

Article

A Linearly Convergent Algorithm for the H -Spectral Radius of Index-Cyclic Symmetric Positive Tensors

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Abstract

A class of index-cyclic symmetric positive tensors is defined, and a diagonal similarity algorithm is proposed to compute the H -spectral radius of such tensors. The linear convergence of the algorithm is rigorously proven. The computational efficiency of the proposed method is compared with that of the classical power method by numerical examples. The results indicate that the proposed algorithm achieves reliable linear convergence and offers competitive computational performance for index-cyclic symmetric positive tensors.

Keywords: nonnegative tensor; H -spectral radius; linear convergence; index-cyclic symmetric positive tensor

1. Introduction

Tensors are natural generalizations of matrices with a broad spectrum of applications. In particular, nonnegative tensors have emerged as powerful tools in the era of big data, drawing growing attention and rigorous study from mathematicians [1–3]. Sharing many desirable properties with nonnegative matrices, nonnegative tensors have found widespread practical use. Among these, the eigenvalue problem of nonnegative tensors has been systematically studied, yielding a wealth of influential results [4–17].

In 2005, Qin and Lim [4,5] respectively defined the eigenvalues of tensors. Assuming $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m, n]}$ is an m -order n -dimensional real tensor, if $\lambda \in \mathbb{R}$ and $0 \neq x \in \mathbb{R}^n$ satisfy

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where

$$\mathcal{A}x^{m-1} = \left(\sum_{i_2, \dots, i_m=1}^n a_{i i_2 \dots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \leq i \leq n}, \quad x^{[m-1]} = (x_i^{m-1})_{1 \leq i \leq n},$$

then λ is called the H -eigenvalue of tensor $\mathcal{A}$, and (λ, x) is called the H -eigenpair of tensor $\mathcal{A}$. Let $\sigma(\mathcal{A})$ be the set of H -eigenvalues of tensor $\mathcal{A}$, and $\rho(\mathcal{A}) = \max_{\lambda \in \sigma(\mathcal{A})} |\lambda|$ be the H -spectral radius of tensor $\mathcal{A}$.

In 2009, Ng, Qi and Zhou [6] proposed the NQZ algorithm for computing the H -spectral radius of irreducible nonnegative tensors, which is a natural extension of the power method for computing the maximum eigenvalue of matrices (see Algorithm 1).



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Algorithm 1: NQZ algorithm

Step 0. Choose $x^{(0)} > 0, x^{(0)} \in \mathbb{R}^n$. Let $y^{(0)} = \mathcal{A}(x^{(0)})^{m-1}$ and set $k := 0$.

Step 1. Compute

$$x^{(k+1)} = \frac{(y^{(k)})^{[\frac{1}{m-1}]}}{\|(y^{(k)})^{[\frac{1}{m-1}]}\|},$$

$$y^{(k+1)} = \mathcal{A}(x^{(k+1)})^{m-1},$$

$$\underline{\lambda}_{k+1} = \min_{(x^{(k+1)})_i > 0} \frac{(y^{(k+1)})_i}{(x^{(k+1)})_i^{m-1}},$$

$$\bar{\lambda}_{k+1} = \max_{(x^{(k+1)})_i > 0} \frac{(y^{(k+1)})_i}{(x^{(k+1)})_i^{m-1}}.$$

Step 2. If $\bar{\lambda}_{k+1} = \underline{\lambda}_{k+1}$, stop. Otherwise, replace k by $k + 1$ and go to Step 1.

Subsequently, scholars have studied the convergence of the NQZ algorithm for computing the H -spectral radius of nonnegative tensors [6–13]. Among them, in 2014, Hu, Huang and Qi [12] provided more general convergence conditions and proved the R-linear convergence of the NQZ algorithm for weak primitive tensors. In 2012, Zhang and Qi [8] proved the linear convergence of the NQZ algorithm on essentially positive tensors. In 2021, Zhang and Bu [10] defined a generalized weakly positive tensor and provided a diagonal similarity algorithm for the H -spectral radius, proving the linear convergence of the algorithm. In 2023, Liu and Lv [13] defined the s -index positive tensor and provided a linear convergence algorithm for the H -spectral radius of the s -index positive tensor. In 2025, Lyu and Chen [18] first introduced the directed hypergraph of tensors into the study of the convergence of algorithms for the H -spectral radius of nonnegative tensors. They applied the directed hypergraph of tensors to investigate the R-linear convergence of the NQZ algorithm and derived general conditions for its linear convergence using the representation matrix of nonnegative tensors. The NQZ algorithm works well for computing the H -spectral radius of weakly primitive tensors, but it fails to converge for some weakly irreducible nonnegative tensors. Therefore, we propose a shifted algorithm for the H -spectral radius of nonnegative tensors, which is applicable to general weakly irreducible nonnegative tensors. An appropriate choice of the shift parameter can further improve the computational efficiency.

Building on these advances, this work extends the existing classes of essentially positive tensors [7], weakly positive tensors [19], generalized weakly positive tensors [10], and s -index positive tensors [13], and introduces the novel class of index-cyclic symmetric positive tensors. A diagonal similarity algorithm is proposed for computing the H -spectral radius of these tensors, and its linear convergence is rigorously established. The computational performance of the proposed algorithm is then compared with that of the NQZ algorithm via comprehensive numerical examples.

This paper is structured as follows. Section 1 outlines the NQZ algorithm and relevant research findings. Section 2 provides relevant definitions and preliminary results. In Section 3, we introduce the class of index-cyclic symmetric positive tensors, present our algorithm for computing the H -spectral radius, and establish its linear convergence. Section 4 contains examples that compare our proposed algorithm with the NQZ one. Finally, Section 5 concludes the paper and discusses potential future work.

