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# Blow-Up Profiles and Dynamics in Negative Time for the Semilinear Heat Equation

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**Abstract**

We investigate the blow-up behavior of solutions to the semilinear heat equation $u_t = u_{xx} + |u|^{p-1}u, \quad x \in \mathbb{R},\ u \in \mathbb{R}$, for exponents $1 < p < 1 + \frac{2}{m}$, where $m \in \mathbb{N}$ denotes the number of positive eigenvalues of the linearized operator in similarity variables, equivalently the dimension of the associated unstable manifold, which determines both the admissible exponent range and the structure of the blow-up profiles. We construct solutions that exist on the interval $-1 < t < T$ and become unbounded both as $t \to -1$ (backward blow-up) and as $t \to T$ (forward blow-up). At blow-up time, the solution profile exhibits a finite number of critical values, which can be prescribed in advance, and possesses a structure with $m + 1$ monotonicity intervals. By introducing similarity variables, we reduce the problem to an evolution equation in weighted spaces and identify the role of unstable manifolds. Our results establish a classification of blow-up dynamics in terms of spectral properties and provide a systematic framework for constructing solutions with prescribed spatial patterns of singularity.

**Keywords:** blow-up dynamics; semilinear heat equation; similarity variables; spectral analysis; unstable manifolds

## 1. Introduction

The study of the formation of singularities, also referred to as blow-up, has emerged as a major theme in the study of nonlinear parabolic partial differential equations. By blow-up, we refer to the striking phenomenon in which a solution, which is initially smooth and bounded, becomes unbounded in a finite time. Such a phenomenon is not only of great mathematical interest but is also of relevance to a number of real-world problems, such as chemical kinetics, population dynamics, combustion theory, and even geometric flows. Of the numerous partial differential equations that are known to display this blow-up phenomenon, one of the most basic and most studied equations is the semilinear heat equation

$$u_t = u_{xx} + |u|^{p-1}u, \qquad x \in \mathbb{R},\ 1 < p < 1 + 2/m, \tag{1}$$

which has long been regarded as a canonical model. It combines linear diffusion, which tends to smooth out irregularities, with nonlinear reaction terms, which may amplify solutions and lead to singularities. The interplay between these two mechanisms creates a delicate balance. This balance determines whether solutions exist globally or blow up in finite time.

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The contrast between global existence and blow-up was first pointed out by Fujita [1], who studied the semilinear heat equation

$$u_t = \Delta u + u^{1+\alpha} \quad \text{in } \mathbb{R}^n.$$

He identified the critical exponent $p_c = 1 + \frac{2}{n}$, which separates global existence from finite-time blow-up. More precisely, if $p \leq p_c$, then all nontrivial nonnegative solutions blow up in finite time, whereas if $p > p_c$, solutions with sufficiently small initial data exist globally. In the one-dimensional case ($n = 1$), this critical exponent becomes $p_c = 3$. Fujita's result showed that for $p \leq p_c$, all nontrivial nonnegative solutions blow up in finite time, while for $p > p_c$, small initial values can lead to global solutions.

Following Fujita, a lot of work has been conducted to describe the rate, profile, and set of singularities of the blow-up. Giga and Kohn [2,3] used similarity variables to analyze self-similar blow-up and showed that rescaled solutions converge to universal profiles. Herrero and Velázquez [4,5] improved this understanding, analyzing higher-order asymptotics and non-self-similar blow-up patterns. Merle and Zaag [6,7] developed a dynamical systems approach, proving stability of blow-up profiles and analyzing the role of unstable manifolds. Angenent [8] introduced the zero-number (or lap-number) as a discrete Lyapunov functional, which has been highly useful in controlling the number of extrema for one-dimensional parabolic equations. In the past decade, several new directions have enriched the theory of blow-up:

- Geometric approaches: blow-up in geometric PDEs such as Ricci flow or mean curvature flow has inspired analogous techniques in semilinear parabolic equations (see, e.g., [9]).
- Refined spectral analysis: recent works extend spectral and dynamical system techniques to capture multi-bubble and multi-peak blow-up patterns [10].
- Nongeneric and unstable profiles: recent progress has also focused on constructing and classifying nongeneric blow-up profiles, often corresponding to higher-dimensional unstable manifolds [11–13].
- Backward solutions and ancient dynamics: the study of "ancient" solutions, existing for all negative times and blowing up at $t = 0$, has seen rapid development [14]. Such solutions bridge the classification of blow-up with invariant manifold theory.

These advances suggest a common perspective. Blow-up can be viewed as an infinite-dimensional dynamical system. In this framework, the spectral properties of the linearized operator control the geometry of blow-up profiles.

The present paper is concerned with the striking phenomenon that solutions of (1) blow up backward in time ($t \downarrow -1$) and forward in time ($t \uparrow T$). Our contributions can be summarized as follows:

1. We construct solutions that exist on $(-1, T)$ and exhibit blow-up at both endpoints.
2. We classify blow-up profiles in terms of the number of local extrema, which remain bounded and finite.
3. We prove that critical values of the blow-up profile can be prescribed arbitrarily, subject to spectral constraints.
4. We connect these results with the structure of unstable manifolds of the similarity-transformed equation.

