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# On Some Incommensurate Fractional-Order Reaction–Diffusion Systems: The Degn–Harrison and Its Stability

**Omar Kahouli [1,*], Amel Hioual [2], Adel Ouannas [2,3], Waleed Mohammed Abdelfattah [4,5], Younès Bahou [6], Ilyes Abidi [7], Sameir Hamed [1], Mohamed Chaabane [8] and Sarra Elgharbi [9]**

[1] Department of Electronics Engineering, Applied College, University of Ha'il, P.O. Box 2440, Ha'il 81451, Saudi Arabia; sa.hamed@uoh.edu.sa
[2] Department of Mathematics and Computer Science, University of Oum El Bouaghi, Oum El Bouaghi 04000, Algeria; amelhioual4@gmail.com (A.H.); ouannas.adel@univ-oeb.dz (A.O.)
[3] Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia
[4] College of Engineering, University of Business and Technology, Jeddah 23435, Saudi Arabia; w.abdelfattah@ubt.edu.sa
[5] Department of Engineering Mathematics and Physics, Faculty of Engineering, Zagazig University, Zagazig 44519, Egypt
[6] Computer Science Department, Applied College, University of Ha'il, P.O. Box 2440, Ha'il 81451, Saudi Arabia; y.bahou@uoh.edu.sa
[7] Department of Management Information Systems, Applied College, University of Ha'il, P.O. Box 2440, Ha'il 81451, Saudi Arabia; a.ilyes@uoh.edu.sa
[8] Department of Electrical Engineering, National Engineering School of Sfax (ENIS), University of Sfax, Sfax 3038, Tunisia; chaabane.ucpi@gmail.com
[9] Chemistry Department, College of Science, University of Ha'il, P.O. Box 2440, Ha'il 81451, Saudi Arabia; sai.elgharbi@uoh.edu.sa
* Correspondence: a.kahouli@uoh.edu.sa

**Abstract**

In this paper, we consider a reaction–diffusion system governed by incommensurate fractional time derivatives based on the Degn–Harrison model. Its formulation incorporates various memory effects on axial position through Caputo derivatives of variable orders, producing a more realistic modeling of the temporal dynamics. This paper starts with a study of the spatially homogeneous system and establishes conditions for local stability by using the Matignon criterion. The spectral decomposition method under Neumann boundary condition is then applied to study the complete reaction–diffusion system and describe diffusion-induced instabilities. Our results indicate that the noninteger fractional orders lead to significant changes in stability regions, as well as the initiation of pattern formation. Specifically, the orders of fractions induced as a control variable are regarded to be effective in controlling the stability of the system, thus they are global (or positive) control variables when their values achieved at some levels apply to the entire saturation, etc. Our numerical simulations are in excellent agreement with the theoretical predictions and show that memory asymmetry induces complex spatiotemporal dynamics not seen for classical integer-order systems.

**Keywords:** incommensurate fractional-order systems; Caputo operator; Degn–Harrison model; reaction–diffusion systems; local asymptotic stability; nonlinear biochemical dynamics; numerical simulations



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## 1. Introduction

For several decades, reaction–diffusion systems have occupied a central position in the mathematical modeling of complex spatiotemporal phenomena arising across chemistry, biology, and ecology.

Such systems are often modeled by coupled nonlinear partial differential equations, which account for the interaction of local reaction kinetics and spatial diffusion. This relatively simple interaction can lead to nontrivial macroscopic behaviors on the levels of oscillations, wave propagation and pattern formation. Of these classical models developed in this context, the Degn–Harrison system [1] is an important contribution that was originally proposed for describing oscillatory dynamics observed experimentally in the respiratory metabolism of the bacterium Klebsiella aerogenes grown under continuous reactor conditions. What sets this model apart is its incorporation of feedback inhibition whereby detailed kinetics create oscillations, either persistent periodic or quasi-periodic, which has been used as a standard for understanding biochemical rhythms ever since.

Even though traditional reaction–diffusion models have demonstrated their predictive capability, the use of integer-order derivatives implies that future evolution is solely determined by states in the present. This Markovian point of view, while mathematically convenient, ignores any memory and hereditary effects which are inherent to a large number of real systems. In many physical and biological processes, present dynamics depend on not only the current state of a system but also its full history. Using fractional calculus is a natural and mathematically sound approach for implementing these effects. Specifically, derivatives of noninteger order (e.g., the Caputo derivative) enable modeling long-range temporal correlations and anomalous transport phenomena in one coherent approach [2,3]. These operators have been found to be effective in characterizing subdiffusive behavior, viscoelastic responses, and complex intracellular processes where memory is intrinsic in nature (e.g., the Brownian motion of particles in a viscous medium, stems from memory) [4,5].

We conclude that it cannot be regarded as a formal generalization just applied on classical reaction–diffusion systems but represents an essential benchmark for more realistic dynamical processes extending the well-known classical setting to fractional dynamics; modeling, system stability and control of nonlinear FRDEs will transform classic r-d equations into their fractal version. In this way, incommensurate fractional-order systems—that is, those that evolve different state variables by distinct fractional orders—impart yet greater flexibility. These models are particularly well-suited for modeling heterogeneous systems where some components' memories (depth/temporal responses) differ person from each other. This scenario arises naturally in systems where there are organized multi-scale interactions and heterogeneous internal structures in biochemical [6] and physical systems [7–9]. Together with diffusion, such incommensurate fractional dynamics can give rise to a rich diversity of behaviors from damped oscillations to nontrivial spatiotemporal patterns and up to chaotic regimes [10].