2. Related Definitions and Preliminary Results

In this paper, let $\mathbb{R}^{[m,n]}$ denote the set of all real m th-order n -dimensional tensors, and $\mathbb{R}_+^{[m,n]}$ denote the set of all real m th-order n -dimensional nonnegative tensors. Let $\mathbb{R}^{n \times n}$ be the set of all real $n \times n$ matrices. Furthermore, $\mathbb{R}^n$, $\mathbb{R}_+^n$, and $\mathbb{R}_{++}^n$ denote, respectively, the set of all real vectors, the set of all nonnegative nonzero vectors, and the set of all positive vectors in the n -dimensional Euclidean space.

In 2008, Chang et al. [14] generalized the concept of irreducible matrices to irreducible tensors.

Definition 1 ([14]). An m th-order n -dimensional tensor $\mathcal{A}$ is said to be reducible if there exists a nonempty proper index subset $N \subset \langle n \rangle$ such that

$$a_{i_1 i_2 \dots i_m} = 0, \quad \forall i_1 \in N, \quad \forall i_2, \dots, i_m \notin N.$$

If $\mathcal{A}$ is not reducible, then $\mathcal{A}$ is irreducible.

In order to further discuss the algorithm for the H -spectral radius of nonnegative tensors, the concepts of essentially positive tensors and generalized weakly positive tensors were introduced in [7,10], respectively.

Definition 2. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}_+^{[m,n]}$.

- (1) ([7]) A tensor $\mathcal{A}$ is called essentially positive if $a_{ij \dots j} > 0$, $i, j \in \langle n \rangle$.
- (2) ([10]) A tensor $\mathcal{A}$ is called generalized weakly positive if there exists $i_0 \in \langle n \rangle$, such that $a_{i_0 j \dots j} > 0$, $a_{j i_0 \dots i_0} > 0$, for all $j \in \langle n \rangle \setminus \{i_0\}$, $j \neq i_0$.

In 2011, Chang et al. [11] defined primitive tensors, and in 2014, Hu et al. [12] generalized this concept by proposing weakly primitive tensors and weakly irreducible tensors.

Definition 3 ([12]). Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}_+^{[m,n]}$; define a matrix $M = (M_{ij}) \in \mathbb{R}^{n \times n}$ with

$$M_{ij} = \sum_{j \in \{i_2, \dots, i_m\}} a_{i i_2 \dots i_m}, \quad i, j \in \langle n \rangle.$$

We call $\mathcal{A}$ weakly reducible if M is a reducible matrix, and weakly primitive if M is a primitive matrix. If $\mathcal{A}$ is not weakly reducible, then it is called weakly irreducible.

In 2008, Chang et al. [14] generalized the Perron–Frobenius Theorem of nonnegative matrices to nonnegative tensors. In 2013, Friedland S. et al. [15] extended it to the class of weakly irreducible nonnegative tensors.

Theorem 1 ([14,15]). Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}_+^{[m,n]}$ be a weakly irreducible nonnegative tensor. Then $\rho(\mathcal{A}) > 0$ is an eigenvalue of $\mathcal{A}$ with a positive eigenvector x corresponding to it. Moreover, if λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \rho(\mathcal{A})$. If λ is an eigenvalue of $\mathcal{A}$, then $|\lambda| \leq \rho(\mathcal{A})$.

In 2010, Yang et al. [16] generalized the classical result of upper and lower bound results for the spectral radius of nonnegative matrices to nonnegative tensors.

Theorem 2 ([16]). Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}_+^{[m,n]}$, $\rho(\mathcal{A})$ be the H -spectral radius of $\mathcal{A}$. Then

$$\min_{i \in \langle n \rangle} \sum_{i_2, \dots, i_m=1}^n a_{i i_2 \dots i_m} \leq \rho(\mathcal{A}) \leq \max_{i \in \langle n \rangle} \sum_{i_2, \dots, i_m=1}^n a_{i i_2 \dots i_m}. \quad (1)$$

In 2013, Shao [17] gave the diagonal similarity transformation of tensors and discussed the nature of eigenvalues.

Definition 4 ([17]). Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m, n]}$, $\mathcal{B} = (b_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m, n]}$. We say that $\mathcal{A}$ and $\mathcal{B}$ are diagonally similar, if there exists some invertible diagonal matrix $D = \text{diag}(d_1, d_2, \dots, d_n)$ of order n such that $\mathcal{B} = D^{-(m-1)} \cdot \mathcal{A} \cdot D$, where $b_{i_1 i_2 \dots i_m} = a_{i_1 i_2 \dots i_m} d_{i_1}^{-(m-1)} d_{i_2} \dots d_{i_m}$.

Theorem 3 ([17]). If the two m th-order n -dimensional tensors $\mathcal{A}$ and $\mathcal{B}$ are diagonally similar, then $\text{spec}(\mathcal{A}) = \text{spec}(\mathcal{B})$.

In 1985, Hron [20] introduced the definition of a directed graph of a matrix, where the directed graph of matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is denoted as $\Gamma(A)$ and its directed edges are denoted as $e_{ij} \in \Gamma(A)$.

Definition 5 ([20]). If $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, then $\Gamma(A)$ is called strongly connected, if for any pair of ordered nodes (i, j) , where $i \neq j$ of $\Gamma(A)$, there exists a directed path $i \rightarrow i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_s \rightarrow j$ to connect them.

Theorem 4 ([20]). If $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, $n \geq 2$, then A is irreducible if and only if the directed graph $\Gamma(A)$ of A is strongly connected.

3. Algorithm and Its Convergence

In this section, we first define a class of index-cyclic symmetric positive tensors.

