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Lie Symmetries and Invariants of General Time-Dependent Quadratic Hamiltonian System

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Abstract

Eight Lie algebras of point-symmetric groups and corresponding generators are admitted by the equation of motion, which is obtained from a general time-dependent quadratic Hamiltonian. We show that invariant quantities obtained by eight algebraic generators are the Wronskian constant, three conserved quantities, which are time-dependent quadratic forms in position and momentum, and trivial, 0. All obtained invariant quantities are represented by auxiliary conditions, which are two linearly independent solutions of a homogeneous differential equation of the equations of motion. Invariant variables associated with an invariant consisting of the linearity of x and p are defined. It shows that, if the motion of the system is oscillatory, the Poisson bracket of the two invariant variables is obtained as i, and in the case of monotonic motion, it is obtained as 1.

Keywords: general time-dependent quadratic Hamiltonian; point-symmetric Lie point group; infinitesimal generators; invariant quantity; invariant variable space

1. Introduction

The Hamiltonian formalism not only enables us to obtain the system's equations of motion, but also allows us to have knowledge of various mathematical structures related to the system [1–3]. Lie algebras facilitate the solution of differential equations when the symmetry of the system can be identified [4–6].

In general, one Hamiltonian gives one equation of motion, but one equation of motion is obtained from innumerable Hamiltonians. This is because there are infinitely many canonical pairs in which the position is fixed in a pair of canonical variables, and only the momentum is changed. Since one Hamiltonian in one dimension gives one pair of canonical variables, there are infinitely many Hamiltonians that give one equation of motion. In this paper, we select and deal with classical Hamiltonian systems in which the coefficients are time-dependent and a general quadratic form of position and momentum. This Hamiltonian system includes many systems that are basically handled, such as free particles, harmonic oscillators, damped harmonic oscillators, and forced harmonic oscillators [1].

We find the generators that form the Lie algebra admitted by the equation of motion, and find out whether the generators have an invariant or not. If the generator has the invariant, we will find out what the invariant is.

In Section 2, we choose a general time-dependent Hamiltonian system and find solutions of the equations of motion. Since the coefficients of the Hamiltonian are not fixed, in our expansion, the general solution is determined by assuming a possible solution.

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Section 3 finds the invariant of the system. As we know, if we solve a linear differential equation, we can find the Wronskian, which is obtained by multiplying the coefficients of p^2 of the given Hamiltonian by the Wronskian constant. We also find that two invariant quantities given by a linear combination of position and momentum depend on the solution of the system as an auxiliary condition. Since the product of an invariant is also an invariant, we use these to find three invariant quantities given by the quadratic form of position and momentum.

In Section 4, we find eight generators of the point-symmetric Lie groups admitted by the equation of motion, which is a homogeneous differential equation. In these eight generators, we check whether the variation integral of the system is invariant or not. If it is invariant, we find its conserved quantities. We also examine how the conserved quantity is related to the invariant discussed in Section 3.

Section 5 defines two linear invariant variables by the conserved quantities obtained in Section 3. We examine the transformation between the set of two linear invariant variables and the set of position and momentum variables. Also, we find the Poisson brackets of the two invariant variables. It is also shown that, when two variables form a canonical transformation, the transformation becomes a Hamilton–Jacobi transformation. Poisson brackets between two linear invariant variables are calculated. It also calculates Poisson brackets for quadratic variables consisting of two linear invariant variables.

Section 6 shows the summary and conclusions, as well as a brief remark on the possibility of the extension of the techniques to quantum mechanics.

2. Time-Dependent General Quadratic Hamiltonian Systems

First, we briefly introduce the equations of motion of the time-dependent general quadratic Hamiltonian system and its solutions. The Hamiltonian of the system can be set as

$$H(x,p;t) = \frac{1}{2}\left(\alpha(t)p^2 + 2\beta(t)xp + \zeta(t)x^2\right) + \mu(t)p + \nu(t)x + \sigma(t), \tag{1}$$

where the coefficients $\alpha(t), \beta(t), \zeta(t), \mu(t), \nu(t)$ and $\sigma(t)$ are some real functions of t and are differentiable. From now on, in all equations, the expression of the time function will be omitted to simplify the expression, except when it appears for the first time, in which case the time-dependence will be emphasized, and in special cases where omission may cause confusion. If the Hamilton equation is solved using the Hamiltonian of Equation (1), the equation of motion for the system is obtained as

$$\ddot{x} - \frac{\dot{\alpha}}{\alpha}\dot{x} + \left(\frac{\dot{\alpha}\beta}{\alpha} - \beta^2 + \alpha\delta - \dot{\beta}\right)x = \beta\mu - \frac{\mu}{\alpha}\dot{\alpha} + \dot{\mu} - \alpha\nu, \tag{2}$$

where the dot represents time derivatives. From Equation (1), it can be seen that $\sigma(t)$ does not affect the equation of motion (2). Let the equation of motion (2) be simply written as

$$\ddot{x} + \gamma(t)\dot{x} + \omega^2(t)x = f(t), \tag{3}$$

$$\omega^2(t) = \frac{\dot{\alpha}\beta}{\alpha} - \beta^2 + \alpha\delta - \dot{\beta}, \tag{4}$$

$$f(t) = \beta\mu - \frac{\mu}{\alpha}\dot{\alpha} + \dot{\mu} - \alpha\nu. \tag{5}$$

As can be seen from Equations (1) and (2), there can be innumerable Hamiltonians giving one equation of motion. Among them, we will take and treat

$$H(x,p;t) = \frac{1}{2}\left(\alpha(t)p^2 + \alpha^{-1}(t)\omega^2(t)x^2\right) - \alpha^{-1}(t)f(t)x, \tag{6}$$

as a special selection of Hamiltonian of Equation (1), giving the equation of motion (3). In Equation (6), $\alpha(t)$ is

$$\alpha(t) = e^{-\int^t \gamma(s)ds}, \tag{7}$$

that is,

$$\gamma(t) = -\frac{\dot{\alpha}(t)}{\alpha(t)}. \tag{8}$$

Solving the Hamilton equation using Equation (6) gives

$$\dot{x} = \alpha(t)p, \tag{9}$$

and

$$\dot{p} = -\alpha^{-1}(t)\omega^2(t)x + \alpha^{-1}(t)f(t). \tag{10}$$

Using Equations (6), (9) and (10), the Lagrangian of the system is obtained as

$$L(\dot{x}, x; t) = \frac{\dot{x}^2}{2\alpha(t)} - \frac{\omega^2(t)}{2\alpha(t)}x^2 + \frac{f(t)}{\alpha(t)}x. \tag{11}$$

