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Semi-Global Symplectic Invariant of the Champagne Bottle

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Abstract

We study a two degrees of freedom Hamiltonian system describing the motion of a particle in a potential field in the form of a S^1 symmetric double well, namely $V = -(x_1^2 + x_2^2) + (x_1^2 + x_2^2)^2$, known also as a champagne bottle potential. This system is completely integrable. The champagne bottle is the simplest member of a class of integrable systems that have no global action variables due to a non-trivial monodromy. Beyond that, the geometric and dynamical properties of the system near the equilibrium are of primary interest. We calculate the Birkhoff normal form and the nontrivial action near the focus–focus singularity and obtain the semi-global symplectic invariant near the focus–focus point, which is introduced by Vũ Ngọc. Examples of such calculations are still few. We compare our result with the semi-global symplectic invariant of the spherical pendulum.

Keywords: semi-global symplectic invariant; champagne bottle; focus–focus singularity; Birkhoff normal form; actions

MSC: 37j35; 37j40; 70H06; 70H08

1. Introduction

In this paper we deal with the Hamiltonian

$$H = \frac{1}{2}(y_1^2 + y_2^2) - (x_1^2 + x_2^2) + (x_1^2 + x_2^2)^2, \tag{1}$$

defined on the phase space $M^4 = \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4\}$ and endowed with the standard symplectic form $\Omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and the standard Poisson structure $\{x_i, y_j\} = \delta_{i,j}$. The Hamiltonian system associated with H

$$\begin{aligned} \dot{x}_1 &= y_1, \qquad \dot{y}_1 = 2x_1[1 - 2(x_1^2 + x_2^2)], \\ \dot{x}_2 &= y_2, \qquad \dot{y}_2 = 2x_2[1 - 2(x_1^2 + x_2^2)] \end{aligned} \tag{2}$$

is completely integrable due to S^1 symmetry. The second integral (the angular momentum) reads

$$J_2 = x_1 y_2 - x_2 y_1. \tag{3}$$

This system describes the motion of a particle in a champagne bottle potential, also referred as a particle in a Mexican hat potential. Such potentials are of interest to field theorists studying the Higgs field. As can be seen, this system does not depend on any parameters. This makes it the simplest member of a class of systems that exhibit Hamiltonian monodromy, which is an obstruction for the existence of global action variables. Bates [1] observed that the energy surface for positive energy $h > 0$ has a topological type of S^3. For

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energy $h \in (-1/4, 0)$, see Figure 1, an energy surface has a topological type of $S^2 \times S^1$. The topological types of these manifolds are distinct, which implies the presence of monodromy, generated by

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

on a suitable basis. It turns out that this topological feature causes a sharp change in the disposition of the quantum-mechanical spectrum near the zero energy in the champagne bottle quantum counterpart; see Child [2].

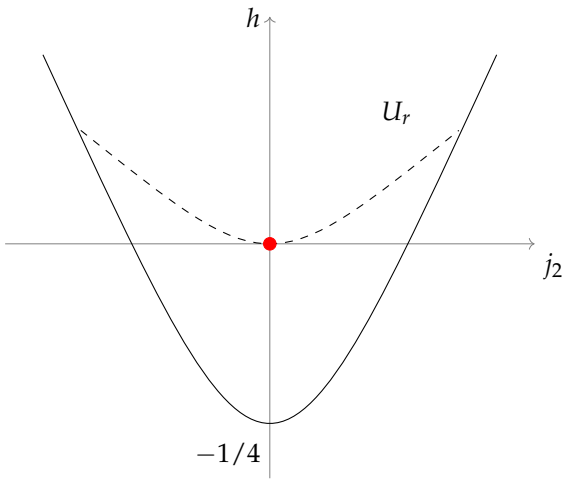


Figure 1. The set U_r of regular values in the momentum map.

We are interested in the geometry and the dynamical properties of the system (2) near the equilibrium $(0, 0)$, and especially in the computation of the semi-global symplectic invariant near this point, which turns out to be a focus–focus point.

In the pursuit of solving the classification problem of integrable systems with singularities green Vũ Ngọc [3] introduced semi-global symplectic invariants for the Liouville foliation of integrable Hamiltonian systems with two degrees of freedom near the separatrix of a simple focus–focus point. This result is an extension of the invariants introduced by Dufour et al. [4] for one-degree-of-freedom systems with simple hyperbolic singularity.

Prior to that, topological and smooth invariants were introduced; see [5,6] and also the survey article [7].

Later, Dullin and Vũ Ngọc [8] introduced the semi-global symplectic invariants for hyperbolic–hyperbolic equilibrium for two degrees of freedom systems and computed this invariant for the Neumann system.

In our treatment of the considered system we follow closely the approach taken by Dullin [9] in his study of semi-global symplectic invariants for the spherical pendulum. It is also worth mentioning the work of Alonso et al. [10] in which some invariants for the spin-oscillator Hamiltonian integrable system (an important example in the theory of semitoric systems) are calculated; see also [11,12]. Further, Alonso et al. [13] made a complete study of the symplectic invariants of the coupled angular momenta, another important semitoric system.

The Hamiltonian system investigated here has a simple focus–focus point and is a generalized semitoric system, since the angular momentum integral fails to be proper (the proof of that fact is exactly the same as in Pelayo et al. [14]).

Finally, we will point out some recent papers on this subject that have provided considerable progress in this theory, as well as offering some open problems.

In the paper cited above, Vũ Ngọc [3] proved that integrable systems in dimension 4 with a single focus–focus point are classified by a formal series. Then he derived a way to generalize this classification in the presence of $k > 1$ focus–focus points. Pelayo and Tang [15] proved this hypothesis by showing that one needs k formal series to classify (up to isomorphism) the integrable systems near a singular fiber containing k focus–focus points.

A closely related paper is that of Palmer et al. [16] which extends the theory of Pelayo and Vũ Ngọc about semitoric integrable 2 d.o.f. systems to systems with more than one isolated singularity in the same energy-momentum level set. Such systems are called non-simple.

In [17,18] Pelayo firstly reviewed the symplectic invariants of semitoric systems. Then, he considered the recent progress on the relations of these integrable systems with their quantum integrable counterparts and, more specifically, the corresponding inverse problem, which asks how much information about the associated classical system can be found in the quantum spectrum. Some open problems about the geometry, topology and dynamics of these systems are proposed.

In connection to these papers we mention a very recent work from Ghori et al. [19] which studies a quantum integrable Hamiltonian system with a focus–focus singularity that comes from a cosmological minisuperspace model. It turns out that the quantum spectrum behaves in a similar way as in Child [2] due to monodromy.

This paper is organized as follows. All necessary concepts and results are outlined in Section 2. In Section 3 the Birkhoff normal form near the focus–focus point is calculated. In Section 4 we compute the expansion of the non-trivial action integral near the focus–focus singularity. Then we find the semi-global symplectic invariant (or Taylor series invariant) as well as other characteristics, such as the period, the rotation number and the twist, in Section 5. We recover some results about the KAM conditions for this system. Finally, we compare the semi-global symplectic invariant of the champagne bottle with that of the spherical pendulum and conclude that the spherical pendulum and the champagne bottle are not symplectically equivalent.

2. Theoretical Background

In this section we discuss well known elements of integrable Hamiltonian systems. Some of them are valid for systems with arbitrary number of degrees of freedom, some of them are valid only for the systems with two degrees of freedom and are in a process of generalization (see, for instance, [5,20,21]).

