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On Generalized Criterion for the Non-Isothermic Spacelike Surfaces

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Abstract

In this work, we examine the generalized criterion to determine a non-isothermic spacelike Bonnet surface in three-dimensional Lorentzian space L^3. For this aim, we generalized the criterion presented by Soyuçok in the Euclidean space to L^3. By applying the generalized criterion, we identify the non-isothermic spacelike Bonnet surfaces among the helicoid surfaces classified by Beneki et al.

Keywords: Bonnet surface; spacelike surface; spacelike helicoid; associate surface; A-net

1. Introduction

Bonnet investigated surfaces admitting non-trivial isometries that leave the principal curvatures invariant [1]. If surfaces S and S' are mapped isometrically onto each other while preserving their principal curvatures, they are associate surfaces, known as Bonnet surfaces. Thus, the curvature lines (C-net) of the surface S correspond to an orthogonal net (B-net) on S' and vice versa [2]. Roussos classified Bonnet surfaces in Euclidean 3-space into three groups: surfaces with constant mean curvature, surfaces with non-constant mean curvature that admit a one-parameter family of isometries, and surfaces with non-constant mean curvature that admit a single isometry; he also demonstrated that helicoidal surfaces in Euclidean 3-space are Bonnet surfaces and realize all three types [3].

The Bonnet problem has generally been considered in terms of isothermic surfaces, for which the lines of curvature form an isothermic orthogonal system (that is, a coordinate system with $E = G$, $F = 0$, $M = 0$ [4]), and non-isothermic surfaces, which do not satisfy this property, and has been studied by many researchers. Fujioka and Inoguchi classified timelike $\pm$ isothermic Bonnet surfaces using Hopf differentials, Lorentz-harmonic conditions, and Gauss–Codazzi equations [5]. Magid studied the isothermic conditions of timelike surfaces [6] and investigated Bonnet pairs of isothermic surfaces [7]. Fujioka and Inoguchi related the generalized Hazzidakis equations for HIMC surfaces defined by $\triangle\left(\frac{1}{H}\right)$, obtained by Bobenko, Eitner and Kitaev to spacelike Bonnet surfaces [8,9]. Fujioka and Inoguchi addressed spacelike surfaces with harmonic inverse mean curvature, introduced a classification of spacelike Bonnet surfaces with constant curvatures [8], and applied a similar approach for timelike surfaces [10] in [11]. Gaussian curvature, Hopf differential conditions, isothermal coordinates, and harmonic functions are given for the classification of spacelike Bonnet surfaces. In [12], spacelike Bonnet surfaces are classified according to specific geodesic curvature conditions. Inalcık and Ersoy investigated the conditions for timelike and spacelike helicoidal surfaces with non-null axis to be Bonnet surfaces in terms of differential equations [13].

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As known in E^3, if a net is parametrized as $E = G,\ F = 0$, it is an isothermic net [4,14]. Any net becomes an isothermic net by a scale transformation under the condition $\left(\ln \frac{E}{G}\right)_{uv} = 0$ [15] and when the curvature lines of the surface form an isothermal coordinate system ($M = 0$), so the surface is an isothermic surface [2,16]. Similarly, in L^3 if a net is parametrized as $E = \varepsilon G,\ F = 0$ on S given by the vectoral equation $\boldsymbol{x} = \boldsymbol{x}(u, v)$ then it is an isothermic net (for spacelike surfaces $\varepsilon = 1$, for timelike surfaces $\varepsilon = -1$) [14,17–19] and a surface is called an isothermic surface if it admits an isothermal system such that shape operator is diagonalized over $\mathbb{R}$ or $\mathbb{C}$ (i.e., $M = 0$) [7].

Soyuçok introduced an interesting criterion for the isothermic net [2]. According to the criterion, the necessary and sufficient condition for a surface to be a Bonnet surface is that the surface has an A-net such that: $E = G,\ F = 0,\ M = c = const. \neq 0$. Moreover, the coefficients of the fundamental forms of the associate surface S' are written directly as $E' = E,\ G' = G,\ F' = 0,\ L' = L,\ M' = -M,\ N' = N$. As a consequence of his criterion, he stated that helicoids are Bonnet surfaces [2,3]. Using this criterion, Kanbay studied ruled Bonnet surfaces in three-dimensional Euclidean space; Kanbay and Yüksekdağ generalized it to timelike surfaces, providing examples of timelike Bonnet helicoids [20–22]. Despite this, a comprehensive classification of spacelike Bonnet surfaces using the A-net criterion has not yet been presented in the literature.

This study contributes to the literature on spacelike Bonnet surfaces both chronologically and methodologically, by extending the A-net criterion, defined by Soyuçok in Euclidean space [2] and later generalized by Kanbay and Yüksekdağ for timelike surfaces in Lorentz space [22], to the case of spacelike surfaces. Although spacelike and timelike cases appear formally similar, the change in the Lorentz metric signature leads to a shift from hyperbolic to trigonometric structures in curvature expressions, preventing the direct application of criteria developed for timelike surfaces to the spacelike case. Therefore, expressing the A-net condition for spacelike surfaces requires a separate geometric approach.