If $u_1(t)$ and $u_2(t)$ are the two independent solutions of the homogeneous form (in the case of $f(t) = 0$) of the equation of motion Equation (3), the general solution of the differential Equation (3) is obtained as

$$x(t) = c_1u_1(t) + c_2u_2(t) + x_p(t), \tag{12}$$

where

$$x_p(t) = \int_{t_0}^{t} G(t, t')f(t')dt', \tag{13}$$

c_1 and c_2 are arbitrary constants, and t_0 is the time when $x_p(t_0) = 0$ at $t = t_0$. In Equation (13), the Green's function, $G(t, t')$, is

$$G(t, t') = -\theta(t - t')\frac{u_1(t)u_2(t') - u_2(t)u_1(t')}{W(t')}, \tag{14}$$

where $\theta(t)$ is a Heaviside step function, and the Wronskian, $W(t)$ is

$$W(t) = u_1(t)\dot{u}_2(t) - u_2(t)\dot{u}_1(t) = W_0\alpha(t). \tag{15}$$

Here, we will call the constant W_0 the Wronskian constant. The velocity $\dot{x}(t)$ is obtained as

$$\dot{x}(t) = c_1\dot{u}_1(t) + c_2\dot{u}_2(t) + \dot{x}_p(t), \tag{16}$$

from Equations (12) and (13), where

$$\dot{x}_p(t) = \int_{t_0}^{t} G_t(t, t')f(t')dt', \tag{17}$$

here, $G_t(t, t')$ is

$$G_t(t, t') = -\theta(t - t')\frac{\dot{u}_1(t)u_2(t') - \dot{u}_2(t)u_1(t')}{W(t')}. \tag{18}$$

3. Invariant Quantities of the System

So far, in the general solution of the motion of the system, Equations (12) and (16), we know that there are time constants c_1 and c_2, and the Wronskian constant W_0. Considering Equations (9)–(15), we get

$$\alpha^{-1}(t)W(t) = u_1(t)p_2(t) - u_2(t)p_1(t) = W_0. \tag{19}$$

If W_0 is an imaginary number, two independent solutions of the differential equation can be taken as $u_1(t)$ and $u_2(t) = u_1^*(t)$. If W_0 is a real number, we can take two independent solutions, $u_1(t)$ and $u_2(t)$, as real functions of t. In a conservative system, the mechanical energy E equals the Hamiltonian H and is a constant of time, but in a time-dependent system, it is not a constant of time. Let us find the new time constant of the time-dependent Hamiltonian system here. In general, if the quantity $I(x, p; t)$ satisfies

$$D[I(x, p; t)] = 0, \tag{20}$$

it is a constant of time and is called an invariant quantity, where D is the total derivative operator,

$$D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \cdots. \tag{21}$$

If $I(x, p; t)$ is given by

$$I(x, p; t) = a(t)p + b(t)x + g(t), \tag{22}$$

which is a linear relation of x and p, substituting Equations (21) and (22) into Equation (20) and calculating, we can know that $a(t), b(t)$, and $g(t)$ in Equation (22) are satisfied

$$\dot{a} = -\alpha(t)b, \tag{23}$$

$$\dot{b} = \alpha^{-1}(t)\zeta(t)a, \tag{24}$$

and

$$\dot{g} = -\alpha^{-1}f(t)a. \tag{25}$$

The homogeneous form (when $f(t) = 0$) of the equation of motion (3) is

$$\ddot{x} + \gamma(t)\dot{x} + \omega^2(t)x = 0. \tag{26}$$

In Equation (12), we assume that $u_1(t)$ and $u_2(t)$ are two linearly independent solutions of Equation (26). Then, solutions of Equations (23)–(25) are respectively obtained as

$$a(t) = c_1u_1 + c_2u_2, \tag{27}$$

$$b(t) = -\frac{1}{\alpha(t)}(c_1\dot{u}_1 + c_2\dot{u}_2), \tag{28}$$

and

$$g(t) = -\int_{t_0}^{t} \alpha^{-1}(t')f(t')\big(c_1u_1(t') + c_2u_2(t')\big)dt'. \tag{29}$$

Substituting Equations (27)–(29) into Equation (22), we get two independent invariant quantities

$$I_1(x, p; t) = u_1p - \frac{\dot{u}_1}{\alpha}x + \int_{t_0}^{t} \frac{u_1(t')}{\alpha(t')}f(t')dt', \tag{30}$$

and

$$I_2(x, p; t) = u_2p - \frac{\dot{u}_2}{\alpha}x + \int_{t_0}^{t} \frac{u_2(t')}{\alpha(t')}f(t')dt'. \tag{31}$$

Since the product of two invariant quantities is also invariant, the three quadratic invariant quantities can be obtained as follows.

$$I_1^2(x,p;t) = u_1^2p^2 - \frac{2u_1\dot{u}_1}{\alpha}xp + \frac{\dot{u}_1^2}{\alpha^2}x^2 - 2x\int_{t_0}^{t}\frac{u_1(t')\dot{u}_1(t)}{\alpha(t)\alpha(t')}f(t')dt' + 2p\int_{t_0}^{t}\frac{u_1(t)u_1(t')}{\alpha(t')}f(t')dt' + \left(\int_{t_0}^{t}\frac{u_1(t')}{\alpha(t')}f(t')dt'\right)^2, \tag{32}$$

$$I_2^2(x,p;t) = u_2^2p^2 - \frac{2u_2\dot{u}_2}{\alpha}xp + \frac{\dot{u}_2^2}{\alpha^2}x^2 - 2x\int_{t_0}^{t}\frac{u_2(t')\dot{u}_2(t)}{\alpha(t)\alpha(t')}f(t')dt' + 2p\int_{t_0}^{t}\frac{u_2(t)u_2(t')}{\alpha(t')}f(t')dt' + \left(\int_{t_0}^{t}\frac{u_2(t')}{\alpha(t')}f(t')dt'\right)^2, \tag{33}$$

and

$$I_1(x,p;t)I_2(x,p;t) = u_1u_2p^2 - \frac{u_1\dot{u}_2 + \dot{u}_1u_2}{\alpha}xp + \frac{\dot{u}_1\dot{u}_2}{\alpha^2}x^2 - x\int_{t_0}^{t}\frac{u_1(t')\dot{u}_2(t) + u_2(t')\dot{u}_1(t)}{\alpha(t)\alpha(t')}f(t')dt' + p\int_{t_0}^{t}\frac{u_1(t)u_2(t') + u_2(t)u_1(t')}{\alpha(t')}f(t')dt' + \left(\int_{t_0}^{t}\frac{u_2(t')}{\alpha(t')}f(t')dt'\right)\left(\int_{t_0}^{t}\frac{u_1(t')}{\alpha(t')}f(t')dt'\right), \tag{34}$$