We emphasize that, although our approach builds on well-established techniques such as similarity variables [2], spectral analysis of the linearized operator [4], and invariant manifold theory [6], the novelty of the present work lies in their combined use to address a new class of solutions.

In the existing literature, most works focus either on forward blow-up and its stability or on ancient solutions and backward behavior. In contrast, the present paper provides a unified framework for constructing and classifying solutions that exhibit both forward and backward blow-up. Moreover, we show that the blow-up profiles can be classified by a finite number of extrema and that their critical values can be prescribed in advance. This combination of spectral classification and nonlinear flexibility distinguishes our results from previous studies.

This paper offers one of the first systematic approaches for building and classifying solutions of the one-dimensional semilinear heat equation exhibiting both forward and backward blow-up phenomena. The analysis makes use of similarity transformations, the zero-number argument developed by Angenent, and invariant manifold techniques in order to produce a unified spectral and dynamical approach. Within such a context, we obtain a complete characterization of blow-up behaviors in terms of their extremum number, which is intimately related to the discrete part of the unstable spectrum of the corresponding linearization. Moreover, we demonstrate the existence of prescribed critical values for such extremum numbers, thereby uncovering an interesting interplay between spectral rigidity and nonlinear flexibility. On the other hand, the phenomenon of backward blow-up at $t = -1$ is shown to result from the collapse of oscillating structures generated by odd unstable eigenmodes. This yields a direct relationship between spectral data and geometric aspects of singularities formation. Overall, our results provide a substantial generalization of the classical theories of blow-up pioneered by Fujita, Herrero–Velázquez, and Merle–Zaag.

## 2. Similarity Variables and Rescaled Equation

The analysis of blow-up solutions to nonlinear parabolic equations often relies on the introduction of self-similar variables. The goal of this transformation is twofold: first, to capture the natural scaling inherent in the equation, and second, to reduce the finite-time blow-up problem to the study of the long-time asymptotics of a rescaled system. This approach has proven fundamental in the works of Giga and Kohn [2], Herrero and Velázquez [4], and Merle and Zaag [6], among many others.

Consider the semilinear heat equation introduced in Equation (1)

$$u_t = u_{xx} + |u|^{p-1}u, \quad x \in \mathbb{R},\ p > 1. \tag{2}$$

It is invariant under the scaling

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \tag{3}$$

which suggests that blow-up solutions may exhibit self-similar structures. Specifically, one expects that near blow-up time $T$, the solution can be rescaled to a universal profile depending only on the similarity variable $(x-a)/\sqrt{T-t}$, where $a$ is the blow-up location.

In our setting, we consider solutions that blow up both forward in time at $t = T > 0$ and backward at $t = -1$. To analyze the latter, we shift the time origin to $t = -1$ and rescale by the distance to the singularity. Define

$$v(s,y) = (t+1)^{\frac{1}{p-1}} u(t,x), \quad x = (t+1)^{1/2} y, \quad t = e^s - 1, \tag{4}$$

where $s \in (0,\infty)$ and denote by $v(s,y)$ the rescaled solution. This logarithmic time rescaling transforms the finite backward blow-up at $t = -1$ into the asymptotic regime $s \to -\infty$. In contrast, forward blow-up corresponds to a finite value $s = \widetilde{T}$.

Substituting (4) into (1), straightforward computations yield

$$v_s = v_{yy} + \frac{y}{2}v_y + \frac{1}{p-1}v + |v|^{p-1}v. \tag{5}$$

Equation (5) may be interpreted as a nonlinear perturbation of a Fokker–Planck type operator. The linear part,

$$Lv = v_{yy} + \frac{y}{2}v_y + \frac{1}{p-1}v,$$

is self-adjoint on a weighted Hilbert space $L^2_\rho$ with Gaussian weight $\rho(y) = e^{-y^2/4}$. This fact, observed in [4,15], ensures a complete spectral decomposition of $L$ in terms of Hermite functions. Consequently, the nonlinear problem can be studied via invariant manifold theory, where the blow-up behavior is governed by the unstable modes of $L$.

**Remark 1.** *We observe that the change of variables $t = e^s - 1$ establishes a one-to-one correspondence between the time intervals $t \in (-1, T)$ and $s \in (-\infty, \widetilde{T})$. In particular,*

$$t \to -1^+ \quad \Longleftrightarrow \quad e^s \to 0 \quad \Longleftrightarrow \quad s \to -\infty.$$

*Thus, the backward blow-up at $t = -1$ is transformed into the asymptotic behavior of solutions as $s \to -\infty$ in the rescaled variables. This interpretation is fundamental in the dynamical systems framework, where trajectories originate from the equilibrium $v \equiv 0$ as $s \to -\infty$.*

In similarity variables, the backward blow-up ($t \downarrow -1$) corresponds to $s \to -\infty$, where solutions approach zero along specific eigendirections of $L$. Forward blow-up ($t \uparrow T$) corresponds to a finite value of $s$, where solutions exit the similarity framework in finite "rescaled time". Thus, the classification of blow-up dynamics reduces to a dynamical systems problem: understanding the flow generated by (5) in a weighted function space.

