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G-α^k-Derivations of Hom-Lie Algebras

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Abstract

In this paper, we introduce a new class of generalized derivations on multiplicative Hom-Lie algebras, called G-α^k-derivations. Such derivations depend on two automorphisms $\theta, \vartheta \in G$ and generalize the classical α^k-derivations of original algebras. We construct a nontrivial Hom-Lie algebra and its (θ, id)-α^k-derivations following a geometric progression rule. Moreover, we prove that two related derivation spaces share the same dimension. Particularly, a Hom-Lie algebra structure is constructed on the direct sum of derivations and θ-derivations. Finally, we study the restriction of G-α^k-derivations to Hom-ideals and establish their relationships with α^k-quasi-derivations.

Keywords: Hom-Lie algebra; Hom-ideal; G-α^k-derivation; α^k-quasi-derivation

1. Introduction

The notion of Hom-Lie algebras originated from the investigation of σ-derivation deformations of the Witt and Virasoro algebras [1]. Hom-Lie algebras exhibit profound intrinsic connections with Hom-quantum groups, the Hom-type Yang-Baxter equation, and Hom-Lie bialgebras [2–4]. Consequently, they have evolved into a significant branch of non-associative algebras. Over the past twenty years, well-developed frameworks have been established for the representation theory [5], as well as the cohomology and deformation theories [6,7] of Hom-Lie algebras. Building on these foundational developments, a wide array of generalizations of Hom-type algebras have been successively developed [8–12].

Derivations are a common type of transformation in algebras, and they play an important role in studying the basic properties of algebras and describing algebraic structures. Starting from the classic derivations, various types such as inner derivations, quasi-derivations, and generalized derivations have appeared later. In recent years, the study of derivation-type mappings has expanded well beyond the classical Lie and associative settings. Various generalizations and characterizations of derivations have been investigated in different non-associative algebraic structures. For instance, generalized derivations of multiplicative n-ary Hom-Ω color algebras were studied in [13]. In the framework of alternative rings, Jordan derivations [14], multiplicative Lie-type derivations [15], additivity of n-multiplicative maps [16], and additivity properties of multiplicative derivations [17] have also been extensively investigated. Among them, as an important generalization of derivations, generalized derivations have so far achieved many valuable results in the study of different algebras, including rings [18], Lie algebras [19], Lie triple systems [20], and Hom-Lie algebras [12]. More related results can refer to references [21–23]. The study of generalized derivations not only deepens the understanding of algebraic structures but also has potential applications in mathematical physics, such as in the analysis of

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symmetries and conservation laws. In 2021, some scholars defined a new type of generalized derivation in Lie algebras [24]. Later, the relevant research results were gradually extended, and corresponding results were also obtained in Lie superalgebras [25] and Lie-Yamaguti algebras [26]. The main novelty of our work is to introduce and systematically study this new type of generalized derivation in the context of Hom-Lie algebras, thereby bridging the gap between the classical theory of derivations and the rapidly developing field of Hom-type algebras. Inspired by these research works, we want to study this new type of generalized derivation problem in Hom-type algebras. Therefore, we did relevant research on Hom-Lie algebras and obtained some conclusions about this type of generalized derivation.

This paper is organized as follows: In Section 2, we recall some necessary concepts and introduce the notion of G-α^k-derivations of Hom-Lie algebras. We present (θ, id)-α^k-derivations following a geometric progression rule on a nontrivial Hom-Lie algebra. In Section 3, we study the Hom-Lie algebraic structures on spaces of such derivations, including dimension relations and the construction of Hom-Lie algebras via direct sums. In Section 4, we discuss the restriction of such derivations to Hom-ideals and establish their relation with α^k-quasi-derivations.

2. Preliminaries

Definition 1 ([1]). *(1) A Hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ consisting of a vector space $\mathfrak{g}$ over a field $\mathbb{K}$, a skew-symmetric bilinear map (bracket) $[\cdot,\cdot] : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{g}$, and a linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$ satisfying the following Hom-Jacobi identity:*

$$[\alpha(x), [y,z]] + [\alpha(y), [z,x]] + [\alpha(z), [x,y]] = 0.$$

(2) A Hom-Lie algebra is called a multiplicative Hom-Lie algebra if for any $x, y \in \mathfrak{g}$, we have $\alpha([x,y]) = [\alpha(x), \alpha(y)]$.

In this paper, all Hom-Lie algebras mentioned are assumed to be multiplicative Hom-Lie algebras.

Definition 2 ([1]). *A subspace I of a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ is called a Hom-ideal of $\mathfrak{g}$ if it satisfies $\alpha(I) \subseteq I$ and $[I, \mathfrak{g}] \subseteq I$.*

For a Hom-ideal I of $\mathfrak{g}$, we call I abelian if $[I, I] = 0$. The derived Hom-subalgebra of $\mathfrak{g}$ is denoted by $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$. If $\mathfrak{g}^{(1)} = \mathfrak{g}$, then $\mathfrak{g}$ is said to be perfect.

Definition 3 ([27]). *The center $Z(\mathfrak{g})$ of a Hom-Lie algebra $\mathfrak{g}$ can be defined as $Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x,y] = 0,\ \forall y \in \mathfrak{g}\}$. We say that $\mathfrak{g}$ is centerless if $Z(\mathfrak{g}) = \{0\}$.*

Definition 4 ([1]). *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ and $(\mathfrak{g}', [\cdot,\cdot]', \beta)$ be Hom-Lie algebras. A linear map $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is called a Hom-Lie algebra homomorphism if*

$$\varphi\alpha = \beta\varphi, \qquad \varphi([x,y]) = [\varphi(x), \varphi(y)]',$$

for all $x, y \in \mathfrak{g}$. Here, $\varphi\alpha = \beta\varphi$ means $\varphi \circ \alpha = \beta \circ \varphi$. In what follows, the composition of linear maps is also written by juxtaposition. A bijective homomorphism from $\mathfrak{g}$ onto itself is called an automorphism. The automorphism group of $\mathfrak{g}$ is denoted by $\mathrm{Aut}(\mathfrak{g})$.