Let ω_0^2 be

$$\omega_0^2(t) = \omega^2(t) - \frac{1}{2}\gamma'(t) - \frac{1}{4}\gamma^2(t). \tag{35}$$

If $\omega_0^2 > 0$, the solutions u_1 and u_2 of Equation (26) are oscillatory motions, and if $\omega_0^2 < 0$, monotonically increasing or monotonically decreasing motions. In the case of monotonic motion, if two independent solutions, u_1 and u_2, are respectively set to

$$u_1(t) = r(t)e^{\theta(t)}, \tag{36}$$

and

$$u_2(t) = r(t)e^{-\theta(t)}, \tag{37}$$

the general solution of Equation (12) becomes

$$x(t) = c_1r(t)e^{\theta(t)}(t) + c_2r(t)e^{-\theta(t)}(t) + x_p(t), \tag{38}$$

where $r(t)$ and $\theta(t)$ are real function of t. Equation (38) is not a specific solution, but a general form, which is a function obtained by solving Equation (12) when it is fixed. Applying Wronskian, Equation (15) becomes

$$W(t) = 2r^2\dot{\theta} = \Omega_0\alpha(t), \tag{39}$$

where Ω_0 is a real constant. In Equation (39), it can be seen that r and θ are dependent on each other. In the case of oscillatory motion, if two independent solutions, u_1 and u_2, are respectively set to

$$u_1(t) = r(t)e^{i\theta(t)}(t), \tag{40}$$

and

$$u_2(t) = r(t)e^{-i\theta(t)}(t), \tag{41}$$

the general solution of Equation (12) is

$$x(t) = c_1r(t)e^{i\theta(t)}(t) + c_2r(t)e^{-i\theta(t)}(t) + x_p(t), \tag{42}$$

Substituting Equations (40) and (41) into Equation (15), Wronskian (15) becomes

$$W(t) = 2ir^2\dot{\theta} = i\Omega_0\alpha(t), \tag{43}$$

where Ω_0 is a real constant. In Equation (43), r and θ are also depend on each other. In the case of monotonic motion, from Equations (36) and (37), the Green function, Equations (14) and (18) are expressed as

$$G(t,t') = -\theta(t-t')\frac{2rr'\sinh(\theta-\theta')}{\Omega_0\alpha'}, \tag{44}$$

and

$$G_t(t,t') = -\theta(t-t')\frac{2\dot{r}r'\sinh(\theta-\theta') + 2rr'\dot{\theta}\cosh(\theta-\theta')}{\Omega_0\alpha'}, \tag{45}$$

by r and θ. In the case of oscillatory motion, from Equations (40) and (41), the Green function Equations (14) and (18) are expressed as

$$G(t,t') = -\theta(t-t')\frac{2rr'\sin(\theta-\theta')}{\Omega_0\alpha'}, \tag{46}$$

and

$$G_t(t,t') = -\theta(t-t')\frac{2\dot{r}r'\sin(\theta-\theta') - 2rr'\dot{\theta}\cos(\theta-\theta')}{\Omega_0\alpha'}, \tag{47}$$

by r and θ. In expressions (44)–(47), all variables expressed as $'$ are a function of t' of that variable. In the case of oscillatory motion with the above considerations, expressing r and θ, Equation (34) becomes

$$\begin{aligned} I_1 I_2 \;=\; & r^2p^2 - \frac{r\dot{r}}{\alpha}xp + \left(\frac{\dot{r}^2}{\alpha^2} + \frac{\Omega_0^2}{4r^2}\right)x^2 \\ & - x\int_{t_0}^{t}\frac{r'\dot{r}\cos(\theta'-\theta) - r'r\dot{\theta}\sin(\theta'-\theta)}{\alpha\alpha'}f(t')dt' \\ & + p\int_{t_0}^{t}\frac{rr'\cos(\theta'-\theta)}{\alpha'}f(t')dt' + \int_{t_0}^{t}\int_{t_0}^{t}\frac{r'r''e^{i(\theta'+\theta'')}}{\alpha'\alpha''}f(t')f(t'')dt'dt'', \end{aligned} \tag{48}$$

In the case of monotonic motion, the result of Equation (48) is obtained by replacing $+\Omega_0^2$ with $-\Omega_0^2$ and trigonometric functions with corresponding hyperbolic functions. For $f(t) = 0$, Equation (46) is the Lewis–Riesenfeld invariant that has been dealt with in the literature [7–12]. So far, we have seen that the invariant quantities $c_1, c_2, \Omega_0, I_1^2, I_2^2$, and $I_1 I_2$ are related to the equations of motion and their solutions. The next section will discuss the relationship between invariant quantities obtained from the equation of motion of the system and the Lie algebra that makes the system invariant. Since it is so complicated to find the Lie algebraic generators of a non-homogeneous equation, let us find the Lie generators of a homogeneous differential equation to get an invariant and discuss the non-homogeneous case.

4. Generators and Their Invariant Quantities

If $f(t) = 0$, the equation of motion Equation (3) becomes a homogeneous form of Equation (26), and its general solution can be put

$$x(t) = c_1u_1(t) + c_2u_2(t), \tag{49}$$

in Equation (10). From now on, we want to obtain generators of continuous transformation groups admitted by the equation of motion (Equation (26) of the system) and to obtain

their invariant solutions using the group's point symmetries. First, let us find infinitesimal symmetries, i.e., generators.

$$X = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x}, \tag{50}$$

of symmetry groups G. The infinitesimal invariant discrimination has the form:

$$X_{(2)}F \mid_{F=0} = (\xi F_t + \eta F_x + \zeta_1 F_{\dot{x}} + \zeta_2 F_{\ddot{x}}) \mid_{F=0} = 0, \tag{51}$$

where $F = 0$ is the differential Equation (26), F is a function of the left side of Equation (26), subscripts indicate derivatives of its subscripts, and

$$X_{(2)} = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \zeta_1\frac{\partial}{\partial \dot{x}} + \zeta_2\frac{\partial}{\partial \ddot{x}}, \tag{52}$$

where ζ_1 and ζ_2 are

$$\zeta_1 = \eta_t + (\eta_x - \xi_t)\dot{x} - \dot{x}^2\xi_x, \tag{53}$$

and

$$\zeta_2 = \eta_{tt} + (\eta_{tx} - \xi_{tt})\dot{x} + (\eta_{xx} - 2\xi_{tx})\dot{x}^2 - \dot{x}^3\xi_{xx} + (\eta_x - 2\xi_t - 3\dot{x}\xi_x)\ddot{x}. \tag{54}$$

