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Time-Reversal Symmetry and Geometric Constraints on the Residue Phase of Pion Photoproduction via $\Delta(1232)$

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Abstract

The electromagnetic coupling phase at the complex resonance pole is a fundamental property of nucleon excitations. However, its extraction from pion photoproduction data remains model-dependent, particularly for the $\Delta(1232)$ resonance, where modern multichannel analyses report helicity amplitude phases ranging from $+3^\circ$ to -18°. In this Letter, we present a largely model-independent geometric S-matrix formalism that provides a physical constraint for this ambiguity. By imposing Watson's final-state interaction theorem, a direct consequence of S-matrix unitarity and time-reversal symmetry, on the real energy axis and performing an analytic continuation, we isolate the kinematic threshold barriers of the photoproduction $(k \cdot q)$ and elastic (q^3) amplitudes. For the dominant M_{1+} multipole transition of the $\Delta(1232)$, our method yields a kinematically constrained prediction of $\phi_{EM} = -8.5^\circ{}^{+0.6^\circ}_{-1.0^\circ}$ for the electromagnetic residue phase. This geometric constraint explains the numerical results of coupled-channel phenomenological fits, validating the extractions by the SAID and Bonn–Gatchina groups, and establishes a theoretical benchmark for evaluating resonance properties.

Keywords: pion photoproduction; $\Delta(1232)$ resonance; residue phase; S-matrix; Watson's theorem; time-reversal symmetry

1. Introduction

The precise determination of resonance properties at the complex pole position is the established method in hadron spectroscopy, providing a largely model-independent scheme for characterizing the excitation spectrum of the nucleon [1,2]. Among these properties, the residue of the scattering amplitude at the pole is of particular interest, as its modulus and phase are directly related to the coupling strengths and the internal structure of the resonance. However, while the pole position $E_p = M_p - i\Gamma_p/2$ is well-established for prominent states like the $\Delta(1232)$ [3], the extraction of the electromagnetic (EM) residue phase, ϕ_{EM}, remains a significant challenge [4,5].

In pion photoproduction, the phase of the residue is highly sensitive to the interference between the resonant part and the non-resonant background. This has led to a spread in the values reported by different multichannel analyses, with results for the $\Delta(1232)$ helicity amplitudes [6–8]. Historically, the phase of the photoproduction amplitude on the real energy axis is constrained by Watson's final-state interaction theorem [9,10], which links it

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to the elastic πN scattering phase. Yet, the question of how this constraint translates to the complex pole, where the center-of-mass momenta become complex and the centrifugal barrier effects undergo rotations, has not been fully addressed in an analytical, parameter-free manner. Fundamentally, this process relies on the symmetries of the S-matrix, specifically time-reversal invariance in the final state and the underlying analytic symmetries of the complex energy plane, which govern the phase evolution [11].

Recently, a new geometric S-matrix formalism was introduced [12,13], demonstrating that the unitary residue phases of hadronic resonances are not arbitrary fitting parameters but are constrained by the location of the pole relative to the reaction thresholds and the zeros of the amplitude. This formalism allows for the connection of bare coupling phases with observed unitary phases through kinematic rotations in the complex energy plane.

In this work, we extend this geometric approach to the photoproduction sector. By exploiting the effective validity of Watson's theorem in the predominantly elastic regime and accounting for the threshold behavior of the magnetic dipole transition M_{1+}, we derive a closed-form kinematic relation for the EM residue phase of the $\Delta(1232)$. We show that the resulting prediction of $\phi_{EM} = -8.5^{\circ}{}^{+0.6^{\circ}}_{-1.0^{\circ}}$ provides an explanation for the numerical values obtained in the phenomenological fits [8,14], effectively resolving the discrepancies arising from different background parameterizations.