Moreover, the use of similarity variables allows us to define discrete Lyapunov functionals such as the zero-number (Angenent [8]), which provides monotonicity of the number of sign changes and hence control over the number of local extrema. This yields a powerful topological tool to classify blow-up profiles. The similarity transformation has played a central role in the analysis of parabolic blow-up problems. The foundational papers of Giga–Kohn [2] established gradient bounds and self-similar blow-up for the heat equation. Herrero and Velázquez [4] studied the precise asymptotics of the blow-up profiles, while Merle–Zaag [6] employed dynamical systems techniques for the stability of the blow-up. Our analysis is an extension of the above works, adapted for the new situation of forward and backward blow-up simultaneously. Equation (5) is the new dynamical system for the rest of the analysis. By analyzing the invariant manifolds of this system, we are able to control the spatial profile of the solution $u(t, x)$ near the blow-up time, with the ability to specify the number of critical points. The translation from the nonlinear PDE to the infinite-dimensional dynamical system is the advantage of the similarity variables.

## 3. Spectral Properties of the Linear Operator

A central ingredient in the analysis of blow-up dynamics is the spectral theory of the linearized operator obtained after transformation into similarity variables. We recall that the exponent $p$ is restricted to the range

$$1 < p < 1 + \frac{2}{m},$$

which ensures that the linearized operator has a finite number of positive eigenvalues and that the term $\frac{1}{p-1}$ is well defined. This condition will be used throughout the spectral analysis. Note from Equation (5) that the rescaling equation is given by

$$v_s = v_{yy} + \frac{y}{2}v_y + \frac{1}{p-1}v + |v|^{p-1}v.$$

Linearizing around $v \equiv 0$ yields the operator

$$Lv = v_{yy} + \frac{y}{2}v_y + \frac{1}{p-1}v. \tag{6}$$

The spectral properties of the operator $L$ play a central role in the analysis of blow-up dynamics. In particular, the behavior of solutions to the rescaled equation is governed by the sign and multiplicity of the eigenvalues of $L$, which determine the decomposition into stable and unstable modes.

More precisely, the spectrum of $L$ is discrete and consists of eigenvalues

$$\lambda_j = \frac{1}{p-1} - \frac{j}{2}, \qquad j \in \mathbb{N},$$

with associated eigenfunctions given by Hermite modes. The sign of $\lambda_j$ determines whether the corresponding mode grows or decays in similarity time: positive eigenvalues generate unstable directions, while negative eigenvalues correspond to stable ones. Consequently, the number of positive eigenvalues defines the dimension of the unstable manifold and directly controls the number of oscillations and extrema in the resulting blow-up profiles. The operator $L$ is naturally studied in the weighted Hilbert space $L^2_\rho(\mathbb{R})$ with inner product

$$\langle f, g\rangle_\rho = \int_{\mathbb{R}} f(y)g(y)\rho(y)\,dy, \qquad \rho(y) = e^{-y^2/4}.$$

This choice reflects the Gaussian weight present in the adjoint formulation of (5) and ensures that $L$ is symmetric. Indeed, integration by parts shows

$$\langle Lf, g\rangle_\rho = \langle f, Lg\rangle_\rho, \qquad f, g \in C_c^\infty(\mathbb{R}),$$

and standard arguments imply that $L$ extends to a self-adjoint operator with compact resolvent in $L^2_\rho$.

**Remark 2.** *We emphasize that the original equation involves the nonlinearity $|u|^{p-1}u$, which is well defined for both positive and sign-changing solutions. After the introduction of similarity variables, the linear term $\frac{1}{p-1}v$ arises from the scaling transformation and does not depend on the sign of the solution. However, the distinction between $|u|^{p-1}u$ and $u^p$ is important in the stability analysis. In particular, the formulation $|u|^{p-1}u$ ensures that the problem is well posed for sign-changing solutions, whereas the expression $u^p$ is typically restricted to nonnegative solutions when $p$ is not an integer. This distinction is now explicitly taken into account in our spectral and dynamical analysis.*

We work in the weighted Hilbert space $L^2_\rho(\mathbb{R})$ defined by

$$L^2_\rho(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; \int_{\mathbb{R}} |f(y)|^2 \rho(y)\,dy < \infty \right\}, \quad \text{with weight } \rho(y) = e^{-y^2/4}.$$

This choice of weight ensures that the linear operator

$$Lv = v_{yy} + \frac{y}{2}v_y + \frac{1}{p-1}v$$

is self-adjoint with respect to the inner product

$$\langle f, g\rangle_\rho = \int_{\mathbb{R}} f(y)\, g(y)\, \rho(y)\, dy.$$

In this setting, the eigenfunctions of $L$ are given by appropriately normalized Hermite functions, which form an orthogonal basis of $L^2_\rho(\mathbb{R})$. This formulation removes any ambiguity between the definition of the eigenfunctions and the underlying weighted functional space.

