbb{A}\right)x,x\rangle|_{\frac{p}{2}}^{\frac{p}{2}}.\tag{19}$$

On the other hand, for $x\in S_1(\mathcal{H})$ and $n\geq1$, it follows that

$$\begin{aligned}\|A_nx\|^p&=|\langle A_n^*A_nx,x\rangle|^{\frac{p}{2}}\\&=|\langle(ReA_n+iImA_n)^*(ReA_n+iImA_n)x,x\rangle|^{\frac{p}{2}}\\&=|\langle(ReA_n-iImA_n)(ReA_n+iImA_n)x,x\rangle|^{\frac{p}{2}}\\&=|\langle\Big((Re^2A_n+Im^2A_n)+i(ReA_nImA_n-ImA_nReA_n)\Big)x,x\rangle|^{\frac{p}{2}}\\&=|\langle(Re^2A_n+Im^2A_n)x,x\rangle+i\langle(ReA_nImA_n-ImA_nReA_n)x,x\rangle|^{\frac{p}{2}}\\&\leq\Big(\langle(Re^2A_n+Im^2A_n)x,x\rangle+|\langle(ReA_nImA_n-ImA_nReA_n)x,x\rangle|\Big)^{\frac{p}{2}}\\&\leq\Big(\langle(Re^2A_n+Im^2A_n)x,x\rangle+\|ReA_nImA_n-ImA_nReA_n\|\Big)^{\frac{p}{2}}\\&=\Big(\langle(Re^2A_n+Im^2A_n)x,x\rangle+\alpha_n\Big)^{\frac{p}{2}}\end{aligned}$$

where $\alpha_n=\|ReA_nImA_n-ImA_nReA_n\|$, $n\geq1$.

Hence, for $x\in S_1(\mathcal{H})$ and $0\leq\delta\leq\|\mathbb{A}x\|_p$ it follows that

$$\begin{aligned}\delta^2&\leq\|\mathbb{A}x\|_p^2\\&=\left(\sum_{n=1}^{\infty}\|A_nx\|^p\right)^{\frac{2}{p}}\\&\leq\left(\sum_{n=1}^{\infty}|\langle\left(Re^2A_n+Im^2A_n\right)x,x\rangle+\alpha_n|^{\frac{p}{2}}\right)^{\frac{2}{p}}\\&\leq\left(\sum_{n=1}^{\infty}|\langle\left(Re^2A_n+Im^2A_n\right)x,x\rangle|^{\frac{p}{2}}\right)^{\frac{2}{p}}+\left(\sum_{n=1}^{\infty}\alpha_n^{\frac{p}{2}}\right)^{\frac{2}{p}}\\&\leq\left(\sum_{n=1}^{\infty}\|\left(Re^2A_n+Im^2A_n\right)x\|^{\frac{p}{2}}\right)^{\frac{2}{p}}+\alpha\\&=\|\left(Re^2\mathbb{A}+Im^2\mathbb{A}\right)x\|_{\frac{p}{2}}+\alpha,\end{aligned}$$

where $\alpha = \left(\sum_{n=1}^{\infty} \alpha_n^{\frac{p}{2}}\right)^{\frac{2}{p}}$.

Consequently, from the last relation it is obtained that for $x \in S_1(\mathcal{H})$

$$\delta^2 - \alpha \leq \| \left(Re^2\mathbb{A} + Im^2\mathbb{A} \right) x \|_{\frac{p}{2}}.$$

As well as this, if $x \in S_1(\mathcal{H})$ and $n \geq 1$, then

$$\| \left(Re^2 A_n + Im^2 A_n \right) x \|^{\frac{p}{2}} \leq \left(\|A_n x\|^2 + \alpha_n \right)^{\frac{p}{2}}.$$

Hence, for $x \in S_1(\mathcal{H})$

$$\begin{aligned} \| \left(Re^2\mathbb{A} + Im^2\mathbb{A} \right) x \|_{\frac{p}{2}} &= \left(\sum_{n=1}^{\infty} \| \left(Re^2 A_n + Im^2 A_n \right) x \|^{\frac{p}{2}} \right)^{\frac{2}{p}} \\ &\leq \left(\sum_{n=1}^{\infty} \left(\|A_n x\|^2 + \alpha_n \right)^{\frac{p}{2}} \right)^{\frac{2}{p}} \\ &\leq \left(\sum_{n=1}^{\infty} \|A_n x\|^p \right)^{\frac{2}{p}} + \left(\sum_{n=1}^{\infty} \alpha_n^{\frac{p}{2}} \right)^{\frac{2}{p}} \\ &= \|\mathbb{A}x\|_p^2 + \alpha. \end{aligned}$$