Definition 6. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}_+^{[m, n]}$. If there are r different integers $i_1, i_2, \dots, i_r \in \langle n \rangle$ such that $a_{i_s N_{s+1}} > 0$, $s = 1, 2, \dots, r$, $i_{s+1} \in N_{s+1}$, $a_{i_r N_{r+1}} = a_{i_r N_1} > 0$, $i_{r+1} = i_1 \in N_1 = N_{r+1}$, and if $N_1 \cup N_2 \cup \dots \cup N_r$ contains exactly $m - 1$ integers i_s , $s = 1, 2, \dots, r$, then $a_{i_s N_{s+1}} > 0$, $s = 1, 2, \dots, r - 1$ and $a_{i_r N_1} > 0$ are called index-cyclic symmetric positive, and each $a_{i_s N_{s+1}}$ is said to lie on an index-cyclic symmetric positive element chain.

Example 1. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_5}) \in \mathbb{R}_+^{[5, 6]}$, where $a_{11222} = \frac{1}{6}$, $a_{23233} = \frac{1}{3}$, $a_{34443} = \frac{1}{2}$, $a_{44555} = \frac{2}{3}$, $a_{56665} = \frac{1}{5}$, $a_{61611} = 1$, and all other entries are nonnegative. Then a_{11222} , a_{23233} , a_{34443} , a_{44555} , a_{56665} , a_{61611} form an index-cyclic symmetric positive element chain, with index subsets $N_1 = \{1222\}$, $N_2 = \{3233\}$, $N_3 = \{4443\}$, $N_4 = \{4555\}$, $N_5 = \{6665\}$, $N_6 = \{1611\}$.

Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}_+^{[m, n]}$, and denote $\mathcal{A}_N = (a_{i_1 i_2 \dots i_m}^N) \in \mathbb{R}_+^{[m, n]}$, where

$$a_{i_1 i_2 \dots i_m}^N = \begin{cases} a_{i_1 i_2 \dots i_m}, & \text{if } a_{i_1 i_2 \dots i_m} \text{ lies on an index-cyclic symmetric positive element chain,} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 7. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}_+^{[m, n]}$. If there exists $i_0 \in \langle n \rangle$, such that $e_{i i_0} \in \Gamma(M(\mathcal{A}_N))$, and $\Gamma(M(\mathcal{A}_N))$ is strongly connected, then $\mathcal{A}$ is called an index-cyclic symmetric positive tensor.

Example 2. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}_+^{[m, n]}$, where $a_{i i_0 i \dots i} = i$, $a_{i_0 i_0 \dots i_0 i} = i$, $i \in \langle n \rangle$, $i \neq i_0$, and all other entries are zero, then $\mathcal{A}$ is an index-cyclic symmetric positive tensor.

Remark 1. Obviously, essentially positive tensors, weakly positive tensors, generalized weakly positive tensors, and s -index positive tensors are all special classes of index-cyclic symmetric positive tensors.

From Remark 1, index-cyclic symmetric positive tensors constitute a broader family of tensors than essentially positive tensors [7], weakly positive tensors [19], generalized weakly positive tensors [10], and s -index positive tensors [13]. The inclusion relationships among these families of nonnegative tensors are illustrated in Figure 1.

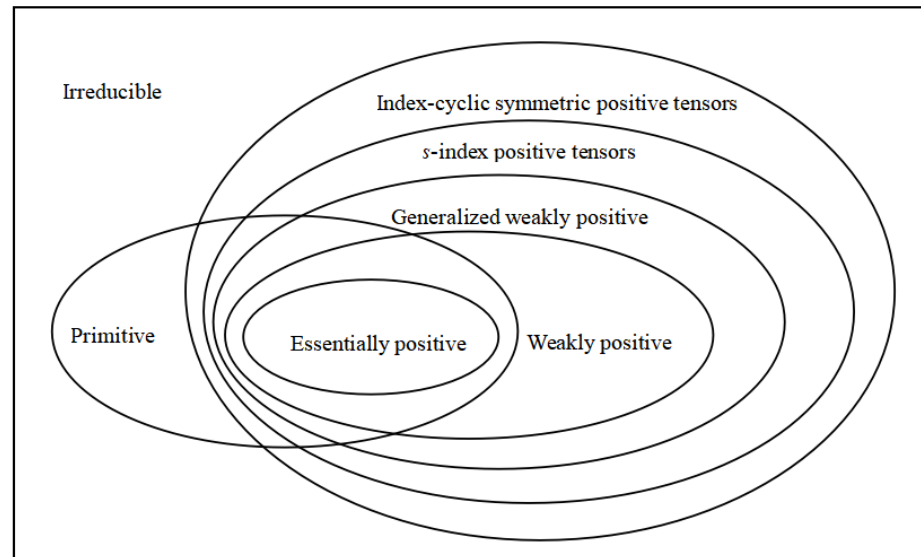


Figure 1. Inclusion relationships of several families of nonnegative tensors.

Applying Theorem 3, we provide a diagonal similarity algorithm for the H -spectral radius of index-cyclic symmetric positive tensors.

Denote $\mathcal{A}^{(k)} + \gamma \mathcal{I} = (a_{i_1 i_2 \dots i_m}^{(k)}(\gamma))$, then $a_{ii \dots i} + \gamma > \varepsilon$, for any $i \in \langle n \rangle$.

Lemma 1. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}_+^{[m, n]}$ be an index-cyclic symmetric positive tensor. In the two sequences defined by Algorithm 2, $\bar{r}^{(k)}$ monotonically decreases with a lower bound, and $\underline{r}^{(k)}$ monotonically increases with an upper bound.

Algorithm 2: Diagonal similarity Algorithm 1

Step 0. Given $\mathcal{A}^{(0)} = \mathcal{A} = (a_{i_1 i_2 \dots i_m})$, $\varepsilon > 0$, $\varepsilon - \min_{i \in \langle n \rangle} a_{ii \dots i} < \gamma \leq \frac{\bar{r}^{(0)} + \underline{r}^{(0)}}{2}$. Set

$k := 0$.