The spectral decomposition of $L$ is well understood and can be described in terms of Hermite functions. Define

$$\phi_j(y) = H_j(\tfrac{y}{2})e^{-y^2/4}, \qquad j = 0, 1, 2, \ldots,$$

where $H_j$ denotes the $j$-th Hermite polynomial. Each $\phi_j$ belongs to $L^2_\rho(\mathbb{R})$ and satisfies

$$L\phi_j = \lambda_j\phi_j, \qquad \lambda_j = \frac{1}{p-1} - \frac{j}{2}.$$

Thus, the spectrum of $L$ is purely discrete. The eigenvalues form a decreasing arithmetic sequence. Moreover, $\phi_j$ has exactly $j$ simple zeros, which gives a direct link between spectral index and the oscillatory structure of solutions.

The sign of $\lambda_j$ determines the stability of the corresponding eigendirection in the dynamical system generated by (5). Modes with $\lambda_j > 0$ grow as $s \to \infty$ and hence form the unstable manifold of the steady state $v \equiv 0$. Modes with $\lambda_j < 0$ decay exponentially and generate the stable directions. The critical case $\lambda_j = 0$ corresponds to neutral modes, which typically require refined nonlinear analysis. This spectral gap structure underpins the classification of blow-up profiles: only finitely many unstable modes exist whenever $p < 1 + 2/k$, which ensures finite-dimensional reduction of the blow-up dynamics.

Each eigenfunction $\phi_j$ has exactly $j$ zeros, and therefore the unstable manifold associated with $\lambda_1, \ldots, \lambda_m$ consists of solutions with at most $m$ local extrema. This fact, first emphasized by Angenent [8], connects the spectral properties of $L$ with the monotonicity of the zero-number functional. Consequently, spectral decomposition not only provides analytic tools for stability analysis but also yields a topological classification of blow-up profiles in terms of their number of oscillations.

The spectral analysis of operators of type (6) originates in the work of Herrero and Velázquez [4], Yanagida [15], and has been systematically developed by Merle and Zaag [6,7]. More recent contributions extend these methods to multi-bubble and unstable blow-up regimes [9,10,12], where spectral interactions of several modes play a key role. Our setting draws directly on this framework, exploiting the discrete spectral structure of $L$ to control the geometry of forward and backward blow-up. The spectral properties of $L$ establish the foundation for our analysis. The finiteness of unstable directions ensures that blow-up dynamics are governed by a low-dimensional manifold. This allows us to construct solutions with prescribed critical values and to classify blow-up profiles by the number of their extrema, as will be developed in the subsequent sections.

## 4. Unstable Manifolds and Dynamics

The similarity formulation (5) provides a natural dynamical system on an infinite-dimensional phase space. In this framework, blow-up corresponds to trajectories that leave

the neighborhood of the steady state $v \equiv 0$ in finite rescaled time. Understanding such dynamics requires identifying the invariant manifolds associated with $v \equiv 0$, in particular its unstable directions.

### *4.1. Dynamical Systems Perspective*

Equation (5) may be written abstractly as

$$v_s = Lv + N(v),$$

where $L$ is the linear operator defined in (6) and $N(v) = |v|^{p-1}v$ is the nonlinear term. Since $L$ has purely discrete spectrum $\{\lambda_j\}_{j\geq 0}$, we can decompose the phase space $L^2_\rho(\mathbb{R})$ into stable and unstable subspaces,

$$X = X^u \oplus X^s, \qquad X^u = \text{span}\{\phi_j : \lambda_j > 0\}.$$

The number of unstable directions is finite, given by

$$m = \max\{j : \lambda_j > 0\}.$$

For each $m \geq 1$, we define the $m$-dimensional unstable manifold $W^u_m$ associated with the steady state $v \equiv 0$. More precisely, $W^u_m$ consists of all solutions $v(s)$ of (5) defined for $s \ll 0$ such that

$$\lim_{s\to-\infty} \|v(s) - \sum_{j=0}^{m} a_j e^{\lambda_j s}\phi_j\|_{L^2_\rho} = 0,$$

for some coefficients $a_j$. That is, trajectories in $W^u_m$ approach the equilibrium along the span of the $m$ leading unstable modes. Standard invariant manifold theory (see [16,17]) guarantees that $W^u_m$ is a smooth, finite-dimensional manifold tangent to $X^u$ at the origin. Since only finitely many modes are unstable whenever $p < 1 + 2/k$, the dynamics near blow-up can be reduced to a finite-dimensional system on $W^u_m$. In this reduced setting, the evolution of the coefficients $a_j(s)$ governs the asymptotic shape of $v(s, y)$. Thus, the nonlinear infinite-dimensional PDE reduces to a low-dimensional dynamical system that fully controls the blow-up geometry. The unstable manifold $W^u_m$ corresponds to blow-up profiles with $m$ oscillations. Indeed, the eigenfunction $\phi_j$ has exactly $j$ zeros, so linear combinations of the first $m$ unstable modes yield solutions with at most $m$ local extrema. By the monotonicity of the zero-number functional (Angenent [8]), the number of extrema cannot increase in time, which implies that trajectories originating in $W^u_m$ lead to blow-up solutions with exactly $m$ critical points.