Consequently, if $x \in S_1(\mathcal{H})$ and $\|\mathbb{A}x\|_p \leq \tau$,

$$\| \left(Re^2\mathbb{A} + Im^2\mathbb{A} \right) x \|_{\frac{p}{2}} \leq \tau^2 + \alpha.$$

Therefore, from these calculations it is found that

$$\Delta_{(p,\delta,\tau)}(\mathbb{A}) \subset \Delta_{\left(\frac{p}{2}, \delta^2 - \alpha, \tau^2 + \alpha\right)} \left(Re^2\mathbb{A} + Im^2\mathbb{A} \right). \tag{20}$$

Thus, from (19) and (20), it is obtained that

$$\omega^p_{(p,\delta,\tau)}(\mathbb{A}) \leq \omega^{\frac{p}{2}}_{\left(\frac{p}{2}, \delta^2 - \alpha, \tau^2 + \alpha\right)} \left(Re^2\mathbb{A} + Im^2\mathbb{A} \right).$$

□

Example 3. *Assume that*

$$A_n x = a_n x,\ a_n \in \mathbb{C},\ A_n : \mathbb{C} \to \mathbb{C},\ n \geq 1$$

where $a_n = b_n + ic_n,\ b_n, c_n \in \mathbb{R},\ n \geq 1$. *Let* $p = 2,\ \mathbb{A} = (A_n)$, *and assume that* $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, $\delta = 0,\ \tau = \|\mathbb{A}\|_2 = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{\frac{1}{2}}$. *Then,*

$$Re^2\mathbb{A} + Im^2\mathbb{A} = (b_n^2 + c_n^2).$$

Hence, from (6) it is obtained that

$$\omega_{(2,0,\|\mathbb{A}\|_2)}(\mathbb{A}) \leq \omega_{(1,0,\|\mathbb{A}\|_2^2)}(b_n^2 + c_n^2) = \left(\sum_{n=1}^{\infty} (b_n^2 + c_n^2) \right)^{\frac{1}{2}}.$$

From Theorem 6, we obtain the next results.

Corollary 4. *Let $\mathbb{A} = (A_n)$ be an operator sequence in $\mathfrak{B}(\mathcal{H})$, the operator $A_n \in \mathfrak{B}(\mathcal{H}_n)$ is normal for any $n \geq 1, 2 \leq p < \infty$, and $0 \leq \delta \leq \tau \leq \|\mathbb{A}\|_p$. Then*

$$\omega^2_{(p,\delta,\tau)}(\mathbb{A}) \leq \omega_{\left(\frac{p}{2},\delta^2,\tau^2\right)}\left(Re^2\mathbb{A} + Im^2\mathbb{A}\right).$$

Remark 2. *For any $n = 1, \ldots$ since*

$$Re^2 A_n + Im^2 A_n = \frac{1}{2}(A_n^* A_n + A_n A_n^*),$$

then from (1) of Theorem 1 and Theorem 6 it can be rewritten in the equivalent form

$$\omega^2_{(p,\delta,\tau)}(\mathbb{A}) \leq \frac{1}{2}\omega_{\left(\frac{p}{2},2(\delta^2-\alpha),2(\tau^2+\alpha)\right)}(\mathbb{A}^*\mathbb{A} + \mathbb{A}\mathbb{A}^*)$$

Then, if we take $n = 1,\ \delta = 0,\ \tau > \|A\|,\ A \in \mathfrak{B}(\mathcal{H})$, then it is true that

$$\omega^2_{(p,0,\tau)}(A) = \omega^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|,$$

and this result has been found in [13].