Step 1. Compute

$$r_i^{(k)} = \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m}^{(k)}, i \in \langle n \rangle,$$

$$\bar{r}^{(k)} = \max_{i \in \langle n \rangle} r_i^{(k)}, \underline{r}^{(k)} = \min_{i \in \langle n \rangle} r_i^{(k)}.$$

Step 2. If $\bar{r}^{(k)} - \underline{r}^{(k)} < \varepsilon$, then $\rho(\mathcal{A}) = \frac{1}{2}(\bar{r}^{(k)} + \underline{r}^{(k)})$ and stop.

Step 3. Set

$$D_k = \left(\text{diag}(r_1^{(k)}, r_2^{(k)}, \dots, r_n^{(k)}) + \gamma I \right)^{\left[\frac{1}{m-1} \right]} / (\bar{r}^{(k)} + \gamma)^{\frac{1}{m-1}},$$

$$\mathcal{A}^{(k+1)} = D_k^{-(m-1)} \cdot \mathcal{A}^{(k)} \cdot D_k,$$

and replace k by $k + 1$, go to Step 1.

Proof. Since $\mathcal{A}^{(k+1)} = D_k^{-(m-1)} \cdot \mathcal{A}^{(k)} \cdot D_k = (a_{i_1 i_2 \dots i_m}^{(k+1)}) \in \mathbb{R}_+^{[m,n]} (k = 0, 1, 2, \dots), \forall i \in \langle n \rangle$, from Algorithm 2, we have

$$\begin{aligned} r_i^{(k+1)} &= \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m}^{(k+1)} = \sum_{i_2, \dots, i_m=1}^n \left\{ \frac{a_{ii_2 \dots i_m}^{(k)}(\gamma)}{r_i^{(k)} + \gamma} \prod_{j=2}^m (r_{i_j}^{(k)} + \gamma)^{\frac{1}{m-1}} \right\} - \gamma \\ &\leq \sum_{i_2, \dots, i_m=1}^n \frac{a_{ii_2 \dots i_m}^{(k)}(\gamma)}{r_i^{(k)} + \gamma} \cdot (\bar{r}^{(k)} + \gamma) - \gamma = \bar{r}^{(k)}, \end{aligned}$$

thus,

$$\bar{r}^{(k+1)} \leq \bar{r}^{(k)} (k = 0, 1, 2, \dots).$$

Similarly, we get

$$\underline{r}^{(k)} \leq \underline{r}^{(k+1)} (k = 0, 1, 2, \dots).$$

From Theorems 2 and 3, we obtain that

$$\underline{r}^{(0)} \leq \underline{r}^{(k)} \leq \rho(\mathcal{A}^{(k)}) = \rho(\mathcal{A}) \leq \bar{r}^{(k)} \leq \bar{r}^{(0)} (k = 1, 2, \dots). \quad \square$$

Lemma 2. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}_+^{[m,n]}$ be an index-cyclic symmetric positive tensor. Thus, for any element $a_{i_s N_{s+1}} > 0$ that lies on an index-cyclic symmetric positive element chain, there exists a positive number $a > 0$, such that $a_{i_s N_{s+1}}^{(k)} \geq a > 0, k = 0, 1, 2, \dots$

Proof. Consider the elements $a_{i_s N_{s+1}} > 0, s = 1, 2, \dots, r, a_{i_r N_{r+1}} = a_{i_r N_1}$ that lie on an index-cyclic symmetric positive element chain. By applying Lemma 2, we get

$$\begin{aligned} \bar{r}^{(0)} &\geq a_{i_s N_{s+1}}^{(k+1)} = a_{i_s N_{s+1}}^{(k)} \frac{\prod_{j \in N_{s+1}} (r_j^{(k)} + \gamma)^{\frac{1}{m-1}}}{r_{i_s}^{(k)} + \gamma} \\ &= \dots = a_{i_s N_{s+1}}^{(0)} \frac{\prod_{t=0}^k \prod_{j \in N_{s+1}} (r_j^{(t)} + \gamma)^{\frac{1}{m-1}}}{\prod_{t=0}^k (r_{i_s}^{(t)} + \gamma)} \\ &\geq \underline{a} \cdot \frac{\prod_{t=0}^k \prod_{j \in N_{s+1}} (r_j^{(t)} + \gamma)^{\frac{1}{m-1}}}{\prod_{t=0}^k (r_{i_s}^{(t)} + \gamma)}, \end{aligned} \quad (2)$$

where $\underline{a} = \min\{a_{i_s N_{s+1}} > 0 : a_{i_s N_{s+1}} \text{ lies on an index-cyclic symmetric positive element chain}\}$, so

$$\frac{\prod_{t=0}^k \prod_{j \in N_{s+1}} (r_j^{(t)} + \gamma)^{\frac{1}{m-1}}}{\prod_{t=0}^k (r_{i_s}^{(t)} + \gamma)} \leq \frac{\bar{r}^{(0)}}{\underline{a}}. \quad (3)$$

By Definition 7, the union $N_1 \cup N_2 \cup \dots \cup N_r$ contains exactly $m - 1$ indices i_s for $s = 1, 2, \dots, r$. Hence we obtain

$$\frac{\prod_{t=0}^k \prod_{j \in N_2} (r_j^{(t)} + \gamma)^{\frac{1}{m-1}}}{\prod_{t=0}^k (r_{i_1}^{(t)} + \gamma)} \cdot \frac{\prod_{t=0}^k \prod_{j \in N_3} (r_j^{(t)} + \gamma)^{\frac{1}{m-1}}}{\prod_{t=0}^k (r_{i_2}^{(t)} + \gamma)} \cdot \dots \cdot \frac{\prod_{t=0}^k \prod_{j \in N_1} (r_j^{(t)} + \gamma)^{\frac{1}{m-1}}}{\prod_{t=0}^k (r_{i_r}^{(t)} + \gamma)} = 1.$$