In this study, we show that the transformation condition $\left(\ln \frac{E}{G}\right)_{uv} = 0$, which characterizes isothermic nets in E^3, remains valid in Lorentzian 3-space L^3. This allows a non-trivial extension of Soyuçok's A-net criterion to spacelike surfaces, thereby characterizing spacelike Bonnet surfaces. Finally, we show that spacelike helicoid pairs related by isometric mappings that preserve principal curvatures within the class classified by Beneki et al. [23] satisfy the generalized A-net criterion and hence form spacelike Bonnet pairs. The structure of the paper is as follows: Section 2 provides preliminary information, Section 3 introduces the basic definitions and theorems, and Section 4 contains examples illustrating the developed theory.

2. Preliminaries

Lorentzian three-space L^3 is defined as a vector space furnished with the Lorentzian inner product, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$ where $\boldsymbol{x}, \boldsymbol{y} \in R^3$. A vector $\boldsymbol{a} \in L^3$ is classified as spacelike when $\langle \boldsymbol{a}, \boldsymbol{a} \rangle > 0$ or $\boldsymbol{a} = 0$, timelike when $\langle \boldsymbol{a}, \boldsymbol{a} \rangle < 0$, and lightlike (null) when $\langle \boldsymbol{a}, \boldsymbol{a} \rangle = 0$ [24]. Unlike the Euclidean setting, the Lorentzian metric plays a fundamental role in geometric structures considered in this study, leading to a natural classification of vectors and surfaces as spacelike, timelike or lightlike. In particular, when the normal vector of a surface is timelike, the induced Lorentzian metric on the surface is positive definite, and hence, the surface is spacelike [25].

Consider the spacelike surface S given by $\boldsymbol{x} = \boldsymbol{x}(u, v)$ and assume that the spacelike coordinate lines $u = const.$ and $v = const.$ are orthogonal. We denote the unit tangent vectors and the arc lengths of the spacelike curves: an arbitrary spacelike curve (c), $v = const.(c_1)$, $u = const.(c_2)$ at the point P by $\boldsymbol{t}$, $\boldsymbol{t}_1$, $\boldsymbol{t}_2$ and s, s_1, s_2, respectively.

Here $\boldsymbol{t} = \boldsymbol{t}_1 \cos\varphi + \boldsymbol{t}_2\, sin\varphi$ where φ is the spacelike angle between $\boldsymbol{t}$ and $\boldsymbol{t}_1$. The normal curvature, geodesic curvature, and geodesic torsion of the coordinate curves satisfy the compatibility (Mainardi–Codazzi) equations, which describe the surface's geometric properties and ensure the consistency of its geometric structure.

At point P, the unit normal vector $\boldsymbol{N}$ varies with the arc length s_1 and s_2, and its derivatives are

$$\boldsymbol{N}_1 = \frac{d\boldsymbol{N}}{ds_1} = \left(-\frac{L}{E}\right)\boldsymbol{x}_1 + \left(-\frac{M}{\sqrt{EG}}\right)\boldsymbol{x}_2 \tag{1}$$

$$\boldsymbol{N}_2 = \frac{d\boldsymbol{N}}{ds_2} = \left(-\frac{M}{\sqrt{EG}}\right)\boldsymbol{x}_1 + \left(-\frac{N}{G}\right)\boldsymbol{x}_2 \tag{2}$$

where $\boldsymbol{N}$ is the unit normal vector and $E = \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle$, $G = \langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle$, $F = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = 0$, $w = \sqrt{EG - F^2}$, $\left(f_1 = \frac{f_u}{\sqrt{E}}, f_2 = \frac{f_v}{\sqrt{G}}\right)$. Let $[\boldsymbol{t}_1, \boldsymbol{g}_1, \boldsymbol{N}]$ and $[\boldsymbol{t}_2, \boldsymbol{g}_2, \boldsymbol{N}]$ be the Darboux trihedrons of the curves (c_1) and (c_2) at P on S. Here $\boldsymbol{N} = \boldsymbol{t}_1 \wedge \boldsymbol{t}_2$, $\boldsymbol{g}_1 = \boldsymbol{t}_2$ and $\boldsymbol{g}_2 = -\boldsymbol{t}_1$ [26]. In this case

$$\boldsymbol{N}_1 = \frac{d\boldsymbol{N}}{ds_1} = \kappa_n \boldsymbol{t}_1 + \tau_g \boldsymbol{g}_1, \boldsymbol{N}_2 = \frac{d\boldsymbol{N}}{ds_2} = \overline{\kappa_n}\boldsymbol{t}_2 + \overline{\tau_g}\boldsymbol{g}_2 \tag{3}$$

are written where κ_n, $\overline{\kappa_n}$ and τ_g, $\overline{\tau_g}$ are the normal curvatures and the geodesic torsions of the curves $v = const.$ and $u = const.$, respectively. From (1), (2) and (3) we get

$$\kappa_n = -\frac{L}{E},\ \tau_g = -\frac{M}{\sqrt{EG}},\ \overline{\kappa_n} = -\frac{N}{G},\ \overline{\tau_g} = \frac{M}{\sqrt{EG}}. \tag{4}$$