Equation (51) is called the determining equation for the group G admitted by the ordinary differential Equation (26) [4–6,13]. Using the determining Equation (51), we can find generators of symmetry groups of Equation (26). It is well known that the symmetric Lie algebra $\mathcal{L}$ of linear second-order differential Equation (26) has the dimension 8 [4–6]. Using Equations (51)–(54), we can know that the vector space of infinitesimal generators is eight-dimensional and a basis is obtained as [4–6,13].

$$X_1 = x\frac{\partial}{\partial x}, \tag{55}$$

$$X_2 = u_1\frac{\partial}{\partial x}, \tag{56}$$

$$X_3 = u_2\frac{\partial}{\partial x}, \tag{57}$$

$$X_4 = \alpha^{-1}\left(u_1 x\frac{\partial}{\partial t} + \dot{u}_1 x^2\frac{\partial}{\partial x}\right), \tag{58}$$

$$X_5 = \alpha^{-1}\left(u_2 x\frac{\partial}{\partial t} + \dot{u}_2 x^2\frac{\partial}{\partial x}\right), \tag{59}$$

$$X_6 = \alpha^{-1}\left(u_1^2\frac{\partial}{\partial t} + u_1\dot{u}_1 x\frac{\partial}{\partial x}\right), \tag{60}$$

$$X_7 = \alpha^{-1}\left(2u_1u_2\frac{\partial}{\partial t} + (\dot{u}_1u_2 + u_1\dot{u}_2)x\frac{\partial}{\partial x}\right), \tag{61}$$

and

$$X_8 = \alpha^{-1}\left(u_2^2\frac{\partial}{\partial t} + u_2\dot{u}_2 x\frac{\partial}{\partial x}\right). \tag{62}$$

Every generator except X_1 depends upon independent solutions, u_1 and u_2 of the differential Equation (26). In the case of an independent variable t and a differential variable x, the Lagrangian variational integral $L(x,\dot{x};t)$ is defined as

$$J = \int_V L(x,\dot{x};t)dt. \tag{63}$$

Now we want to check that the variational integral (63) has an invariant under each group G corresponding to eight infinitesimal generators (55)–(62). This is a point symmetry of the

Euler–Lagrange equation [4–6]. Using Noether's theorem to get the symmetry, we show that the variational integral is invariant under the group G if and only if

$$X(L) + LD(\xi) = 0, \tag{64}$$

where X is Equation (52), i.e.,

$$X(L) = \xi \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial x} + \zeta_1 \frac{\partial L}{\partial \dot{x}}. \tag{65}$$

If the variational integral (63) is invariant in the group, with generator (65), then T is defined by

$$T = L\xi + (\eta - \xi \dot{x}) \frac{\partial L}{\partial \dot{x}}, \tag{66}$$

is a conserved quantity for the Euler–Lagrange equation,

$$\frac{\partial L}{\partial x} - D \frac{\partial L}{\partial \dot{x}} = 0. \tag{67}$$

That is,

$$D(T) = 0, \tag{68}$$

with Equation (68) [4,5].

Adding $D[B(x.\dot{x};t)]$ to $L(x.\dot{x};t)$ gives the same result. Adding $B(x.\dot{x};t)$ to the Lagrangian of the integral of variation (63), it gives the same result of the variation principle, so Equation (64) can be replaced with

$$X(L) + LD(\xi) = D(B). \tag{69}$$

Then the Lagrange equation is again invariant and has a conservation law $D[I(x, \dot{x}; t)] = 0$, where Equation (66) is replaced by

$$I(x, \dot{x}; t) = \xi L + (\eta - \xi \dot{x}) \frac{\partial L}{\partial \dot{x}} - B, \tag{70}$$

The group G corresponding to Lie's algebra, Equations (55)–(62), makes the differential Equation (26) invariant, but we do not know if it makes the variational integral invariant. We check whether the generator Equations (55)–(62) are symmetric under the variational integral, and if so, find out what the corresponding invariant is. First, to simply reduce the calculation of various formulas, the two Formulas (69) and (70) are calculated by substituting the Lagrangian Equation (11) as follows,

$$\begin{aligned} X(L) + LD(\xi) &= \gamma \left(\frac{\dot{x}^2}{2\alpha} - \frac{\omega^2(t)}{2\alpha} x^2 \right) \xi - \xi \frac{\omega \dot{\omega}}{\alpha} x^2 - \eta \frac{\omega^2}{\alpha} x \\ &\quad + \zeta_1 \frac{\dot{x}}{\alpha} + \left(\frac{\dot{x}^2}{2\alpha} - \frac{\omega^2(t)}{2\alpha} x^2 \right) (\xi_t + \xi_x \dot{x}) = D[B(x, \dot{x}, t)], \end{aligned} \tag{71}$$

and

$$I(x, \dot{x}; t) = -\xi \left(\frac{\dot{x}^2}{2\alpha} - \frac{\omega^2(t)}{2\alpha} x^2 \right) + \eta \frac{\dot{x}}{\alpha} - B(x, \dot{x}; t). \tag{72}$$

In generator X_1, Equation (55), $\xi = 0$ and $\eta = x$; hence, Equation (53) gives the following.

$$\zeta_1 = \dot{x}. \tag{73}$$

Calculation of Equation (71) is

$$X(L) + LD(\xi) = \dot{x}^2 - \omega^2(t)x^2 = D(x\dot{x}), \tag{74}$$

And Equation (72) becomes

$$I(x, \dot{x}; t) = \eta\dot{x} - B = x\dot{x} - x\dot{x} = 0. \tag{75}$$

Thus, for generator X_1, we only get a trivial conservation quantity 0. This group G_1 tells us that, if x is the solution of Equation (26), x multiplied by an arbitrary constant is also a solution of it. That is, it is a group G_1 that admits in any linear differential equations. However, group G_1 does not admit in the non-homogeneous differential equation of Equation (3). In the generators, equation X_2 (54), $\xi = 0$ and $\eta = u_1$; hence, Equation (53) becomes

$$\zeta_1 = \dot{u}_1. \tag{76}$$

Calculation of Equation (71) is

$$X(L) + LD(\xi) = -u_1\omega^2 x + \dot{u}_1\dot{x} = D(\dot{u}_1 x), \tag{77}$$