Definition 5 ([5]). *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra. For any nonnegative integer k, an α^k-derivation on $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \to \mathfrak{g}$ such that*

$$\begin{aligned} D\alpha &= \alpha D, \\ D([x,y]) &= [D(x), \alpha^k(y)] + [\alpha^k(x), D(y)], \end{aligned}$$

for all $x, y \in \mathfrak{g}$.

Proposition 1 ([12]). *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra. Define a subspace $\mho$ of the vector space $\mathrm{End}(\mathfrak{g})$ and a linear map σ on $\mho$ satisfying*

$$\mho = \{u \in \mathrm{End}(\mathfrak{g}) \mid u\alpha = \alpha u\}, \quad \sigma : \mho \to \mho;\ \sigma(u) = \alpha u.$$

Then $\mho$, equipped with the operation

$$[D, D'] = DD' - D'D, \quad \forall D, D' \in \mho,$$

forms a Hom-Lie algebra.

We denote the set of all α^k-derivations by $\mathrm{Der}_{\alpha^k}(\mathfrak{g})$, and let $\mathrm{Der}(\mathfrak{g}) := \bigoplus_{k\geq 0} \mathrm{Der}_{\alpha^k}(\mathfrak{g})$. Then $\mathrm{Der}(\mathfrak{g})$, equipped with the above bracket operation and the linear map

$$\tilde{\alpha} : \mathrm{Der}(\mathfrak{g}) \to \mathrm{Der}(\mathfrak{g}), \quad \tilde{\alpha}(D) = D\alpha$$

is a Hom-subalgebra of $\mho$, which is called the **derivation algebra** of $\mathfrak{g}$.

Definition 6 ([5]). *For any $x \in \mathfrak{g}$ satisfying $\alpha(x) = x$, define $D_k(x) : \mathfrak{g} \to \mathfrak{g}$ by*

$$D_k(x)(y) = [\alpha^k(y), x], \quad \forall y \in \mathfrak{g}.$$

Then $D_k(x)$ is an α^{k+1}-derivation, which we call an inner α^{k+1}-derivation.

Definition 7 ([28]). *For any nonnegative integer k, a linear map $D : \mathfrak{g} \to \mathfrak{g}$ is called an α^k-quasi-derivation if there exists $D' \in \mathrm{End}(\mathfrak{g})$, such that*

$$\begin{aligned} D'\alpha &= \alpha D', \\ D\alpha &= \alpha D, \\ D'([x,y]) &= [D(x), \alpha^k(y)] + [\alpha^k(x), D(y)], \end{aligned}$$

for all $x, y \in \mathfrak{g}$. The set of all α^k-quasi-derivations is denoted by $\mathrm{QDer}_{\alpha^k}(\mathfrak{g})$.

Definition 8 ([28]). *For any nonnegative integer k, the set $C_{\alpha^k}(\mathfrak{g}) = \{D \in \mathrm{End}(\mathfrak{g}) \mid D\alpha = \alpha D, D([x,y]) = [D(x), \alpha^k(y)] = [\alpha^k(x), D(y)], \forall x, y \in \mathfrak{g}\}$ is called the α^k-centroid of $\mathfrak{g}$.*

Definition 9. *Let G be a subgroup of $\mathrm{Aut}(\mathfrak{g})$. For any nonnegative integer k, a linear map $D : \mathfrak{g} \to \mathfrak{g}$ is called a G-α^k-derivation of $\mathfrak{g}$ if there exist $\theta, \vartheta \in G$ such that the following conditions hold:*

$$\begin{aligned} &D\alpha = \alpha D, \qquad \theta\alpha = \alpha\theta, \qquad \vartheta\alpha = \alpha\vartheta, \\ &D([x,y]) = [D(x), \theta\alpha^k(y)] + [\vartheta\alpha^k(x), D(y)], \end{aligned}$$

for all $x, y \in \mathfrak{g}$. The automorphisms θ and ϑ are called the associated automorphisms of D.

The set of all G-α^k-derivations of $\mathfrak{g}$ is denoted by $\mathrm{Der}^G_{\alpha^k}(\mathfrak{g})$. For fixed $\theta, \vartheta \in G$, we write $\mathrm{Der}^G_{\theta,\vartheta,\alpha^k}(\mathfrak{g})$ for the subspace consisting of all G-α^k-derivations associated with θ and ϑ. If $\vartheta = \mathrm{id}$, we simply write $\mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g}) = \mathrm{Der}^G_{\theta,\mathrm{id},\alpha^k}(\mathfrak{g})$. In particular, $\mathrm{Der}^G_{\mathrm{id},\mathrm{id},\alpha^k}(\mathfrak{g}) = \mathrm{Der}_{\alpha^k}(\mathfrak{g})$. When $\alpha = \mathrm{id}$, the above definition reduces to the usual G-derivation of a Lie algebra.

Definition 10. *Let G be a subgroup of $\mathrm{Aut}(\mathfrak{g})$. For any nonnegative integer k, a linear map $D : \mathfrak{g} \to \mathfrak{g}$ is called a G-α^k-quasi-derivation of $\mathfrak{g}$ if there exist $\theta, \vartheta \in G$ and $D' \in \mathrm{End}(\mathfrak{g})$ such that the following conditions hold:*

$$D\alpha = \alpha D, \qquad D'\alpha = \alpha D', \qquad \theta\alpha = \alpha\theta, \qquad \vartheta\alpha = \alpha\vartheta,$$

$$D'([x,y]) = [D(x), \theta\alpha^k(y)] + [\vartheta\alpha^k(x), D(y)],$$

for all $x, y \in \mathfrak{g}$. The automorphisms θ and ϑ are called the associated automorphisms of D. The set of all G-α^k-quasi-derivations is denoted by $\mathrm{QDer}^G_{\theta,\vartheta,\alpha^k}(\mathfrak{g})$.