Moreover

$$\kappa_g = -\frac{\left(\sqrt{E}\right)_v}{\sqrt{EG}}, \overline{\kappa_g} = \frac{\left(\sqrt{G}\right)_u}{\sqrt{EG}} \tag{5}$$

are the geodesic curvatures of the coordinate lines $v = const.$ and $u = const.$, respectively [26]. From $f_{12} - f_1(\kappa_g) = f_{21} + f_2(\overline{\kappa_g})$ and Equations (1)–(3), (5), we can write the Mainardi–Codazzi Equations as follows:

$$(\kappa_n)_2 = \kappa_g(\kappa_n - \overline{\kappa_n}) + (\tau_g)_1 + 2\overline{\kappa_g}\tau_g, (\overline{\kappa_n})_1 = \overline{\kappa_g}(\kappa_n - \overline{\kappa_n}) + (\tau_g)_2 - 2\kappa_g\tau_g. \tag{6}$$

In addition, for a function $\phi(u, v)$, its Beltrami (Laplacian) second differential parameter, important for surface analysis, is described as [24]:

$$\Delta\phi = \phi_{11} + \phi_{22} + \phi_1\overline{\kappa_g} - \phi_2\kappa_g \tag{7}$$

3. Material and Method

As an initial step, we begin by considering the bisector network of the C-net, representing curvature lines, together with that of the B-net (Bonnet net), as the coordinate system on S. Let $\frac{\phi}{2}$ be the spacelike angle formed between the C-net and the curve $v = const.$. Similarly let us take $-\frac{\phi}{2}$ be the spacelike angle at which the B-net intersects the curve $v = const.$ Hence

$$\kappa_n = -H - J\cos\phi,\ \overline{\kappa_n} = -H + J\cos\phi,\ \tau_g = -\overline{\tau_g} = J\sin\phi \tag{8}$$

can be written. Here Gaussian curvature K and mean curvature H are $H = -\left(\frac{r+\bar{r}}{2}\right)$, $K = -r\bar{r}$, $J = -\left(\frac{r-\bar{r}}{2}\right)$ where r and $\bar{r}$ are principal curvatures. If we replace the spacelike angle ϕ by $-\phi$ in (8), this allows us to determine the values of the relevant quantities for the associate surface S'. So

$$\kappa_n' = \kappa_n,\ \overline{\kappa_n}' = \overline{\kappa_n},\ \tau_g{}' = -\tau_g. \tag{9}$$

From (6) and (9), we get

$$(\kappa_n)_2 = \kappa_g(\kappa_n - \overline{\kappa}_n),\ (\overline{\kappa}_n)_1 = \overline{\kappa}_g(\kappa_n - \overline{\kappa}_n),\ \left(\ln|\tau_g|\right)_1 = -2\overline{\kappa}_g,\ \left(\ln|\tau_g|\right)_2 = 2\kappa_g.$$

From (8) and (9), it is seen that the results in R^3 which were given in [2,4,15,20] are also valid in the Lorentzian 3-space for spacelike surface $\boldsymbol{x} = \boldsymbol{x}(u, v)$.

Definition 1. *On a spacelike surface S in L^3, if the coefficients of S for an orthogonal system (u, v) are $E = G,\ F = 0,\ M = const \neq 0$, then the orthogonal system is said to be A-net.*

Corollary 1. *The condition that is both necessary and sufficient for defining an orthogonal net on a spacelike surface in L^3 as an isothermal coordinate system is*

$$\left(\ln\frac{E}{G}\right)_{uv} = 0 \text{ or } (\kappa_g)_1 = -(\overline{\kappa_g})_2 \tag{10}$$

Proof. It is obvious when $E = G$. For $E \neq G$, we get $\frac{E}{(f'(u))^2} = \frac{G}{(h'(v))^2}$ by using equations $u_1 = f(u), v_1 = h(v)$. Thus $\ln\frac{E}{G} = \ln(f'(u))^2 - \ln(h'(v))^2$, so we have $\left(\ln\frac{E}{G}\right)_{uv} = 0$.

Conversely, if $\left(\ln\frac{E}{G}\right)_{uv} = 0$ then we can use $u_1 = \int\sqrt{A(u)}du$, $v_1 = \int\sqrt{B(v)}dv$; $\left(\text{with } (f'(u))^2 = A(u) \text{ and } (h'(v))^2 = B(v)\right)$ so $\frac{E}{(f'(u))^2} = \frac{G}{(h'(v))^2}$ and this implies $\overline{E} = \overline{G}$. By replacing $u_1 \to u,\ v_1 \to v$; the fundamental quantities of S become $E = G,\ F = 0$. According to (5), we see that $(\kappa_g)_1 = -(\overline{\kappa_g})_2$ $\left(f_1 = \frac{f_u}{\sqrt{E}},\ f_2 = \frac{f_v}{\sqrt{G}}\right)$.