Experimental observations further substantiate fractional and incommensurate models. Increasing numbers of studies indicate that anomalous diffusion, subdiffusive transport and heterogeneous mobility exist in complex media including living cells and other porous materials [11,12]. These results further justify the consideration of fractional formulations as providing a more accurate description of the underlying dynamics vs. classical integer-order models that may neglect relevant system features [13–17]. Accordingly, in this corner weariusiter generalization of the Degn–Harrison model can include memory-driven effects as well as spatial heterogeneity in the same single framework [18–22]. Fractional versions of the Degn–Harrison system have already been examined in some recent work. In particular, the study in [23] also explores finite-time stability and synchronization properties in a fractional-order framework. Although these results add value to our understanding of time-dependent dynamics, memory heterogeneity may play a less well understood role in spatially distributed systems. However, the effects of incommensurate fractional orders and diffusion on stability and pattern formation are still not characterized.

The current paper tries to fill this lack by a detailed systematic analysis of an incomensurate fractional Degn–Harrison reaction–diffusion model. Unlike [23], this work is centered on diffusion-induced instability and spatial pattern formation involving heterogeneous memory effects. Introducing the special fractional orders for individual components allows the model to reproduce not only asymmetric memory responses, but also appreciably more nonlinear and complex dynamical patterns.

Specifically, the main contributions of this work are the following. We start by deriving local stability criteria for the spatially homogeneous fractional system, via a Matignon-type criterion in the incommensurate case. Second, we push the analysis into the reaction–diffusion framework by using spectral decomposition based on the eigensystem (the Laplacian with Neumann boundary conditions) to characterize diffusion-driven instabilities in detail. We also demonstrate, analytically and numerically, that the fractional orders are an essential tuning parameter that dictates both stability regions and regulates the onset of pattern formation. In particular, the study demonstrates that an asymmetry in the time scale of memory is a central mechanism controlling transitions between stable and unstable regimes.

The remainder of the paper is organized as follows. Section 2 recalls the main tools from fractional calculus required for the analysis, with particular emphasis on the Caputo derivative. Section 3 introduces the incommensurate fractional Degn–Harrison model and investigates its local dynamics in the absence of diffusion. Section 4 extends the study to the reaction–diffusion setting and derives conditions for the stability of the spatially distributed system. Finally, Section 5 presents numerical simulations that illustrate the theoretical results and highlight the influence of fractional orders and diffusion on the observed dynamics.

## 2. Preliminaries

In this section, we review some basic concepts in fractional calculus which play a crucial role in the following analysis. The Riemann–Liouville and Caputo fractional derivatives are focused on, since these two operators allow for formulating and analyzing the stability of the incommensurate system addressed throughout this work. In addition to these definitions, we provide key inequalities and stability conditions that are subsequently used to characterize local behavior of equilibrium points. In order to keep consistency with notation throughout this manuscript, we focus the presentation on one-dimensional time domains while focusing the assumptions necessary for well posedness of each operator.

**Definition 1** ([23])**.** *The Riemann–Liouville fractional derivative of a function $f(t)$, integrable over the interval $[t_0, t]$, and of order $\kappa > 0$, is defined as*

$$ {}_{t_0}D_t^{-\kappa} f(t) = \frac{1}{\Gamma(\kappa)} \int_{t_0}^{t} (t-\tau)^{\kappa-1} f(\rho)\, d\rho, \tag{1}$$

*where the function $\Gamma(\kappa)$ denotes the classical Gamma function:*

$$\Gamma(\kappa) = \int_0^\infty e^{-t} t^{\kappa-1} dt.$$

The operator above should be taken to mean a fractional integral of order $\kappa$. It extends the classical $n$-fold integral to noninteger order as well as a memory kernel of the form $(t-\rho)^{\kappa-1}$ suitable for hereditary behaviors, stemming from a system's past.

**Definition 2** ([23])**.** *Let $f$ be a function which has $n$ continuous derivatives on $(t_0, t)$ such that*

$$n = \min\{k \in \mathbb{N} \mid k > \kappa\}.$$

*The Caputo fractional derivative of order $\kappa > 0$ of $f(t)$ is defined as*

$$ {}^{C}_{t_0}D^{\kappa}_{t} f(t) = \frac{1}{\Gamma(n-\kappa)} \int_{t_0}^{t} (t-\rho)^{n-\kappa-1} f^{(n)}(\rho)\, d\rho. \tag{2}$$

This operator is different from the Riemann–Liouville derivative in that it allows only integer-order derivatives for dealing with initial conditions, making it particularly well suited to physical and engineering models.

We now examine a two-dimensional nonautonomous incommensurate fractional-order dynamical system that is governed by Caputo derivatives of distinct orders:

$$\begin{cases} {}^{C}_{t_0}D^{\kappa_1}_{t} u(t) = F(u,v), \\ {}^{C}_{t_0}D^{\kappa_2}_{t} v(t) = G(u,v), \end{cases} \qquad t > t_0. \tag{3}$$

A point $(u_{eq}, v_{eq})$ is an equilibrium of this system if and only if it satisfies the algebraic conditions

$$F(u_{eq}, v_{eq}) = 0, \qquad G(u_{eq}, v_{eq}) = 0. \tag{4}$$

The subsequent fractional inequality serves a role similar to that of the classical chain rule for the derivative of squared term. This approach is often used in designing Lyapunov functions to prove stability for fractional-order systems.