Remark 1. *If $\vartheta = \mathrm{id}$, we simply write $\mathrm{QDer}_{\theta,\alpha^k}(\mathfrak{g}) = \mathrm{QDer}^G_{\theta,\mathrm{id},\alpha^k}(\mathfrak{g})$. In particular, $\mathrm{QDer}^G_{\mathrm{id},\mathrm{id},\alpha^k}(\mathfrak{g}) = \mathrm{QDer}_{\alpha^k}(\mathfrak{g})$.*

Example 1. *Let $\mathfrak{g}$ be a three-dimensional vector space over $\mathbb{K}$ with a basis $\{e_1, e_2, e_3\}$. Define a skew-symmetric bracket on $\mathfrak{g}$ by*

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = [e_2, e_3] = 0.$$

Let $\alpha : \mathfrak{g} \to \mathfrak{g}$ be given by

$$\alpha(e_1) = 2e_1, \qquad \alpha(e_2) = 3e_2, \qquad \alpha(e_3) = 6e_3.$$

Then $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ is a multiplicative Hom-Lie algebra.

Define an automorphism θ of $\mathfrak{g}$ by

$$\theta(e_1) = 2e_1, \qquad \theta(e_2) = 2e_2, \qquad \theta(e_3) = 4e_3.$$

Then $\theta\alpha = \alpha\theta$. Here we define a linear map $D : \mathfrak{g} \to \mathfrak{g}$ by

$$D(e_1) = 2e_1, \qquad D(e_2) = 3e_2, \qquad D(e_3) = 18e_3.$$

We claim that $D \in \mathrm{Der}^G_{\theta,\alpha}(\mathfrak{g})$. Clearly, $D\alpha = \alpha D$. Since e_3 is central, only the basis $\{e_1, e_2\}$ need to be considered. We have

$$D([e_1, e_2]) = D(e_3) = 18e_3.$$

On the other hand,

$$[D(e_1), \theta\alpha(e_2)] + [\alpha(e_1), D(e_2)] = [2e_1, 6e_2] + [2e_1, 3e_2] = 18e_3.$$

Thus D is a G-α-derivation associated with θ and id.

Example 2. *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra and θ be an automorphism given in Example 1. Indeed, for every $k \geq 1$ (here $k \neq 0$ ensures that the Hom-Lie algebra is nontrivial), define a linear map $D^{(k)} : \mathfrak{g} \to \mathfrak{g}$ by*

$$D^{(k)}(e_1) = 2^k e_1, \qquad D^{(k)}(e_2) = 3^k e_2, \qquad D^{(k)}(e_3) = 3 \cdot 6^k e_3.$$

Since $D^{(k)}$ and α are both diagonal with respect to the basis $\{e_1, e_2, e_3\}$, we have

$$D^{(k)}\alpha = \alpha D^{(k)}.$$

We claim that $D^{(k)} \in \mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g})$. That is, we need to verify

$$D^{(k)}([e_i, e_j]) = [D^{(k)}(e_i), \theta\alpha^k(e_j)] + [\alpha^k(e_i), D^{(k)}(e_j)],$$

for all $e_i, e_j \in \mathfrak{g}, i, j = 1, 2, 3$. On the one hand,

$$D^{(k)}([e_1, e_2]) = D^{(k)}(e_3) = 3 \cdot 6^k e_3.$$

On the other hand,

$$\alpha^k(e_1) = 2^k e_1, \qquad \alpha^k(e_2) = 3^k e_2,$$

and

$$\theta\alpha^k(e_2) = \theta(3^k e_2) = 2 \cdot 3^k e_2.$$

Therefore,

$$\begin{aligned} &[D^{(k)}(e_1), \theta\alpha^k(e_2)] + [\alpha^k(e_1), D^{(k)}(e_2)] \\ =&[2^k e_1, 2 \cdot 3^k e_2] + [2^k e_1, 3^k e_2] \\ =&2 \cdot 6^k e_3 + 6^k e_3 \\ =&3 \cdot 6^k e_3. \end{aligned}$$

Since e_3 is centralizer, hence, for every $k \geq 1$,

$$D^{(k)}([e_i, e_j]) = [D^{(k)}(e_i), \theta\alpha^k(e_j)] + [\alpha^k(e_i), D^{(k)}(e_j)], \qquad i, j = 1, 2, 3.$$

That is, $D^{(k)} \in \mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g})$. These (θ, id)-α^k-derivations have a geometric progression property with respect to k; that is, the coefficients of $D^{(k)}(e_1)$, $D^{(k)}(e_2)$ and $D^{(k)}(e_3)$ form geometric sequences in k, with common ratios 2, 3 and 6, respectively.

3. Hom-Lie Structures on Derivation Spaces

Proposition 2. *For any $\theta, \vartheta \in G$ and nonnegative integer k, the spaces $\mathrm{Der}^G_{\theta,\vartheta,\alpha^k}(\mathfrak{g})$ and $\mathrm{Der}^G_{\vartheta^{-1}\theta,\alpha^k}(\mathfrak{g})$ are linearly isomorphic. Consequently,* $\dim \mathrm{Der}^G_{\theta,\vartheta,\alpha^k}(\mathfrak{g}) = \dim \mathrm{Der}^G_{\vartheta^{-1}\theta,\alpha^k}(\mathfrak{g})$.