From Equation (3), it can be concluded that

$$\frac{\prod_{t=0}^k \prod_{j \in N_{s+1}} (r_j^{(t)} + \gamma)^{\frac{1}{m-1}}}{\prod_{t=0}^k (r_{i_s}^{(t)} + \gamma)} \left(\frac{\bar{r}^{(0)}}{a} \right)^{r-1} \geq 1.$$

From Equation (2), we can obtain

$$a_{i_s N_{s+1}}^{(k+1)} \geq a \left(\frac{a}{\bar{r}^{(0)}} \right)^{r-1} \geq a \left(\frac{a}{\bar{r}^{(0)}} \right)^{n-1} > 0.$$

Denote $a = \min\{a \left(\frac{a}{\bar{r}^{(0)}} \right)^{n-1}, \varepsilon\}$; for any $a_{i_s N_{s+1}} > 0$, $a_{i_s N_{s+1}}^{(k)} > a$ can be obtained, and $a_{i \dots i}^{(k)} + \gamma = a_{i \dots i} + \gamma > a$ can be obtained for any $i \in \langle n \rangle$. $\square$

Below we present the convergence results of Algorithm 2 for computing the H -spectral radius of an index-cyclic symmetric positive tensor.

Theorem 5. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}_+^{[m, n]}$ be an index-cyclic symmetric positive tensor, and let $\rho(\mathcal{A})$ denote its H -spectral radius. By applying Algorithm 2, we have

$$\bar{r}^{(k)} - \underline{r}^{(k)} \leq \alpha (\bar{r}^{(k-1)} - \underline{r}^{(k-1)}), \quad (4)$$

where $\alpha = 1 - \frac{a}{4(m-1)\bar{r}^{(0)}}$. Moreover,

$$\lim_{k \rightarrow \infty} \bar{r}^{(k)} = \lim_{k \rightarrow \infty} \underline{r}^{(k)} = \rho(\mathcal{A}).$$

It thus follows from Equation (4) that Algorithm 2 converges linearly for index-cyclic symmetric positive tensors.

Proof. Given that $\mathcal{A}$ is an index-cyclic symmetric positive tensor and Definition 7, there exists i_0 , and for any $i \in \langle n \rangle$, $e_{i i_0} \in \Gamma(M(\mathcal{A}_N))$ can be obtained, i.e., $i_0 \in N_{i+1}$ exists such that $a_{i N_{i+1}} > 0$.

(I) Let $\underline{r}^{(0)} < r_{i_0}^{(0)} < \bar{r}^{(0)}$. We discuss two cases:

(i) If $\bar{r}^{(0)} - r_{i_0}^{(0)} \geq \frac{1}{2}(\bar{r}^{(0)} - \underline{r}^{(0)})$, we have

$$\begin{aligned} r_{i_0}^{(1)} &= \sum_{i_2, \dots, i_m=1}^n a_{i_0 i_2 \dots i_m}^{(0)} \frac{\prod_{j=2}^m (r_{i_j}^{(0)} + \gamma)^{\frac{1}{m-1}}}{r_{i_0}^{(0)} + \gamma} \\ &= \bar{r}^{(0)} - \sum_{i_2, \dots, i_m=1}^n a_{i_0 i_2 \dots i_m}^{(0)} (\gamma) \frac{(\bar{r}^{(0)} + \gamma) - \prod_{j=2}^m (r_{i_j}^{(0)} + \gamma)^{\frac{1}{m-1}}}{r_{i_0}^{(0)} + \gamma} \\ &\leq \bar{r}^{(0)} - \frac{(\bar{r}^{(0)} + \gamma) - (r_{i_0}^{(0)} + \gamma)}{r_{i_0}^{(0)} + \gamma} a_{i_0 i_0 \dots i_0}^{(0)} (\gamma) \\ &\leq \bar{r}^{(0)} - \frac{a}{4\bar{r}^{(0)}} (\bar{r}^{(0)} - \underline{r}^{(0)}). \end{aligned} \quad (5)$$

For any $i \neq i_0$, we can obtain that

$$\begin{aligned}
r_i^{(1)} &= \sum_{i_2, \dots, i_m=1}^n a_{i i_2 \dots i_m}^{(0)} \frac{\prod_{j=2}^m (r_{i_j}^{(0)} + \gamma)^{\frac{1}{m-1}}}{r_i^{(0)} + \gamma} \\
&= \bar{r}^{(0)} - \sum_{i_2, \dots, i_m=1}^n a_{i i_2 \dots i_m}^{(0)}(\gamma) \frac{(\bar{r}^{(0)} + \gamma) - \prod_{j=2}^m (r_{i_j}^{(0)} + \gamma)^{\frac{1}{m-1}}}{r_i^{(0)} + \gamma} \\
&\leq \bar{r}^{(0)} - \frac{(\bar{r}^{(0)} + \gamma) - [(\bar{r}^{(0)} + \gamma)^{m-2} (r_{i_0}^{(0)} + \gamma)]^{\frac{1}{m-1}}}{r_i^{(0)} + \gamma} a_{i N_{i+1}}^{(0)}(\gamma) \\
&\leq \bar{r}^{(0)} - \frac{a}{2\bar{r}^{(0)}} (\bar{r}^{(0)} + \gamma)^{\frac{m-2}{m-1}} \left[(\bar{r}^{(0)} + \gamma)^{\frac{1}{m-1}} - (r_{i_0}^{(0)} + \gamma)^{\frac{1}{m-1}} \right] \\
&\leq \bar{r}^{(0)} - \frac{a}{2\bar{r}^{(0)}} \cdot \frac{1}{m-1} (\bar{r}^{(0)} - r_{i_0}^{(0)}) \\
&\leq \bar{r}^{(0)} - \frac{a}{4(m-1)\bar{r}^{(0)}} (\bar{r}^{(0)} - \underline{r}^{(0)}). \tag{6}
\end{aligned}$$