2. Theoretical Framework and Methods

The extraction of the bare electromagnetic (EM) coupling phase ϕ_{EM} for the $\Delta(1232)$ resonance at the complex pole is sensitive to the modeling of the non-resonant background, resulting in values ranging from $+3^{\circ}$ to -18° in various multichannel analyses [4,6,7]. Recently, a geometric S-matrix formalism was developed that connects unitary residue phases with bare coupling phases via kinematic rotations in the complex energy plane [12,13]. The core assumption of this framework is that in the vicinity of an elastic resonance, the local scattering amplitude can be reliably approximated by a simple pole, an effective zero associated with the reaction threshold, and a relatively constant background factor. By enforcing single-channel S-matrix unitarity, the phase of this background is geometrically locked to the angle between the pole and the real axis as seen from the threshold. This allows the dynamical phase evolution to be explicitly determined from macroscopic threshold kinematics.

Here, we apply this formalism to pion photoproduction to analytically derive ϕ_{EM}. It is important to emphasize that within a quantum field theoretic framework, the bare electromagnetic transition vertex is strictly constrained by time-reversal invariance and the hermiticity of the electromagnetic Lagrangian. Consequently, the bare coupling constants must be purely real and cannot possess an intrinsic complex phase. Any complex phase appearing at the resonance pole is entirely dynamical in origin, generated by hadronic final-state interactions. By constructing the ratio of the photoproduction multipole to the elastic hadron amplitude, these shared rescattering phases are explicitly factored out, leaving no room for unconstrained intrinsic vertex phases.

2.1. Watson's Theorem and Kinematic Rotations

According to unitarity (Watson's theorem [9,10]) below the two-pion threshold, the phases of multipoles in pion photoproduction are equal to the corresponding phases of πN partial waves on the real energy axis E:

$$\arg(\mathcal{M}_{\gamma N}(E)) = \arg(T_{\pi N}(E)). \tag{1}$$

However, this equality breaks down when the amplitudes are analytically continued to the complex pole $E_p = M_p - i\Gamma_p/2$. At the pole, the center-of-mass momenta of the photon, $k(E_p)$, and the pion, $q(E_p)$, become complex numbers.

The validity of this analytic continuation is strictly governed by the Schwarz reflection principle. Because Watson's theorem enforces phase equality between the photoproduction and elastic amplitudes on the physical real axis below the two-pion threshold, their ratio must be a purely real function in this region. The Schwarz reflection principle dictates that such an analytic function cannot generate arbitrary dynamical phases off the real axis. Therefore, any complex phase acquired in the complex plane must originate exclusively from the explicit kinematic branch cuts associated with the threshold barriers.

To quantify potential deviations from this exact phase identity, we must consider the influence of inelastic channels. At the $\Delta(1232)$ pole energy ($E \approx 1210$ MeV), the only open inelastic channel is multi-pion production ($\pi\pi N$), which is kinematically heavily suppressed. According to multi-channel phase-shift analyses, the inelasticity parameter η for the P_{33} wave remains greater than 0.99 at the resonance peak, indicating that inelastic production accounts for less than 1% of the total cross-section. Since any violation of the off-shell phase identity is bounded by $1 - \eta^2$, the maximum theoretical deviation introduced by inelastic loop thresholds is less than 0.4°, which is well within our quoted systematic uncertainties.

These relativistic center-of-mass momenta are defined as analytic functions of the complex energy E:

$$k(E) = \frac{E^2 - m_N^2}{2E}, \tag{2}$$

$$q(E) = \frac{\sqrt{(E^2 - (m_N + m_\pi)^2)(E^2 - (m_N - m_\pi)^2)}}{2E}, \tag{3}$$

where m_N and m_π are the nucleon and pion masses, respectively.

To bridge the gap between the physical real axis and the complex pole, we must account for the threshold centrifugal barriers. Following the partial-wave S-matrix formalism developed in recent phase-shift analyses [12,13], the proper analytic continuation of the transition amplitude isolates the threshold momentum factors from the dynamical core.