Proof. Define two maps F_ϑ and G_ϑ by

$$F_\vartheta : \mathrm{Der}^G_{\theta,\vartheta,\alpha^k}(\mathfrak{g}) \longrightarrow \mathrm{Der}^G_{\vartheta^{-1}\theta,\alpha^k}(\mathfrak{g}), \qquad F_\vartheta(D) = \vartheta^{-1}D,$$

$$G_\vartheta : \mathrm{Der}^G_{\vartheta^{-1}\theta,\alpha^k}(\mathfrak{g}) \longrightarrow \mathrm{Der}^G_{\theta,\vartheta,\alpha^k}(\mathfrak{g}), \qquad G_\vartheta(D) = \vartheta D.$$

Let $D \in \mathrm{Der}^G_{\theta,\vartheta,\alpha^k}(\mathfrak{g})$. Since both D and ϑ commute with α, we have

$$F_\vartheta(D)\alpha = \vartheta^{-1}D\alpha = \vartheta^{-1}\alpha D = \alpha\vartheta^{-1}D = \alpha F_\vartheta(D).$$

Moreover, applying ϑ^{-1} to the defining identity of D gives

$$\begin{aligned} F_\vartheta(D)([x, y]) &= \vartheta^{-1}D([x, y]) \\ &= \vartheta^{-1}\big([D(x), \theta\alpha^k(y)] + [\vartheta\alpha^k(x), D(y)]\big) \\ &= [F_\vartheta(D)(x), \vartheta^{-1}\theta\alpha^k(y)] + [\alpha^k(x), F_\vartheta(D)(y)]. \end{aligned}$$

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Therefore $F_\vartheta(D) \in \mathrm{Der}^G_{\vartheta^{-1}\theta,\alpha^k}(\mathfrak{g})$. It is clear that the maps F_ϑ and G_ϑ are linear, $F_\vartheta^{-1} = G_\vartheta$. The conclusion holds, i.e., $\dim \mathrm{Der}^G_{\theta,\vartheta,\alpha^k}(\mathfrak{g}) = \dim \mathrm{Der}^G_{\vartheta^{-1}\theta,\alpha^k}(\mathfrak{g})$. □

Corollary 1. *For any $\theta \in G$ and nonnegative integer k, $\mathrm{Der}^G_{\theta,\theta,\alpha^k}(\mathfrak{g}) \cong \mathrm{Der}_{\alpha^k}(\mathfrak{g})$.*

Now we consider $\mathcal{D}_{\theta,\theta}(\mathfrak{g}) := \bigoplus_{k\geq 0} \mathrm{Der}^G_{\theta,\theta,\alpha^k}(\mathfrak{g})$. Namely, for $D_1, D_2 \in \mathcal{D}_{\theta,\theta}(\mathfrak{g})$, define

$$[D_1, D_2]_\theta = F_\theta^{-1}([F_\theta(D_1), F_\theta(D_2)]),$$

$$\widetilde{\alpha}_\theta(D) = D\theta^{-1}\alpha\theta.$$

Since $\theta\alpha = \alpha\theta$, this can also be written as

$$\widetilde{\alpha}_\theta(D) = D\alpha.$$

It can be proved that $(\mathcal{D}_{\theta,\theta}(\mathfrak{g}), [\cdot,\cdot]_\theta, \widetilde{\alpha}_\theta)$ is a Hom-Lie algebra. We prove, in a convenient way, that the Hom-Jacobi identity holds on $\mathcal{D}_{\theta,\theta}(\mathfrak{g})$; the remaining properties are obvious. By Proposition 2, for any nonnegative integer k, the map

$$F_\theta : \mathrm{Der}^G_{\theta,\theta,\alpha^k}(\mathfrak{g}) \longrightarrow \mathrm{Der}_{\alpha^k}(\mathfrak{g}), \qquad F_\theta(D) = \theta^{-1}D,$$

is a linear isomorphism. Hence it induces a linear isomorphism $F_\theta : \mathcal{D}_{\theta,\theta}(\mathfrak{g}) \longrightarrow \mathrm{Der}(\mathfrak{g})$. Its inverse is given by

$$F_\theta^{-1}(T) = \theta T, \qquad T \in \mathrm{Der}(\mathfrak{g}).$$

By definition,

$$[D_1, D_2]_\theta = F_\theta^{-1}([F_\theta(D_1), F_\theta(D_2)]),$$

for all $D_1, D_2 \in \mathcal{D}_{\theta,\theta}(\mathfrak{g})$. Therefore,

$$\begin{aligned} F_\theta([D_1, D_2]_\theta) &= F_\theta F_\theta^{-1}([F_\theta(D_1), F_\theta(D_2)]) \\ &= [F_\theta(D_1), F_\theta(D_2)]. \end{aligned}$$

Since $\theta\alpha = \alpha\theta$, we have

$$\begin{aligned} F_\theta(\widetilde{\alpha}_\theta(D)) &= F_\theta(D\theta^{-1}\alpha\theta) \\ &= \theta^{-1}D\theta^{-1}\alpha\theta \\ &= \theta^{-1}D\alpha \\ &= (\theta^{-1}D)\alpha \\ &= \widetilde{\alpha}(F_\theta(D)). \end{aligned}$$

It remains to verify the Hom-Jacobi identity. For any $D_1, D_2, D_3 \in \mathcal{D}_{\theta,\theta}(\mathfrak{g})$, set

$$\begin{aligned} J_\theta(D_1, D_2, D_3) = {} & [\widetilde{\alpha}_\theta(D_1), [D_2, D_3]_\theta]_\theta + [\widetilde{\alpha}_\theta(D_2), [D_3, D_1]_\theta]_\theta \\ & + [\widetilde{\alpha}_\theta(D_3), [D_1, D_2]_\theta]_\theta. \end{aligned}$$

