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Statistical Inference of Stress–Strength Reliability for Multi-State System Based on Exponentiated Pareto Distribution Using Generalized Survival Signature

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Abstract

The stress–strength reliability model is widely applied in various fields such as mechanical engineering, materials science, and aerospace engineering to identify weak links in systems and thereby improve system reliability. This paper analyzes the stress–strength reliability for multi-state systems composed of multi-state components. One of the main contributions is the derivation of a multi-state stress–strength reliability model under combined stresses based on the generalized survival signature theory. In the model analysis, it is assumed that each component of the system is subjected to two different stresses corresponding to two different strengths, and that the stress variables and strength variables are mutually independent and all follow the exponentiated Pareto distribution with the common second shape parameter. Another contribution is the use of maximum likelihood estimation, empirical Bayesian estimation, and weakly informative Bayesian estimation to estimate the variable parameters and the stress–strength reliability under the progressive first-failure censoring scheme. In addition, the asymptotic confidence intervals for the stress–strength reliability model are derived, and the Bayesian credible intervals are constructed based on MCMC sampling. Finally, through MCMC simulation of a three-state consecutive 3-out-of-5: G system, the accuracy of the variable parameters and the stress–strength reliability under the aforementioned point estimation and interval estimation methods is analyzed, and the performance of these estimation methods is compared under different sample sizes. In addition, sensitivity analyses were conducted on the common shape parameter w and the hyperparameters of the weakly informative prior distributions. Furthermore, a real data set is applied to illustrate the proposed procedures.

Keywords: multi-state system; stress–strength reliability; exponentiated Pareto distribution; generalized survival signature; progressive first failure censoring

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1. Introduction

The stress–strength model serves as a core foundational theory in reliability engineering and was first proposed by Birnbaum [1]. Subsequently, with the continuous advancement of relevant disciplines, the stress–strength model has been widely applied in various engineering and natural science fields where uncertainty and safety considerations are paramount. In this model, stress refers to the various external loads borne by a device during its service life, while strength denotes the inherent capacity of the device to withstand these external loads. The stress–strength reliability model assumes that both the strength X and the stress Y are random variables. Consequently, the reliability R is defined as the probability that strength exceeds stress; that is, $R = P(X > Y)$. Given this probabilistic framework, the analysis of the model essentially reduces to the statistical inference of the distribution parameters of X and Y based on sample data. Therefore, selecting an appropriate probability distribution to characterize the random nature of stress and strength is of paramount importance.

Existing research on statistical inference for stress–strength reliability models mainly focuses on three perspectives: parameter estimation methods, lifetime distribution types, and sampling scheme designs. Zou Q. et al. proposed a reliability analysis framework based on probability weighted moments to address the limitations of maximum likelihood estimation (MLE) for the stress–strength reliability model under the three-parameter Weibull distribution [2]. Mathew A. et al. proposed two nonparametric estimators based on the empirical distribution function and the smoothed kernel distribution function for quasi-relative data derived from stress and strength observations, and evaluated their performance [3]. Ahmadi V. M. et al. constructed a stress–strength model for multi-state systems under the assumption that the stress and strength variables follow two independent Weibull distributions, and studied the estimation of the multi-state stress–strength reliability function using various estimation methods [4]. Oommen N. P. et al. compared and elaborated on the concepts, model descriptions, and applications of conventional and hybrid censoring schemes [5]. Sharma S. et. al. discussed the scenario where both stress and strength variables follow the weighted Exponential-Lindley lifetime model with different parameters under progressively censored data. The model parameters and the reliability of the multicomponent stress–strength system were evaluated based on MLE and Bayesian estimation [6]. Hassan S. A. et al. compared the performance of various point estimation and interval estimation methods under the Lomax distribution for strength and stress variables, where data were generated using a progressive Type-II censoring scheme based on a partially accelerated life test [7]. Kohansal A. et al. evaluated the performance of MLE and Bayesian estimation methods for the Lomax distribution under the progressive first failure censoring scheme, considering two scenarios in which the common parameters of the stress and strength variables are known and unknown [8]. Kohansal A. et al. studied the statistical estimation of the stress–strength reliability of multi-component systems under progressive first failure censoring samples, where the lifetime distribution of each component follows the modified Kumaraswamy distribution, and compared the performance of different estimators under various parameter estimation methods [9]. Singh K. et al. estimated the multicomponent stress–strength reliability under classical and Bayesian approaches, assuming that both stress and strength components follow lognormal distributions, based on progressive censoring schemes [10]. However, existing research on stress–strength models has predominantly focused on binary-state systems, with limited attention devoted to multi-state systems. In practical engineering, many systems are not confined to merely two conditions of perfect functioning and complete failure, but rather exhibit intermediate degraded or derated operating states. Moreover, the existing research has rarely addressed scenarios in which system components are subjected to multiple stresses with corresponding multiple strengths.