The A-net in Definition 1 and isothermal condition in Corollary 1 provide the framework for the next theorem. □

Theorem 1. *A non-isothermic spacelike surface S is characterized as a Bonnet surface if and only if it admits an A-net.*

Proof. Suppose that S is a non-isothermic spacelike Bonnet surface. Applying (6) and (9)

$$(\kappa_n{}')_2 = \kappa_g'(\kappa_n{}' - \overline{\kappa_n}') - (\tau_g)_1 - 2\overline{\kappa_g}'\tau_g,\ (\overline{\kappa_n}')_1 = \overline{\kappa_g}'(\kappa_n{}' - \overline{\kappa_n}') - (\tau_g)_2 + 2\kappa_g{}'\tau_g \tag{11}$$

and from (6) and (11)

$$\left(\ln|\tau_g|\right)_1 = -2\overline{\kappa_g},\ \left(\ln|\tau_g|\right)_2 = 2\kappa_g \tag{12}$$

are written. According to the compatibility condition $f_{12} - f_1(\kappa_g) = f_{21} + f_2(\overline{\kappa_g})$, from (12) we can write

$$(\kappa_g)_1 = -(\overline{\kappa_g})_2. \tag{13}$$

Equation (13) is identical to $\left(\ln\frac{E}{G}\right)_{uv} = 0$. By Corollary 1, the orthogonal net (u, v) defines an isothermal coordinate system. Consequently,

$$E = G, F = 0. \tag{14}$$

Using (14), (12) and (5), we get $\tau_g = \frac{c}{E}$ and $M = -c = const. \neq 0$. According to Definition 1, there is an A-net on S.

Conversely, assume that there is an A-net on S. We have $M = \varepsilon$ after scaling $\frac{1}{\sqrt{|c|}}$. By (4),

$$E = G = \frac{1}{\tau_g} = \frac{1}{|J|\sin\phi}, \quad F = 0, \quad L = \frac{1}{|J|\sin\phi}(H + J\cos\phi)$$
$$M = \varepsilon = -sgn(J), \qquad N = \frac{1}{|J|\sin\phi}(H - J\cos\phi) \tag{15}$$

can be written. Using (6) and (8), the compatibility equations and Gauss equations, which play a crucial role in ensuring the consistency of the geometric structure on the surface, are given by

$$\frac{H_v}{J} = \frac{\phi_v}{\sin\phi} = \left(\ln\left|\tan\frac{\phi}{2}\right|\right)_v, \frac{H_u}{J} = \frac{-\phi_u}{\sin\phi} = -\left(\ln\left|\tan\frac{\phi}{2}\right|\right)_u \tag{16}$$

and $2K = 2(H^2 - J^2) = -\Delta\ln(|J|\sin\phi) = -\Delta(\ln E)$. From (8), (9) and (15), the coefficients of S' are obtained as

$$E' = G' = \frac{1}{|J|\sin\phi}, \quad F' = 0, \quad L' = \frac{1}{|J|\sin\phi}(H + J\cos\phi)$$
$$M' = -\varepsilon, \qquad N' = \frac{1}{|J|\sin\phi}(H - J\cos\phi) \tag{17}$$

Thus, a non-isothermic spacelike surface S possessing an A-net is a spacelike Bonnet surface, and the Euclidean results presented in [2] remain valid in L^3.

Theorem 1, which provides a characterization of non-isothermic spacelike Bonnet surfaces via A-nets, plays a key role in the identification of spacelike Bonnet surfaces and their associated surfaces. □

Corollary 2. *From (16),* $\left(\frac{H_u}{J}\right)_v + \left(\frac{H_v}{J}\right)_u = 0$.

Corollary 3. *H remains constant precisely when ϕ is constant.*

Corollaries 2 and 3 indicate the relationship between the given coordinate net and the curvature quantities of the surface.

Theorem 2. *Let θ be the spacelike angle between an orthogonal nets (u', v') and (u, v) with geodesic curvatures κ_g and $\overline{\kappa_g}$. The net (u', v') is isothermic precisely when $\Delta\theta + (\kappa_g)_1 + (\overline{\kappa_g})_2 = 0$, where Δ denotes the second Beltrami differential parameter (Laplacian) of the function θ.*

Proof. Starting with the following equations:

$$\kappa_g{}^{\circ} = -(\kappa_g + \theta_1)\sin\theta + (\overline{\kappa_g} + \theta_2)\cos\theta, \kappa_g{}^{*} = (\kappa_g + \theta_1)\cos\theta + (\overline{\kappa_g} + \theta_2)\sin\theta \tag{18}$$

with $\kappa_g{}^{*}$, $\kappa_g{}^{\circ}$ representing the geodesic curvatures of the curves belonging to the orthogonal net (u', v') [26]. By combining the formulas $q_{1'} = q_1\cos\theta + q_2\sin\theta$ and $q_{2'} = -q_1\sin\theta + q_2\cos\theta$ with Equation (18), we thus find $(\kappa_g{}^{*})_{1'} + (\kappa_g{}^{\circ})_{2'} = \kappa_{g1} + \overline{\kappa}_{g2} + \Delta\theta$; and according to (13), for an isothermic net (u', v'), we have $(\kappa_g{}^{*})_{1'} = -(\kappa_g{}^{\circ})_{2'}$. This implies that $\kappa_{g1} + \overline{\kappa_{g2}} + \Delta\theta = 0$. □