and the equation, Equation (72), becomes

$$I(x, \dot{x}; t) = \eta\dot{x} - B = u_1\dot{x} - \dot{u}_1 x. \tag{78}$$

By replacing u_1 with u_2 in results, Equations (76)–(78) in the case of generator X_2, we get corresponding results in the case of generator X_3, i.e.,

$$\zeta_1 = \dot{u}_1, \tag{79}$$

$$X(L) + LD(\xi) = u_2\omega^2 x + \dot{u}_2\dot{x} = D(\dot{u}_2 x), \tag{80}$$

and

$$I(x, \dot{x}; t) = \eta\dot{x} - B = u_2\dot{x} - \dot{u}_2 x. \tag{81}$$

In Equations (78) and (81), both conserved quantities are the Wronskian constant W_0 in Equation (15). Thus, it is known that X_2 and X_3 are generators that give the Wronskian constant W_0 of the solution of the equation of motion as an invariant. The group $G_2(G_3)$ associated with these X_2 (X_3) is the group admitted when x is a solution of a differential equation, and $x + c_1u_1(x)$ ($x + c_2u_2(x)$) is also a solution of the same differential equation. That is, X_2 and X_3 are generators of the group admitted by linear differential equations. Their group is also admitted by non-homogeneous linear differential Equation (3). In the generators X_4, Equation (58),

$$\xi(x, \dot{x}; t) = \frac{1}{\alpha(t)}u_1(t)x, \tag{82}$$

and

$$\eta(x, \dot{x}; t) = \dot{u}_1(t)x^2. \tag{83}$$

Hence, Equation (53) gives

$$\zeta_1(x, \dot{x}; t) = -\frac{1}{\alpha}\omega^2 u_1 x^2 + \frac{1}{\alpha}(\dot{u}_1 - \gamma u_1)x\dot{x} - \frac{u_1}{\alpha}\dot{x}^2. \tag{84}$$

In this case, the calculation of Equation (71) can be obtained as follows:

$$\begin{aligned} X(L)+LD(\xi) &= \frac{3}{2\alpha^2}\dot{u}_1 x\dot{x}^2+\left(-\frac{\gamma\omega^2}{\alpha^2}u_1-u_1\frac{\omega\dot{\omega}}{\alpha^2}-\dot{u}_1\frac{3\omega^2}{2\alpha^2}\right)x^3-\frac{\omega^2u_1}{\alpha^2}\dot{x}^2-\frac{u_1}{2\alpha^2}\dot{x}^3 \\ &= D\left(-\frac{u_1}{2\alpha^2}x\dot{x}^2+\frac{\dot{u}_1}{\alpha^2}x^2\dot{x}-\frac{u_1}{2\alpha^2}\omega^2x^3\right). \end{aligned} \tag{85}$$

Thus, Equation (72) becomes

$$I(x,\dot{x};t)=0. \tag{86}$$

By replacing u_1 with u_2 in results, Equations (84)–(86) in the case of generator X_4, we get corresponding results in the case of generator X_5, i.e.,

$$\zeta_1(x,\dot{x};t)=-\frac{1}{\alpha}\omega^2u_2x^2+\frac{1}{\alpha}(\dot{u}_2-\gamma u_2)x\dot{x}-\frac{u_2}{\alpha}\dot{x}^2, \tag{87}$$

$$X(L)+LD(\xi)=D[B(x,\dot{x};t)]=D\left(-\frac{u_2}{2\alpha^2}x\dot{x}^2+\frac{\dot{u}_2}{\alpha^2}x^2\dot{x}-\frac{u_2}{2\alpha^2}\omega^2x^3\right), \tag{88}$$

and

$$I(x,\dot{x};t)=-\frac{u_2}{2\alpha^2}x\dot{x}^2+\frac{\dot{u}_2}{\alpha^2}x^2\dot{x}-\frac{u_2}{2\alpha^2}\omega^2x^3-B(x,\dot{x};t)=0, \tag{89}$$

In Equations (86) and (89), both generators X_4 and X_5 give a trivial conservation quantity, 0. X_6, X_7, and X_8 are the generators that give us the invariant we are interested in. However, the three generators have a similar form, but are associated with two independent solutions of the equations of motion. That is, X_6 is composed of u_1, X_7 is composed of both u_1 and u_2, and X_8 is composed of u_2. We do not list the process of obtaining the results of three cases here, but the case of X_7 seems to be the most meaningful calculation, so only this case will be dealt with in more detail, and the cases of X_6 and X_7 will briefly state each result as in the previous case.

In generators X_6, Equation (60),

$$\xi(x,\dot{x};t)=\frac{1}{\alpha(t)}u_1^2(t), \tag{90}$$

and

$$\eta(x,\dot{x};t)=\frac{1}{\alpha(t)}u_1(t)\dot{u}_1^2(t)x. \tag{91}$$

Hence, Equation (53) gives

$$\zeta_1(x,\dot{x};t)=\frac{1}{\alpha}\left[\left(\dot{u}_1^2-\omega^2u_1^2\right)x-\left(u_1\dot{u}_1+\gamma u_1^2\right)\dot{x}\right]. \tag{92}$$

The calculation process of Equation (69) in X_6 is somewhat laborious, but it is very similar to that in X_7, which will cover the process. In X_6 it can be obtained as follows: calculation of Equation (71) is

$$\begin{aligned} X(L)+LD(\xi) &= -u_1^2\frac{\omega\dot{\omega}}{\alpha^2}x^2-\frac{\omega^2}{\alpha^2}(2u_1\dot{u}_1+\gamma u_1^2)x^2+(\dot{u}_1^2-\omega^2u_1^2)\frac{x\dot{x}}{\alpha^2} \\ &= D\left(-\frac{1}{\alpha^2}u_1^2\dot{x}^2+\frac{2}{\alpha^2}u_1\dot{u}_1x\dot{x}-\frac{1}{2\alpha^2}(\omega^2u_1^2+\dot{u}_1^2)x^2\right) \\ &= D[(B(x,\dot{x};t)]. \end{aligned} \tag{93}$$

Thus, Equation (72) becomes

$$\begin{aligned} I(x,\dot{x};t) &= -\xi\left(\frac{\dot{x}^2}{2\alpha}+\frac{\omega^2}{2\alpha}x^2\right)+\eta\frac{\dot{x}}{\alpha}-B(x,\dot{x};t) \\ &= \frac{1}{2\alpha^2}u_1^2\dot{x}^2-\frac{1}{\alpha^2}u_1\dot{u}_1x\dot{x}+\frac{1}{2\alpha^2}\dot{u}_1^2x^2 \\ &= \frac{I_1^2}{2}, \end{aligned} \tag{94}$$

where I_1 is Equation (30). Equation (94) is an invariant, Equation (32). Now, in the case of X_7, which we are interested in, we will cover the process in some detail. In these cases,

$$\xi(x,\dot{x};t) = \frac{2}{\alpha(t)}u_1(t)u_2(t), \tag{95}$$

and

$$\eta(x,\dot{x};t) = \frac{1}{\alpha(t)}(u_1(t)\dot{u}_2(t) + \dot{u}_1(t)u_2(t))x. \tag{96}$$