A Hamiltonian system related to a Hamiltonian H, defined on a symplectic manifold (M^{2n}, Ω) is called completely integrable if there exist n first integrals $f_1 = H, f_2, \ldots, f_n$, functionally independent almost everywhere and Poisson commuting, that is, $\{f_i, f_j\} = 0$ for all i, j. These first integrals define the momentum map $F := (f_1, f_2, \ldots, f_n) : M^{2n} \to \mathbb{R}^n$.

A point $m \in M^{2n}$ is called a regular point if $rankDF = n$ and a singular point if $rankDF < n$. A value $a \in \mathbb{R}^n$ is called regular if $F^{-1}(a)$ consists only of regular points and critical otherwise. Equivalently independency of the integrals almost everywhere means that $rankDF = n$ almost everywhere.

Let $m \in M^{2n}$ be an equilibrium point such that $df_1(m) = df_2(m) = \ldots = df_n(m) = 0$ ($rankDF = 0$). Consider $A_j = (\Omega)^{-1} d^2 f_j(m)$ as a linear operator: $A_j \in sp(T_m M, \Omega)$. Since $f_1, f_2, \ldots, f_n$ commute, $A_1, A_2, \ldots, A_n$ generate a commutative subalgebra K in $sp(T_m M, \Omega)$. The point $m \in M^{2n}$ is *non-degenerate*, if K is a Cartan subalgebra.

The map F defines a singular Liouville fibration on M^{2n}, whose fibers are connected components of $F^{-1}(c), c \in \mathbb{R}^n$ which we assume to be compact. A fibre $F^{-1}(c)$ is called regular if c is a regular value and singular if it contains at least one singular point.

Remark 1. *Clearly, for a given integrable Hamiltonian H, the moment map is not unique. However, under certain non-resonance condition (see the end of this section), the regular level sets of the moment map are uniquely determined by the system.*

If c is a regular value of F, we have near any compact connected component Λ_c of $F^{-1}(c)$ the Liouville–Arnold theorem [20,21] which says that Λ_c is diffeomorphic to an n-dimensional torus $\mathbb{T}^n$. The neighborhood of this torus is symplectomorphic to the standard model $\mathbb{T}^n \times D^n$ with $\Omega = \sum d\varphi_j \wedge dI_j$, where $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ are coordinates on $\mathbb{T}^n$ called *angles* and $\mathbf{I} = (I_1, I_2, \ldots, I_n)$ are coordinates on D^n called *actions*. The dynamics are governed by extremely simple equations

$$\dot{\mathbf{I}} = 0, \quad \dot{\boldsymbol{\varphi}} = \frac{\partial H}{\partial \mathbf{I}}, \quad H = H(\mathbf{I}). \tag{4}$$

The actions can be computed as follows

$$I_j = \frac{1}{2\pi} \int_{\gamma_j} \delta, \qquad j = 1, \ldots, n, \tag{5}$$

where δ is the Liouville 1-form $d\delta = \Omega$ and γ_j are cycles on $\mathbb{T}^n$.

In general, the actions are not globally defined due to some geometric obstructions as Hamiltonian monodromy; see Duistermaat [22]. The system considered in this paper is an example of an integrable system without global actions.

The action variables are the most natural symplectic invariants of integrable systems. In other words, the Liouville–Arnold theorem gives a classification of integrable systems (up to the dimension) locally near regular fibers and all such systems look the same.

However, the Hamiltonian systems studied here do have singularities. Therefore, the dynamics near singular fibers have to be studied to find distinctions between integrable systems. In this way a general problem arises: that of how to describe invariants and classify singular Liouville fibrations.

We mentioned some works on the different types of classifications of non-degenerate singularities in the Introduction. Now, we give a more detailed description of the symplectic invariant for focus–focus singularities. To do that, we will consider integrable Hamiltonian systems with two degrees of freedom, namely, those in which there are two independent, commuting first integrals f_1 and f_2.

Focus–focus singularities are one of the four types of singularities of the Morse–Bott type in dimension 4 [23,24]. A fixed point $m \in M^4$ ($df_1(m) = df_2(m) = 0$) is called a focus–focus point if there are symplectic coordinates (q_i, p_i) near m (that is, $\Omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$) such that

$$f_1 = a(q_1p_1 + q_2p_2) + b(q_1p_2 - q_2p_1) + h.o.t.,$$

$$f_2 = c(q_1p_1 + q_2p_2) + d(q_1p_2 - q_2p_1) + h.o.t.,$$

where a, b, c, d are constants satisfying $ad - bc \neq 0$. Equivalently, the focus–focus equilibrium has eigenvalues $\pm\nu \pm i\omega$, where ν, ω are non-zero real numbers. Such eigenvalues are also called loxodromic.

Observe that, at least for those systems with two degrees of freedom, the focus–focus singularities and non-trivial monodromy are connected [1,9,19,22,25,26].

Further, it is also assumed that the singular Liouville fibration has a unique singular point m, which is of a focus–focus type and the fiber containing it is a compact, simple focus–focus singularity.

The fibration is defined by the two integrals f_1 and f_2. They can be chosen in such a way that the actions are

$$\begin{aligned} I_1 &= \mathrm{Re}((f_1 + if_2)\ln(f_1 + if_2)) + c(f_1, f_2), \\ I_2 &= f_2, \end{aligned} \tag{6}$$

where $c(f_1, f_2)$ is a smooth formal series.

Theorem 1 (San Vũ Ngọc, 2003 [3]). *The formal series $c(f_1, f_2)$ is a symplectic invariant of the simple focus–focus singularity.*

We will make the expression of $c(f_1, f_2)$ more precise in Section 5. It is seen that the symplectic invariant is closely related to the same actions as in the regular case. Moreover, one can consider $c(f_1, f_2)$ to be the regularized part of the action.

As an important result, the introduced invariant classifies the singular Liouville fibration in the open vicinity of the singular fiber, up to a symplectomorphism.

We finish this section with some remarks about the KAM theory conditions for the champagne bottle. In 1995, Georgiev [27] showed that the Kolmogorov condition [21]

$$\det(\partial^2 H/\partial I_i \partial I_j) \neq 0,$$

where H is an integrable Hamiltonian and I_j are the action variables, is fulfilled everywhere out of the bifurcation diagram of the momentum map using the complex-analytic approach developed by Horozov [28,29]. In general, if the Hamiltonian of a given integrable system H satisfies the Kolmogorov condition (almost everywhere), then H is non-resonant.

Georgiev also showed that the iso-energetic non-degeneracy condition (twist condition)

$$\det \begin{pmatrix} \frac{\partial^2 H}{\partial I^2} & \frac{\partial H}{\partial I} \\ \frac{\partial H}{\partial I} & 0 \end{pmatrix} \neq 0$$

is violated along the curve through the focus–focus point (see the dashed curve in Figure 1).

A year later, Zung [30] proved that for an integrable two-degrees-of-freedom Hamiltonian system with a simple focus–focus value, the Kolmogorov condition is always satisfied in a neighborhood of the focus–focus singularity.

Several years later, Dullin and Vũ Ngọc [31] improved Zung's result using the theory developed in [3] and described above. Their results show that in the neighborhood of a simple focus–focus point, the Arnold–Moser condition is violated while the Kolmogorov condition is satisfied.

At almost same time, Rink [32] presented a new proof of Zung's result with a technique quite similar to that of [31].

3. Normal Form

In this section we will compute the Birkhoff normal form for the Hamiltonian (1) near the equilibrium. The Birkhoff normal form is a simplified Hamiltonian, and that simplification is achieved by a series of near-identity canonical transformations obtained by a generating function. For this purpose, the Lie series approach developed by Deprit [33] will be used, which we will recall at the beginning briefly (for full details, see also [34]).