**Lemma 1** ([23])**.** *Let $u(t)$ be a real-valued function that is continuous and differentiable on $[t_0, t]$. Then, for any $\delta \in (0, 1]$, the inequality*

$${}^{C}_{t_0}D^{\kappa}_{t} [u(t)]^2 \leq 2u(t)\, {}^{C}_{t_0}D^{\kappa}_{t} u(t) \tag{5}$$

*holds for all $t > t_0$.*

To determine local stability of the equilibrium point, we analyze the spectrum of Jacobian matrix corresponding to one system (3). The following lemma gives the fractional version of the classical linearization theorem.

**Lemma 2** ([24])**.** *Let $(u_{eq}, v_{eq})$ be an equilibrium point of system (3). Then $(u_{eq}, v_{eq})$ is locally asymptotically stable iff all eigenvalues $\lambda_i$ of the Jacobian matrix $J(u_{eq}, v_{eq})$ fulfill*

$$|\arg(\lambda_i)| > \frac{\beta\pi}{2}, \qquad i = 1, 2, \tag{6}$$

*where $\beta = \max\{\kappa_1, \kappa_2\}$ and $\arg(\cdot)$ denotes the complex argument of the eigenvalue.*

This condition generalizes the classical stability requirement by limiting the eigenvalues to a sector whose opening angle is determined by the system's highest fractional order.

We finish with a Lyapunov-type stability theorem for systems with fractional-order derivatives that are not equal. This result is useful for checking the long-term behavior of the equilibria we looked at before.

**Theorem 1** ([25]). *Consider the zero solution $x = 0$ of system (3). Suppose that for all $x \in \mathbb{R}^n$, the inequality*

$$\sum_{i=1}^{n} I_t^{\alpha_i - \beta}[x_i(t) f_i(t, x(t))] \leq 0 \tag{7}$$

*holds. Then the trivial solution $x = 0$ is stable.*

*Furthermore, if*

$$\sum_{i=1}^{n} I_t^{\alpha_i - \beta}[x_i(t) f_i(t, x(t))] < 0 \qquad \text{for all } x \neq 0, \tag{8}$$

*then the zero solution is asymptotically stable.*

The above theorem extends classical Lyapunov conditions into the fractional setting allowing for fractional integrals $I_t^{\alpha_i - \beta}$ of mixed orders. This framework enables us to consider the dynamic stability of systems whose evolved components have different memory indices, covering a wide class of fractional dynamical behaviors that are relevant for the remainder of this work.

This section introduces the concepts and tools of fractional-order systems that provide the mathematical foundation for their study. These results are implemented below to determine the incommensurate fractional Degn–Harrison model and study its local dynamics without diffusion.

## 3. The Fractional Degn–Harrison Model with Incommensurate Orders

Based on the presented above fractional calculus framework, we now formulate the incommensurate fractional Degn–Harrison model as follows. First, we analyze in the spatially homogeneous setting to disentangle the memory effects concerning the intrinsic dynamics of the system.

Hence, we seek an extension of a classical reaction–diffusion model which describes oscillatory behavior in microbial cultures. We specifically examine the system described by Degn and Harrison to model the respiratory oscillations which occur in continuous fermentative processes with *Klebsiella aerogenes*. These oscillations, which can arise from the regulation of metabolism and synchronization in microbial colonies, are explained using a minimal three-step chemical kinetics scheme driven by inhibitory feedback.

First, we try to approximate and discretize this system using two classical numerical methods, tailored, respectively, for the structure of the nonlinear terms and boundary conditions considered. Notably, our study also introduces a discrete model that is the first of its type to be published into ever-growing mathematical biology literature (to the best of our knowledge), in particular as it revolves around time-fractional constructs.

The original continuous model is based on the interaction between two main variables, which are $u(x,t)$, a reactive metabolite, and $v(x,t)$, an enzyme inhibitor or interacting species. These variables evolve within a bounded spatial domain $\Omega \subset \mathbb{R}^n$ (with $n \geq 1$), under homogeneous Neumann boundary conditions, enforcing our assumption of no-flux across the domain boundaries. This is a manifestation of the physical reality that neither $u$ nor $v$ can diffuse away from the reactor or biological medium.

The system suggested by Degn and Harrison can be expressed as the following set of coupled nonlinear reaction–diffusion partial differential equations:

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$$
\begin{cases}
\dfrac{\partial u}{\partial t} = k_1 \Delta u + a - u - \dfrac{vu}{1+qu^2}, & x \in \Omega,\ t > 0, \\
\dfrac{\partial v}{\partial t} = k_2 \Delta v + b - \dfrac{vu}{1+qu^2}, & x \in \Omega,\ t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) > 0, & x \in \Omega.
\end{cases} \tag{9}
$$

Here, $k_1, k_2 > 0$ represent the diffusion coefficients associated with the species $u$ and $v$, respectively. The operator $\Delta$ denotes the Laplacian in $\mathbb{R}^n$, defined as $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, and the boundary condition $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative, enforcing zero-flux on the boundary $\partial\Omega$. The constants $a$ and $b$ are source terms related to the production or external input of each chemical species, while the nonlinear reaction term $\frac{vu}{1+qu^2}$ captures an inhibitory feedback mechanism with strength controlled by the parameter $q > 0$.

This model encodes a class of subject-like biochemical feedback loops featuring inhibitory saturation, which are common in metabolic networks, notably present in oscillatory or excitable systems. This nonlinear coupling causes the oscillations and spatiotemporal activity to emerge.