Remark 3. *If $\mathbb{A} = (A_n)$, $\inf\limits_{n\geq 1}\|A_n\| > 0$, for any $n \geq 1$ $A_n \geq 0$ and $2 \leq p < \infty$ in Theorem 6, then from Lemma 1 it is clear that*

$$\begin{aligned}
\delta^2 &\leq \|\mathbb{A}x\|_p^2 \\
&= \left(\sum_{n=1}^{\infty}\|A_n x\|^p\right)^{\frac{2}{p}} \\
&\leq \left(\sum_{n=1}^{\infty}\|A_n\|^{\frac{p}{2}}|\langle A_n x, x\rangle|^{\frac{p}{2}}\right)^{\frac{2}{p}} \\
&\leq \sup_{n\geq 1}\|A_n\|\left(\sum_{n=1}^{\infty}|\langle A_n x, x\rangle|^{\frac{p}{2}}\right)^{\frac{2}{p}} \\
&\leq \lambda\left(\sum_{n=1}^{\infty}|\langle A_n x, x\rangle|^{\frac{p}{2}}\right)^{\frac{2}{p}} \\
&\leq \lambda\left(\sum_{n=1}^{\infty}\langle A_n x, x\rangle\right) \\
&= \lambda\|\langle\sum_{n=1}^{\infty}A_n x, x\rangle\| \\
&= \lambda\|\sum_{n=1}^{\infty}A_n x\|,\ x \in S_1(\mathcal{H})
\end{aligned}$$

where $\lambda = \sup\limits_{n\geq 1}\|A_n\|$. Hence, if for any $x \in S_1(\mathcal{H})$, then the following holds

$$\delta^2 \leq \|\mathbb{A}x\|_p^2 \leq \lambda\|\sum_{n=1}^{\infty}A_n x\|. \tag{21}$$

On the other hand, for any $x \in S_1(\mathcal{H})$ *and* $\|\mathbb{A}x\| \leq \tau$

$$\inf_{n\geq 1}\|A_n\|\sum_{n=1}^{\infty}\|A_nx\| \leq \sum_{n=1}^{\infty}\|A_n\|\|A_nx\| \leq \left(\sum_{n=1}^{\infty}\|A_n\|^q\right)^{\frac{1}{q}}\left(\sum_{n=1}^{\infty}\|A_nx\|^p\right)^{\frac{1}{p}}$$

i.e.,

$$\|\sum_{n=1}^{\infty}A_nx\| \leq \left(\inf_{n\geq 1}\|A_n\|\right)^{-1}\left(\sum_{n=1}^{\infty}\|A_n\|^q\right)^{\frac{1}{q}}\|\mathbb{A}x\|_p \leq \sigma\tau, \tag{22}$$

where $\sigma = \left(\inf_{n\geq 1}\|A_n\|\right)^{-1}\left(\sum_{n=1}^{\infty}\|A_n\|^q\right)^{\frac{1}{q}}$ *and* $q = \dfrac{p}{p-1}$.

Thus, from (21) and (22) it is obtained that

$$\Delta_{(p,\delta,\tau)}(\mathbb{A}) \subset \Delta_{\left(1,\frac{\delta^2}{\lambda},\sigma\tau\right)}\left(\sum_{n=1}^{\infty}A_n\right). \tag{23}$$

In addition, since $A_n \geq 0$, $n \geq 1$, *then for any* $2 \leq p < \infty$ *and* $x \in \mathcal{H}$

$$|\langle \mathbb{A}x, x\rangle|_p^p = \sum_{n=1}^{\infty}|\langle A_nx, x\rangle|^p \leq \left(\sum_{n=1}^{\infty}\langle A_nx, x\rangle\right)^p = \langle\sum_{n=1}^{\infty}A_nx, x\rangle^p. \tag{24}$$

Then, from (23) and (24) we obtain

$$\omega_{(p,\delta,\tau)}(\mathbb{A}) \leq \omega_{\left(1,\frac{\delta^2}{|\lambda|},\sigma\tau\right)}\left(\sum_{n=1}^{\infty}A_n\right).$$

Now, we will present an important result.

Theorem 7. *Let* $\mathbb{A} = (A_n)$ *be an operator sequence in* $\mathfrak{B}(\mathcal{H})$. *Then,*

$$\omega_{(p,\delta,\tau)}(Re\mathbb{A}) \leq \omega_{(p,0,\|\mathbb{A}\|)}(\mathbb{A}),\ for\ \delta \leq \tau \leq \|Re\mathbb{A}\|$$

and

$$\omega_{(p,\delta,\tau)}(Im\mathbb{A}) \leq \omega_{(p,0,\|\mathbb{A}\|)}(\mathbb{A}),\ for\ \delta \leq \tau \leq \|Im\mathbb{A}\|.$$