Suppose, without loss of generality, that the equilibrium is $(0,0)$. The scaling $(x,H) \to (\varepsilon x, \varepsilon^{-2}H)$ "zooms in" to the equilibrium. Here, ε is a formal parameter, which we can set as $\varepsilon = 1$ eventually.

We are given a Hamiltonian in the form

$$H(x,\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} H_i^0(x) \tag{7}$$

Via a symplectic, near-identity change of variables $x \to y$, which does not transform ε, the above Hamiltonian becomes

$$\overline{H}(y,\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} H_0^i(y) \tag{8}$$

Usually $\overline{H}$ is called the Lie transform of H, generated by W

$$W(x,\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} W_{i+1}(x) \tag{9}$$

The new Hamiltonian has to be simpler than the original, so it is said to be in normal form. Notice that this normal form need not be convergent.

To facilitate the calculations, the intermediate Hamiltonians $\{H_j^i\}$, $i = 1,2,\ldots$, $j = 0,1,\ldots$ are introduced by the recursive identities

$$H_j^i = H_{j+1}^{i-1} + \sum_{k=0}^{j} \binom{j}{k} \{H_{j-k}^{i-1}, W_{k+1}\} \tag{10}$$

The algorithm starts with a given Hamiltonian, that is, all H_i^0 are known (we follow [34] closely). Assume all the rows of the Lie–Deprit triangle

$$\begin{array}{ccccccc}
H_0^0 & & & & & & \\
\downarrow & & & & & & \\
H_1^0 & \longrightarrow & H_0^1 & & & & \\
\downarrow & & \downarrow & & & & \\
H_2^0 & \longrightarrow & H_1^1 & \longrightarrow & H_0^2 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
H_3^0 & \longrightarrow & H_2^1 & \longrightarrow & H_1^2 & \longrightarrow & H_0^3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow
\end{array}$$

have been computed down to the $(N-1)$st row, thus $W_1,\ldots,W_{N-1}$ have been determined and the desired members of the normal form $H_0^1,\ldots,H_0^{N-1}$ are obtained. To compute the Nth row we perform the following steps:

Step 1. Assume $W_N = 0$ and compute all the terms of Nth row using (10) and denote them by $L_j^i, i+j = N$.

Step 2. Solve the equation $H_0^N = L_0^N + \{H_0^0, W_N\}$ for W_N and H_0^N, so that H_0^N is in normal form.

Step 3. Calculate $H_j^i = L_j^i + \{H_0^0, W_N\}$ for all $i+j = N$.

Step 4. Repeat for the next row.

The definition of normal form depends on the equation $H_0^N = L_0^N + \{H_0^0, W_N\}$, which in turn depends on H_0^0. This equation is called the homological or the Lie equation.

This algorithm can be used to establish the following

Theorem 2 (Theorem 10.3.1, [34]). *Let $\{\mathcal{P}_i\}_{i=0}^{\infty}$, $\{\mathcal{Q}_i\}_{i=1}^{\infty}$ and $\{\mathcal{R}_i\}_{i=1}^{\infty}$ be sequences of linear spaces of smooth functions defined on a common domain O in $\mathbb{R}^n$ with the following properties:*

1. $\mathcal{Q}_i \subset \mathcal{P}_i, \quad i = 1, 2, \ldots$
2. $H_i^0 \in \mathcal{P}_i, \quad i = 0, 1, 2, \ldots$
3. $\{\mathcal{P}_i, \mathcal{R}_j\} \subset \mathcal{P}_{i+j}, \quad i + j = 1, 2, \ldots$
4. *for any $D \in \mathcal{P}_i, i = 1, 2, \ldots$, there exists $B \in \mathcal{Q}_i$ and $C \in \mathcal{R}_i$, such that*

$$B = D + \{H_0^0, C\}. \tag{11}$$

Then, there exists a function W of the form (9) with $W_i \in \mathcal{R}_i, i = 1, 2, \ldots$, which generates a near-identity symplectic change of variables $x \to y$ such that the Hamiltonian in the new variables has a series of expansions given by (8) with $H_0^i \in \mathcal{Q}_i, i = 1, 2, \ldots$

For the uniqueness of the normal form one needs two more conditions (Theorem 10.3.3 [34]):

- The linear operator $\mathcal{H}_i = \{H_0^0, .\} : \mathcal{P}_i \to \mathcal{P}_i$ must be simple, that is,

$$\mathcal{P}_i = \text{kernel}(\mathcal{H}_i) \oplus \text{range}(\mathcal{H}_i)$$

  and
- $\{\mathcal{Q}_i, \mathcal{Q}_j\} = 0, \quad i, j = 1, 2, \ldots$

Returning to our system, we define the momentum mapping $F = (H, J_2)$ on some open subset V of M^4 with values in $\mathbb{R}^2$

$$F : V \to (h, j_2) \in \mathbb{R}^2. \tag{12}$$

Next, we find that the equilibrium points of (2) are $(0, 0)$ and $y_1 = y_2 = 0, x_1^2 + x_2^2 = \frac{1}{2}$. We are interested mainly in the dynamics near $(0, 0)$. The linearization around $(0, 0)$ gives eigenvalues $\pm\sqrt{2}$ with multiplicity two. Therefore, this equilibrium is a degenerate saddle–saddle point. On the other hand, from the perspective of the foliation of the integrable system, it is a focus–focus point; see Zung [26]. Clearly, any function of the first integrals H and J_2 is a first integral, and hence, has the same foliation. Then, the linear combination $H + \omega J_2$ for arbitrary $\omega \neq 0$ has an equilibrium of focus–focus type with eigenvalues $\pm\sqrt{2} \pm i\omega$.

Further, we want to transform the quadratic part of the Hamiltonian (1) into Williamson normal form [24]. The symplectic rescaling (preserves Ω)

$$x_{1,2} \to \frac{1}{\sqrt[4]{2}} x_{1,2}, \qquad y_{1,2} \to \sqrt[4]{2}\, y_{1,2} \tag{13}$$

brings (1) to the form

$$H = \frac{\sqrt{2}}{2}\left(y_1^2 - x_1^2\right) + \frac{\sqrt{2}}{2}\left(y_2^2 - x_2^2\right) + \frac{1}{2}(x_1^2 + x_2^2)^2. \tag{14}$$

Further, we make another symplectic change

$$x_j = \frac{1}{\sqrt{2}}(q_j - p_j), \qquad y_j = \frac{1}{\sqrt{2}}(q_j + p_j), \qquad j = 1, 2. \tag{15}$$

The symplectic form becomes

$$\Omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2. \tag{16}$$

Denote also

$$J_1 = p_1 q_1 + p_2 q_2. \tag{17}$$

In these coordinates the second integral reads $J_2 = q_1 p_2 - q_2 p_1$ and the Hamiltonian (8) is

$$\begin{aligned} H =& H_2 + H_4, \\ & H_2 = \sqrt{2} J_1, \quad H_4 = \frac{1}{8}\Big(q_1^2 + q_2^2 + p_1^2 + p_2^2 - 2J_1\Big)^2. \end{aligned} \tag{18}$$

Scaling $(q, p, H) \to (\varepsilon q, \varepsilon p, \frac{H}{\varepsilon^2})$, where ε is a formal small parameter that we can set as $\varepsilon = 1$ at the end, the Hamiltonian reads

$$\begin{aligned} H = H_0^0 + \frac{\varepsilon^2}{2} H_2^0, \qquad & H_1^0 \equiv H_j^0 \equiv 0, \quad j > 2, \\ H_0^0 = \sqrt{2} J_1, \qquad & H_2^0 = \frac{1}{4}(q_1^2 + q_2^2 + p_1^2 + p_2^2 - 2J_1)^2 \end{aligned} \tag{19}$$