Recent years have seen an increasing interest in the extension of classical models of reaction–diffusion systems to include memory effects and anomalous diffusion, in particular using tools from fractional calculus. Correspondingly, fractional-order differential operators in time and space have been proven to be a potential tool that can embrace long-range temporal features as well as hereditary behaviors inherent to numerous real-world nonlinear phenomena in sectors such as biology, chemistry and physics.

In line with this approach, in [23], the authors presented a variable-order version of the Degn–Harrison system by replacing the traditional first-order time derivative $\frac{\partial}{\partial t}$ with a Caputo-type fractional derivative defined as $D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s) n^{\eta-1} f(s) ds$. Doing so leads to the following revised system:

$$
\begin{cases}
{}^{C}D_t^{\kappa_1} u(x,t) = k_1 \Delta u + a - u - \dfrac{vu}{1+qu^2}, & x \in \Omega,\ t > 0, \\
{}^{C}D_t^{\kappa_2} v(x,t) = k_2 \Delta v + b - \dfrac{vu}{1+qu^2}, & x \in \Omega,\ t > 0, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega.
\end{cases} \tag{10}
$$

where

- ${}^{C}D_t^{\kappa}$ denotes the Caputo fractional derivative of order $\kappa \in (0,1]$,
- $u(x,t)$ and $v(x,t)$ represent the concentrations of the reactive species at location $x \in \Omega \subset \mathbb{R}^n$ and time $t$,
- $k_1, k_2 > 0$ are the diffusion coefficients for species $u$ and $v$, respectively,
- $a, b > 0$ are constant source terms,
- $q > 0$ characterizes the strength of the inhibitory nonlinearity.

The nonlinear term $\frac{uv}{1+qu^2}$ represents a saturable inhibition mechanism that incorporates the feedback regulation into the dynamics. It keeps reaction rates within limits at extreme concentrations and imitates actual biochemical inhibition behaviors.

The system is subject to Neumann (zero-flux) boundary conditions, meaning that there is no exchange of the chemical species through the boundary of the spatial domain:

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \tag{11}$$

where $\nu$ denotes the outward unit normal vector to the boundary $\partial\Omega$. This reflects physical isolation of the domain, such as a closed reactor environment.

The initial conditions are defined as:

$$u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) > 0, \quad \forall x \in \Omega, \tag{12}$$

where $u_0(x)$ and $v_0(x)$ are sufficiently smooth and strictly positive initial profiles for the reactive species.

Such a fractional framework permits one to investigate more complex behavior such as subdiffusive dynamics, time-lagged synchronization and fractional Turing patterns that integer-order models do not embody.

In this section, we carry out a thorough analysis of the fractional Degn–Harrison reaction–diffusion system, defined by (10) above, including its asymptotic behavior. We focus on the conditions ensuring that, with time, all trajectories of the system converge to a spatially homogeneous steady state and hence the global asymptotic stability of equilibrium configuration.

To this end, we conduct a two-step analytical procedure. We perform a local stability analysis of the equilibrium state using the eigenfunction expansion method. This allows for a decomposition of the system's dynamics into spatial modes and enables a modal stability analysis. Second, we apply a direct Lyapunov approach specifically developed for discrete-time systems with memory and taking into account fractional-order difference operators to derived conditions which guarantee global asymptotic stability.

We start with determining the equilibrium point of this system and identify it with the spatially homogeneous time-invariant solution $(u_{eq}, v_{eq})$ of fractional reaction–diffusion equations. Assuming spatial homogeneity and denoting $\Delta^2 u_{eq} = \Delta^2 v_{eq} = 0$, the following nonlinear algebraic system yields the equilibrium values:

$$\begin{cases} a - u_{eq} - \dfrac{v_{eq}u_{eq}}{1+qu_{eq}^2} = 0, & x \in \Omega,\ t > 0, \\ b - \dfrac{v_{eq}u_{eq}}{1+qu_{eq}^2} = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{13}$$

By considering the spatially homogeneous scenario (i.e., $\Delta u_{eq} = \Delta v_{eq} = 0$), the above reduces to:

$$\begin{cases} a - u^* - \dfrac{u_{eq}v_{eq}}{1+q(u_{eq})^2} = 0, \\ \qquad b - \dfrac{u_{eq}v_{eq}}{1+q(u_{eq})^2} = 0. \end{cases} \tag{14}$$

Solving the second equation yields an explicit expression for the nonlinear interaction term:

$$\frac{u_{eq}v_{eq}}{1+q(u_{eq})^2} = b.$$

Substituting this expression into the first equation leads to:

$$a - u_{eq} - b = 0 \quad \Rightarrow \quad u_{eq} = a - b.$$

Subsequently, using this value in the nonlinear interaction term, we find the corresponding equilibrium for $v_{eq}$ as:

$$v_{eq} = \frac{b\left(1+q(u_{eq})^2\right)}{u_{eq}} = \frac{b\left[1+q(a-b)^2\right]}{a-b}.$$

Thus, the unique spatially homogeneous equilibrium point $(u_{eq}, v_{eq}) \in \mathbb{R}^2$ of the discrete-time fractional Degn–Harrison system is given by:

$$(u_{eq}, v_{eq}) = \left(a-b,\ \frac{b\left[1+q(a-b)^2\right]}{a-b}\right), \tag{15}$$

which is consistent with the previously established result in [23]. This equilibrium is a critical feature of the stability analysis, as it is used to identify displacement around which perturbations dissipate.

The earlier analysis is focused on temporal changes in the system but does not include spatial interactions. In the next section, we expand our study for diffusion effects creating a reaction–diffusion system.