(ii) If $\bar{r}^{(0)} - r_{i_0}^{(0)} < \frac{1}{2}(\bar{r}^{(0)} - \underline{r}^{(0)})$, we have $r_{i_0}^{(0)} - \underline{r}^{(0)} > \frac{1}{2}(\bar{r}^{(0)} - \underline{r}^{(0)})$. Similarly, we can obtain

$$\begin{aligned}
r_{i_0}^{(1)} &= \sum_{i_2, \dots, i_m=1}^n a_{i_0 i_2 \dots i_m}^{(0)} \frac{\prod_{j=2}^m (r_{i_j}^{(0)} + \gamma)^{\frac{1}{m-1}}}{r_{i_0}^{(0)} + \gamma} \\
&= \underline{r}^{(0)} + \sum_{i_2, \dots, i_m=1}^n a_{i_0 i_2 \dots i_m}^{(0)}(\gamma) \frac{\prod_{j=2}^m (r_{i_j}^{(0)} + \gamma)^{\frac{1}{m-1}} - (\underline{r}^{(0)} + \gamma)}{r_{i_0}^{(0)} + \gamma} \\
&\geq \underline{r}^{(0)} + \frac{(r_{i_0}^{(0)} + \gamma) - (\underline{r}^{(0)} + \gamma)}{r_{i_0}^{(0)} + \gamma} a_{i_0 i_0 \dots i_0}^{(0)}(\gamma) \\
&\geq \underline{r}^{(0)} + \frac{a}{4\bar{r}^{(0)}} (\bar{r}^{(0)} - \underline{r}^{(0)}), \tag{7}
\end{aligned}$$

and for any $i \neq i_0$,

$$\begin{aligned}
r_i^{(1)} &\geq \underline{r}^{(0)} + \frac{[(\underline{r}^{(0)} + \gamma)^{m-2} (r_{i_0}^{(0)} + \gamma)]^{\frac{1}{m-1}} - (\underline{r}^{(0)} + \gamma)}{r_i^{(0)} + \gamma} a_{i N_{i+1}}^{(0)}(\gamma) \\
&\geq \underline{r}^{(0)} + \frac{a}{2\bar{r}^{(0)}} (\underline{r}^{(0)} + \gamma)^{\frac{m-2}{m-1}} \left[(r_{i_0}^{(0)} + \gamma)^{\frac{1}{m-1}} - (\underline{r}^{(0)} + \gamma)^{\frac{1}{m-1}} \right] \\
&\geq \underline{r}^{(0)} + \frac{a}{2\bar{r}^{(0)}} \cdot \frac{1}{m-1} (r_{i_0}^{(0)} - \underline{r}^{(0)}) \\
&\geq \underline{r}^{(0)} + \frac{a}{4(m-1)\bar{r}^{(0)}} (\bar{r}^{(0)} - \underline{r}^{(0)}). \tag{8}
\end{aligned}$$

Thus, from Lemma 1 and Equations (5)–(8), we conclude that

$$\begin{aligned}
\bar{r}^{(1)} - \underline{r}^{(1)} &\leq \max_{i \in \langle n \rangle} r_i^{(1)} - \underline{r}^{(0)} \\
&\leq \bar{r}^{(0)} - \frac{a}{4(m-1)\bar{r}^{(0)}} (\bar{r}^{(0)} - \underline{r}^{(0)}) - \underline{r}^{(0)} \\
&= \left(1 - \frac{a}{4(m-1)\bar{r}^{(0)}} \right) (\bar{r}^{(0)} - \underline{r}^{(0)}) \\
&= \alpha (\bar{r}^{(0)} - \underline{r}^{(0)}),
\end{aligned}$$

or

$$\begin{aligned}\bar{r}^{(1)} - \underline{r}^{(1)} &\leq \bar{r}^{(0)} - \min_{i \in \langle n \rangle} r_i^{(1)} \\ &\leq \bar{r}^{(0)} - \underline{r}^{(0)} - \frac{a}{4(m-1)\bar{r}^{(0)}} (\bar{r}^{(0)} - \underline{r}^{(0)}) \\ &= \alpha (\bar{r}^{(0)} - \underline{r}^{(0)}).\end{aligned}$$

Combining the above discussions, we obtain

$$\bar{r}^{(1)} - \underline{r}^{(1)} \leq \alpha (\bar{r}^{(0)} - \underline{r}^{(0)}).$$

(II) Using a proof technique similar to Case (I), for any $k \in \mathbb{Z}_+$, we have

$$\bar{r}^{(k)} - \underline{r}^{(k)} \leq \alpha (\bar{r}^{(k-1)} - \underline{r}^{(k-1)}).$$

Combining Case (I) and Case (II), for any $k = 1, 2, \dots$, we obtain

$$\begin{aligned}\bar{r}^{(k)} - \underline{r}^{(k)} &\leq \alpha (\bar{r}^{(k-1)} - \underline{r}^{(k-1)}) \\ &\leq \alpha^2 (\bar{r}^{(k-2)} - \underline{r}^{(k-2)}) \\ &\leq \dots \\ &\leq \alpha^k (\bar{r}^{(0)} - \underline{r}^{(0)}).\end{aligned}\tag{8}$$

Since $0 < \alpha = 1 - \frac{a}{4(m-1)\bar{r}^{(0)}} < 1$, it follows that

$$\lim_{k \rightarrow \infty} (\bar{r}^{(k)} - \underline{r}^{(k)}) \leq \lim_{k \rightarrow \infty} \alpha^k (\bar{r}^{(0)} - \underline{r}^{(0)}) = 0.$$

From Theorem 2 and Lemma 1, we have

$$\lim_{k \rightarrow \infty} \bar{r}^{(k)} = \lim_{k \rightarrow \infty} \underline{r}^{(k)} = \rho(\mathcal{A}).$$

By Equation (8), Algorithm 2 is linearly convergent.