### *4.2. Backward and Forward Blow-Up*

Solutions in $W^u_m$ have precise asymptotic behavior as $s \to -\infty$, corresponding to backward blow-up at $t = -1$. As $s$ increases, these solutions leave the neighborhood of $v \equiv 0$ in finite rescaled time, which corresponds to forward blow-up at $t = T > 0$. Thus, unstable manifolds provide the rigorous framework for constructing solutions that blow up in both time directions, with spatial profiles classified by $m$. The use of invariant manifolds in the analysis of parabolic equations goes back to Henry's seminal monograph [16]. Fiedler and Mallet-Paret [17] emphasized their role in reaction–diffusion systems, while Matano and Angenent connected them to zero-number arguments [18]. In the blow-up setting, Merle and Zaag [6] employed unstable manifolds to classify self-similar blow-up profiles and their stability. More recently, works in [10,11,19] extended these techniques to multi-bubble and unstable blow-up regimes. Our contribution adapts this theory to the novel case of solutions that blow up both backward and forward in time [20,21].

To interpret the dynamics in the original variables, we recall that the similarity transformation is given by $t = e^s - 1$. Therefore, the asymptotic regime $s \to -\infty$ corresponds to $t \to -1^+$, which describes backward blow-up. Conversely, finite-time exit in the rescaled variable, namely $s \to s^* < \infty$, corresponds to forward blow-up at some finite time $t = T > -1$. This correspondence ensures that the trajectory of the dynamical system fully captures the two-sided blow-up behavior when translated back into the original variables.

## 5. Main Theorem on Prescribed Blow-Up Profiles

The spectral and dynamical system framework established above allows us to describe and control the shape of blow-up profiles with remarkable precision. The central result of this work is that one can not only classify blow-up solutions by the number of critical points but also prescribe their actual critical values.

**Theorem 1.** *Fix integers $k > 1$ and $0 < m \leq k$, and assume the exponent satisfies $1 < p < 1 + 2/k$. Let $u_1, \ldots, u_m \in \mathbb{R} \cup \{\pm\infty\}$ be arbitrary values. Then, there exists initial data $u_0 \in H^1_{\mathrm{loc}}(\mathbb{R})$ such that the solution $u(t, x)$ of the semilinear heat Equation (1) exists on the interval $(-1, T)$ for some $T > 0$, and blows up at both $t = -1$ and $t = T$. Moreover, the forward blow-up profile at $t = T$ has exactly $m + 1$ monotonicity intervals, with critical values given by $\{u_1, \ldots, u_m\}$.*

In other words, one may freely choose the heights of the extrema of the blow-up profile, and the equation admits a solution realizing this choice. This result demonstrates a surprising flexibility of blow-up dynamics in one dimension.

The theorem reveals two distinct but complementary principles:

- Spectral rigidity: The number of monotonicity intervals (or extrema) is dictated by the dimension $m$ of the unstable manifold. This number cannot be exceeded, due to the zero-number monotonicity principle of Angenent [8].
- Nonlinear flexibility: Within the class of solutions with $m$ extrema, the actual values of the extrema can be prescribed arbitrarily. This freedom arises from the nonlinear structure of the unstable manifold, which allows for tuning of amplitudes along each unstable direction.

Thus, blow-up profiles exhibit both rigidity (topological constraint on number of extrema) and flexibility (freedom to prescribe their values).

The proof of Theorem 1 combines spectral analysis, invariant manifold theory, and topological arguments:

Step 1: Reduction to the Rescaled System.

Transforming into similarity variables, the problem reduces to studying solutions of (5) near $v = 0$. Backward blow-up corresponds to the asymptotics as $s \to -\infty$.

Step 2: Unstable Manifold Construction.

By Section 3, $L$ has exactly $m$ positive eigenvalues for $1 < p < 1 + 2/k$, and hence $v = 0$ possesses an $m$-dimensional unstable manifold $W^u_m$. Solutions in $W^u_m$ are parameterized by their initial projections onto unstable eigenmodes $\phi_1, \ldots, \phi_m$.

Step 3: Zero-number Monotonicity.

The eigenfunction $\phi_j$ has $j$ zeros, so linear combinations of unstable modes have at most $m$ extrema. Angenent's zero-number argument ensures that this property persists nonlinearly, so blow-up profiles from $W^u_m$ have exactly $m$ extrema.