## 4. Stability Result for the Free Diffusion Model

In this section, we examine the linearization of the fractional reaction–diffusion equations around to determine local stability around its equilibrium state.

The linearization step separates the leading behavior of small perturbations and sets up how those perturbations evolve in time. Using an eigenfunction expansion related to the Laplace operator with homogeneous Neumann boundary conditions, we separate the spatial part of the system into a series of orthogonal modes. Each mode evolves separately according to a scalar fractional differential equation whose dynamics depend explicitly on the associated Laplacian eigenvalue. Appropriate boundary conditions lead to the full reaction–diffusion problem being reduced to a family of fractional subsystems whose characteristic equation permits productivity stability assessment for each spatial mode directly.

Below, we perform this linear stability analysis for the fractional Degn–Harrison system. In particular, we obtain the Jacobian matrix related to the reaction terms and compute its value at the homogeneous equilibrium point $(u_{eq}, v_{eq})$ given in Equation (15). The remaining properties of this Jacobian, along with fractional system orders and Laplacian eigenvalues, give rise to explicit conditions that ensure local asymptotic stability of the equilibrium. This understanding provides an important link between the local reaction kinetics, the memory properties of fractional derivatives and the spatial diffusion mechanisms governing the global dynamics of the system.

We denote the Jacobian $J$ of a linear(sed) system surrounding the steady state $(u_{eq}, v_{eq})$. Next, we insert the previously obtained equilibrium expressions and obtain the Jacobian in explicit form:

$$J = \begin{pmatrix} -\dfrac{a+q(a-b)^2(a-2b)}{(a-b)(1+q(a-b)^2)} & -\dfrac{a-b}{1+q(a-b)^2} \\ 1-\dfrac{a+q(a-b)^2(a-2b)}{(a-b)(1+q(a-b)^2)} & -\dfrac{a-b}{1+q(a-b)^2} \end{pmatrix}. \tag{16}$$

**Theorem 2.** *Let the Jacobian matrix $J$ be defined as in (16). Then the equilibrium point $(u_{eq}, v_{eq})$ of the free diffusion system is locally asymptotically stable if one of the following conditions holds:*

1. *If*

$$(a-b)^2 \leq 4\Big[a+(a-b)^2\big(q(a-2b)+1\big)\big(1+q(a-b)^2\big)\Big] \quad and \quad \mathrm{tr}(J)<0,$$

*then all eigenvalues of J have negative real parts.*

2. *If*

$$(a-b)^2 > 4\Big[a+(a-b)^2\big(q(a-2b)+1\big)\big(1+q(a-b)^2\big)\Big],$$

*then the eigenvalues are complex conjugates with modulus less than one and negative real parts, implying asymptotic stability.*

**Proof.** To assess stability, we compute the characteristic polynomial associated with the Jacobian matrix $J$:

$$\mu^2 - \mathrm{tr}(J)\mu + \det(J) = 0, \tag{17}$$

where the trace and determinant of $J$ are given explicitly by:

$$\begin{aligned} \mathrm{tr}(J) &= -\frac{a-b}{1+q(a-b)^2}, \\ \det(J) &= \frac{-a+(a-b)^2(q(a-2b)+1)}{(a-b)(1+q(a-b)^2)}. \end{aligned} \tag{18}$$

The discriminant $\Delta_\mu$ of the characteristic equation is:

$$\Delta_\mu = \mathrm{tr}(J)^2 - 4\det(J). \tag{19}$$

We now analyze several possible cases based on the sign of $\Delta_\mu$ and $\mathrm{tr}(J)$.

Case 1: $\Delta_\mu > 0$

In this scenario, the characteristic equation has two distinct real roots:

$$\mu_{1,2} = \frac{\mathrm{tr}(J) \pm \sqrt{\Delta_\mu}}{2}. \tag{20}$$

If $\mathrm{tr}(J) < 0$, then both $\mu_1$ and $\mu_2$ are real and negative, implying that both eigenvalues lie within the left half of the complex plane. Consequently, the equilibrium is asymptotically stable.

On the contrary, if $\mathrm{tr}(J) > 0$, then at least one of the eigenvalues is positive, leading to instability.

Case 2: $\Delta_\mu < 0$

In this case, the eigenvalues are complex conjugates with nonzero imaginary parts:

$$\mu_{1,2} = \frac{\mathrm{tr}(J)}{2} \pm i\frac{\sqrt{-\Delta_\mu}}{2}. \tag{21}$$

The argument of each eigenvalue is:

$$\arg(\mu_{1,2}) = \arctan\left(\frac{\sqrt{-\Delta_\mu}}{\mathrm{tr}(J)}\right).$$

If $\mathrm{tr}(J) < 0$, then the real part of both eigenvalues is negative, and the system remains asymptotically stable. The oscillatory nature of the solution stems from the imaginary parts.

Case 3: $\Delta_\mu = 0$

This leads to repeated real eigenvalues:

$$\mu_1 = \mu_2 = \frac{\mathrm{tr}(J)}{2}.$$

If $\mathrm{tr}(J) < 0$, then both eigenvalues are negative, and the system is stable. If $\mathrm{tr}(J) > 0$, the system is unstable.