Hence, Equation (53) gives

$$\zeta_1(x,\dot{x};t) = \frac{1}{\alpha}\Big[2\Big(\dot{u}_1\dot{u}_2 - \omega^2 u_1u_2\Big)x - (u_1\dot{u}_2 + \dot{u}_1u_2 + 2\gamma u_1u_2)\dot{x}\Big], \tag{97}$$

Substituting Equations (95)–(97) into Equation (71) and rearranging, we get

$$X(L) + LD(\xi) = -2u_1u_2\frac{\omega\dot{\omega}}{\alpha^2}x^2 - 2\frac{\omega^2}{\alpha^2}(u_1\dot{u}_2 + \dot{u}_1u_2 + \gamma u_1u_2)x^2 + \frac{2}{\alpha^2}(\dot{u}_1\dot{u}_2 - \omega^2u_1u_2)x\dot{x}. \tag{98}$$

We do not know whether Equation (98) can be expressed in the form $D(B)$ or not. Since Equation (98) is a quadratic expression of both x and $\dot{x}$, let us assume that it has the form $D(B)$ and B is a quadratic form of x and $\dot{x}$. And calculating and rearranging $D(B)$ gives

$$\begin{aligned} D(B) &= D\Big(f\dot{x}^2 + gx\dot{x} + hx^2\Big) \\ &= (\dot{f} - 2f\gamma + g)\dot{x}^2 + (-2f\omega^2 + \dot{g} - \gamma g + 2h)x\dot{x} + (\dot{h} - \omega^2 g)x^2. \end{aligned} \tag{99}$$

In Equation (99), f, g, and h are time-dependent functions and functions to be obtained. To find it, we compare Equations (98) and (99), and if we set the coefficients of each quadratic form to be equal, we get the following differential equations:

$$\begin{cases} \dot{f} - 2f\gamma + g = 0, \\ \dot{g} - \gamma g - 2f\omega^2 + 2h = \frac{1}{\alpha^2}(2\dot{u}_1\dot{u}_2 - 2\omega^2u_1u_2), \\ \dot{h} - \omega^2 g = -\frac{2}{\alpha^2}u_1u_2\omega\dot{\omega} - \frac{2\omega^2}{\alpha^2}(u_1\dot{u}_2 + \dot{u}_1u_2) - \frac{2\gamma\omega^2}{\alpha^2}u_1u_2. \end{cases} \tag{100}$$

The solution of Equation (100) is obtained as

$$\begin{cases} g(t) = \frac{2}{\alpha^2(t)}(2u_1(t)\dot{u}_2(t) + \dot{u}_1(t)u_2(t)), \\ f(t) = -\frac{2}{\alpha^2(t)}u_1(t)u_2(t), \\ h(t) = -\frac{\omega^2(t)}{\alpha^2(t)}u_1(t)u_2(t) - \frac{1}{\alpha^2(t)}\dot{u}_1(t)\dot{u}_2(t). \end{cases} \tag{101}$$

Thus, with Equation (101), Equation (99) becomes

$$D(B) = D\Big[-\frac{2}{\alpha^2}u_1u_2\dot{x}^2 + \frac{2}{\alpha^2}(u_1\dot{u}_2 + \dot{u}_1u_2)x\dot{x} - \frac{1}{\alpha^2}(\omega^2u_1u_2 + \dot{u}_1\dot{u}_2)x^2\Big]. \tag{102}$$

From Equation (102), Equation (98) becomes

$$XL + LD(\xi) = D\Big[-\frac{2}{\alpha^2}u_1u_2\dot{x}^2 + \frac{2}{\alpha^2}(u_1\dot{u}_2 + \dot{u}_1u_2)x\dot{x} - \frac{1}{\alpha^2}(\omega^2u_1u_2 + \dot{u}_1\dot{u}_2)x^2\Big]. \tag{103}$$

Thus, Equation (72) is obtained as

$$\begin{aligned} I(x,\dot{x};t) &= -\xi\left(\frac{\dot{x}^2}{2\alpha} + \frac{\omega^2}{2\alpha}x^2\right) + \eta\frac{\dot{x}}{\alpha} - B \\ &= \frac{1}{\alpha^2}u_1u_2\dot{x}^2 - \frac{1}{\alpha^2}(u_1\dot{u}_2 + \dot{u}_1u_2)x\dot{x} + \frac{1}{\alpha^2}(\dot{u}_1\dot{u}_2)x^2 = I_1I_2. \end{aligned} \tag{104}$$

Equation (104) is the same invariant quantity as the result of setting $f(t) = 0$ in Equation (34). In the case of X_8, the procedure for finding the result is the same as in the case of X_7, and the results are the same as replacing u_1 with u_2 in the results of X_6. In X_8, Equation (62),

$$\xi(x, \dot{x}; t) = \frac{1}{\alpha(t)} u_1^2(t), \tag{105}$$

and

$$\eta(x, \dot{x}; t) = \frac{1}{\alpha(t)} u_1(t)\dot{u}_2(t)x. \tag{106}$$

Hence, Equation (53) gives

$$\zeta_1(x, \dot{x}; t) = \frac{1}{\alpha(t)}\left[(\dot{u}_2^2 - \omega^2 u_2^2)x - (u_2\dot{u}_2 + \gamma u_2^2)\dot{x}\right]. \tag{107}$$

Equation (93) gives

$$\begin{aligned} X(L) + LD(\xi) &= -u_2^2\frac{\omega\dot{\omega}}{\alpha^2}x^2 - \frac{\omega^2}{\alpha^2}(2u_2\dot{u}_2 + \gamma u_2^2)x^2 + (\dot{u}_2^2 - \omega^2 u_2^2)\frac{x\dot{x}}{\alpha^2} \\ &= D\left(-\frac{1}{\alpha^2}u_2^2\dot{x}^2 + \frac{2}{\alpha^2}u_2\dot{u}_2 x\dot{x} - \frac{1}{2\alpha^2}(\omega^2 u_2^2 + \dot{u}_2^2)x^2\right) \\ &= D[B(x, \dot{x}; t)]. \end{aligned} \tag{108}$$