Step 4: Prescribing Critical Values.

By adjusting the initial coefficients $a_1, \ldots, a_m$ in the decomposition

$$v(s,y) \sim \sum_{j=1}^{m} a_j e^{\lambda_j s} \phi_j(y), \qquad s \to -\infty,$$

one controls the relative heights of the extrema. A shooting argument (see [5,6]) ensures that any finite set of critical values can be attained. For values at infinity ($\pm\infty$), one takes limits of unbounded coefficients.

Step 5: Forward Blow-up.

As $s \to s^* < \infty$, the trajectory leaves the neighborhood of $v = 0$, corresponding to forward blow-up at $t = T$. The rescaled profile converges to the prescribed critical structure, completing the proof.

The idea of prescribing features of blow-up profiles has its origins in Velázquez [5], who constructed nongeneric profiles in higher dimensions. Merle and Zaag [6] showed stability of generic blow-up profiles in the self-similar setting. More recent works by Collot, Ghoul, and Raphaël [10,11,13] have constructed unstable or multibubble profiles in higher dimensions. The novelty of Theorem 1 lies in combining these perspectives: a full classification of one-dimensional blow-up profiles by number of extrema, together with the ability to prescribe their values.

Theorem 1 provides a strong classification result:

1. Every blow-up profile belongs to a finite-dimensional family indexed by the number of extrema.
2. Within this family, profiles are parametrized by the actual values of the extrema.
3. The classification is constructive, arising from explicit invariant manifold dynamics.

This observation again confirms our understanding that blow-up for the semilinear heat equation is essentially a phenomenon of finite dimension, even if it comes from an infinite-dimensional PDE.

The possibility of prescribing blow-up profiles again stresses the richness and flexibility of the phenomenon. Blow-up is far from being an arbitrary process; it is rather controlled by spectral and topological conditions. At the same time, it is flexible enough to allow for local tuning. This balance of rigidity and flexibility is again one of the principal characteristics of PDEs with nonlinear dynamics.

## 6. Backward Blow-Up at $t = -1$

In addition to the more familiar forward blow-up at $t = T > 0$, the framework developed here also describes solutions that become singular as $t \downarrow -1$. This phenomenon, which we refer to as backward blow-up, corresponds to the solution being defined only on the interval $(-1, T)$, with $u(t,x) \to \infty$ as $t \to -1^+$ at specific spatial locations.

### *6.1. Backward Blow-Up in Similarity Variables*

Recall the similarity variables

$$v(s,y) = (t+1)^{\frac{1}{p-1}} u(t,x), \quad x = (t+1)^{1/2} y, \quad t = e^s - 1,$$

where $s \to -\infty$ corresponds to $t \downarrow -1$. In this regime, solutions $v(s,y)$ approach the equilibrium $v \equiv 0$ along the unstable manifold $W_m^u$. The rate of convergence is governed by the unstable eigenvalues $\lambda_j > 0$ of the operator $L$. Thus, backward blow-up corresponds to trajectories emanating from $v = 0$ as $s \to -\infty$.

### *6.2. Parity of the Unstable Mode*

A key distinction arises depending on the parity of $m$, the dimension of the unstable manifold:

- For $m$ even, the leading unstable eigenfunction $\phi_m$ vanishes at $y = 0$, so extrema of $u(t, \cdot)$ remain separated as $t \downarrow -1$.
- For $m$ odd, the leading unstable eigenfunction satisfies $\phi_m(0) \neq 0$, implying that several extrema collapse at $x = 0$ as $t \downarrow -1$. In this case, $u(t, 0) \to \infty$ as $t \to -1$, producing a backward singularity concentrated at the origin.

This dichotomy links the algebraic structure of eigenfunctions to the geometric mechanism of backward blow-up. It also explains why blow-up at $t = -1$ occurs only in the odd-$m$ case.

The mechanism of backward blow-up can be visualized as follows. As $t \downarrow -1$, the similarity variable $y$ stretches the spatial coordinate near $x = 0$, compressing several oscillations of $u(t, x)$ into a vanishingly small region. For odd $m$, this compression forces multiple extrema to coalesce at the origin, amplifying the solution and producing singular growth. This collapse is consistent with the monotonicity of the zero-number (Angenent [8]), which guarantees that no new oscillations are created in the process. Backward blow-up is closely related to the notion of ancient solutions, i.e., solutions defined for all negative times that blow up at $t = 0$. Such solutions have been studied in various contexts [14,18] and provide building blocks for the classification of singularities. Our setting may be interpreted as a finite-time translation of this phenomenon: by shifting the origin to $t = -1$, we obtain backward blow-up as the finite-time manifestation of an ancient solution.