We obtain

$$\begin{aligned} F_\theta(J_\theta(D_1, D_2, D_3)) = {} & [F_\theta(\widetilde{\alpha}_\theta(D_1)), F_\theta([D_2, D_3]_\theta)] \\ & + [F_\theta(\widetilde{\alpha}_\theta(D_2)), F_\theta([D_3, D_1]_\theta)] \\ & + [F_\theta(\widetilde{\alpha}_\theta(D_3)), F_\theta([D_1, D_2]_\theta)] \\ = {} & [\widetilde{\alpha}(F_\theta(D_1)), [F_\theta(D_2), F_\theta(D_3)]] \\ & + [\widetilde{\alpha}(F_\theta(D_2)), [F_\theta(D_3), F_\theta(D_1)]] \\ & + [\widetilde{\alpha}(F_\theta(D_3)), [F_\theta(D_1), F_\theta(D_2)]]. \end{aligned}$$

Since $(\mathrm{Der}(\mathfrak{g}), [\cdot,\cdot], \widetilde{\alpha})$ is a Hom-Lie algebra by Proposition 1, the last expression is equal to zero. Thus $F_\theta(J_\theta(D_1, D_2, D_3)) = 0$. We get $J_\theta(D_1, D_2, D_3) = 0$. Hence, the Hom-Jacobi identity holds on $\mathcal{D}_{\theta,\theta}(\mathfrak{g})$.

Therefore, $(\mathcal{D}_{\theta,\theta}(\mathfrak{g}), [\cdot,\cdot]_\theta, \widetilde{\alpha}_\theta)$ is a Hom-Lie algebra.

Theorem 1. *The Hom-Lie algebras* $(\mathcal{D}_{\theta,\theta}(\mathfrak{g}), [\cdot,\cdot]_\theta, \widetilde{\alpha}_\theta)$ *and* $(\mathrm{Der}(\mathfrak{g}), [\cdot,\cdot], \widetilde{\alpha})$ *are isomorphic.*

Proof. By Corollary 1, F_θ is a bijective linear map. We have

$$F_\theta([D_1, D_2]_\theta) = F_\theta F_\theta^{-1}([F_\theta(D_1), F_\theta(D_2)]) = [F_\theta(D_1), F_\theta(D_2)].$$

For the twisting maps, using $\theta\alpha = \alpha\theta$ gives

$$F_\theta(\widetilde{\alpha}_\theta(D)) = \theta^{-1}D\theta^{-1}\alpha\theta = \theta^{-1}D\alpha = (\theta^{-1}D)\alpha = \widetilde{\alpha}(F_\theta(D)),$$

which implies F_θ is again a homomorphism of Hom-Lie algebras. Hence, F_θ is an isomorphism. □

To simplify notation, we write $\mathcal{D}_\theta(\mathfrak{g}) = \bigoplus_{k\geq 0} \mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g})$ and present other results.

Lemma 1. *Assume that* θ *commutes with* α *and with the operators under consideration. Let* $D \in \mathrm{Der}^G_{\theta^i,\alpha^k}(\mathfrak{g})$, $T \in \mathrm{Der}^G_{\theta^j,\alpha^s}(\mathfrak{g})$, *where* $i, j \in \{0, 1\}$. *Then* $[D, T] \in \mathrm{Der}^G_{\theta^{i+j},\alpha^{k+s}}(\mathfrak{g})$.

Proof. Since $D\alpha = \alpha D$ and $T\alpha = \alpha T$, it is immediate that

$$[D, T]\alpha = \alpha[D, T].$$

For $x, y \in \mathfrak{g}$, we obtain

$$\begin{aligned} D(T([x,y])) &= D([T(x), \theta^j\alpha^s(y)] + [\alpha^s(x), T(y)]) \\ &= [DT(x), \theta^i\alpha^k(\theta^j\alpha^s(y))] + [\alpha^k(T(x)), D(\theta^j\alpha^s(y))] \\ &\quad + [D(\alpha^s(x)), \theta^i\alpha^k(T(y))] + [\alpha^k(\alpha^s(x)), DT(y)] \\ &= [DT(x), \theta^{i+j}\alpha^{k+s}(y)] + [\alpha^k T(x), \theta^j\alpha^s D(y)] \\ &\quad + [\alpha^s D(x), \theta^i\alpha^k T(y)] + [\alpha^{k+s}(x), DT(y)]. \end{aligned}$$

Similarly,

$$\begin{aligned} T(D([x,y])) &= [TD(x), \theta^{i+j}\alpha^{k+s}(y)] + [\alpha^s D(x), \theta^i\alpha^k T(y)] \\ &\quad + [\alpha^k T(x), \theta^j\alpha^s D(y)] + [\alpha^{k+s}(x), TD(y)]. \end{aligned}$$

Subtracting the second equality from the first one yields

$$[D, T]([x,y]) = [[D, T](x), \theta^{i+j}\alpha^{k+s}(y)] + [\alpha^{k+s}(x), [D, T](y)].$$

Thus $[D, T] \in \mathrm{Der}^G_{\theta^{i+j},\alpha^{k+s}}(\mathfrak{g})$. □

Proposition 3. *Let* θ *be an involutive automorphism of* $\mathfrak{g}$, *that is,* $\theta^2 = \mathrm{id}_\mathfrak{g}$. *Suppose that* θ *commutes with* α *and each element of* $\mathrm{Der}(\mathfrak{g}) \oplus \mathcal{D}_\theta(\mathfrak{g})$. *Define* $\widehat{\alpha} : \mathrm{Der}(\mathfrak{g}) \oplus \mathcal{D}_\theta(\mathfrak{g}) \longrightarrow \mathrm{Der}(\mathfrak{g}) \oplus \mathcal{D}_\theta(\mathfrak{g})$ *by* $\widehat{\alpha}(D + T) = D\alpha + T\theta^{-1}\alpha\theta$. *Then* $\mathrm{Der}(\mathfrak{g}) \oplus \mathcal{D}_\theta(\mathfrak{g})$ *is a Hom-Lie algebra under the commutator bracket and the twisting map* $\widehat{\alpha}$.