Following Dullin [9], we introduce "almost action-angle" coordinates by

$$\begin{aligned} q_1 &= e^{\theta_1} \cos\theta_2, & p_1 &= (J_1 \cos\theta_2 - J_2 \sin\theta_2) e^{-\theta_1}, \\ q_2 &= e^{\theta_1} \sin\theta_2, & p_2 &= (J_1 \sin\theta_2 + J_2 \cos\theta_2) e^{-\theta_1}. \end{aligned} \tag{20}$$

Since

$$\theta_1 = \ln \sqrt{q_1^2 + q_2^2}, \qquad \theta_2 = \arctan \frac{q_2}{q_1} \tag{21}$$

these coordinates are symplectic when $(q_1, q_2) \neq (0, 0)$,

$$\Omega = d\theta_1 \wedge dJ_1 + d\theta_2 \wedge dJ_2, \qquad \{\theta_i, J_j\} = \delta_{ij}$$

and only θ_2 is an angle.

With these variables H_2^0 becomes

$$H_2^0 = \frac{1}{2}(3J_1^2 + J_2^2) + \frac{1}{4}\Big(e^{4\theta_1} + e^{-4\theta_1}(J_1^2 + J_2^2)^2 - 4J_1 e^{2\theta_1} - 4J_1(J_1^2 + J_2^2)e^{-2\theta_1}\Big). \tag{22}$$

Notice that by construction, H_0^i are functions of (J_1, J_2), since J_1, J_2 generate the algebra of functions that commute with H_0^0. Furthermore, the original Hamiltonian is invariant under the discrete symmetry $J_2 \to -J_2$. It can be easily turned into a discrete symplectic symmetry $S : (J_2, \theta_2) \to -(J_2, \theta_2)$. Due to a Theorem from Gaeta [35] the normal form also enjoys this symmetry, that is, it depends on J_2^2.

Define the class of functions $\mathcal{P}_i, i > 0$

$$\sum_{k=0}^{2i} P_k(J_1, J_2) e^{2(i-k)\theta_1},$$

where $P_k(J_1, J_2)$ is a homogeneous polynomial of degree k. The operator $\mathcal{H}_i = \{H_0^0, .\} : \mathcal{P}_i \to \mathcal{P}_i$ just gives $W_i \to -\sqrt{2}\partial W_i/\partial\theta_1$. Moreover, this operator is simple: $\mathcal{P}_i = \text{kernel}(\mathcal{H}_i) \oplus \text{range}(\mathcal{H}_i)$. Obviously, the kernel consists of functions $\mathcal{Q}_i$, which are independent of θ_1. The range consists of functions $\mathcal{R}_i = \mathcal{P}_i \setminus \mathcal{Q}_i$.

For the uniqueness we need to verify two more conditions: $\{\mathcal{P}_i, \mathcal{R}_j\} \subset \mathcal{P}_{i+j}$ and $\{\mathcal{Q}_i, \mathcal{Q}_j\} = 0$. The second condition is fulfilled, because $\{J_1, J_2\} = 0$. For the first condition we have

$$\{\mathcal{P}_i, \mathcal{R}_j\} \subset \{\mathcal{P}_i, \mathcal{P}_j\} \subset \mathcal{P}_{i+j-1} \subset \mathcal{P}_{i+j}.$$

Now, we apply Theorem 1 and the iterative procedure given above. We give the first steps of the calculation of the normal form, which are easy.

From (10) it follows that

$$H_0^1 = H_1^0 + \{H_0^0, W_1\}. \tag{23}$$

Since $H_1^0 \equiv 0$, we choose $H_0^1 = W_1 \equiv 0$. Next,

$$H_0^2 = H_1^1 + \{H_0^0, W_1\} = H_1^1. \tag{24}$$

Again from (10) for H_1^1 we have

$$H_1^1 = H_2^0 + \{H_0^0, W_2\} = H_2^0 + \sqrt{2}\{J_1, W_2\} = H_2^0 - \sqrt{2}\frac{\partial W_2}{\partial \theta_1}. \tag{25}$$

Combining the last two identities yields

$$\begin{aligned} H_0^2 = H_1^1 =& \frac{1}{2}(3J_1^2 + J_2^2) \\ &+ \frac{1}{4}\left(e^{4\theta_1} + e^{-4\theta_1}(J_1^2 + J_2^2)^2 - 4J_1 e^{2\theta_1} - 4J_1(J_1^2 + J_2^2)e^{-2\theta_1}\right) - \sqrt{2}\frac{\partial W_2}{\partial \theta_1}. \end{aligned} \tag{26}$$

We choose

$$H_0^2 = \frac{1}{2}(3J_1^2 + J_2^2) \tag{27}$$

and then we solve the homological equation

$$\sqrt{2}\frac{\partial W_2}{\partial \theta_1} = \frac{1}{4}\left(e^{4\theta_1} + e^{-4\theta_1}(J_1^2 + J_2^2)^2 - 4J_1 e^{2\theta_1} - 4J_1(J_1^2 + J_2^2)e^{-2\theta_1}\right) \tag{28}$$

for W_2 after trivial integration to get

$$W_2 = \frac{1}{4\sqrt{2}}\left(\frac{1}{4}e^{4\theta_1} - 2J_1 e^{2\theta_1} + 2J_1(J_1^2 + J_2^2)e^{-2\theta_1} - \frac{1}{4}e^{-4\theta_1}(J_1^2 + J_2^2)^2\right). \tag{29}$$

To perform the next steps, computer algebra is used to calculate numerous brackets. Convergency is not an issue, since it is well-known that for Liouville integrable systems the normal form is convergent [36,37]. Skipping the details, we obtain

Theorem 3. *The normal form at the focus–focus point* $(0,0)$ *of the champagne bottle is*

$$\begin{aligned} \overline{H} =& \sqrt{2}J_1 + \frac{1}{4}(3J_1^2 + J_2^2) - \frac{\sqrt{2}}{32}J_1(17J_1^2 + 9J_2^2) + \frac{1}{256}(375J_1^4 + 258J_1^2J_2^2 + 11J_2^4) \\ &- \frac{\sqrt{2}}{4096}J_1(10689J_1^4 + 8910J_1^2J_2^2 + 909J_2^4) + O(J^6). \end{aligned} \tag{30}$$

For certain reasons further on, we will need the inverse of $\overline{H}(J_1, J_2) = h$ with respect to J_1 ($J_2 = j_2$), so that $\overline{H}(J_1(h, j_2), j_2) = h$ is an identity. Since the normal form $\overline{H}$ depends on j_2^2, so does $J_1(h, j_2)$. Therefore, we search for J_1 in the following form

$$J_1(h, j_2) = a_1 h + a_2 h^2 + c_1 j_2^2 + a_3 h^3 + b_1 h j_2^2 + a_4 h^4 + b_2 h^2 j_2^2 + c_2 j_2^4 + \ldots,$$

where a_j, b_j, c_j are unknown coefficients. Then we insert this expression into the normal form and compare the coefficients. Straightforward computations yield that

$$J_1(h, j_2) = \frac{1}{\sqrt{2}}\left[h - \frac{1}{8}(3h^2 + 2j_2^2) + \frac{h}{64}(35h^2 + 30j_2^2) - \frac{5}{1024}(231h^4 + 252h^2j_2^2 + 28j_2^4)\right] + \dots \tag{31}$$

4. Action Integrals

In this section, the calculation of the action integrals for the champagne bottle is carried out. First, we have to find where they exist. Since the coordinates used in the previous section do not fit well for this purpose, we introduce the polar coordinates

$$\begin{aligned} x_1 &= r\cos\varphi, & y_1 &= p_r\cos\varphi - \frac{p_r}{r}\sin\varphi, \\ x_2 &= r\sin\varphi, & y_2 &= p_r\sin\varphi + \frac{p_r}{r}\cos\varphi. \end{aligned} \tag{32}$$

Then the Hamiltonian (1) takes the form

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + r^4 - r^2 \tag{33}$$

and the other integral becomes $J_2 = p_\varphi = j_2$.