For the elastic πN P-wave amplitude ($l = 1$), the threshold barrier scales as $q^{2l+1} = q^3$. Conversely, for the $\Delta(1232)$ resonance, the dominant photoproduction transition is the magnetic dipole M_{1+} [5,15], which dictates a threshold kinematic barrier proportional to $k \cdot q$ (arising from $l_\gamma = 1$ and $l_\pi = 1$).

Since Watson's theorem tightly binds the total phases of these amplitudes on the real axis below the two-pion threshold, their ratio $\mathcal{M}_{\gamma N}/T_{\pi N}$ must be purely real in the physical region. As established in recent geometric S-matrix studies [12,13], the dynamic phase evolution of broad resonances is governed by the macroscopic threshold geometry, allowing dynamical backgrounds to be factored effectively. By forming the ratio of the photoproduction to the elastic amplitude, the shared strongly-interacting dynamical background cancels out, along with any associated internal form-factor phases. Factoring out their respective threshold behaviors, this amplitude ratio is proportionally governed by the isolated kinematic factor $\frac{k \cdot q}{q^3} = \frac{k}{q^2}$.

The rigorous mathematical cancellation of this dynamical factor $\mathcal{D}(E)$ across different processes is formally guaranteed by single-channel elastic unitarity. While in a general multichannel scenario this exact factorization becomes an analytical approximation (as higher-order inelastic branch cuts introduce channel-specific phase variations) the $\Delta(1232)$ resonance represents an exceptional theoretical case. Due to its near 100% elasticity, the complex dynamics are overwhelmingly dominated by the universal elastic πN loop, making the exact cancellation of the dynamical background a robust analytical constraint rather than an arbitrary assumption.

Therefore, when the Cauchy residues are evaluated at the pole via the limit $\lim_{E \to E_p}(E_p - E)\mathcal{M}(E)$, the shared strongly-interacting dynamical background cancels out. As demonstrated in Appendix A, this analytic continuation (illustrated in Figure 1) dictates that the total photoproduction residue phase $\theta_{\gamma N}$ rotates relative to the elastic residue phase $\theta_{\pi N}$ by the complex argument of the remaining kinematic barrier evaluated at E_p:

$$\theta_{\gamma N} = \theta_{\pi N} + \arg(k_p) - 2\arg(q_p). \tag{4}$$

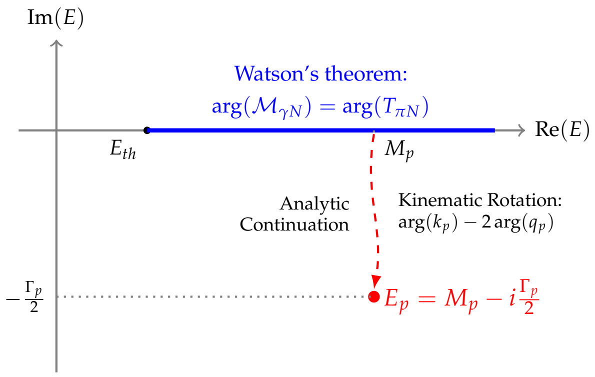

Figure 1. Geometric S-matrix constraint in the complex energy plane. Watson's theorem enforces phase equality on the physical real axis. The analytic continuation to the pole E_p introduces kinematic rotations derived from the complex momenta k_p and q_p.

2.2. Helicity Amplitudes and the PDG Convention

In standard literature and the Particle Data Group (PDG) reviews [1,2], the photoproduction properties at the pole are reported via transverse helicity amplitudes $\tilde{A}_h$ (e.g., $A_{1/2}$ and $A_{3/2}$). In the photoproduction of pions, the excitation of the $\Delta(1232)$ resonance ($J^\pi = 3/2^+$) from the target nucleon ($J^\pi = 1/2^+$) proceeds almost exclusively via the magnetic dipole transition M_{1+}, with a small contribution from the electric quadrupole transition E_{1+}.