Proof. Because $\theta^2 = \mathrm{id}$, Lemma 1 implies that the commutator of two homogeneous elements of $\mathrm{Der}(\mathfrak{g}) \oplus \mathcal{D}_\theta(\mathfrak{g})$ still belongs to $\mathrm{Der}(\mathfrak{g}) \oplus \mathcal{D}_\theta(\mathfrak{g})$. Indeed, if $i + j$ is even, $[D, T] \in$

$\mathrm{Der}_{\alpha^{k+s}}(\mathfrak{g})$; if $i+j$ is odd, $[D,T] \in \mathrm{Der}^G_{\theta,\alpha^{k+s}}(\mathfrak{g})$. It remains to check that the space is stable under $\widehat{\alpha}$. Let $D \in \mathrm{Der}_{\alpha^k}(\mathfrak{g})$, $T \in \mathrm{Der}^G_{\theta,\alpha^s}(\mathfrak{g})$. Then

$$\begin{aligned}(D\alpha)([x,y]) &= D([\alpha(x),\alpha(y)])\\ &= [D\alpha(x),\alpha^k\alpha(y)] + [\alpha^k\alpha(x), D\alpha(y)]\\ &= [(D\alpha)(x),\alpha^{k+1}(y)] + [\alpha^{k+1}(x),(D\alpha)(y)],\end{aligned}$$

and

$$\begin{aligned}(T\alpha)([x,y]) &= T([\alpha(x),\alpha(y)])\\ &= [T\alpha(x),\theta\alpha^s\alpha(y)] + [\alpha^s\alpha(x), T\alpha(y)]\\ &= [(T\alpha)(x),\theta\alpha^{s+1}(y)] + [\alpha^{s+1}(x),(T\alpha)(y)].\end{aligned}$$

We obtain

$$D\alpha \in \mathrm{Der}_{\alpha^{k+1}}(\mathfrak{g}), \qquad T\alpha \in \mathrm{Der}^G_{\theta,\alpha^{s+1}}(\mathfrak{g}).$$

Since $\theta^{-1}\alpha\theta = \alpha$, we have

$$\widehat{\alpha}(D+T) = D\alpha + T\alpha \in \mathrm{Der}(\mathfrak{g}) \oplus \mathcal{D}_\theta(\mathfrak{g}).$$

Therefore, this direct sum is a Hom-Lie algebra by Proposition 1. □

4. Restrictions to Hom-Ideals and Quasi-Derivations

In this section, we study G-α^k-derivations under some additional conditions. In particular, we discuss their restrictions to Hom-ideals and their relationship with α^k-quasi-derivations.

Proposition 4. *Let $\mathfrak{g}$ be a Hom-Lie algebra, and suppose that $D \in \mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g})$. If $[D,\theta](\mathfrak{g}) \subseteq Z(\mathfrak{g})$, then $\mathfrak{g}^{(1)} \subseteq \ker[D,\theta]$. In particular, if $\mathfrak{g}$ is perfect, then $D\theta = \theta D$.*

Proof. It is enough to evaluate $[D,\theta]$ on the linear generators of $\mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}]$. For $x,y \in \mathfrak{g}$, we have

$$\begin{aligned}(D\theta)([x,y]) &= D([\theta(x),\theta(y)])\\ &= [D\theta(x),\theta^2\alpha^k(y)] + [\theta\alpha^k(x), D\theta(y)],\end{aligned}$$

whereas

$$\begin{aligned}(\theta D)([x,y]) &= \theta\big([D(x),\theta\alpha^k(y)] + [\alpha^k(x), D(y)]\big)\\ &= [\theta D(x),\theta^2\alpha^k(y)] + [\theta\alpha^k(x),\theta D(y)].\end{aligned}$$

Therefore,

$$\begin{aligned}[D,\theta]([x,y]) &= (D\theta - \theta D)([x,y])\\ &= [[D,\theta](x),\theta^2\alpha^k(y)] + [\theta\alpha^k(x),[D,\theta](y)].\end{aligned}$$

By the assumption $[D,\theta](\mathfrak{g}) \subseteq Z(\mathfrak{g})$, both $[D,\theta](x)$ and $[D,\theta](y)$ belong to $Z(\mathfrak{g})$. Thus $[D,\theta]([x,y]) = 0$ for all $x,y \in \mathfrak{g}$, which gives $\mathfrak{g}^{(1)} \subseteq \ker[D,\theta]$. If $\mathfrak{g}$ is perfect, then $\ker[D,\theta] = \mathfrak{g}$, and hence, $[D,\theta] = 0$. □

Proposition 5. *Let $D \in C_{\alpha^k}(\mathfrak{g}) \cap \mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g})$, where $C_{\alpha^k}(\mathfrak{g})$ denotes the α^k-centroid introduced in Definition 8. Then $[D(x),\theta\alpha^k(y)] = 0$, $\forall\, x,y \in \mathfrak{g}$. In addition, if α is a surjective homomorphism and $\mathfrak{g}$ is centerless, then $C_{\alpha^k}(\mathfrak{g}) \cap \mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g}) = 0$.*