It is shown in Bates [1] that the critical values of the momentum mapping F (12) of the champagne bottle are $(0, 0)$ and the curve is parameterized by

$$(h, j_2) := (3r^4 - 2r^2, \pm\sqrt{4r^6 - 2r^4}), \qquad r \geq \frac{1}{\sqrt{2}}.$$

Denote the set of regular values of F by U_r; see Figure 1. For the points $(h, j_2) \in U_r$, the connected components of $F^{-1}(h, j_2)$ (that is, the connected components of the level surfaces determined by the equations $H = h, J_2 = j_2$) are two-tori T_{h,j_2}.

Let us choose a basis of the homology group $\mathrm{H}_1(\mathrm{T}_{\mathrm{h,j_2}}, \mathbb{Z})$ with the following representatives: for γ_1 we fix p_φ and φ and let r, p_r make one circle on the curve

$$\frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + r^4 - r^2 = h;$$

for γ_2 we fix r, p_r and make φ to cover $[0, 2\pi]$. Then the action integrals are

$$I_1 = \frac{1}{2\pi}\oint_{\gamma_1} p_r dr = \frac{1}{2\pi}\oint_{\gamma_1} \frac{\sqrt{2(hr^2 + r^4 - r^6) - j_2^2}}{r} dr. \tag{34}$$

$$I_2 = J_2 = \frac{1}{2\pi}\oint_{\gamma_2} p_\varphi d\varphi = j_2 \tag{35}$$

It is convenient to scale the action integrals as

$$\tilde{I}_1 = \sqrt{2}I_1, \quad \tilde{J}_1 = \sqrt{2}J_1, \quad \tilde{J}_2 = \sqrt{2}J_2, \quad \tilde{j}_2 = \sqrt{2}j_2 \tag{36}$$

and drop the tildes thereafter. We can always go back from (36) when necessary. Observe that, in view of the above scaling, the normal form (30) and its inverse (31) become

$$\overline{H} = J_1 + \frac{1}{8}(3J_1^2 + J_2^2) - \frac{1}{64}J_1(17J_1^2 + 9J_2^2) + \frac{1}{1024}(375J_1^4 + 258J_1^2J_2^2 + 11J_2^4) \quad (37)$$
$$- \frac{1}{16{,}384}J_1(10{,}689J_1^4 + 8910J_1^2J_2^2 + 909J_2^4) + \ldots = h,$$

$$J_1(h, j_2) = h - \frac{1}{8}(3h^2 + j_2^2) + \frac{h}{64}(35h^2 + 15j_2^2) - \frac{1}{1024}(1155h^4 + 630h^2j_2^2 + 35j_2^4) + \ldots \quad (38)$$

Further, we transform (34) by putting

$$z = r^2, \quad dz = 2rdr, \quad w^2 = P(z) := -2z^3 + 2z^2 + 2hz - \frac{j_2^2}{2}. \quad (39)$$

Denote the real oval of the elliptic curve

$$\Gamma = \left\{(z, w): \quad w^2 = P(z) = -2z^3 + 2z^2 + 2hz - \frac{j_2^2}{2}\right\}, \quad (40)$$

which exists for all $(h, j_2) \in U_r$ by β. Then (34) becomes

$$I_1(h, j_2) = \frac{\sqrt{2}}{4\pi}\oint_\beta \frac{w}{z}dz. \quad (41)$$

Let the roots of $P(z)$ be z_j, $j = 1, 2, 3$. They are real and located as follows: $z_1 \le 0 \le z_2 < z_3$. The β-cycle encloses the interval $[z_2, z_3]$ along which $w^2 \ge 0$; see Figure 2. On the other hand, the α-cycle encloses the interval $[z_1, z_2]$ along which $w^2 \le 0$. That is why it is called an *imaginary* cycle. For a nice introduction of the *imaginary* cycles of the elliptic curves, see, for example, [11]. Evidently, when $h, j_2 \to 0$, then $z_1, z_2 \to 0$, so this cycle becomes arbitrarily small; therefore, it is also called the *vanishing* cycle.

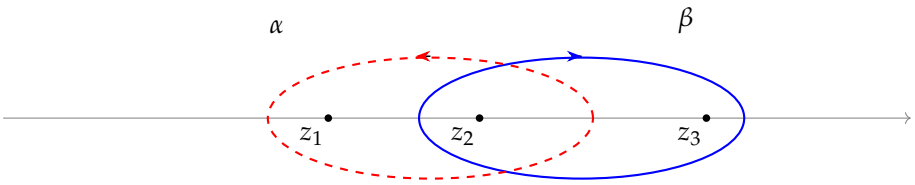


Figure 2. The β-cycle (blue) and the vanishing α-cycle (red).

Before stating the next lemma we recall that the functions

$$F(x;\kappa) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2t^2)}} = \int_0^\theta \frac{dt}{\sqrt{1-\kappa^2\sin^2 t}} =: F(\theta;\kappa),$$
$$E(x;\kappa) = \int_0^x \sqrt{\frac{1-\kappa^2t^2}{1-t^2}}dt = \int_0^\theta \sqrt{1-\kappa^2\sin^2 t}\,dt =: E(\theta;\kappa),$$
$$\Pi(n;x;\kappa) = \int_0^x \frac{dt}{(1-nt^2)\sqrt{(1-t^2)(1-\kappa^2t^2)}}$$
$$= \int_0^\theta \frac{dt}{(1-n\sin^2 t)\sqrt{1-\kappa^2\sin^2 t}} =: \Pi(n;\theta,\kappa)$$

are called incomplete elliptic integrals in the Legendre canonical forms of the first, second and third kind, respectively. The number κ is called the modulus and n is said to be characteristic.

For the particular value $x = 1$, we have the notations $K(\kappa) := F(1;\kappa)$, $E(\kappa) := E(1;\kappa)$ and $\Pi(n,\kappa) := \Pi(n;1,\kappa)$, which are known as complete elliptic integrals of the first, second and third kinds, respectively.