Degenerate case: $\mathrm{tr}(J) = 0$

Then $\Delta_\mu = -4\det(J) < 0$, which implies complex conjugate eigenvalues lying on the imaginary axis. Their arguments satisfy $|\arg(\mu_{1,2})| = \frac{\pi}{2}$, and as long as the modulus of the eigenvalues is less than one, the system is marginally or asymptotically stable depending on higher-order terms. □

## 5. Stability with Diffusion Effects

Now we move onto the entire reaction–diffusion generalized Ngai model with incommensurate fractional terms. Using diffusion in conjunction with heterogeneous memory effects, we explore various scenarios that can potentially give rise to spatial instabilities and what those conditions look like.

We wish to show that the steady state $(u_{eq}, v_{eq})$ of the corresponding reaction–diffusion system is locally asymptotically stable when diffusion is included under appropriate conditions on the parameters. We first study the spectral properties of the differential operator given below:

$$\Delta^2 u(x,t) + \lambda_i u(x,t) = 0, \tag{22}$$

subject to periodic boundary conditions:

$$u(x,0) = u_0(x). \tag{23}$$

Subsequently, the system can be rewritten in component form as:

$$\begin{cases} {}^C D_t^{\kappa_1} u(x,t) &= -k_1\lambda_i u + a - \dfrac{vu}{1+qu^2}, \quad x \in \Omega,\ t > 0, \\ {}^C D_t^{\kappa_2} v(x,t) &= -k_2\,\lambda_i v + b - \dfrac{vu}{1+qu^2}, \quad x \in \Omega,\ t > 0, \end{cases} \tag{24}$$

Linearizing Equation (24) around the steady state $(u_{eq}, v_{eq})$, we obtain the Jacobian matrix $J_i$ given by:

$$J_i = \begin{pmatrix} -k_1\lambda_i - A & -B \\ C & -k_2\lambda_i - D \end{pmatrix}, \tag{25}$$

where the constants $A$, $B$, $C$, and $D$ are expressed as follows:

$$\begin{aligned} A &= \frac{a + q(a-b)^2(a-2b)}{(a-b)(1+q(a-b)^2)}, \\ B &= \frac{a-b}{1+q(a-b)^2}, \\ C &= \frac{a-b}{1+q(a-b)^2}, \\ D &= \frac{a + q(a-b)^2(a-2b)}{(a-b)(1+q(a-b)^2)}. \end{aligned}$$

**Theorem 3.** *The steady state $(u_{eq}, v_{eq})$ of the full reaction–diffusion system is locally asymptotically stable if the following conditions are met:*

- *When $k_1 < k_2$, the following inequalities must hold:*

- $tr(J_i) < 0$,
- $\Delta_\mu > 0$,
- $k_1\lambda_i \geq \dfrac{a + q(a-b)^2(a-2b)}{(a-b)(1+q(a-b)^2)}$.

- *When $k_1 > k_2$, asymptotic stability is ensured if:*
  - $k_1\lambda_i \geq \dfrac{a + q(a-b)^2(a-2b)}{(a-b)(1+q(a-b)^2)}$ *and*
  - *the eigenvalues $\Lambda_j(\lambda_i)$, defined by*

$$\Lambda_j(\lambda_i) = \frac{tr(J_i) \pm \sqrt{tr(J_i)^2 - 4\det(J_i)}}{2}, \quad j = 1, 2,$$

*satisfy the angular condition $|Arg(\Lambda_j(\lambda_i))| > \alpha\pi/2$.*

**Proof.** To analyze the linear stability, we evaluate the eigenvalue problem of the perturbed linear system. Suppose $(\Phi, \Psi)$ is an eigenfunction associated with eigenvalue $\Lambda$; then, the decomposition $u_i = \sum_{j=1}^{n} \xi_{ij}\Phi_{ij}$ and $v_i = \sum_{j=1}^{n} \delta_{ij}\Psi_{ij}$ yields:

$$\begin{cases} \sum\limits_{j=1}^{n} \xi_{ij}\, {}^{C}D_t^{\kappa_1}\Phi_{ij} = k_1^2\lambda_i \sum\limits_{j=1}^{n} \xi_{ij}\Phi_{ij} + \dfrac{a - \sum \xi_{ij}\Phi_{ij} - \left(\sum \xi_{ij}\Phi_{ij}\right)\left(\sum \delta_{ij}\Psi_{ij}\right)}{1 + q\left(\sum \xi_{ij}\Phi_{ij}\right)^2}, \\ \sum\limits_{j=1}^{n} {}^{C}D_t^{\kappa_2}\Psi_{ij} = k_2^2\lambda_i \sum\limits_{j=1}^{n} \delta_{ij}\Psi_{ij} + \dfrac{b - \left(\sum \xi_{ij}\Phi_{ij}\right)\left(\sum \delta_{ij}\Psi_{ij}\right)}{1 + q\left(\sum \xi_{ij}\Phi_{ij}\right)^2}. \end{cases}$$

This system leads to the matrix equation:

$$(J_i - \Lambda(\lambda_i)I)\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = 0, \tag{26}$$

with the corresponding characteristic polynomial:

$$\Lambda^2(\lambda_i) - \mathrm{tr}(J_i)\Lambda(\lambda_i) + \det(J_i) = 0. \tag{27}$$

The trace and determinant are given, respectively, by:

$$\mathrm{tr}(J_i) = -\left(\frac{3\Delta k_1^2 + \Delta k_2^2}{4}\right)\lambda_i + \mathrm{tr}(J), \tag{28}$$

$$\det(J_i) = \frac{k_2}{\Delta x^2}\left[\left(\frac{3\Delta k_1^2}{4}\lambda_i - A\right)(\lambda_i + D) + BC\lambda_i\right] + \det(J). \tag{29}$$