Proof. Since $D \in C_{\alpha^k}(\mathfrak{g}) \cap \mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g})$, we have

$$D([x,y]) = [D(x), \alpha^k(y)] = [\alpha^k(x), D(y)],$$

$$D([x,y]) = [D(x), \theta\alpha^k(y)] + [\alpha^k(x), D(y)].$$

Substituting $[\alpha^k(x), D(y)] = D([x,y])$ into the above equation, we obtain

$$D([x,y]) = [D(x), \theta\alpha^k(y)] + D([x,y]).$$

Hence,

$$[D(x), \theta\alpha^k(y)] = 0.$$

If α is surjective, then $\theta\alpha^k$ is also surjective. Thus, for any $z \in \mathfrak{g}$, there exists $u \in \mathfrak{g}$ such that

$$z = \theta\alpha^k(u).$$

Therefore,

$$[D(x), z] = [D(x), \theta\alpha^k(u)] = 0, \qquad \forall x, z \in \mathfrak{g},$$

that is,

$$\mathrm{Im}D \subseteq Z(\mathfrak{g}).$$

If $Z(\mathfrak{g}) = 0$, then $D = 0$. □

Let $\mathfrak{g}$ be a Hom-Lie algebra and let I be a Hom-ideal of $\mathfrak{g}$. Consider an automorphism θ of $\mathfrak{g}$ such that I is invariant under θ, i.e., $\theta(I) \subseteq I$. We denote by $\mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g})$ the set of all θ-α^k-derivations of $\mathfrak{g}$ that leave I invariant, namely,

$$\mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g}) = \{D \in \mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g}) \mid D(I) \subseteq I\}.$$

Proposition 6. *With the above notation, we have* $\mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g}) \subseteq \mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g})$. *Furthermore, when* I *is a perfect Hom-ideal of* $\mathfrak{g}$, $\mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g}) = \mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g})$.

Proof. Clearly, $\mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g}) \subseteq \mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g})$. It remains to verify that $\mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g})$ is closed under linear combinations. Let $a, b \in \mathbb{K}$ and $D, T \in \mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g})$. Then, for any $u \in I$, we have $(aD + bT)(u) = aD(u) + bT(u) \in I$, since $D(u), T(u) \in I$. Hence, $aD + bT$ also leaves I invariant and, therefore, $aD + bT \in \mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g})$. Consequently, $\mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g})$ is a subspace of $\mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g})$.

To prove the second assertion, it suffices to show that $\mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g}) \subseteq \mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g})$. Let $D \in \mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g})$ and $u \in I$. Since I is perfect, for any $u \in I$, there exist $u_{1i}, u_{2i} \in I$, $1 \leq i \leq n$, satisfying $u = \sum_{i=1}^n [u_{1i}, u_{2i}]$. Moreover, as θ leaves I invariant, we have

$$\begin{aligned} D(u) &= \sum_{i=1}^n D([u_{1i}, u_{2i}]) \\ &= \sum_{i=1}^n ([D(u_{1i}), \theta\alpha^k(u_{2i})] + [\alpha^k(u_{1i}), D(u_{2i})]). \end{aligned}$$

Since I is a Hom-ideal, it follows that $D(u) \in I$. Hence, $D(I) \subseteq I$, which shows that $\mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g}) \subseteq \mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g})$. Consequently, $\mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g}) = \mathrm{Der}^G_{\theta,\alpha^k}(\mathfrak{g})$. □

As I is invariant under θ, the restriction $\theta|_I$ is an automorphism of I. Furthermore, any $D \in \mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g})$ leaves I invariant and hence restricts to a $\theta|_I$-α^k-derivation of I. Therefore, we obtain a natural linear map:

$$\delta : \mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g}) \longrightarrow \mathrm{Der}^G_{\theta|_I,\alpha^k}(I) \text{ given by } D \mapsto D|_I.$$

Assume that the inner α^{k+1}-derivation $D_k(y) \in \mathrm{Aut}(\mathfrak{g})$ for some $y \in \mathfrak{g}$. Naturally, for any $u \in \mathfrak{g}$, since $D_k(y) \in \mathrm{Aut}(\mathfrak{g})$, there exists a unique $v \in \mathfrak{g}$ such that

$$u = D_k(y)(v) = [\alpha^k(v), y].$$

For notational convenience, denoting $\alpha^k(v)$ by x, we have $u = [x, y]$. Consequently, for $D \in \mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g})$, we can define a map $\widehat{D} : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\widehat{D}(u) = 2D(u) + [D(y), \theta\alpha^k(x)] + [\theta\alpha^k(y), D(x)].$$

That is

$$\widehat{D}([x, y]) = 2D([x, y]) + [D(y), \theta\alpha^k(x)] + [\theta\alpha^k(y), D(x)].$$

Here, $x = \alpha^k(v)$, where v is determined by u, and $D_k(y)$ is an automorphism on $\mathfrak{g}$. Thus, $\widehat{D}$ is well-defined.

Next we present some properties of the above map.

Proposition 7. *Assume that $D_k(x) \in \mathrm{Aut}(I)$ for some $x \in \mathfrak{g}$, where $D_k(x)$ is an inner α^{k+1}-derivation. Then $\widehat{D}$ restricts to a homogeneous linear map $\widehat{D}|_I : I \longrightarrow I$.*