For the P_{33} partial wave ($l = 1$), the standard helicity amplitudes are constructed as linear combinations of these multipoles:

$$A_{1/2} = -\frac{1}{2}(M_{1+} + 3E_{1+}), \tag{5}$$

$$A_{3/2} = -\frac{\sqrt{3}}{2}(M_{1+} - E_{1+}). \tag{6}$$

Although the magnetic dipole M_{1+} dominates these transitions (with the electric quadrupole ratio $E_{1+}/M_{1+} \approx -2.5\%$), Watson's theorem dictates that both multipoles must share the same phase δ_{33} on the real axis. Because they also share the same $k \cdot q$ threshold barrier, their analytic continuations undergo the same kinematic rotation in the complex plane. Consequently, the underlying electromagnetic phases of both helicity amplitudes $A_{1/2}$ and $A_{3/2}$ must be equal, irrespective of the magnitude of the E_{1+} contribution.

To extract this EM vertex phase, the PDG convention demands factoring out the strong pion vertex and the phase-space kinematics, defining the helicity amplitude proportionally to:

$$\tilde{A}_h \propto \sqrt{\frac{q_p}{k_p}} \frac{R_{\gamma N}}{\sqrt{R_{\pi N}}}, \tag{7}$$

where R are the respective residues. Taking the complex argument of this expression yields the officially reported electromagnetic phase ϕ_{EM}:

$$\phi_{EM} = \frac{1}{2}\arg(q_p) - \frac{1}{2}\arg(k_p) + \theta_{\gamma N} - \frac{1}{2}\theta_{\pi N}. \tag{8}$$

By substituting the kinematically rotated phase $\theta_{\gamma N}$ from Equation (4) into the PDG convention Equation (8), we can explicitly evaluate the phase cancellations:

$$\begin{aligned} \phi_{EM} = & \frac{1}{2}\arg(q_p) - \frac{1}{2}\arg(k_p) \\ & + \left[\theta_{\pi N} + \arg(k_p) - 2\arg(q_p)\right] - \frac{1}{2}\theta_{\pi N}. \end{aligned} \tag{9}$$

Grouping the respective terms leaves a relation connecting the sought electromagnetic phase solely to the measurable elastic phase and the complex threshold kinematics of the pole:

$$\phi_{EM} = \frac{1}{2}\theta_{\pi N} + \frac{1}{2}\arg(k_p) - \frac{3}{2}\arg(q_p). \tag{10}$$

3. Results

To evaluate our geometric constraint in Equation (10), we compute the complex momenta using the definitions from Section 2.1. Here we use the proton mass $m_N = 938.27$ MeV and the neutral pion mass $m_\pi = 134.97$ MeV. We evaluate these momenta at the standard complex pole position of the $\Delta(1232)$ resonance, given by the PDG evaluation as $E_p = 1210 - i\,50$ MeV [1].

This analytic continuation directly yields the complex momenta k_p and q_p, and their respective kinematic rotations: $\arg(k_p) \approx -9.39°$ and $\arg(q_p) \approx -12.81°$. Furthermore, the elastic unitary residue phase is well-established from the SAID $\pi N \to \pi N$ partial-wave analysis, yielding $\theta_{\pi N} \approx -46°$ [16,17]. It may be useful to contrast this value with the behavior on the physical real energy axis, where the partial-wave phase shift δ_{33} passes through $90°$ at the Breit-Wigner mass. However, when the amplitude is analytically continued to the complex pole E_p, the residue phase acquires a negative value due to the macroscopic geometry of the threshold and non-resonant background interference. The value of $-46°$ is not an arbitrary model assumption, but the internationally accepted value for the πN elastic pole residue cited by PDG. The input parameters and the resulting kinematic rotations are summarized in Table 1.

Table 1. Input parameters for the $\Delta(1232)$ resonance and the resulting complex center-of-mass momenta evaluated at the pole $E_p = M_p - i\Gamma_p/2$. The calculated kinematic phases $\arg(k_p)$ and $\arg(q_p)$ dictate the geometric rotation of the photoproduction residue in the complex energy plane.