Proof. For any $x \in \mathcal{H}$, it is clear that

$$|\langle Re\mathbb{A}x, x\rangle|_p = \left(\sum_{n=1}^{\infty}|\langle ReA_nx, x\rangle|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty}|\langle A_nx, x\rangle|^p\right)^{\frac{1}{p}} = |\langle \mathbb{A}x, x\rangle|_p,$$

and similarly

$$|\langle Im\mathbb{A}x, x\rangle|_p \leq |\langle \mathbb{A}x, x\rangle|_p.$$

On the other hand, it is easy to see that

$$\Delta_{(p,\delta,\tau)}(Re\mathbb{A}) \subset S_1(\mathcal{H}) \text{ for } \delta \leq \tau \leq \|Re\mathbb{A}\|_p,$$

$$\Delta_{(p,\delta,\tau)}(Im\mathbb{A}) \subset S_1(\mathcal{H}) \text{ for } \delta \leq \tau \leq \|Im\mathbb{A}\|_p,$$

From these relations, we obtain

$$\omega_{(p,\delta,\tau)}(Re\mathbb{A}) \leq \omega_{(p,0,\|\mathbb{A}\|)}(\mathbb{A}),\ for\ \delta \leq \tau \leq \|Re\mathbb{A}\|$$

and

$$\omega_{(p,\delta,\tau)}(Im\mathbb{A}) \leq \omega_{(p,0,\|\mathbb{A}\|)}(\mathbb{A}),\ for\ \delta \leq \tau \leq \|Im\mathbb{A}\|.$$

□

From Theorem 7, we obtain the next results.

Corollary 5. *Let* $\mathbb{A} = (A, 0, 0, \ldots)$, $A \in \mathfrak{B}(\mathcal{H})$, $\delta = 0$ *and* $\tau \geq \max\{\|ReA\|_p, \|ImA\|_p\}$. *Then,*

$$\omega(Re^2A + Im^2A) \leq \omega(Re^2A) + \omega(Im^2A) \leq \omega^2(A) + \omega^2(A) \leq 2\omega^2(A).$$

Therefore, from the last relation it is found that for $A \in \mathfrak{B}(\mathcal{H})$

$$\frac{1}{4}\|A^*A + AA^*\| \leq \omega^2(A).$$

This result has been found by Kittaneh in [13].

3. Sectorial Case of Coordinate Operators

In this section, the sequential (p, δ, τ)-numerical radius function will be investigated in the case when the coordinate operators are sectorial.

Theorem 8. *Let* $\mathbb{A} = (A_n)$ *be an operator sequence in* $\mathfrak{B}(\mathcal{H})$ *and* $0 \leq \delta \leq \tau \leq \|\mathbb{A}\|_p$.

(1) If $A_n \in Sec_{\alpha_n}(\mathcal{H})$, $0 \leq \alpha_n < \frac{\pi}{2}$, $n \geq 1$ *and* $\alpha_{\sup} = \sup_{n\geq 1} \alpha_n < \frac{\pi}{2}$, *then*

$$\omega_{(p,\delta,\tau)}(\mathbb{A}) \leq \sec(\alpha_{\sup})\omega_{(p,\delta_r,\tau_r)}(Re\mathbb{A}),$$

where $\delta_r = \delta - \|Im\mathbb{A}\|_p$ *and* $\tau_r = \tau + \|Im\mathbb{A}\|_p$.

(2) If $A_n \in Sec_{\alpha_n}(\mathcal{H})$, $0 < \alpha_n < \frac{\pi}{2}$, $n \geq 1$ *and* $\alpha_{\inf} = \inf_{n\geq 1} \alpha_n > 0$, *then*

$$\omega_{(p,\delta,\tau)}(\mathbb{A}) \leq \csc(\alpha_{\inf})\omega_{(p,\delta_i,\tau_i)}(Im\mathbb{A}),$$

where $\delta_i = \delta - \|Re\mathbb{A}\|_p$ *and* $\tau_i = \tau + \|Re\mathbb{A}\|_p$.

Proof.