Proof. First, observe that $I = D_k(x)(I) = [\alpha^k(I), x] \subseteq [\mathfrak{g}, \mathfrak{g}]$. Hence, for any $u \in I$, there exists $v \in I$ such that $u = [\alpha^k(v), x]$, and $\widehat{D}(u) = \widehat{D}([\alpha^k(v), x]) = 2D([\alpha^k(v), x]) + [D(x), \theta\alpha^k(\alpha^k(v))] + [\theta\alpha^k(x), D(\alpha^k(v))] \in D(I) + [\mathfrak{g}, I] \subseteq I$. Therefore, $\widehat{D}(I) \subseteq I$. It remains to verify that the map $\widehat{D}|_I$ is linear. Take another element $u' \in I$ with $u' = [\alpha^k(v'), x]$ for some $v' \in I$. Then, for $a, b \in \mathbb{K}$,

$$\begin{aligned}
\widehat{D}(au + bu') &= \widehat{D}([a\alpha^k(v), x] + [b\alpha^k(v'), x]) \\
&= \widehat{D}([a\alpha^k(v) + b\alpha^k(v'), x]) \\
&= 2D([a\alpha^k(v) + b\alpha^k(v'), x]) + [D(x), \theta\alpha^k(a\alpha^k(v) + b\alpha^k(v'))] \\
&\quad + [\theta\alpha^k(x), D(a\alpha^k(v) + b\alpha^k(v'))] \\
&= 2D([a\alpha^k(v), x]) + [D(x), \theta\alpha^k(a\alpha^k(v))] + [\theta\alpha^k(x), D(a\alpha^k(v))] \\
&\quad + 2D([b\alpha^k(v'), x]) + [D(x), \theta\alpha^k(b\alpha^k(v'))] + [\theta\alpha^k(x), D(b\alpha^k(v'))] \\
&= a\{2D([\alpha^k(v), x]) + [D(x), \theta\alpha^k(\alpha^k(v))] + [\theta\alpha^k(x), D(\alpha^k(v))]\} \\
&\quad + b\{2D([\alpha^k(v'), x]) + [D(x), \theta\alpha^k(\alpha^k(v'))] + [\theta\alpha^k(x), D(\alpha^k(v'))]\} \\
&= a\widehat{D}(u) + b\widehat{D}(u').
\end{aligned}$$

Thus, $\widehat{D}$ is a linear map. □

According to the notations, $\mathrm{QDer}_{\theta,\alpha^k}(\mathfrak{g}) = \mathrm{QDer}^G_{\theta,\mathrm{id},\alpha^k}(\mathfrak{g})$ in Remark 1. Then, the following result holds.

Proposition 8. *Let $x_1 \in \mathfrak{g}$ be such that the inner α^{k+1}-derivation $D_k(x_1)$ of $\mathfrak{g}$ belongs to $\mathrm{Aut}(I)$, $\theta(x_1) = x_1$, and $D(x_1) \in Z(\mathfrak{g})$ for all $D \in \mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g})$. Then $\delta\big(\mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g})\big) \subseteq \mathrm{QDer}_{\theta,\alpha^k}(I)$.*

Proof. Take arbitrary elements $y_1, y_2 \in I$ and $D \in \mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g})$. By the definition of an inner α^{k+1}-derivation of $\mathfrak{g}$, the element x_1 satisfies $\alpha(x_1) = x_1$. Therefore, $\alpha^k(x_1) = x_1$. Since $D(x_1) \in Z(\mathfrak{g})$ and $\theta(x_1) = x_1$, we have

$$\begin{aligned}
\widehat{D}|_I D_k(x_1)([y_1,y_2]) &= \widehat{D}([\alpha^k([y_1,y_2]),x_1]) \\
&= 2D([\alpha^k([y_1,y_2]),x_1]) + [D(x_1),\theta\alpha^{2k}([y_1,y_2])] \\
&\quad + [\theta\alpha^k(x_1), D(\alpha^k([y_1,y_2]))] \\
&= 2D([\alpha^k([y_1,y_2]),x_1]) + [x_1, D(\alpha^k([y_1,y_2]))] \\
&= 2([D(\alpha^k([y_1,y_2])),\theta\alpha^k(x_1)] + [\alpha^k(\alpha^k([y_1,y_2])), D(\alpha^k(x_1))]) \\
&\quad + [x_1, D(\alpha^k([y_1,y_2]))] \\
&= 2[D(\alpha^k([y_1,y_2])),x_1] + [x_1, D(\alpha^k([y_1,y_2]))] \\
&= [D(\alpha^k([y_1,y_2])),x_1] \\
&= [\alpha^k(D([y_1,y_2])),x_1] \\
&= D_k(x_1)(D([y_1,y_2])) \\
&= D_k(x_1)([D(y_1),\theta\alpha^k(y_2)],[\alpha^k(y_1),D(y_2)]).
\end{aligned}$$

As $D_k(x_1) \in \mathrm{Aut}(I)$, there exists $D_k(x_1)^{-1}$, we have $[D(y_1),\theta\alpha^k(y_2)] + [\alpha^k(y_1),D(y_2)] = (D_k(x_1)^{-1})\widehat{D}|_I D_k(x_1)([y_1,y_2])$. By Definition 10 and Remark 1, $D \in \mathrm{QDer}_{\theta,\alpha^k}(I)$. It is clear that the restriction of D to I is an α^k-quasi-derivation of I. Hence, $\delta\left(\mathrm{Der}^G_{\theta,I,\alpha^k}(\mathfrak{g})\right) \subseteq \mathrm{QDer}_{\theta,\alpha^k}(I)$. □

5. Conclusions

This paper introduces G-α^k-derivations of multiplicative Hom-Lie algebras, extending the concept of α^k-derivations through the incorporation of automorphisms. We presented explicit examples exhibiting a geometric progression pattern and established several structural properties of the corresponding derivation spaces. In particular, we proved dimension-preserving isomorphisms between related derivation spaces and constructed Hom-Lie algebra structures on these spaces. We also investigated the restriction of G-α^k-derivations to Hom-ideals and clarified their relationship with α^k-quasi-derivations. These results enrich the theory of derivations in Hom-Lie algebras and provide a foundation for further studies on generalized symmetries, representations, and Hom-type algebraic structures.

Based on these results, one can also define G-α^k-derivations on other algebraic structures, such as (Hom-)left-symmetric algebras and 3-Lie algebras. Further studies may explore their structural properties, classify low-dimensional cases, and thereby deepen and enrich the theory of derivations across diverse algebraic systems.

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