Conversely, from (7) and (18) $(\kappa_g{}^{*})_{1'} + (\kappa_g{}^{\circ})_{2'} = 0$ is obtained, which shows that the net (u', v') is isothermic.

Theorem 2 gives a geometric characterization of isothermic nets via the Laplacian operator.

Corollary 4. *If two orthogonal nets (u, v) and (u', v') on spacelike surface S are isothermic coordinates, then $\Delta\theta = 0$.*

Definition 2. *For the non-isothermic helicoids $H(\gamma, c)$ and $H(\gamma', -c)$ with the same axis, if the curve γ' is symmetric to the curve γ with respect to a meridian plane of $H(\gamma, c)$, then two helicoids are said to be adjoint helicoids. Here c, $-c$ are the pitches, and γ, γ' are their generating curves.*

Theorem 3. *Adjoint spacelike helicoids form an associated pair.*

This concept, which establishes a geometric relationship between two spacelike helicoids, allows for the construction of associated (adjoint) surfaces and understanding their behavior under the Lorentzian metric structure. Figures were generated using MATLAB R2021b.

4. Spacelike Bonnet Helicoids and Their Associated Pairs

In this section, we illustrate the theoretical results obtained in the previous sections by considering spacelike helicoidal surfaces together with their associated pairs.

4.1. Spacelike Helicoids with a Timelike Axis of Revolution

Assuming conditions $1-(f'(u))^2>0$; $u^2-c^2>0$ and $u^2-c^2-u^2(f'(u))^2>0$ hold, the spacelike helicoid S with a timelike axis of revolution admits the following parametrization in terms of (u,v):

$$\boldsymbol{x}(u,v)=(-u\sin v,u\cos v,f(u)+cv),u>0,c\in R^+ \tag{19}$$

with the generating curve γ defined as $\gamma=(u,0,f(u))$ [23,27]. The first and second fundamental forms of S are $E=1-(f')^2>0$, $F=-cf'$, $G=u^2-c^2>0$, $L=-\frac{uf''}{w}$, $M=\frac{c}{w}$, $N=\frac{-u^2f'}{w}$ and $\langle \boldsymbol{x}_u\times\boldsymbol{x}_v,\boldsymbol{x}_u\times\boldsymbol{x}_v\rangle=-w^2=-u^2+c^2+u^2(f'(u))^2<0$. Applying the transformation $\overline{u}=u$, $\overline{v}=v+\int\frac{-cf'(u)}{u^2-c^2}du$, the metric elements of the coordinate mesh composed of the $\overline{u}=const.$ helices and their orthogonal trajectories can be written as $\overline{E}=\frac{w^2}{G}$, $\overline{F}=0$, $\overline{G}=u^2-c^2$, $\overline{L}=\frac{-uf''(u)\left(u^2-c^2\right)^2+2c^2w^2f'+u^2c^2f'^3}{w(u^2-c^2)^2}$, $\overline{M}=\frac{cw}{u^2-c^2}$, $N=\overline{N}=-\frac{u^2f'(u)}{w}$. Upon applying $\overline{\overline{u}}=\int\frac{w}{G}d\overline{u}$, $\overline{\overline{v}}=\overline{v}$ to get an A-net on the surface, then $\overline{\overline{E}}=\overline{\overline{G}}=u^2-c^2$, $\overline{\overline{F}}=0$, $\overline{\overline{M}}=c\neq0$, $\overline{\overline{N}}=-\frac{u^2f'(u)}{w}$, $\overline{\overline{L}}=\frac{-uf''(u)\left(u^2-c^2\right)^2+2c^2w^2f'+u^2c^2f'^3}{w^3}$.

Hence, the curves $\overline{\overline{v}}=const.$, together with their orthogonal trajectories, constitute an A-net. According to the fundamental theorem, the spacelike helicoid S given in Equation (19) is a spacelike Bonnet surface. Consequently, the spacelike associate helicoid S' are written as

$$\boldsymbol{z}(u,v)=(-u\sin v,u\cos v,f(u)-cv) \tag{20}$$

Figures 1 and 2 illustrate a spacelike helicoidal surface S and its associated surface S', respectively. Both surfaces, given by (19) and (20), are generated by a helicoidal motion about a common timelike axis of revolution with pitches c and $-c$, respectively. This illustrates the formation of a spacelike Bonnet pair under the change in the pitch parameter.