(1) For each $x \in S_1(\mathcal{H})$ it follows that

$$\begin{aligned}\delta \leq \|\mathbb{A}x\|_p &= \left(\sum_{n=1}^{\infty} \|A_m x\|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=1}^{\infty} |\ \|ReA_n x\| + \|ImA_n x\|\ |^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=1}^{\infty} \|ReA_n x\|\|^p\right)^{\frac{1}{p}} + \|Im\mathbb{A}\|_p.\end{aligned}$$

Then, it is clear that for each $x \in S_1(\mathcal{H})$

$$\delta - \|Im\mathbb{A}\|_p \leq \|Re\mathbb{A}x\|_p,$$

and setting $\delta_r = \delta - \|Im\mathbb{A}\|_p$ yields

$$\delta_r \leq \|Re\mathbb{A}x\|_p.$$

On the other hand, for $x \in S_1(\mathcal{H})$

$$\|Re\mathbb{A}x\|_p \leq \|\mathbb{A}x\|_p + \|Im\mathbb{A}\|_p \leq \tau + \|Im\mathbb{A}\|_p = \tau_r.$$

Hence,

$$\Delta_{(p,\delta,\tau)}(\mathbb{A}) \subset \Delta_{(p,\delta_r,\tau_r)}(Re\mathbb{A}). \tag{25}$$

As well as this, for $x \in \mathcal{H}$

$$\begin{aligned} |\langle \mathbb{A}x, x\rangle|_p &= \left(\sum_{n=1}^{\infty} |\langle A_n x, x\rangle|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{n=1}^{\infty} \left(|\langle ReA_n x, x\rangle|^2 + |\langle ImA_n x, x\rangle|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=1}^{\infty} \left(\left(1+\tan^2(\alpha_n)\right)|\langle ReA_n x, x\rangle|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\ &\leq \left(1+\tan^2(\alpha_{\sup})\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |\langle ReA_n x, x\rangle|^p\right)^{\frac{1}{p}} \\ &= \sec(\alpha_{\sup})|\langle Re\mathbb{A}x, x\rangle|_p, \end{aligned} \tag{26}$$

where $\alpha_{\sup} = \sup_{n\geq 1} \alpha_n$.
Thus, from (25) and (26) it is obtained that

$$\omega_{(p,\delta,\tau)}(\mathbb{A}) \leq \sec(\alpha_{\sup})\omega_{(p,\delta_r,\tau_r)}(Re\mathbb{A}),$$

where $\delta_r = \delta - \|Im\mathbb{A}\|_p$ and $\tau_r = \tau + \|Im\mathbb{A}\|_p$.

(2) Assume that $x \in \Delta_{(p,\delta,\tau)}(\mathbb{A})$. For any $x \in S_1(\mathcal{H})$, it follows that

$$\begin{aligned} \delta \leq \|\mathbb{A}x\|_p &= \left(\sum_{n=1}^{\infty} \|A_n x\|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=1}^{\infty} (\|ReA_n x\| + \|ImA_n x\|)^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=1}^{\infty} \|ReA_n x\|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} \|ImA_n x\|^p\right)^{\frac{1}{p}} \\ &\leq \|Re\mathbb{A}\|_p + \|Im\mathbb{A}x\|_p. \end{aligned}$$

Hence, from the last calculations it is obtained that for $x \in S_1(\mathcal{H})$

$$\delta - \|Re\mathbb{A}\|_p \leq \|Im\mathbb{A}x\|_p,$$

and setting $\delta_i = \delta - \|Re\mathbb{A}\|_p$ yields

$$\delta_i \leq \|Im\mathbb{A}x\|_p. \tag{27}$$

On the other hand, for any $x \in S_1(\mathcal{H})$

$$\|Im\mathbb{A}x\|_p \leq \|\mathbb{A}x\|_p + \|Re\mathbb{A}x\|_p \leq \tau + \|Re\mathbb{A}x\|_p$$

and setting $\tau_i = \tau + \|Re\mathbb{A}\|_p$ yields

$$\tau_i \geq \|Im\mathbb{A}x\|_p. \tag{28}$$

Consequently, from (27) and (28) it is established that

$$\Delta_{(p,\delta,\tau)}(\mathbb{A}) \subset \Delta_{(p,\delta_i,\tau_i)}(Im\mathbb{A}) \tag{29}$$