If $r_{i_0}^{(0)} = \bar{r}^{(0)}$ or $r_{i_0}^{(0)} = \underline{r}^{(0)}$, the same convergence result holds, and Algorithm 2 is linearly convergent. $\square$

In Algorithm 2, the tensor $\mathcal{A}^{(k)}$ requires storage at every iteration. To cut down redundant memory consumption, Algorithm 2 can be revised as stated below.

Remark 2. If $\mathcal{A}$ is an m th-order n -dimensional positive tensor, Algorithm 2 performs n^m more division operations per iteration than Algorithm 3, so Algorithm 3 possesses far superior stability and computational efficiency.

Algorithm 3: Diagonal similarity Algorithm 2

Step 0. Given $\mathcal{A}^{(0)} = \mathcal{A} - \min_{i \in \langle n \rangle} a_{i \dots i} \mathcal{I} = (a_{i_1 i_2 \dots i_m}^{(0)})$, $F_0 = I$, $\varepsilon + a < \gamma \leq \frac{1}{2} (\bar{r}^{(0)} + \underline{r}^{(0)}) - \min_{i \in \langle n \rangle} a_{i \dots i}$. Set $k := 0$.

Step 1. Compute

$$\begin{aligned}f_i^{(k+1)} &= \sum_{i_2, \dots, i_m=1}^n (\mathcal{A}^{(0)} \cdot F_k)_{i i_2 \dots i_m}, \quad i \in \langle n \rangle, \\ \bar{r}^{(k+1)} &= \max_{i \in \langle n \rangle} \frac{f_i^{(k+1)}}{f_i^{(k)}}, \quad \underline{r}^{(k+1)} = \min_{i \in \langle n \rangle} \frac{f_i^{(k+1)}}{f_i^{(k)}}.\end{aligned}$$

Step 2. If $\bar{r}^{(k+1)} - \underline{r}^{(k+1)} < \varepsilon$, then $\rho(\mathcal{A}) = \frac{1}{2} (\bar{r}^{(k+1)} + \underline{r}^{(k+1)}) + \min_{i \in \langle n \rangle} a_{i \dots i}$ and stop.

Step 3. Set

$$F_{k+1} = \left(\text{diag}(f_1^{(k+1)}, f_2^{(k+1)}, \dots, f_n^{(k+1)}) + \gamma I \right)^{\left[\frac{1}{m-1} \right]} / (\bar{r}^{(k+1)} + \gamma)^{\frac{1}{m-1}},$$

and replace k by $k + 1$, go to Step 1.

Below we provide the calculation method for the eigenvector of $\mathcal{A}$ corresponding to its spectral radius $\rho(\mathcal{A})$.

Theorem 6. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}_+^{[m, n]}$ be an index-cyclic symmetric positive tensor. Denote

$$D_t = \prod_{k=0}^t D^{(k)}, \quad \tilde{D}_t = D_t / \|D_t\|, \quad x^{(t)} = \tilde{D}_t e,$$

where $e = (1, 1, \dots, 1)^T \in \mathbb{R}_{++}^n$. Then from Algorithm 2 we have $x = \lim_{t \rightarrow \infty} x^{(t)}$, and $\mathcal{A}x^{m-1} = \rho(\mathcal{A})x^{[m-1]}$.

Proof. By the construction of $\tilde{D}_t$, the sequence of positive diagonal matrices $\tilde{D}_t$ converges as $t \rightarrow \infty$, which implies $\lim_{t \rightarrow \infty} x^{(t)} = x \geq 0$. From Algorithm 2, we have

$$\begin{aligned} \mathcal{A}^{(t+1)} &= (D_t)^{-(m-1)} \cdot \mathcal{A}^{(t)} \cdot D_t \\ &= \dots = \left(\prod_{k=0}^t D_k \right)^{-(m-1)} \cdot \mathcal{A} \cdot \prod_{k=0}^t D_k \\ &= \tilde{D}_t^{-(m-1)} \cdot \mathcal{A} \cdot \tilde{D}_t, \end{aligned}$$

then

$$\mathcal{A}^{(t+1)} e^{m-1} = \left(\tilde{D}_t^{-(m-1)} \cdot \mathcal{A} \cdot \tilde{D}_t \right) e^{m-1} = (\tilde{D}_t^{-1})^{m-1} (\mathcal{A}(\tilde{D}_t e)^{m-1}),$$

where $e = (1, 1, \dots, 1)^T \in \mathbb{R}_{++}^n$.

According to Theorem 1, as $t \rightarrow \infty$, we obtain

$$\rho(\mathcal{A})e = \lim_{t \rightarrow \infty} (\mathcal{A}^{(t+1)} e^{m-1}) = \lim_{t \rightarrow \infty} (\tilde{D}_t^{-1})^{m-1} \lim_{t \rightarrow \infty} (\mathcal{A}(\tilde{D}_t e)^{m-1}),$$

which yields

$$\rho(\mathcal{A})x^{[m-1]} = \mathcal{A}x^{m-1}.$$

By Theorem 1, $x > 0$, and thus x is the eigenvector of $\mathcal{A}$ corresponding to the eigenvalue λ . $\square$

4. Numerical Examples

In this section, numerical experiments are conducted to compare the performance of the NQZ algorithm, and Algorithms 2 and 3. All experiments are conducted on a laptop equipped with an Intel Core i5-9500 CPU (3.0 GHz) and 8 GB of RAM, using the Matlab R2018b environment. The tolerance is set to $\varepsilon = 10^{-8}$. Let $r(\mathcal{A})$ denote the H -spectral radius of $\mathcal{A}$, ‘Iter’ the number of iterations, and ‘Time(s)’ the CPU time (in seconds).