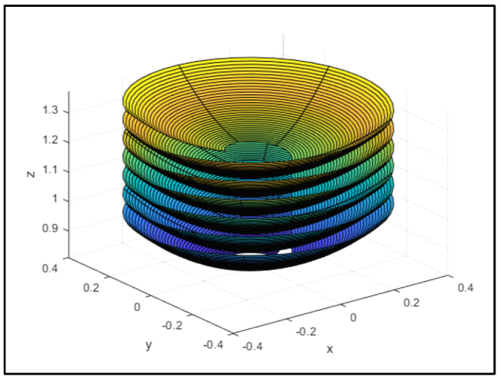


Figure 1. Spacelike helicoid S, generated by a helicoidal motion about a timelike axis with pitch (c).

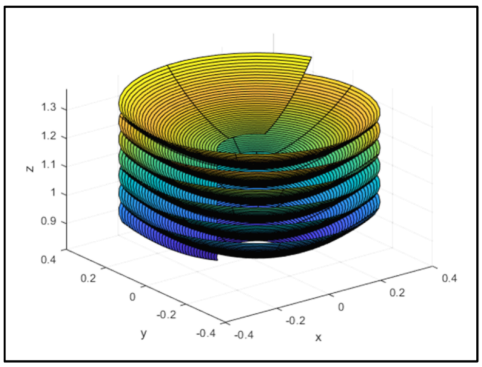


Figure 2. Spacelike associate helicoid S', generated by a helicoidal motion about a timelike axis with pitch $(-c)$.

4.2. Spacelike Helicoid (Type 1) with a Spacelike Axis of Revolution

Assuming conditions $c^2 - u^2 > 0$ and $c^2 - u^2 - u^2(f'(u))^2 > 0$ hold, the spacelike helicoid S with a spacelike axis of revolution (type 1) admits the following parametrization in terms of (u, v):

$$\boldsymbol{x}(u,v) = (f(u) + cv, u\cosh v,\ u\sinh v), u > 0,\ c \in R^+ \tag{21}$$

with the generating curve γ defined as $\gamma = (f(u), u, 0)$ [23,27]. The first and second fundamental form elements of the spacelike helicoidal surface are $E = 1 + (f')^2 > 0$, $F = cf'$, $G = c^2 - u^2 > 0$, $w^2 = c^2 - u^2 - u^2 f'^2 > 0$, $L = -\frac{uf''}{w}$, $M = \frac{c}{w}$, $N = \frac{u^2 f'}{w}$. We change the parameters in accordance with the equations $\overline{u} = u$, $\overline{v} = v + \int \frac{cf'}{c^2 - u^2} du$, so we have $\overline{E} = \frac{w^2}{G}$, $\overline{F} = 0$, $G = \overline{G} = c^2 - u^2$, $\overline{L} = \frac{-uf''(u)\left(c^2-u^2\right)^2 - 2c^2w^2f' - u^2c^2f'^3}{w\left(c^2-u^2\right)^2}$, $\overline{M} = \frac{cw}{G}$, $N = \overline{N} = \frac{u^2 f'(u)}{w}$. Upon applying $\overline{\overline{u}} = \int \frac{w}{G} d\overline{u}$, $\overline{\overline{v}} = \overline{v}$ to get an A-net, then $\overline{\overline{E}} = \overline{\overline{G}} = c^2 - u^2$, $\overline{\overline{F}} = 0$, $\overline{\overline{M}} = c(\neq 0)$, $\overline{\overline{N}} = \frac{u^2 f'(u)}{w}$, $\overline{\overline{L}} = \frac{-uf''(u)\left(c^2-u^2\right)^2 - 2c^2w^2f' - u^2c^2f'^3}{w^3}$ are obtained. Hence, the curves $\overline{\overline{u}} = const.$ together with their orthogonal trajectories constitute an A-net. According to the fundamental theorem, the spacelike helicoid defined by (21) is a spacelike Bonnet surface. Thus, the spacelike associate helicoid S' is represented as

$$\boldsymbol{z}(u,v) = (f(u) - cv, u\cosh v, u\sinh v) \tag{22}$$

The spacelike helicoidal surface (type 1) S and its associated surface S' are illustrated in Figures 3 and 4, respectively. The spacelike helicoid (type 1), given by (21), is generated by a helicoidal motion about a spacelike axis of revolution with pitch c, while its associate surface, given by (22), corresponds to pitch $-c$.

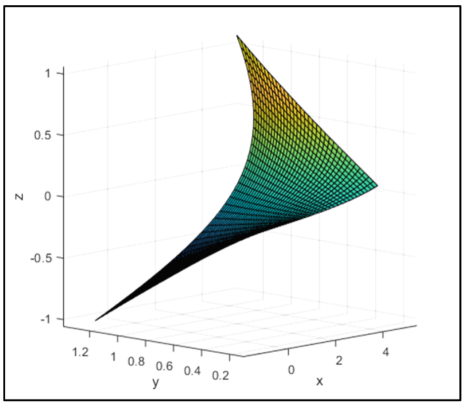


Figure 3. Spacelike helicoid S (type 1), generated by a helicoidal motion about a spacelike axis with pitch (c).