Quantity	Symbol	Value
Input Parameters		
Pole mass	M_p	1210 ± 1 MeV
Pole width	Γ_p	100 ± 2 MeV
Elastic residue phase	$\theta_{\pi N}$	$-46°\,^{+1°}_{-2°}$
Calculated Kinematics at the Pole		
Photon momentum phase	$\arg(k_p)$	$-9.39°$
Pion momentum phase	$\arg(q_p)$	$-12.81°$

By substituting these center-of-mass kinematics and the elastic unitary phase into the derived S-matrix geometric formalism (Equation (10)), the electromagnetic residue phase is determined. To establish the robustness of this geometric constraint, we propagate the uncertainties of the input parameters. Treating the experimental uncertainties as independent contributions to the variance, we calculate the partial derivatives of Equation (10) with respect to the pole mass M_p, the pole width Γ_p, and the elastic phase $\theta_{\pi N}$. The propagation reveals that the kinematic rotations $\arg(k_p)$ and $\arg(q_p)$ are highly stable against small variations in the pole position, contributing a combined systematic uncertainty of only $\approx\pm 0.3°$. Adding this kinematic variance in quadrature with the asymmetric uncertainty of the elastic phase ($\Delta\theta_{\pi N}/2 = +0.5°/-1.0°$), we obtain a kinematically constrained prediction of:

$$\phi_{EM} = -8.5°{}^{+0.6°}_{-1.0°}. \tag{11}$$

The small magnitude of our uncertainty, compared to the $\pm 5°$ reported by the SAID analysis, reflects the fundamentally different nature of a deterministic kinematic constraint versus a data-driven fit. The SAID uncertainty is a statistical error derived from a global fit to tens of thousands of experimental data points, which naturally encompasses large parameter correlations across multiple partial waves and background polynomials. Conversely, our result is derived strictly via linear error propagation of the well-established PDG properties of the $\Delta(1232)$ pole. Because the complex threshold kinematics are mathematically highly stable against small variations in the pole position, the systematic error of our constraint is naturally tight.

This derivation provides a practical analytical tool for the understanding of the numerical results extracted from coupled-channel fits. As shown in Table 2, our parameter-constrained result matches the values obtained by the SAID group ($\approx -9° \pm 5°$) [14] and the Bonn–Gatchina analysis ($\approx -8.7° \pm 2.0°$) [8,18].

Table 2. Compilation of the electromagnetic residue phases ϕ_{EM} for the $\Delta(1232)$ helicity amplitudes $A_{1/2}$ and $A_{3/2}$ at the complex pole. Due to the dominance of the M_{1+} multipole, some analyses inherently yield identical phases for both states, while others report distinct values reflecting differences in background modeling. The analytical prediction from our geometric S-matrix formalism is shown in the last row.

Analysis	$\boldsymbol{\phi_{1/2}}$ (°)	$\boldsymbol{\phi_{3/2}}$ (°)
SAID (2012) [14]	-9 ± 5	-9 ± 5
BnGa (2015) [8]	-8.7 ± 2.0	-8.7 ± 2.0
Jülich–Bonn (2015) [6]	-6.6	$+2.8$
Jülich–Bonn (2022) [7]	-18.0 ± 2.0	-0.7 ± 0.9
BnGa (2025) [1]	-17 ± 3	-10 ± 2
Geometric Model (This work)	$-8.5^{+0.6}_{-1.0}$	$-8.5^{+0.6}_{-1.0}$

It is worth noting a widespread convention discrepancy in the literature regarding the modulus of the complex residue. The EM residue phase ϕ_{EM} is frequently reported alongside a negative modulus to align the phase close to 0°, reflecting the negative sign of the underlying Born coupling (as seen in recent PDG evaluations [1]). Conversely, mathematical S-matrix extractions [2] restrict the modulus to positive values ($|R| > 0$). This mathematical choice shifts the reported unitary phases by exactly 180°, meaning a reported phase of $\approx 171°$ is entirely equivalent to $-8.5°$. In this work, and in Table 2, we adopt the former convention for direct comparison with the standard PDG listings.