Also, since $A_n \in Sec_{\alpha_n}(\mathcal{H})$, $0 < \alpha_n < \frac{\pi}{2}$, $n \geq 1$ it is clear that for $x \in \mathcal{H}$

$$|\langle ReA_nx, x\rangle| \leq \cot(\alpha_n)|\langle ImA_nx, x\rangle|.$$

Using this fact for $x \in \mathcal{H}$ it is obtained that

$$\begin{aligned}
|\langle \mathbb{A}x, x\rangle|_p &= \left(\sum_{n=1}^{\infty} |\langle A_nx, x\rangle|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{n=1}^{\infty} \left(|\langle ReA_nx, x\rangle|^2 + |\langle ImA_nx, x\rangle|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{n=1}^{\infty} \left(\left(1 + \cot^2(\alpha_n)\right)|\langle ImA_nx, x\rangle|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
&\leq \left(1 + \cot^2(\alpha_{\inf})\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |\langle ImA_nx, x\rangle|^p\right)^{\frac{1}{p}} \\
&= \csc(\alpha_{\inf})|\langle Im\mathbb{A}x, x\rangle|_p,
\end{aligned} \tag{30}$$

where $\alpha_{\inf} = \inf_{n\geq 1} \alpha_n$.
Thus, from (29) and (30) it is found that

$$\omega_{(p,\delta,\tau)}(\mathbb{A}) \leq \csc(\alpha_{\inf})\omega_{(p,\delta_i,\tau_i)}(Im\mathbb{A}),$$

where $\delta_i = \delta - \|Re\mathbb{A}\|_p$ and $\tau_i = \tau + \|Re\mathbb{A}\|_p$.

□

Example 4. *Assume that*

$$A_n = \begin{pmatrix} a_n + ib_n & 0 \\ 0 & a_n + ib_n \end{pmatrix} : \mathbb{C}^2(\mathbb{C}),\ n \geq 1,$$

where $a_n, b_n \in \mathbb{R}$, $a_n \geq 1$, *and* $|b_n| \leq \sqrt{3}$. *Let* $\mathbb{A} = (A_n)$, $p = 2$. *Then,*

$$\|\mathbb{A}\|_2 = \left(\sum_{n=1}^{\infty} a_n^2 + b_n^2\right)^{\frac{1}{2}}$$

and

$$\frac{|\langle Im\mathbb{A}x, x\rangle|_2}{\|\langle Re\mathbb{A}x, x\rangle\|_2} \leq \sqrt{3} = \tan\left(\frac{\pi}{3}\right),$$

which implies that $\mathbb{A} \in Sec_{\left(\frac{\pi}{3}\right)}(\mathbb{C}^2)$. *Define*

$$\delta_r = \delta - \|Im\mathbb{A}\|_p \text{ and } \tau_r = \tau + \|Im\mathbb{A}\|_p$$

where $\|Im\mathbb{A}\|_p = \left(\sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}}$. *Therefore, by (1) of Theorem 8 it is obtained that*

$$\omega_{(2,\delta,\tau)}(\mathbb{A}) \leq 2\omega_{(2,\delta_r,\tau_r)}(Re\mathbb{A}) = 2\left(\sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{2}}.$$

If we take $p = 2$, $\delta = 0$, and $\tau \geq \|\mathbb{A}\|_2$ in Theorem 8, we obtain the following corollary.

Corollary 6. *Let* $\mathbb{A} = (A_n)$ *be an operator sequence in* $\mathfrak{B}(\mathcal{H})$.

(1) If $A_m = 0$, $m > n$, $A_k \in Sec_{\alpha_k}(\mathcal{H})$, $0 \leq \alpha_k < \frac{\pi}{2}$, $k = 1, \ldots, n$, *then*

$$\omega(\mathbb{A}) \leq \sec(\alpha_{\sup})\omega(Re\mathbb{A}),$$

where $\alpha_{\sup} = \sup_{1\leq k\leq n} \alpha_k$.

(2) If $A_m = 0$, $m > n$, $A_k \in Sec_{\alpha_k}(\mathcal{H})$, $0 < \alpha_k < \frac{\pi}{2}$, $k = 1, \ldots, n$, *then*

$$\omega(\mathbb{A}) \leq \csc(\alpha_{inf})\omega(Im\mathbb{A}),$$

where $\alpha_{inf} = \inf_{1\leq k\leq n} \alpha_k$.

Remark 4. *If we take* $n = 1$ *in Corollary 6, then for* $A \in Sec_{\alpha}(\mathcal{H})$, $0 < \alpha < \frac{\pi}{2}$

$$\omega(A) \leq \sec(\alpha)\omega(ReA) \text{ and } \omega(A) \leq \csc(\alpha)\omega(ImA),$$

which was proved in [9].