Lemma 1. *The action integral (41) has the following representation in terms of the standard Legendre's integrals*

$$2\pi I_1(h, j_2) = g_1 K(\kappa) + g_2 E(\kappa) + g_3 \Pi(n,\kappa), \tag{42}$$

where

$$g_1 = \frac{4(2h + z_1)}{3\sqrt{z_3 - z_1}}, \quad g_2 = \frac{4\sqrt{z_3 - z_1}}{3}, \quad g_3 = -\frac{j_2^2}{z_3\sqrt{z_3 - z_1}}$$

and

$$\kappa^2 = \frac{z_3 - z_2}{z_3 - z_1}, \qquad n = \frac{z_3 - z_2}{z_3}. \tag{43}$$

Proof. We transform I_1 as follows

$$\frac{4\pi I_1}{\sqrt{2}} = \oint_\beta \frac{w}{z} dz = \oint_\beta \frac{w^2}{zw} dz = -2\oint_\beta \frac{z^2}{w} dz + 2\oint_\beta \frac{z}{w} dz + 2h\oint_\beta \frac{dz}{w} - \frac{j_2^2}{2}\oint_\beta \frac{dz}{zw}. \tag{44}$$

Remembering that the β-cycle encircles the interval $[z_2, z_3]$ we get

$$\begin{aligned} 2\pi I_1 = &-2\int_{z_2}^{z_3} \frac{z^2 dz}{\sqrt{(z_3 - z)(z - z_2)(z - z_1)}} + 2\int_{z_2}^{z_3} \frac{z dz}{\sqrt{(z_3 - z)(z - z_2)(z - z_1)}} \\ &+ 2h\int_{z_2}^{z_3} \frac{dz}{\sqrt{(z_3 - z)(z - z_2)(z - z_1)}} - \frac{j_2^2}{2}\int_{z_2}^{z_3} \frac{dz}{z\sqrt{(z_3 - z)(z - z_2)(z - z_1)}}. \end{aligned} \tag{45}$$

Now, we can transform these integrals in a standard way to their Legendre canonical form or use appropriate formulas from Byrd & Friedman [38]. For example,

$$\int_{z_2}^{z_3} \frac{dz}{z\sqrt{(z_3 - z)(z - z_2)(z - z_1)}} \overset{z = z_3 - t^2}{=\!=} 2\int_0^{\sqrt{z_3 - z_2}} \frac{dt}{(z_3 - t^2)\sqrt{(z_3 - z_2 - t^2)(z_3 - z_1 - t^2)}}$$

$$\overset{t = \sqrt{z_3 - z_2}u}{=\!=} \frac{2}{z_3\sqrt{z_3 - z_1}} \int_0^1 \frac{du}{(1 - nu^2)\sqrt{(1 - u^2)(1 - \kappa^2 u^2)}} = \frac{2}{z_3\sqrt{z_3 - z_1}}\Pi(n,\kappa).$$

(compare with the formula 226.02 in [38]).

In a similar way we get

$$\int_{z_2}^{z_3} \frac{dz}{\sqrt{(z_3 - z)(z - z_2)(z - z_1)}} \overset{z = z_3 - t^2}{=\!=} 2\int_0^{\sqrt{z_3 - z_2}} \frac{dt}{\sqrt{(z_3 - z_2 - t^2)(z_3 - z_1 - t^2)}}$$

$$\overset{t = \sqrt{z_3 - z_2}u}{=\!=} \frac{2}{\sqrt{z_3 - z_1}} \int_0^1 \frac{du}{\sqrt{(1 - u^2)(1 - \kappa^2 u^2)}} = \frac{2}{\sqrt{z_3 - z_1}}K(\kappa),$$

$$\int_{z_2}^{z_3} \frac{z dz}{\sqrt{(z_3 - z)(z - z_2)(z - z_1)}} \overset{z = z_3 - t^2}{=\!=} 2\int_0^{\sqrt{z_3 - z_2}} \frac{(z_3 - t^2)dt}{\sqrt{(z_3 - z_2 - t^2)(z_3 - z_1 - t^2)}}$$

$$\overset{t = \sqrt{z_3 - z_2}u}{=\!=} \frac{2}{\sqrt{z_3 - z_1}} \int_0^1 \frac{[z_1 + (z_3 - z_1) - (z_3 - z_2)u^2]du}{\sqrt{(1 - u^2)(1 - \kappa^2 u^2)}}$$

$$\frac{2z_1}{\sqrt{z_3 - z_1}}K(\kappa) + 2\sqrt{z_3 - z_1}\int_0^1 \frac{1 - \kappa^2}{\sqrt{(1 - u^2)(1 - \kappa^2 u^2)}} = \frac{2z_1}{\sqrt{z_3 - z_1}}K(\kappa) + 2\sqrt{z_3 - z_1}E(\kappa).$$

The last integral is a bit lengthly, so we only give the result

$$\int_{z_2}^{z_3} \frac{z^2 dz}{\sqrt{(z_3-z)(z-z_2)(z-z_1)}} = \frac{2h+4z_1}{3\sqrt{z_3-z_1}}K(\kappa) + \frac{4}{3}\sqrt{z_3-z_1}E(\kappa)$$

(or use 235.00, 236.01 and 230.01 from [38] alternatively). Combining all these expressions, we obtain the needed result (42). □

Lemma 2. *The non-trivial action of the champagne bottle near the focus–focus point* $(0,0)$ *as a function of* h *and* j_2 *has the expansion*

$$\begin{aligned} 2\pi I_1(h,j_2) = \frac{4}{3} - \pi|j_2| + j_2 \arctan\frac{j_2}{h} + \mathcal{J}(h,j_2)\ln\frac{16}{\sqrt{h^2+j_2^2}} \\ + h + \frac{17h^2+3j_2^2}{16} - \frac{h(118h^6+125h^4j_2^2+194h^2j_2^4+39j_2^6)}{64(h^2+j_2^2)^2} + \dots, \end{aligned} \tag{46}$$

where the coefficient in front of the logarithm is

$$\mathcal{J}(h,j_2) = h - \frac{1}{8}(3h^2+j_2^2) + \frac{h}{64}(35h^2+15j_2^2) - \frac{1}{1024}(1155h^4+630h^2j_2^2+35j_2^4) + \dots \tag{47}$$

Proof. Clearly when $h, j_2 \to 0$ then $z_1 \to 0 \leftarrow z_2$ and $z_3 \to 1$; moreover, κ and n tend toward 1. Hence, we need the expansions of the complete elliptic integrals in the singular limit $\kappa \to 1$. To facilitate the calculations, we put $h \to \mu h$ and $j_2 \to \mu j_2$, where μ is a formal small parameter, and develop μ. Eventually, we set $\mu = 1$, but keep in mind that h and j_2 are close to zero.

The roots of $P(z)$ have the following expansions

$$\begin{aligned} z_1 &= -\mu\frac{\sqrt{h^2+j_2^2}+h}{2} + \mu^2 c_2 + \dots, \\ z_2 &= +\mu\frac{\sqrt{h^2+j_2^2}-h}{2} + \mu^2 b_2 + \dots, \\ z_3 &= 1 + \mu h - \mu^2(h^2+\tfrac{j_2^2}{4}) + \dots, \end{aligned} \tag{48}$$

where

$$b_2 = \frac{(4h^2+j_2^2)\sqrt{h^2+j_2^2} - 4h^3 - 3hj_2^2}{8\sqrt{h^2+j_2^2}}, \qquad c_2 = \frac{(4h^2+j_2^2)\sqrt{h^2+j_2^2} + 4h^3 + 3hj_2^2}{8\sqrt{h^2+j_2^2}}.$$