4. Discussion

The resulting kinematically constrained prediction, $\phi_{EM} = -8.5^{\circ}{}^{+0.6^{\circ}}_{-1.0^{\circ}}$, is in agreement with the values extracted by phenomenological models, notably the SAID ($-9^{\circ} \pm 5^{\circ}$) [14] and Bonn–Gatchina ($-8.7^{\circ} \pm 2.0^{\circ}$) [8] analyses. This result not only validates these specific numerical extractions but also provides a clear physical explanation for their origin: the observed phase shift is a direct, measurable consequence of kinematic threshold rotations evaluated at the complex pole.

The 9.5° discrepancy between our geometric baseline and the Jülich–Bonn (2022) Dynamical Coupled-Channel analysis ($\phi_{1/2} = -18.0^{\circ} \pm 2.0^{\circ}$) highlights the impact of explicit background modeling. While SAID and BnGa parameterize the background via smooth, data-driven functions that effectively conform to the macroscopic threshold geometry, the Jülich–Bonn framework explicitly evaluates t-channel vector meson (ρ, ω) and u-channel baryon exchanges. These localized exchange diagrams introduce left-hand cuts and subthreshold singularities in the complex energy plane. When analytically continued to the pole, the propagators of these exchange mesons carry an explicit complex phase that shifts the extracted background away from the minimal-surface geometric baseline.

Furthermore, the apparent splitting of the helicity phases reported in the BnGa (2025) analysis ($\phi_{1/2} = -17^{\circ} \pm 3^{\circ}$, $\phi_{3/2} = -10^{\circ} \pm 2^{\circ}$) reflects a known consequence of multi-channel phenomenological fitting procedures rather than a physical splitting of the vertex. While the $\Delta(1232)$ excitation is dominated by the M_{1+} transition—which mathematically mandates identical phase evolution for both helicity states at the pole—practical fits parameterize non-resonant backgrounds independently for different helicity combinations to match polarization observables. These independent, channel-specific background parameterizations can naturally lead to a numerical spreading of the extracted phases. Our geometric constraint successfully isolates the fundamental vertex phase prior to the application of these specific background parameterizations.

Ultimately, this geometric constraint offers a useful benchmark for evaluating background parameterizations. The fundamental requirement for this geometric formalism is the applicability of Watson's theorem and elastic unitarity. As we have found in parallel studies of purely elastic meson resonances—such as the $\rho(770)$, $f_0(500)$, $K_0^*(700)$, and $K^*(892)$—the exact cancellation of dynamical phases is firmly protected by single-channel unitarity. Applying this framework to the nearly 100% elastic $\Delta(1232)$ baryon resonance is therefore a mathematically robust step. However, extending this specific formalism to broader, highly inelastic excitations—such as the $N(1440)$, $N(1520)$, or $N(1535)$—presents a significant theoretical challenge. Above inelastic thresholds, Watson's theorem breaks down and no longer provides a simple phase identity. Consequently, any rigorous extension to these states must explicitly account for coupled-channel dynamics (e.g., the opening of $\pi\pi N$, ηN, and $K\Lambda$ channels), complex Riemann sheet dependence, and potential subthreshold structures (such as Adler zeros). For such states, the geometric phase serves merely as a baseline, where observed deviations quantify the underlying dynamical corrections and channel couplings. Consequently, this formalism provides a consistent framework to evaluate resonance properties reported across different multichannel analyses.

5. Conclusions

We have addressed the model-dependent ambiguity surrounding the electromagnetic residue phase, ϕ_{EM}, of the $\Delta(1232)$ resonance. By applying a geometric S-matrix formalism grounded in fundamental analytic and time-reversal symmetries, we demonstrated that this phase is not an arbitrary fitting parameter, but a physical quantity constrained by analyticity and threshold kinematics.