Theorem 9. *Let* $\mathbb{A} = (A_n)$ *be an operator sequence in* $\mathfrak{B}(\mathcal{H})$ *and* $A_n \in \mathfrak{B}(\mathcal{H})$, $n \geq 1$ *be an accretive–dissipative operator such that*

$$W(A_n) \subset \left\{re^{i\varphi} : r \geq 0,\ 0 < \gamma_m \leq \varphi \leq \alpha_m < \frac{\pi}{2}\right\}, \text{ for each } n \geq 1.$$

Then,

(1) for any $\delta \leq \tau \leq \|Re\mathbb{A}\|_p$,

$$\omega_{(p,\delta,\tau)}(Re\mathbb{A}) \leq \cos(\gamma_{\inf})\omega_{(p,0,\|\mathbb{A}\|_p)}(\mathbb{A}),$$

where $\gamma_{\inf} = \inf_{n\geq 1} \gamma_n$,

(2) for any $\delta \leq \tau \leq \|Im\mathbb{A}\|_p$,

$$\omega_{(p,\delta,\tau)}(Im\mathbb{A}) \leq \sin(\alpha_{\sup})\omega_{(p,0,\|\mathbb{A}\|_p)}(\mathbb{A}),$$

where $\alpha_{\sup} = \sup_{n\geq 1} \alpha_n$.

Proof.

(1) For any $x \in \mathcal{H}$ it is clear that

$$|\langle \mathbb{A}x, x\rangle|_p^p = \sum_{n=1}^{\infty} |\langle A_n x, x\rangle|^p \leq \sum_{n=1}^{\infty} \left(|\langle ReA_n x, x\rangle|^2 + |\langle ImA_n x, x\rangle|^2\right)^{\frac{p}{2}}.$$

On the other hand, since

$$|\langle ImA_nx, x\rangle| \geq \tan(\gamma_n)|\langle ReA_nx, x\rangle|,\ n \geq 1$$

and the tangent function is monotonically increasing on $\left[0, \frac{\pi}{2}\right)$, then

$$|\langle \mathbb{A}x, x\rangle|_p^p \geq \sum_{n=1}^{\infty}(1+\tan^2(\gamma_n))^{\frac{p}{2}}|\langle ReA_nx, x\rangle|^p \geq (1+\tan^2(\gamma_{\inf}))^{\frac{p}{2}}|\langle Re\mathbb{A}x, x\rangle|_p,$$

i.e., for $x \in \mathcal{H}$

$$|\langle \mathbb{A}x, x\rangle|_p \geq \sec(\gamma_{\inf}))|\langle Re\mathbb{A}x, x\rangle|_p, \tag{31}$$

where $\gamma_{\inf} = \inf_{n\geq 1}\delta_n$.
As well as this, for any $\delta \leq \tau \leq \|Re\mathbb{A}x\|_p$ it is clear that

$$\Delta_{(p,\delta,\tau)}(Re\mathbb{A}) \subset S_1(\mathcal{H}). \tag{32}$$

Hence, from (31) and (32) it is found that

$$\omega_{(p,\delta,\tau)}(Re\mathbb{A}) \leq \cos(\gamma_{\inf})\omega_{(p,0,\|\mathbb{A}\|_p)}(\mathbb{A}),$$

(2) For any $x \in \mathcal{H}$ it is clear that

$$|\langle \mathbb{A}x, x\rangle|_p^p = \sum_{n=1}^{\infty}|\langle A_nx, x\rangle|^p = \sum_{n=1}^{\infty}\Big(|\langle ReA_nx, x\rangle|^2 + |\langle ImA_nx, x\rangle|^2\Big)^{\frac{p}{2}}.$$

On the other hand, since

$$|\langle ImA_nx, x\rangle| \leq \tan(\alpha_n)|\langle ReA_nx, x\rangle|,\ n \geq 1$$

and the cotangent function is monotonically decreasing on $\left(0, \frac{\pi}{2}\right)$, then

$$|\langle \mathbb{A}x, x\rangle|_p^p \geq \sum_{n=1}^{\infty}(1+\cot^2(\alpha_m))^{\frac{p}{2}}|\langle ImA_nx, x\rangle|^p \geq \csc^p(\alpha_{\sup})|\langle Im\mathbb{A}x, x\rangle|_p^p,$$