The complementary modulus κ' becomes

$$\kappa'^2 = 1 - \kappa^2 = \mu d_1 + \mu^2 d_2 + \dots,$$

where

$$d_1 = b_1 - c_1, \qquad d_2 = b_2 - c_2 + (b_1 - c1)(c_1 - a_1).$$

Denote $\Lambda := \ln\frac{\kappa'^2}{16}$. The above formula for κ' enables us to find that

$$\Lambda = \ln\frac{\mu d_1}{16} + \frac{d_2}{d_1}\mu + \dots \tag{49}$$

The following expansions near $\kappa = 1$ ($\kappa' = 0$) for $K(\kappa)$ and $E(\kappa)$ can be found in Cayley [39]

$$K(\kappa) = -\frac{1}{2}\Lambda\left[1 + \frac{1}{4}\kappa'^2 + \frac{9}{64}(\kappa'^2)^2 + \frac{25}{256}(\kappa'^2)^3 + \dots\right] - \frac{1}{4}\kappa'^2 - \frac{21}{128}(\kappa'^2)^2 - \frac{185}{1536}(\kappa'^2)^3 + \dots \tag{50}$$

$$E(\kappa) = -\Lambda\Big[\frac{1}{4}\kappa'^2 + \frac{3}{32}(\kappa'^2)^2 + \frac{15}{256}(\kappa'^2)^3 + \dots\Big] + 1 - \frac{1}{4}\kappa'^2 - \frac{13}{64}(\kappa'^2)^2 - \frac{9}{64}(\kappa'^2)^3 + \dots \tag{51}$$

Since $\kappa^2 < n < 1$ (positive circular case), the elliptic integral $\Pi(n,\kappa)$ reduces to Heuman's Lambda function Λ_0 (see Byrd & Friedman [38], p. 227)

$$\Pi(n,\kappa) = K(\kappa) + \frac{\pi}{2}\sqrt{\frac{n}{(n-\kappa^2)(1-n)}}(1 - \Lambda_0(\theta,\kappa)). \tag{52}$$

Then,

$$g_3\Pi(n,\kappa) = -\frac{j_2^2}{z_3\sqrt{z_3 - z_1}}\mu^2 - \pi|j_2|\mu + \pi|j_2|\mu\Lambda_0(\theta,\kappa).$$

The function $\Lambda_0(\theta,\kappa)$ is expressed via incomplete elliptic integrals $F(\theta,\kappa)$ and $E(\theta,\kappa)$ as

$$\Lambda_0(\theta,\kappa) = \frac{2}{\pi}\Big[K(\kappa)E(\theta,\kappa') - (K(\kappa) - E(\kappa))F(\theta,\kappa')\Big] \tag{53}$$

with

$$\theta = \arcsin\sqrt{\frac{1-n}{(\kappa')^2}}. \tag{54}$$

Taking into account the expansions of K and E, we obtain the Heuman's Lambda function as (see also [11])

$$\begin{aligned}\Lambda_0(\theta,\kappa) = \frac{2}{\pi}\theta - \sin\theta\cos\theta\Big[\frac{\kappa'^2}{2\pi} + \left(\frac{13}{32\pi} + \frac{3\sin^2\theta}{16\pi}\right)(\kappa'^2)^2 + \dots\Big] \\ - \Lambda\sin\theta\cos\theta\Big[\frac{\kappa'^2}{2\pi} + \left(\frac{3}{16\pi} + \frac{\sin^2\theta}{8\pi}\right)(\kappa'^2)^2 + \dots\Big].\end{aligned} \tag{55}$$

The leading term in the expansion of 2θ is

$$2\theta = 2\arcsin\sqrt{\frac{\sqrt{h^2 + j_2^2} - h}{2\sqrt{h^2 + j_2^2}}} + \dots$$

can be transformed using the relation [40]

$$2\arcsin\delta = \arctan\gamma \;\Rightarrow\; \gamma = 2\frac{\sqrt{1-\delta^2}\delta}{1 - 2\delta^2}$$

into the form

$$2\theta = \arctan\frac{|j_2|}{h} + \dots \tag{56}$$

Finally, combining all expansions given above we get the desired result. □

Remark 2. *As noticed by Dullin [9] the coefficient $\mathcal{J}(h, j_2)$ in front of the logarithm is also a complete elliptic integral, but along the α-cycle (imaginary cycle), which is vanishing as $h, j_2 \to 0$. To calculate it, the residue theorem can be used. Recall that the α-cycle encircles the interval $[z_1, z_2]$ along which $w^2 \le 0$ twice. Then*

$$\oint_\alpha \frac{w}{z}dz = 2\pi i\, res_{z=0}\frac{w}{z}.$$

Since only the values of the integrand $\frac{w}{z}$ *close to* $(h, j_2) = (0,0)$ *matter, we formally scale* $h \to \mu h, j_2 \to \mu j_2$ *and find the expansion of* $\frac{w}{z}$ *with respect to* $\mu << 1$. *As a result we get*

$$res_{z=0}\frac{w}{z} = \frac{1}{\sqrt{2}}\Big[h\mu - \frac{1}{8}(3h^2 + j_2^2)\mu^2 + \frac{h}{64}(35h^2 + 15j_2^2)\mu^3 - \frac{1}{1024}(1155h^4 + 630h^2j_2^2 + 35j_2^4)\mu^4\Big] + \dots$$

After putting $\mu = 1$ *as before and keeping in mind that* h *and* j_2 *are small, we obtain exactly the inverse of the Birkhoff normal form (38)*

$$\mathcal{J}(h, j_2) = h - \frac{(3h^2 + j_2^2)}{8} + \frac{h(35h^2 + 15j_2^2)}{64} - \frac{1155h^4 + 630h^2j_2^2 + 35j_2^4}{1024} + \dots := \frac{\sqrt{2}}{2\pi i}\oint_\alpha \frac{w}{z}dz. \tag{57}$$

The geometric interpretation of $\mathcal{J}$ *is also given in [9].*

5. Semi-Global Symplectic Invariant

In our case the critical value of the simple focus–focus point is $(h, j_2) = (0,0)$. By our assumptions, the singular fiber $F^{-1}(0)$ contains only one critical point m and the component of $F^{-1}(0)$ containing m is compact; see Figure 3. In the neighborhood of the focus–focus point, we can consider a momentum map $J = (J_1, J_2)$ consisting of momenta J_1 and J_2 of the quadratic normal form. In this neighborhood, we may assume that H and the second integral (in this case J_2) are functions of J_1 and J_2. This is a general result due to Eliasson [23] and we have demonstrated it in practice by constructing the normal form $\overline{H}$ via near-identity transformations. So, in this neighborhood we can use $\overline{H}$ instead H. Let the point $j = (j_1, j_2) \in \mathbb{R}^2$ be the image of the momentum map $J = (J_1, J_2)$. It can be identified by the complex number $\hat{j} = j_1 + ij_2$. Therefore, we can use (j_1, j_2) as coordinates instead of (h, j_2) in the considered neighborhood of $(0,0)$.

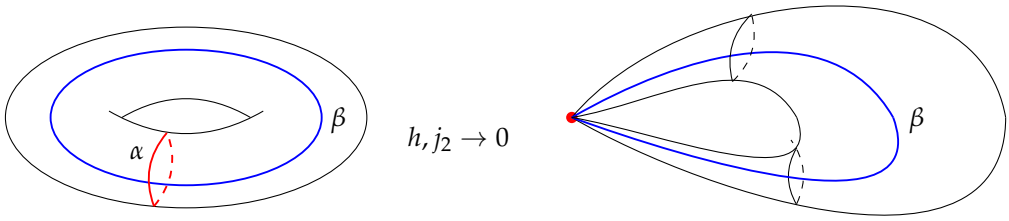


Figure 3. The singular fiber (pinched torus) at the focus–focus point.