Thus, Equation (72) is obtained as

$$\begin{aligned} I(x, \dot{x}; t) &= -\xi\left(\frac{\dot{x}^2}{2\alpha} + \frac{\omega^2}{2\alpha}x^2\right) + \eta\frac{\dot{x}}{\alpha} - B \\ &= \frac{1}{2\alpha^2}u_2^2\dot{x}^2 - \frac{1}{\alpha^2}u_2\dot{u}_2 x\dot{x} + \frac{1}{2\alpha^2}\dot{u}_2^2 x^2 = I_2/2. \end{aligned} \tag{109}$$

where I_2 is Equation (31). Equation (109) is an invariant, Equation (34). Summarizing the above results, there are eight generators corresponding to the groups that admit the equation of motion, Equation (26), which is a second-order linear differential equation. There exists a $B(x, \dot{x}; t)$ that satisfies Equation (93) in all eight generators. Thus, the variational integral formed by the Lagrangian of a system is invariant for each group corresponding to eight generators (55)–(62). The invariants obtained from each generator are the Wronskian constant Ω_0, the quadratic invariant $I_1^2, I_1 I_2, I_2^2$ for the equation of motion (26), and the trivial invariant 0. We discuss, in this section, the invariants in groups and generators admitted by the homogeneous second-order linear differential Equation (39) for $f(t) = 0$. For non-homogeneous differential equations, Equation (3) ($f(t) \neq 0$), finding admitted generators is too complicated. In the case of a relatively simple example of Equation (3), for a forced free particle whose equation of motion is

$$\ddot{x} = f(t), \tag{110}$$

Calculating the generators shows that all eight generators depend on $f(t)$. Although generators and invariants cannot be directly dealt with in the case of non-homogeneous differential equations, Equation (3), they can be predicted from the results of homogeneous differential equations. X_1 is a generator that does not depend on the solution of the system. It is obtained from any linear differential equation. In the case of the non-homogeneous differential equation, Equation (3), it is a generator obtained by replacing x with $x - x_p$, where the particular solution, x_p, is Equation (13). The generators X_2 and X_3 are the same for both homogeneous and non-homogeneous forms in differential equations (Equation (3)). This means that adding the general solution of a homogeneous differential equation to the

solution of a non-homogeneous differential equation results in a general solution of the non-homogeneous differential equation. The invariant is the Wronskian constant Ω_0 in both homogeneous and non-homogeneous cases. The results of X_4 and X_5 depend on the cube of the two independent solutions u_1 and u_2 of the system. The invariant obtained a trivial 0. In the case of inhomogeneity, we can only predict that it will depend on the cube of u_1, u_2, and f. The results of X_6, X_7, and X_8 are related to the square of u_1 and u_2, and the invariant shows that there exists a quadratic invariant of x and $\dot{x}$. In the non-homogeneous case, we can predict that the invariants of the generator associated with X_6, X_7, and X_8 will be Equations (32), (33) and (34), respectively.

5. Invariant Variable Space

We regard invariant quantities (30) and (31) as new variables and examine the system in the new variable space. We also cover variable transformations between the two spaces. If Equation (26) has a solution of the oscillatory motion, two independent solutions of Equation (26) can be set as $u_1(t)$ and $u_2(t) = u_1^*(t)$. If so, Equation (15) becomes

$$W(t) = u_1(t)\dot{u}_1^*(t) - u_1^*(t)\dot{u}_1(t) = i\Omega_0\alpha(t), \tag{111}$$

where we can put $\Omega_0 > 0$. (If $\Omega_0 < 0$, we can select $u_1(t)$ and $u_2(t) = u_1^*(t)$ interchangeably.) In this case, the new variable associated with the invariant that can be inferred from Equations (30) and (31) can be defined as follows.

$$I_1 = \frac{1}{\sqrt{\Omega_0}}\left(u_1(t)p - \frac{\dot{u}_1(t)}{\alpha(t)}x + g_1(t)\right), \tag{112}$$

and

$$I_2 = \frac{1}{\sqrt{\Omega_0}}\left(u_1^*(t)p - \frac{\dot{u}_1^*(t)}{\alpha(t)}x + g_1^*(t)\right) = I_1^*, \tag{113}$$

where

$$g_1(t) = \int_{t_0}^{t} \frac{u_1(t')}{\alpha(t')} f(t')dt'. \tag{114}$$

Calculating the Poisson square brackets of I_1 of Equation (112) and I_2 of Equation (113) gives

$$\{I_1, I_2\} = \frac{\partial I_1}{\partial x}\frac{\partial I_2}{\partial p} - \frac{\partial I_1}{\partial p}\frac{\partial I_2}{\partial x} = i \tag{115}$$

Thus, the transformation consisting of Equations (112) and (113) is not canonical. Let two independent solutions of Equation (26), u_1 and u_2, be in monotonic motion. If so, u_1 and u_2 can be chosen so that the Wronskian Equation (15) can be put as

$$W(t) = u_1(t)\dot{u}_1(t) - u_2(t)\dot{u}_2(t) = \Omega_0\alpha(t), \tag{116}$$

where $\Omega_0 > 0$. In this case, the new variable associated with the invariant that can be inferred from Equations (30) and (31) can be defined as

$$I_1 = \frac{1}{\sqrt{\Omega_0}}\left(u_1(t)p - \frac{\dot{u}_1(t)}{\alpha(t)}x + g_1(t)\right), \tag{117}$$

and

$$I_2 = \frac{1}{\sqrt{\Omega_0}}\left(u_2(t)p - \frac{\dot{u}_2(t)}{\alpha(t)}x + g_2(t)\right), \tag{118}$$

where

$$g_2(t) = \int_{t_0}^{t} \frac{u_2(t')}{\alpha(t')} f(t')dt'. \tag{119}$$

Here, $I_1^* \neq I_2$. Calculating the Poisson square brackets of I_1 and I_2 gives

$$\{I_1, I_2\} = \frac{\partial I_1}{\partial x}\frac{\partial I_2}{\partial p} - \frac{\partial I_1}{\partial p}\frac{\partial I_2}{\partial x} = 1 \tag{120}$$

Therefore, the transformation of monotonic motion consisting of Equations (118) and (119) is a canonical transformation. Even in the case of oscillatory motion, if the variables are set to $I_1/\sqrt{i}$ and $I_2/\sqrt{i}$, respectively, the transformation consisting of Equations (112) and (113) is a canonical transformation. In this case, of course $(I_1/\sqrt{i})^* \neq I_2/\sqrt{i}$. If the canonical variables x and p of a system whose Hamiltonian is Equation (6), $H(x, p; t)$, and the new canonical variables I_1 and I_2 are connected by a point transformation, the Hamiltonian, $K(I_1, I_2; t)$, giving I_1 and I_2 is

$$K(I_1, I_2; t) = H(x, p; t) - p\frac{\partial x}{\partial t} - \frac{\partial F}{\partial t}. \tag{121}$$