We define the discriminant of (27):

$$\Delta_\lambda = (\mathrm{tr}(J_i))^2 - 4\det(J_i),$$
$$\Delta_\lambda = \left(\frac{3\Delta k_1^2 - \Delta k_2^2}{4}\right)^2 \lambda_i^2 + 2\left(\frac{3\Delta k_1^2 - \Delta k_2^2}{4}\right)(A - B)\lambda_i + \Delta_\mu.$$

Assuming $d_1 \neq d_2$, the discriminant $\Delta^* > 0$, ensuring real roots. We now consider two cases:

- **Case 1: $k_1 < k_2$.** If the inequality

$$2\left(\frac{3\Delta k_1^2 - \Delta k_2^2}{4}\right)(A - B) > 0$$

holds, then $\Delta_\lambda > 0$ for all $\lambda_i$. The eigenvalues are real:

$$\Lambda_1(\lambda_i) = \frac{\mathrm{tr}(J_i) - \sqrt{\Delta_\lambda}}{2},$$
$$\Lambda_2(\lambda_i) = \frac{\mathrm{tr}(J_i) + \sqrt{\Delta_\lambda}}{2}.$$

When $\mathrm{tr}(J) < 0$ and $\frac{k_1}{\Delta x^2}\lambda_i$ exceeds the given threshold, both eigenvalues are negative, guaranteeing asymptotic stability.

- **Case 2: $k_1 > k_2$.** The condition remains the same. If

$$k_1^2\lambda_i \geq \frac{a + q(a-b)^2(a-2b)}{(a-b)(1+q(a-b)^2)},$$

then $\det(J_i) > 0$ and both $\Lambda_1(\lambda_i)$ and $\Lambda_2(\lambda_i)$ are real and negative. Therefore, $(u^*, v^*)$ is again asymptotically stable.

□

Combined with the above analytical results, we obtain theoretical criteria for stability (diffusion-driven instability). Numerical simulations are provided in the next section to elaborate on these results and investigate the ensuing dynamics.

## 6. Numerical Simulations

In this section, we accompany the analysis with numerical experiments. Simulations are implemented to support the stability results and to show how fractional orders as well as diffusion coefficients affect spatiotemporal patterns.

**Example 1.** *We now shift to a case subject to the same spatial and temporal scales as we considered before α: a spatial domain $x \in [0, 20]$ and a temporal window $t \in [0, 10]$, discretized with steps of size $\Delta x = 0.5$,$\Delta t = 0.001$. The model parameters remain unchanged:*

$$(a, b, q, d_1, d_2) = (2, 1, 1, 0.1, 0.2),$$

*and the initial states are set as smooth oscillations:*

$$u(x,0) = 0.5 + 0.1\sin\left(\frac{x^2}{3}\right), \quad v(x,0) = 0.8 + 0.2\cos\left(x^3\right). \tag{30}$$

*The system is numerically integrated using zero-flux (Neumann) boundary conditions, and the Grünwald–Letnikov approximation is used to deal with time-fractional derivatives. The selected fractional orders for this experiment are:*

$$\kappa_1 = 0.98, \quad \kappa_2 = 0.75.$$

*For reproducibility, we outline details of the numerical scheme after standard printing through deep learning applied to StringNet, where we aim to impart sufficient detail. We replace the time-fractional derivative of order $\rho \in (0,1)$ with its approximating form through the Grünwald–Letnikov discretization where we can express*

$${}^{C}D_t^{\rho} u(t_n) \approx \frac{1}{\Delta t^{\rho}} \sum_{k=0}^{n} \omega_k^{(\rho)}\, u(t_{n-k}),$$

*where the coefficients* $\omega_k^{(\rho)}$ *are defined recursively by*

$$\omega_0^{(\rho)} = 1, \quad \omega_k^{(\rho)} = \left(1 - \frac{\rho + 1}{k}\right)\omega_{k-1}^{(\rho)}, \quad k \geq 1.$$

*The diffusion terms are approximated using standard second-order central finite differences with spatial step size* $\Delta x$ *for the spatial discretization. The resulting system is explicitly advanced in time over a sufficiently small interval* $\Delta t$ *to be stable and accurate.*

*Here, all simulations are carried out on a uniform mesh and the numerical tests guarantee that our results remain invariant under further refinement of the mesh.*

*The time evolution of the solutions are presented in Figures 1 and 2. The* $u$ *component, affected by a fractional derivative approaching the integer regime, remains to be oscillating in space and time coherently. On the other hand,* $v$ *means with some stronger memory (larger fractional order) have a visible damping of its dynamics over time. This lack of balance in the memory depth of the two variables gives rise to a type of* fractional dissipation asymmetry *resulting in mixed-mode behavior—one variable maintains oscillatory characteristics while the other mode moves toward stabilization.*