Example 3. Consider the tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}_+^{[4, n]}$, where $a_{3ii3} = 1$, $a_{iiii} = n$, $a_{iiii} = 1$ for all $i \in \langle n \rangle$, and all other entries are zero. Then $\mathcal{A}$ is a basic weakly positive tensor. For Algorithms 2 and 3, if the shift parameter is set as $\gamma = -\min_{i \in \langle n \rangle} a_{iiii} + \frac{a}{\bar{r}(0)}^{n-1}$, the comparison results with the NQZ algorithm are listed in Table 1. As can be observed from Table 1, both Algorithms 2 and 3 require only one-third of the iteration steps needed by the NQZ algorithm, and the CPU runtime is reduced by approximately 60%.

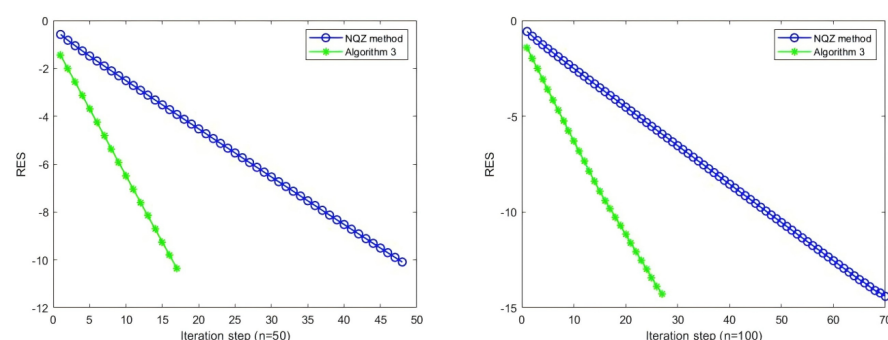
Table 1. Comparison of H -spectral radii calculated by our algorithms and the NQZ algorithm for Example 1.

NQZ Algorithm			Algorithm 2		Algorithm 3		$\rho(\mathcal{A})$
n	Iter	Time (s)	Iter	Time (s)	Iter	Time (s)	
5	10	0.0129	1	0.0106	1	0.0053	5.6415
10	7	0.0186	1	0.0062	1	0.0056	10.6549
20	6	0.0525	1	0.0150	1	0.0083	20.6610
40	4	0.3384	1	0.1261	1	0.1083	40.6638
60	4	1.1127	1	0.5039	1	0.4905	60.66488
80	4	4.9960	1	2.4432	1	2.2400	80.6653
100	3	10.1841	1	5.1075	1	4.1152	100.6655

Example 4. Consider the tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}_+^{[4,n]}$, where $a_{ijij} = \frac{i+j}{n}$, $a_{iiii} = n$ for all $i, j \in \langle n \rangle$, and all other entries are zero. Then $\mathcal{A}$ is a basic weakly positive tensor. For Algorithms 2 and 3, if the shift parameter is set as $\gamma = -\min_{i \in \langle n \rangle} a_{iiii} + \underline{a} \left(\frac{\underline{a}}{\bar{r}(0)} \right)^{n-1}$, numerical comparisons with the NQZ algorithm are listed in Table 2. As shown in Table 2, both Algorithms 2 and 3 require roughly half the number of iterations of the NQZ algorithm, and Algorithm 3 saves nearly 60% CPU runtime compared with the NQZ scheme. Meanwhile, we plot the logarithmic residual error $\log_{10} \|\mathcal{A}(x^{(k)})^{m-1} - \bar{r}^{(k)}(x^{(k)})^{[m-1]}\|_\infty$ (denoted as RES) on the vertical axis against the iteration index k on the horizontal axis, where $x^{(k)} = \tilde{D}_k e$. This plot compares the performance of the NQZ algorithm and Algorithm 3, as illustrated in Figure 2. It can be clearly observed from Figure 2 that Algorithm 3 substantially outperforms the NQZ algorithm.

Table 2. Comparison of H -spectral radii calculated by our algorithms and the NQZ algorithm for Example 2.

NQZ Algorithm			Algorithm 2		Algorithm 3		$\rho(\mathcal{A})$
n	Iter	Time (s)	Iter	Time (s)	Iter	Time (s)	
5	40	0.0121	14	0.0177	12	0.0132	9.9184
10	45	0.0355	17	0.0288	16	0.0216	20.3616
20	47	0.1742	18	0.3167	17	0.0998	41.1631
40	48	1.1164	18	5.1004	17	0.5672	82.7242
60	49	7.1058	18	27.2739	17	2.8202	124.2759
80	49	37.0259	19	96.7450	18	15.1006	165.8253
100	50	93.2079	19	235.7482	18	36.9113	207.3737

**Figure 2.** Comparison of residual errors between the NQZ algorithm and Algorithm 3.

Numerical examples show that Algorithm 3 outperforms the NQZ algorithm, while the relative merits of Algorithm 2 compared with the NQZ algorithm differ across cases. Here, we cannot theoretically derive a reasonable optimal range for the selection of parameter γ in Algorithms 2 and 3. In Examples 1 and 2 of this paper, we set $\gamma = -\min_{i \in \langle n \rangle} a_{iiii} + \underline{a} \left(\frac{\underline{a}}{\bar{r}(0)} \right)^{n-1}$, and Algorithm 3 achieves high computational efficiency under this setting.

5. Conclusions

The basic weakly positive tensor defined in this paper generalizes the essential positive tensor, weakly positive tensor and generalized weakly positive tensor. We prove that Algorithms 2 and 3 converge linearly when computing the H -spectral radius of basic weakly positive tensors. Numerical examples demonstrate that Algorithm 3 outperforms the NQZ algorithm, while the relative performance between Algorithm 2 and the NQZ scheme varies across different test cases. A worthy research issue is how to select the shift parameter γ to improve computational efficiency.

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