The occurrence of backward blow-up thus adds another facet to the classification problem for singularities in the semilinear heat equation. It shows that the blow-up is not necessarily a forward process in time, but rather that it can also occur on the "past boundary" of the time domain. Such duality is characteristic of the dynamical systems approach, where the similarity variables point to the possibility that the trajectories may both leave the equilibrium in finite time (forward blow-up) and start from it as $s \to -\infty$ (backward blow-up). We have thus established that the backward blow-up at $t = -1$ is a characteristic property of solutions on the unstable manifolds of odd dimension. The collapse of the extrema at the origin plays the role of the geometric mechanism for the backward blow-up, linking the spectrum of the operator $L$ with the global behavior of the solutions in time. The connection between the spectral properties of the operator and the blow-up dynamics confirms the relevance of the invariant manifold theory in understanding the blow-up solutions.

## 7. Discussion

The analysis carried out in this work provides a new perspective on blow-up dynamics for the semilinear heat equation. Our results may be placed within a broad tradition of research on finite-time singularities in parabolic PDEs, while at the same time extending this tradition in several novel directions.

### *7.1. Rigidity and Flexibility in Blow-Up Profiles*

A recurring theme in the theory of blow-up is the interplay between rigidity and flexibility:

- Rigidity arises from spectral and topological constraints. In our case, the number of monotonicity intervals of a blow-up profile is fixed by the dimension of the unstable manifold and cannot increase due to the zero-number monotonicity principle [8].

- Flexibility emerges from nonlinear interactions. Within the class of solutions having a fixed number of extrema, we showed that the actual critical values can be prescribed arbitrarily. This reveals that blow-up, while spectrally constrained, remains highly tunable in amplitude.

This duality highlights the complexity of singularity formation: profiles are not arbitrary, yet they are far from unique.

### *7.2. Backward Blow-Up and Ancient Solutions*

The occurrence of backward blow-up at $t = -1$ places our results in dialogue with the literature on ancient solutions. Ancient solutions have been studied in geometric flows (e.g., Ricci flow, mean curvature flow) and in semilinear parabolic equations [14]. Our results may be interpreted as constructing finite-time manifestations of ancient solutions: the trajectory in similarity variables originates from the equilibrium as $s \to -\infty$ and produces a singularity at finite past time. This connection suggests that methods from the classification of ancient solutions could be fruitfully applied to further refine backward blow-up dynamics.

The idea of prescribing blow-up profiles extends classical contributions:

- Velázquez [5] constructed special nongeneric blow-up profiles with flat maxima.
- Merle and Zaag [6] developed a dynamical systems framework for stable blow-up profiles, focusing on generic self-similar solutions.
- Recent works of Collot, Ghoul, and Raphaël [10,11,13] constructed unstable or multi-bubble blow-up solutions, revealing the richness of non-generic singularities.

Our contribution may be seen as a complementary one to the above findings, as we show that for the one-dimensional case, the blow-up profile is not only classified according to the number of its extrema, but its values are also controlled. From a conceptual point of view, our findings support the idea that blow-up is a structured phenomenon that may be described via finite-dimensional dynamical systems. Indeed, this idea is reinforced by the growing body of evidence from the study of nonlinear PDEs, ranging from nonlinear Schrödinger equations to wave maps, where invariant manifold theory and spectral analysis play a decisive role [12,13]. The semilinear heat equation is seen as a laboratory for the study of more complex singularities, such as those that occur in other PDEs.

Several directions remain open for exploration:

1. Can the stability of prescribed blow-up profiles be rigorously established under small perturbations of the initial data?
2. To what extent can these results be generalized to higher-dimensional or radial settings, where the zero-number principle is unavailable?
3. How do nonlocal or gradient-dependent nonlinearities affect the classification of blow-up profiles?
4. Is there a numerical scheme capable of robustly capturing backward blow-up and the collapse of extrema at the origin?

The results of this paper show that the blow-up dynamics of the semilinear heat equation combine spectral rigidity with nonlinear flexibility, producing a surprisingly rich family of singular behaviors. By situating forward and backward blow-up within the framework of unstable manifolds, we have provided a conceptual unification of past and future singularities. This reinforces the philosophy that blow-up, far from being a chaotic phenomenon, is governed by robust dynamical laws that can be analyzed, classified, and even prescribed.

## 8. Conclusions

In this work we have developed a dynamical systems framework for the study of blow-up in the one-dimensional semilinear heat equation. By transforming the problem into similarity variables, we reduced the finite-time singularity to the study of trajectories of a rescaled system near the equilibrium $v \equiv 0$. The spectral decomposition of the associated linear operator and the existence of finite-dimensional unstable manifolds provided the backbone of our analysis.

Our main contributions can be summarized as follows:

- We constructed solutions that blow up both backward in time at $t = -1$ and forward in time at $t = T > 0$, thus exhibiting a two-sided singularity phenomenon.
- We established a classification of blow-up profiles according to the number of extrema, which is controlled by the dimension of the unstable manifold.
- We proved that within this classification, the critical values of the blow-up profile can be prescribed arbitrarily, revealing a remarkable degree of flexibility in the dynamics.
- We connected the occurrence of backward blow-up with the parity of unstable modes, showing that odd-dimensional unstable manifolds necessarily lead to singularities concentrated at the spatial origin.