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Our derivation utilizes Watson's theorem to fix the photoproduction phase to the elastic πN scattering phase on the real energy axis. Through analytic continuation to the complex pole, we accounted for the distinct centrifugal barriers: the $k \cdot q$ dependence of the dominant magnetic dipole transition (M_{1+}) and the q^3 dependence of the elastic P-wave amplitude. The cancellation of intermediate kinematic factors within the PDG helicity amplitude convention yields a simple, purely kinematic relation connecting ϕ_{EM}, the known elastic residue phase, and the complex pole momenta.

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Abbreviations

The following abbreviations are used in this manuscript:

EM	Electromagnetic
PDG	Particle Data Group
SAID	Scattering Analysis Interactive Dial-in
BnGa	Bonn–Gatchina

Appendix A. Derivation of the Kinematic Phase Rotation

To show the cancellation of the dynamical background and the origin of the kinematic rotation, we relate the physical scattering amplitudes to their bare coupling limits. Following the standard S-matrix partial-wave formalism, the elastic P-wave amplitude $T_{\pi N}$ and the magnetic dipole photoproduction amplitude $\mathcal{M}_{\gamma N}$ can be factorized into a kinematic threshold barrier, a real-valued bare coupling constant g, and a shared dynamical propagator $\mathcal{D}(E)$ encompassing the resonant denominator and the background rescattering:

$$T_{\pi N}(E) \propto q(E)^3\, g_{\pi N}^2\, \mathcal{D}(E), \tag{A1}$$

$$\mathcal{M}_{\gamma N}(E) \propto k(E) q(E)\, g_{\gamma N} g_{\pi N}\, \mathcal{D}(E). \tag{A2}$$

Watson's theorem ensures that below the two-pion threshold on the real axis, the complex phase of $\mathcal{D}(E)$ is identical for both processes. To extract the unitary residue phase at the complex pole E_p, we apply the Cauchy limit $R = \lim_{E \to E_p}(E_p - E)T(E)$. Evaluating this limit yields the complex residues $R_{\pi N}$ and $R_{\gamma N}$:

$$|R_{\pi N}| e^{i\theta_{\pi N}} = q_p^3\, g_{\pi N}^2 \lim_{E \to E_p} (E_p - E)\mathcal{D}(E), \tag{A3}$$

$$|R_{\gamma N}| e^{i\theta_{\gamma N}} = k_p q_p\, g_{\gamma N} g_{\pi N} \lim_{E \to E_p} (E_p - E)\mathcal{D}(E), \tag{A4}$$

where $k_p \equiv k(E_p)$ and $q_p \equiv q(E_p)$ are the complex center-of-mass momenta evaluated at the pole.

By forming the ratio of Equation (A4) to Equation (A3), the complex dynamical limit $\lim_{E \to E_p}(E_p - E)\mathcal{D}(E)$ cancels out, leaving:

$$\frac{|R_{\gamma N}|}{|R_{\pi N}|} e^{i(\theta_{\gamma N} - \theta_{\pi N})} = \frac{k_p}{q_p^2} \frac{g_{\gamma N}}{g_{\pi N}}. \tag{A5}$$

Since the bare couplings $g_{\gamma N}$ and $g_{\pi N}$ are real, taking the complex argument of this ratio yields the phase rotation introduced by the threshold kinematics:

$$\theta_{\gamma N} - \theta_{\pi N} = \arg(k_p) - 2\arg(q_p). \tag{A6}$$

This shows that the transformation of the unitary residue phase from the elastic channel to the photoproduction channel depends entirely on kinematic factors.

We emphasize that in a general hadronic process, higher-order inelastic loop corrections generate additional complex phases via multi-particle branch cuts. However, the $\Delta(1232)$ resonance represents an exceptional case due to its near 100% elasticity in the πN channel. Under the restriction of single-channel elastic unitarity, the complex phases generated by the dominant elastic πN loops are strictly universal and are fully absorbed into the shared dynamical propagator. Any asymmetric, higher-order inelastic loops that could potentially break the amplitude factorization employed in Equations (A1) and (A2) are severely suppressed in this specific energy regime, justifying the omission of explicit multi-particle channel couplings in this baseline constraint.

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