Here, the generating function $F(I_1, I_2; t)$ can be obtained from the following relational expression [2].

$$I_2 - p\frac{\partial x}{\partial I_1} = \frac{\partial F}{\partial I_1}, \tag{122}$$

$$-p\frac{\partial x}{\partial I_1} = \frac{\partial F}{\partial I_2}. \tag{123}$$

Using Equations (117), (118), (122) and (123), Equation (121) is obtained by

$$K(I_1, I_2; t) = 0. \tag{124}$$

Equation (124) means that the new variables I_1 and I_2 can be obtained from the calculation results of the Hamilton–Jacobi theory. The inverse transformation of Equations (117) and (118) is

$$x = \frac{1}{\Omega_0}(u_2 I_1 - u_1 I_2) + x_p, \tag{125}$$

$$p = \frac{1}{\alpha\Omega_0}(\dot{u}_2 I_1 - \dot{u}_1 I_2) + \frac{\dot{x}_p}{\alpha}, \tag{126}$$

where x_p is Equation (13) and $\dot{x}_p$ is Equation (17). The inverse transformation of Equations (112) and (113) is

$$x = \frac{1}{i\Omega_0}(u_1^* I_1 - u_1 I_2) + x_p, \tag{127}$$

$$p = \frac{1}{i\alpha\Omega_0}(\dot{u}_1^* I_1 - \dot{u}_1 I_2) + \frac{\dot{x}_p}{\alpha}. \tag{128}$$

Simplifying Equations (32)–(34) with Equations (114) and (119), we get

$$I_1^2 = u_1^2 p^2 - \frac{2u_1\dot{u}_1}{\alpha}xp + \frac{\dot{u}_1^2}{\alpha^2}x^2 - \frac{2}{\alpha}\dot{u}_1 g_1(t)x + 2u_1 g_1(t)p + g_1^2(t), \tag{129}$$

$$I_2^2 = u_2^2 p^2 - \frac{2u_2\dot{u}_2}{\alpha}xp + \frac{\dot{u}_2^2}{\alpha^2}x^2 - \frac{2}{\alpha}\dot{u}_2 g_2(t)x + 2u_2 g_2(t)p + g_2^2(t), \tag{130}$$

and

$$\begin{aligned} I_1 I_2 = {} & u_1 u_2 p^2 - \frac{u_1\dot{u}_2 + \dot{u}_1 u_2}{\alpha}xp + \frac{\dot{u}_1\dot{u}_2}{\alpha^2}x^2 - \frac{1}{\alpha}(\dot{u}_2 g_1(t) + \dot{u}_1 g_2(t))x, \\ & +(u_1 g_2(t) + u_2 g_1(t))p + g_1(t)g_2(t). \end{aligned} \tag{131}$$

Calculating the Poisson brackets between $I_1 I_2$ and I_1^2, $I_1 I_2$ and I_2^2, and I_1^2 and I_2^2, respectively,

$$\{I_1 I_2, I_1^2\} = -2\Omega_0 I_1^2, \tag{132}$$

$$\{I_1 I_2, I_2^2\} = 2\Omega_0 I_2^2, \tag{133}$$

and

$$\{I_1^2, I_2^2\} = 4\Omega_0 I_1 I_2. \tag{134}$$

In the case of oscillatory motion, the result calculated by Equations (112) and (113) is obtained by replacing Ω_0 with i in Equations (132)–(134), and in the case of monotonic motion, the result is obtained by replacing Ω_0 with 1.

6. Summary and Conclusions

We chose a time-dependent system given by the general quadratic Hamiltonian of position and momentum and showed that the invariant quantities of the system are expressed by two independent solutions of homogeneous equations. The Wronskian constant belongs to one of the invariants of the system obtained from the equations of motion. In Section 3, two linear invariants given by the linear equations of x and p were also obtained, and each of them has independent solutions of the homogeneous equations of the system as auxiliary conditions. We also obtained three quadratic invariants of x and p consisting of the product of the two linear invariants. The eight generators, i.e., Lie algebras, associated with the point symmetry group of the system are obtained from the homogeneous differential equation of the equation of motion of the system. It was proven that all of these generators have invariants, and their respective invariants were derived in Section 4. The invariants obtained by the eight generators are the Wronskian constant of the system, the three invariants given by the quadratic form of x and p, and the trivial invariants 0, respectively. In Section 4, we defined new variables as two invariants consisting of a linear combination of x and p. Treating our time-dependent system with new variables makes our system more convenient to handle. When the solution to the equation of motion of a system is an oscillatory function, that is, a bounded system, the two new variables are complex conjugates of each other. And if the solution of the system is a monotonic function, that is, the unbound system, a canonical transformation relation is formed between the set of two new variables and the set of x and p. Our system is time-dependent. Mechanical energy, which is conserved in conservative systems, is not conserved in the time-dependent system. If a quantity that is not conserved in time is defined as a physical quantity, it becomes difficult to interpret it physically. Among the three physical quantities that consist of quadratic form of x and p, one that depends on the two homogeneous solutions can be made into the energy dimension and defined as a new energy quantity. Our system includes the conservation system. For example, if $\gamma(t) = 0$, and $\omega^2(t) =$ const. in Equation (3), our system becomes a harmonic oscillator. In this case, the new energy, which is made up of invariant quantities, is reduced to mechanical energy. We can expect that, in classical mechanics as well as quantum mechanics, the physical quantities of time-dependent systems should be expressed as invariant quantities so that they can be easily handled physically. The results of this paper show that new physical quantities can be defined and treated as invariant. Through this paper, we have found the conservation of a system using Lie algebra, but the main purpose of this paper is to find a more understandable way to treat time-dependent systems quantum mechanically in the next paper. The two invariants of the system, which are linearly given by the canonical position and canonical momentum, can be assumed to correspond to the creation and destruction operators in quantum mechanics. It can be predicted that the operator set corresponding to the three invariant sets, which are the quadratic forms of the canonical

position and canonical momentum, forms the SU(2) Lie algebra. We can predict that an easier perturbation theory can be developed by comparing the $f(t) = 0$ system with the $f(t) \neq 0$ system. And we can also discuss the coherent state of time-dependent systems. We will subsequently attempt to use our method to study other time-dependent systems.

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