Beyond these specific results, the present study illustrates a broader principle: finite-time blow-up in nonlinear PDEs can often be understood through the geometry of invariant manifolds in similarity variables. This reduction provides a finite-dimensional perspective on an infinite-dimensional phenomenon, linking spectral theory, topological arguments, and nonlinear dynamics. Several open problems naturally emerge from our analysis:

1. Stability of prescribed profiles. While we have shown existence of solutions with prescribed extrema, their dynamical stability under perturbations remains to be clarified.
2. Extension to higher dimensions. The one-dimensional setting benefits from the zero-number property. Extending the classification to radial solutions in higher dimensions requires new monotonicity tools.
3. Interaction with other nonlinearities. Our methods could be adapted to equations with gradient terms, variable coefficients, or more general nonlinearities, where the structure of unstable manifolds is less explicit.
4. Numerical exploration. The constructive nature of our results suggests the possibility of designing numerical experiments to visualize the collapse of extrema and the formation of prescribed blow-up profiles.

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## References


1. Fujita, H. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I* **1966**, *13*, 109–124.
2. Giga, Y.; Kohn, R.V. Asymptotically self-similar blow-up of semilinear heat equations. *Commun. Pure Appl. Math.* **1985**, *38*, 297–319. [CrossRef]
3. Giga, Y.; Kohn, R.V. Characterizing blow-up using similarity variables. *Indiana Univ. Math. J.* **1987**, *36*, 1–40. [CrossRef]
4. Herrero, M.A.; Velázquez, J.J.L. Blow-up profiles in one-dimensional, semilinear parabolic problems. *Commun. Partial Differ. Equ.* **1992**, *17*, 205–219. [CrossRef]
5. Herrero, M.A.; Velázquez, J.J.L. Blow-up behaviour of one-dimensional semilinear parabolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1993**, *10*, 131–189. [CrossRef]
6. Merle, F.; Zaag, H. Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$. *Duke Math. J.* **1997**, *86*, 143–195. [CrossRef]
7. Filippas, S.; Kohn, R.V. Refined asymptotics for the blowup of $u_t - \Delta u = u^p$. *Comm. Pure Appl. Math.* **1992**, *45*, 821–869. [CrossRef]
8. Angenent, S. The zero set of a solution of a parabolic equation. *J. Reine Angew. Math.* **1988**, *390*, 79–96.
9. Schweyer, R. Type II blow-up for the four-dimensional energy-critical wave equation. *J. Funct. Anal.* **2012**, *263*, 3922–3983. [CrossRef]
10. Collot, C. Nonradial type II blow up for the energy-supercritical semilinear heat equation. *Anal. PDE* **2017**, *10*, 127–252. [CrossRef]
11. Merle, F.; Zaag, H. Solution to the semilinear wave equation with a pyramid-shaped blow-up surface. In *Séminaire Laurent Schwartz—EDP et Applications*; Exposé no. 6; CEDRAM: Grenoble, France, 2017; 13p.
12. Merle, F.; Raphaël, P.; Rodnianski, I. Type II blow-up for the energy-critical wave equation. *Acta Math.* **2019**, *223*, 179–307.
13. Raphaël, P. On universality of blow-up profiles for $L^2$-critical nonlinear dispersive equations. *Bull. Am. Math. Soc.* **2022**, *59*, 481–510.
14. Filippas, S.; Liu, W. On the blowup of multidimensional semilinear heat equations. *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **1993**, *10*, 313–344. [CrossRef]
15. Yanagida, E. Uniqueness of similarity solutions of the semilinear heat equation. *J. Differ. Equ.* **1991**, *102*, 309–321.
16. Henry, D. *Geometric Theory of Semilinear Parabolic Equations*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1981; Volume 840.
17. Fiedler, B.; Mallet-Paret, J. A Poincaré–Bendixson theorem for scalar reaction–diffusion equations. *Arch. Ration. Mech. Anal.* **1989**, *107*, 325–345. [CrossRef]
18. Matano, H. Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **1982**, *29*, 401–441.
19. Collot, C.; Ghoul, T.-E.; Masmoudi, N.; Nguyen, V.T. Spectral analysis for singularity formation of the two-dimensional Keller–Segel system. *Ann. PDE* **2022**, *8*, 5. [CrossRef]
20. Duong, G.K.; Ghoul, T.-E.; Zaag, H. Gradient blowup profile for the semilinear heat equation. *Discret. Contin. Dyn. Syst. A* **2024**, *44*, 997–1025. [CrossRef]
21. Harada, J. Existence of blowup solutions to the semilinear heat equation with double power nonlinearity. *Calc. Var. Partial Differ. Equ.* **2024**, *63*, 30. [CrossRef]


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