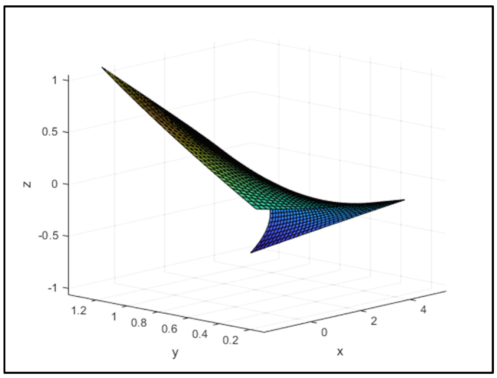


Figure 4. Spacelike associate helicoid S' (type 1), generated by a helicoidal motion about a spacelike axis with pitch $(-c)$.

4.3. Spacelike Helicoid (Type 2) with a Spacelike Axis of Revolution

Assuming conditions $(f'(u))^2 - 1 > 0$ and $u^2(f'(u))^2 - c^2 - u^2 > 0$ hold, the spacelike helicoid S with a spacelike axis of revolution (type 2) admits the following parametrization in terms of (u, v):

$$x(u,v) = (f(u) + cv, u\sinh v, u\cosh v),\ u > 0,\ c \in R^+ \tag{23}$$

with the generating curve γ defined as $\gamma = (f(u), 0, u)$ [23,27]. The first and second fundamental form elements of the spacelike helicoidal surface are $E = f'^2 - 1 > 0$, $F = cf'$, $G = c^2 + u^2 > 0$, $L = \frac{uf''}{w}$, $M = \frac{-c}{w}$, $N = \frac{-u^2 f'}{w}$, $w^2 = u^2 f'^2 - u^2 - c^2 > 0$.

Using the transformation $\overline{u} = u$, $\overline{v} = v + \int \frac{cf'}{c^2+u^2} du$; we get $\overline{E} = \frac{w^2}{G}$, $\overline{F} = 0$, $G = \overline{G} = c^2 + u^2 > 0$, $\overline{M} = \frac{cw}{G}$, $N = \overline{N} = -\frac{u^2 f'(u)}{w}$, $\overline{L} = \frac{uf''(u)\left(c^2+u^2\right)^2 - 2c^2 w^2 f' + u^2 c^2 f'^3}{w\left(c^2+u^2\right)^2}$.

Upon applying $\overline{\overline{u}} = \int \frac{w}{G} d\overline{u}$, $\overline{\overline{v}} = \overline{v}$ to get an A-net, then we obtain $\overline{\overline{E}} = \overline{\overline{G}} = c^2 + u^2 > 0$, $\overline{\overline{F}} = 0$, $\overline{\overline{M}} = c(\neq 0)$, $\overline{\overline{N}} = -\frac{u^2 f'(u)}{w}$, $\overline{\overline{L}} = \frac{uf''(u)\left(c^2+u^2\right)^2 - 2c^2 w^2 f' + u^2 c^2 f'^3}{w^3}$.

Hence, the curves $\overline{\overline{u}} = const.$ together with their orthogonal trajectories constitute an A-net. According to the fundamental theorem, the spacelike helicoid defined by (23) is a spacelike Bonnet surface. Thus, the spacelike associate helicoid S' is represented as

$$z(u,v) = (f(u) - cv, u\sinh v, u\cosh v) \tag{24}$$

The spacelike helicoidal surface S (type 2), illustrated in Figure 5, is obtained via a helicoidal motion about a spacelike axis of revolution with pitch c, and its associated surface, illustrated in Figure 6, corresponds to pitch $-c$.

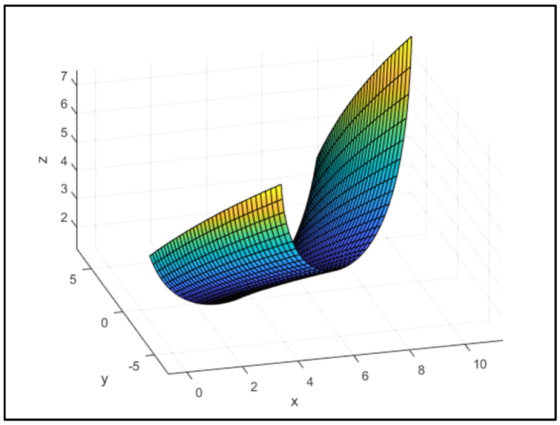


Figure 5. Spacelike helicoid S (type 2), generated by a helicoidal motion about a spacelike axis with pitch (c).