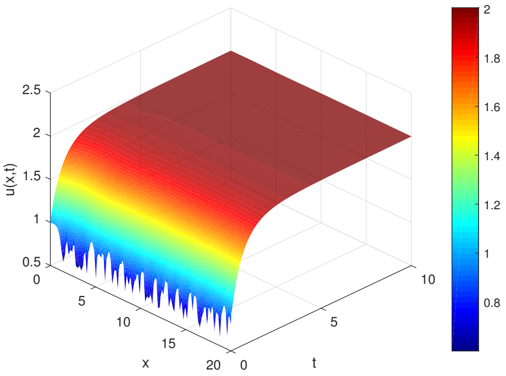


**Figure 1.** Spatio-temporal evolution of $u(x,t)$ with $\kappa_1 = 0.98$.

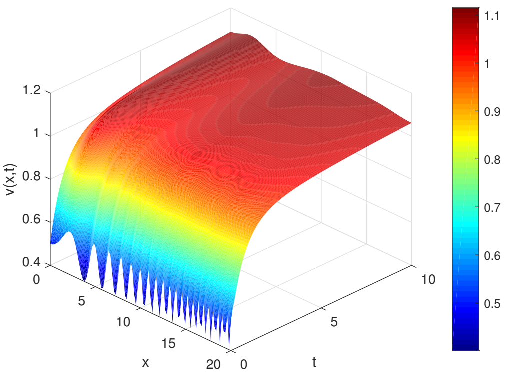


**Figure 2.** Spatio-temporal evolution of $v(x,t)$ with $\kappa_2 = 0.75$.

**Example 2.** *We now work with a 2D world and study how exactly incommensurate fractional orders affect Turing instability and the resultant pattern dynamics. The domain we consider is a square of length* 20*, and its discretized grid coincides with that from the one-dimensional simulations described earlier. The model parameters now look as follows:*

$$(a, b, q, d_1, d_2) = (0.05, 0.04, 0.2, 0.1, 0.3),$$

*with initial conditions composed of structured perturbations:*

$$u(x,0) = 5 + 3\cos(4x), \quad v(x,0) = 10 + 6\sin\left(x^2\right). \tag{31}$$

*The system is resolved utilizing a fractional numerical method on a two-dimensional grid with zero-flux boundary conditions. For the initial experiment, we choose:*

$$\kappa_1 = 0.6, \quad \kappa_2 = 0.7.$$

*Figures 3 and 4 display the corresponding evolution of $v(x,t)$. In the v-component, the emergent architectures appear more erratic and seem sequentially unsynchronized compared to classical integer-order case ($\kappa = 1$). These smeared patches are not the tessellated spandrels with oscillating peaks typical of normal stripes or dots, but rather spread out in space, and their centers drift slowly over time—an illusion of spatial incoherence due to fractional memory asymmetry. This instability remains, but not as a species of proliferating modes (that instigate diffusion powered instabilities), instead a time evolving and diffused embodiment.*

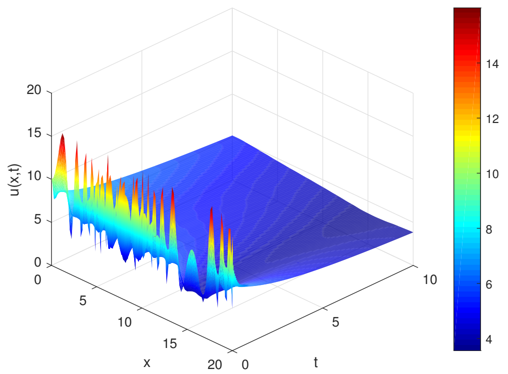


**Figure 3.** Emergent pattern of $u(x,t)$ for $\kappa_1 = 0.6$, $\kappa_2 = 0.7$.

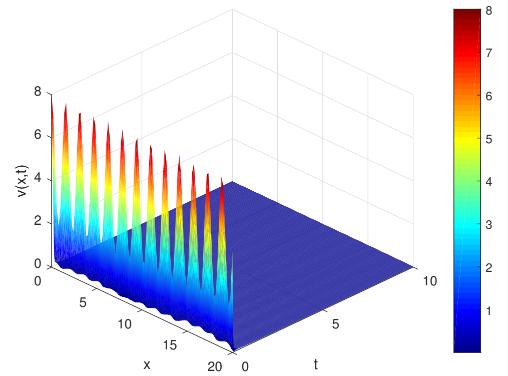


**Figure 4.** Alternative view of $v(x,t)$ under incommensurate fractional orders.

*These 2D experiments provide additional exemplification for the impact of fractional-order mismatch on pattern formation. This stems from the fact that incommensurate differentiation adds an entirely new degree of freedom, decoupling memory effects between components while modulating coherence, persistence and pattern morphology. These results suggest that fractional orders can act additionally as tunable parameters on the pattern selection and synchronization in complex reaction–diffusion systems.*

## 7. Conclusions and Perspectives

In this work, we presented a fractional-order Degn–Harrison reaction–diffusion model through Caputo's derivative which represents the incommensurate system. It is a generalization of the classical framework of superposition principle that accounts for distinct fractional order for each equation, thereby representing a nonuniform memory effect over all system dynamic. We established the existence and uniqueness of a homogeneous steady state as well as explicit conditions that guarantee local asymptotic stability, theoretically. Note: The results were made available through the temporal system and also as a spatially distributed sanctuary In particular. The analysis shows that diffusion has a strong impact on altering the spectral properties of the system and consequently giving rise to different stability thresholds. It resembles the subtle interplay of diffusion mechanisms and origami fractional memory.

The results compare favorably with the numerical and analytical results of the simulations, which show how change in fractional orders or diffusion coefficients affects this system. The experiments validate the results that demonstrate how the model can capture stable phases and transitions in dynamics owing to parameter perturbations, thus providing an understanding of memory-dependent system sensitivity. We provide a unifying theory for the stability of fractional-order reaction–diffusion systems of the Degn–Harrison type, which encompasses many nontrivial applications in modeling the spatial effects and built-in memory of biochemical reactions. Based on this work, some possible future directions would be to study the mechanism of pattern formation; whether to add a better approximation for variable-order fractional operators in the modeling; or to consider more realistic chemical/biological systems and apply our model.

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