i.e., for $x \in \mathcal{H}$

$$|\langle Im\mathbb{A}x, x\rangle|_p \leq \sin(\alpha_{\sup})|\langle \mathbb{A}x, x\rangle|_p, \tag{33}$$

where $\alpha_{\sup} = \sup_{n\geq 1}\alpha_n$.
As well as this, for any $\delta \leq \tau \leq \|Im\mathbb{A}x\|_p$ it is clear that

$$\Delta_{(p,\delta,\tau)}(Im\mathbb{A}) \subset S_1(\mathcal{H}). \tag{34}$$

Hence, from (33) and (34) it is found that

$$\omega_{(p,\delta,\tau)}(Im\mathbb{A}) \leq \sin(\alpha_{\sup})\omega_{(p,0,\|\mathbb{A}\|_p)}(\mathbb{A}).$$

□

Corollary 7. *Under the condition of Theorem 9, if* $\delta \leq \tau \leq \min\{\|Re\mathbb{A}\|_p, \|Im\mathbb{A}\|_p\}$, *then*

$$\omega^2_{(p,\delta,\tau)}(Re\mathbb{A}) + \omega^2_{(p,\delta,\tau)}(Im\mathbb{A}) \leq \Big(\cos^2(\gamma_{\inf}) + \sin^2(\alpha_{\sup})\Big)\omega^2_{(p,0,\|\mathbb{A}\|_p)}(\mathbb{A}).$$

Corollary 8. *Under the condition of Theorem 9, if* $A_m = 0$, $m > n$, $\delta = 0$, *and* $\tau \geq \min\{\|Re\mathbb{A}\|_p, \|Im\mathbb{A}\|_p\}$, *then from Corollary 7*

$$\frac{\|Re\mathbb{A}\|_p^2 + \|Im\mathbb{A}\|_p^2}{\cos^2(\gamma_{\inf}) + \sin^2(\alpha_{\sup})} \leq \omega_p^2(\mathbb{A}).$$

In addition, if $\mathbb{A} = (A, 0, 0, \ldots)$, $A \in \mathfrak{B}(\mathcal{H})$, *then from the last relation it is obtained that for* $0 < \gamma < \alpha < \frac{\pi}{2}$

$$\frac{\|ReA\|^2 + \|ImA\|^2}{\cos^2(\gamma) + \sin^2(\alpha)} \leq \omega^2(A).$$

On the other hand, it is well known that from [8]

$$\|A\|^2 \leq \|ReA\|^2 + \|ImA\|^2.$$

Therefore, from the previous inequality it is obtained that

$$\|A\| \leq \sqrt{\cos^2(\gamma) + \sin^2(\alpha)}\omega(A). \tag{35}$$

It must be noted that in the matrix case of $A \in Sec_\alpha(\mathcal{H})$, $0 \leq \alpha < \frac{\pi}{2}$ *the following main result was obtained in [8],*

$$\|A\| \leq \sqrt{1 + \sin^2(\alpha)}\omega(A).$$

Therefore, the previous estimate shows that in the accretive–dissipative case the last inequality can be improved.

It must be noted that for A, $B \in \mathfrak{B}(\mathcal{H})$ *with condition*

$$W(A),\ W(B) \subset \left\{re^{i\varphi} : r \geq 0,\ 0 < \gamma < \alpha < \frac{\pi}{2}\right\}$$

from (35), the following interpolated inequality is implied

$$\omega(AB) \leq (\cos^2\gamma + \sin^2\alpha)\omega(A)\omega(B).$$

This shows that the analogous result in [8] can be improved.

4. Conclusions

In this study, we investigated several fundamental properties of the sequential (p, δ, τ)-numerical radius function. First, we examined the relationships between the sequential (p, δ, τ)-numerical radius of an operator sequence and the corresponding numerical radii of its coordinate operators. We further analyzed the connections between the sequential (p, δ, τ)-numerical radius and the numerical radii of the real and imaginary parts of the associated operator sequences.

Finally, these results were extended to the case where the coordinate operators are sectorial, which provided additional structural insights into the behavior of the sequential (p, δ, τ)-numerical radius function. The obtained results generalize several well-known classical results on the numerical radius function in the recent literature and contribute new and useful perspectives to the existing theory.

Future research may focus on extending these results to more general classes of operators, as well as exploring possible applications of the sequential (p, δ, τ)-numerical radius in other areas of functional analysis and operator theory.

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