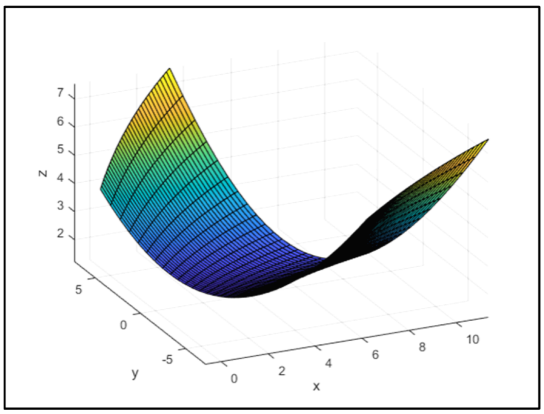


Figure 6. Spacelike associate helicoid S' (type 2), generated by a helicoidal motion about a spacelike axis with pitch $(-c)$.

4.4. Spacelike Helicoid with a Null Axis of Revolution

Assuming conditions $u \neq f(u)$, $1 - f'^2 > 0$ and $(u-f)^2\left(1-f'^2\right) - (1-f')^2c^2 > 0$ hold, the spacelike helicoid S with a null axis of revolution admits the following parametrization in terms of (u, v):

$$\boldsymbol{x}(u,v) = \left((u-f(u))v, \left(1-\frac{v^2}{2}\right)u + \frac{v^2}{2}f(u) + cv, -\frac{v^2}{2}u + \left(1+\frac{v^2}{2}\right)f(u) + cv\right) \quad (25)$$

with the generating curve γ defined as γ is $\gamma = (0, u, f(u))$ [23,27]. The quantities are $E = 1-f'^2 > 0$, $F = c(1-f')$, $G = (u-f)^2$, $L = \frac{(u-f)f''}{w}$, $M = \frac{-c(1-f')^2}{w}$, $N = \frac{(u-f)^2(f'-1)}{w}$, $w^2 = (u-f)^2\left(1-f'^2\right) - (1-f')^2c^2$.

Upon applying $\overline{u} = u$, $\overline{v} = v + \int \frac{c(1-f')}{(u-f)^2}du$, then we get $\overline{E} = \frac{w^2}{G}$, $\overline{F} = 0$, $\overline{G} = (u-f)^2$, $\overline{M} = 0$, $\overline{L} = \frac{f''(u-f)^3 + c^2(1-f')^3}{w(u-f)^2}$, $\overline{N} = \frac{(u-f)^2(f'-1)}{w}$. To obtain an A-net, we apply the transformation $\overline{\overline{u}} = \int \frac{w}{\overline{G}}d\overline{u}$, $\overline{\overline{v}} = \overline{v}$, thus $\overline{\overline{E}} = \overline{\overline{G}} = (u-f)^2$, $\overline{\overline{F}} = 0$, $\overline{\overline{M}} = 0$, $\overline{\overline{L}} = \frac{(u-f)^2\left((u-f)^3f'' + c^2(1-f')^3\right)}{w^3}$, $\overline{\overline{N}} = \frac{(u-f)^2(f'-1)}{w}$.

This implies that the curves $\overline{\overline{v}} = const.$ and their orthogonal trajectories form an isothermic net. Since $\overline{\overline{F}} = 0$ and $\overline{\overline{M}} = 0$, the coordinate curves coincide with the principal curvature lines. Moreover, the coefficients of the first and second fundamental forms of S and S' coincide. Therefore, S and S' represent the same surfaces.

5. Conclusions

The Bonnet problem has been investigated by several researchers using various methods in different spaces [28–31]. There are many studies on timelike Bonnet pairs [5,10,22,32] and spacelike Bonnet pairs [8,11]. Although most existing results focused on isothermic surfaces [6,7,11,33,34], typically in relation to curvature properties [8–10,12,13], the present study differs from the literature by constructing non-isothermic spacelike Bonnet pairs. For this purpose, a generalization of the criterion introduced by Soyuçok for Euclidean space is developed [2], and the problem of constructing non-isothermic spacelike pairs is addressed via a method analogous to that used in the timelike case [22]. Unlike the work of Kanbay and Yüksekdağ [22], which focuses on timelike Bonnet surfaces in Lorentzian 3-space, the present study develops a criterion-based construction of spacelike non-isothermic Bonnet pairs. In particular, the novelty lies not only in the extension of known methods, but in identifying explicit spacelike helicoidal Bonnet pairs outside the isothermic frameworks. The helicoid pairs illustrated in Figures 1–6 demonstrate that changing the sign of the pitch

parameter generates the associated Bonnet pair while preserving the underlying metric structure. Moreover, reversing the pitch parameter preserves the A-net structure of the spacelike helicoid.

Although the present work does not address the global Bonnet problem in full generality, it contributes to its local geometric structure by providing explicit spacelike examples in Lorentzian space, which may serve as building blocks for more global investigations. This work provides explicit geometric realizations of spacelike Bonnet pairs via helicoidal surfaces, highlighting the role of A-net structures in Lorentzian geometry. Moreover, in 2025, Bobenko, Hoffmann and Sageman-Furnas provided the first examples of compact Bonnet pairs using the relationship between Bonnet pairs and isothermal surfaces [33,34]. This demonstrates that interest in Bonnet surfaces remains active and suggests potential avenues for exploring these results in broader geometric contexts.

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