According to [3] non-trivial action near a non-degenerate focus–focus point can be written as follows

$$2\pi I_1(j_1, j_2) = 2\pi I_{10} - \mathrm{Re}(\hat{j}\ln\hat{j} - \hat{j}) + S(j_1, j_2), \tag{58}$$

where I_{10} is a constant and $S(j_1, j_2)$ is the semi-global symplectic invariant, which is in fact a smooth formal series. In our case we have

Theorem 4. *The nontrivial action* I_1 *of the champagne bottle near the focus–focus point (0, 0) is given by*

$$2\pi I_1 = \frac{4}{3} - \pi|j_2| + j_2 arg\hat{j} - j_1\ln|\hat{j}| + j_1 + S(j_1, j_2),$$

where

$$S(j_1, j_2) = j_1 \ln 16 + \frac{1}{16}(17j_1^2 + 3j_2^2) - \frac{j_1}{128}(125j_1^2 + 43j_2^2) + \dots \tag{59}$$

Proof. We only need to substitute the normal form (37) into the expansion of the action I_1 (46) bearing in mind that $\mathcal{J}(h(j_1, j_2), j_2) = j_1$. □

Choosing a different branch of the complex logarithm changes the action I_1 by an integer multiple of j_2, and hence, does not influence the invariant $S(j_1, j_2)$ since it depends only on the foliation.

Next, we use the obtained expansions to calculate the important dynamical quantities, such as the period T and the rotation number W.

The period of the reduced system with respect to the S^1-action induced by J_2 is defined as

$$T(h, j_2) = 2\pi \frac{\partial I_1}{\partial h}(h, j_2). \tag{60}$$

The period can also be viewed as a function of j_1, j_2 near the focus–focus point: $T(j_1, j_2) = T(\overline{H}(j_1, j_2), j_2)$. Then, the result of Theorem 4 yields

$$T(j_1, j_2) = 2\pi \frac{\partial I_1/\partial j_1}{\partial \overline{H}/\partial j_1} = \frac{-\ln|\hat{j}| + S_1}{\partial \bar{H}/\partial j_1} = \frac{\ln \frac{16}{|\hat{j}|} + \frac{17}{8} j_1 + O(2)}{1 + \frac{3}{4} j_1 + O(2)},$$

where $S_k = \partial S/\partial j_k, k = 1, 2$. As expected, the period goes to infinity as $\hat{j} \to 0$, i.e., when approaching the focus–focus point. More detailed expansion can be obtained merely by differentiating (46) in Lemma 2 with respect to h.

The rotation number is defined as follows

$$W(h, j_2) = -\frac{\partial I_1}{\partial j_2}(h, j_2). \tag{61}$$

Clearly, the easiest way to get it is to differentiate (46) in Lemma 2 with respect to j_2. Alternatively, we consider W as a function of j_1, j_2. Then, [9]

$$2\pi W(j_1, j_2) = -2\pi \frac{\partial I_1}{\partial j_2}(j_1, j_2) = \pi sgn(j_2) - arg\hat{j} - A \ln|\hat{j}| + AS_1 - S_2,$$

where $A = A(j) := \partial_{j_2}\overline{H}/\partial_{j_1}\overline{H}$, which expansion reads as follows

$$A(j) = \frac{j_2}{4} - \frac{15 j_1 j_2}{32} + \frac{j_2(135j_1^2 + 10j_2^2)}{128} + \dots$$

From here $W(j_1, j_2)$ can be easily found to be

$$\begin{aligned} 2\pi W(j_1, j_2) = \pi sgn(j_2) - arctan\frac{j_2}{j_1} + \ln \frac{16}{\sqrt{j_1^2 + j_2^2}} \left(\frac{j_2}{4} - \frac{15}{32} j_1 j_2 + \frac{j_2}{128}(135j_1^2 + 10j_2^2) + \dots \right) \\ - \frac{3}{8} j_2 + \frac{77}{64} j_1 j_2 - \frac{j_2}{512}(885j_1^2 + 43j_2^2) + \dots \end{aligned} \tag{62}$$

Notice that W is odd with respect to j_2.

Another important dynamical quantity is the twist $\mathcal{T}$, which is defined as

$$\mathcal{T}(h, j_2) = \frac{\partial W}{\partial j_2}(h, j_2) \tag{63}$$

for constant h. $\mathcal{T}$ is even with respect to j_2. The twist condition $\partial W/\partial j_2 \neq 0$ is equivalent to the iso-energetic non-degeneracy condition (or Arnold–Moser condition), which ensures the survival of invariant tori under small perturbations for the same energy. Using the developed above technique Dullin and Vũ Ngọc [31] showed that when the focus–focus point is loxodromic, there exist tori with vanishing twist for each value of h close to the critical one. In our case the eigenvalues are $\pm\sqrt{2}$, see Section 3. Nevertheless, the result remains the same.

From [9,31] we have a representation of $\mathcal{T}$, namely

$$\mathcal{T}(j_1, j_2) = -A(j)\frac{\partial W}{\partial j_1} + \frac{\partial W}{\partial j_2}. \tag{64}$$

Straightforward calculations give that

$$2\pi\mathcal{T}(j) = \frac{j_1(A(j)^2-1)}{j_1^2+j_2^2} - \frac{2j_2A(j)}{j_1^2+j_2^2} - \frac{3}{8} + \frac{77}{64}j_1 + O(2) + \ln\frac{16}{\sqrt{j_1^2+j_2^2}}\left(\frac{1}{4} - \frac{15}{32}j_1 + O(2)\right).$$

After introducing the mapping $\tilde{\mathcal{T}}(j) = |j|^2 2\pi\mathcal{T}(j)$ the above expression becomes

$$\tilde{\mathcal{T}}(j) = (A^2(j)-1)j_1 - 2A(j)j_2 + O(|j|^2\ln|j|).$$

Since $|j|^2\ln|j|$ is C^1 at the origin, it follows that $\tilde{\mathcal{T}}(0) = 0, \partial\tilde{\mathcal{T}}/\partial j_1(0) = -1, \partial\tilde{\mathcal{T}}/\partial j_2(0) = 0$. Then $\mathcal{T}^{-1}(0)$ is a C^1 curve through origin with a tangent at origin $j_1 = 0$. In a neighborhood of the origin $j_1 \sim h$, hence we recover the shape of the curve along which the twist condition is violated near the origin $(h, j_2) = (0, 0)$, see Figure 1. There is no need to comment on the Kolmogorov condition, because it has been verified in several ways [27,30–32].

6. Concluding Remarks

This paper is inspired by [9], where the gave author the first example of explicit computation of the semi-global symplectic invariant near a focus–focus singularity in the spherical pendulum. The spherical pendulum and the system studied here share many common features; for instance, they are generalized semi-toric systems in the terminology of Pelayo et al. [14]. Here we make use of Dullin's ideas to calculate the semi-global symplectic invariant for the champagne bottle. Prior to that we have calculated the Birkhoff normal form and the non-trivial action near the focus–focus point and after that certain important dynamical quantities. Taylor series invariant for the coupled spin-oscillators is computed in [10], however it is a semitoric system.

This work offers no fundamentally new mathematical contribution. Rather, it can be seen as an extension of mathematical knowledge.

In a view of the classification of the integrable systems we compare our result (59) with the semi-global symplectic invariant of the spherical pendulum

$$S_{sp}(j_1, j_2) = j_1\ln 32 + \frac{3(j_1^2+3j_2^2)}{32} - \frac{j_1(5j_1^2+51j_2^2)}{512} + \frac{(55j_1^4+1230j_1^2j_2^2+271j_2^4)}{32{,}768} + \dots \tag{65}$$

Apparently they do not coincide, so we can conclude that the spherical pendulum and the champagne bottle are not symplectically equivalent.

There is no general symplectic classification for the generalized semitoric systems so far [17]. But we hope that collecting more examples like the considered here will help the classification program to be reevaluated.

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