The exponentiated Pareto distribution (EPD) has garnered increasing attention due to its capacity to simultaneously capture the peakedness, heavy tails, and non-monotonic hazard rates commonly observed in complex system reliability data [11], and was first proposed by Gupta et al. [12]. The EPD represents an exponentiated generalization of the Pareto distribution. Compared with the exponential distribution, which assumes a constant hazard rate, the EPD can accommodate decreasing, increasing, and even unimodal hazard rates, thereby overcoming the memoryless property inherent in the exponential distribution [13]. In contrast to the standard Pareto distribution, which fails to adequately model long-tailed data characterized by an initially increasing and subsequently decreasing hazard rate, the EPD resolves this limitation through the introduction of a second shape parameter w [14]. Furthermore, while the Weibull distribution is restricted to fitting monotonic hazard rates and tends to severely underestimate tail probabilities in the presence of extreme observations, the EPD is precisely equipped to surmount this shortcoming [15]. The Log-normal distribution, although capable of capturing peakedness, heavy tails, and non-monotonic hazard rates that initially rise before falling, suffers from a hazard rate function that ultimately declines to 0. Moreover, its mathematical form is considerably more complex, involving the standard normal cumulative distribution function Φ. Consequently, the Log-normal distribution lacks both the concise extreme value theoretic foundation and the elementary functional form characteristic of the EPD [16]. These attributes render the EPD distinctly advantageous in terms of theoretical tractability and computational efficiency. Equation (1) presents its cumulative distribution function (CDF) and probability density function (PDF), where v and w are two shape parameters. Figure 1 displays the corresponding CDF and PDF curves for various combinations of these two shape parameters. It can be seen from the figure that when $v < 1$, the PDF is monotonically decreasing and trends to infinity at 0. When $v = 1$, the PDF is still monotonically decreasing, but attains a finite value w at 0. When $v > 1$, the PDF first increases and then decreases.

$$\left.\begin{aligned} F(x) &= \left[1-(1+x)^{-w}\right]^{v} \\ f(x) &= wv(1+x)^{-w-1}\left[1-(1+x)^{-w}\right]^{v-1} \end{aligned}\right\} x, v, w > 0. \tag{1}$$

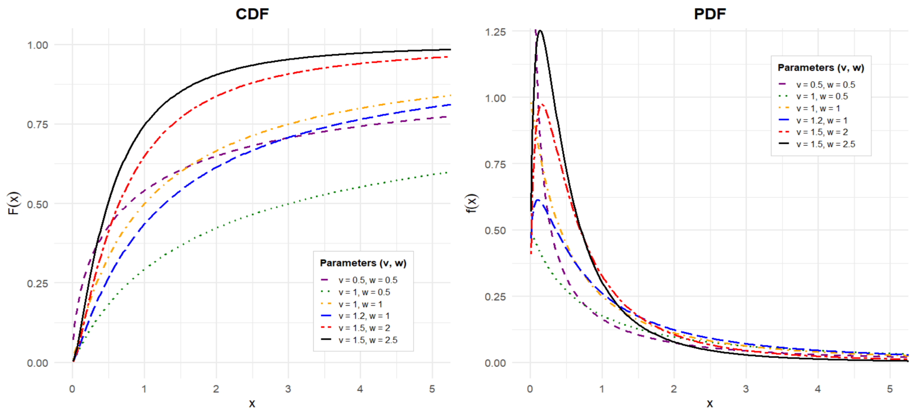

Figure 1. CDF and PDF of exponentiated Pareto distribution.

The EPD has been extensively employed in the context of stress–strength reliability. Hassan A. S. et al. assumed that both stress and strength follow EPDs with known and unequal shape parameters, and investigated the Bayesian and non-Bayesian estimation of reliability for a k-out-of-n system with identical component strengths subjected to a common stress using five non-Bayesian estimation methods [11]. Mahmoud E. A. M. et al. derived the best linear unbiased estimates and maximum likelihood estimates (MLEs) of the parameters of the EPD based on progressively Type-II right censored order statistics [17]. Jinyuan C. et al. discussed the system reliability when both stress and strength follow EPDs with different parameters, and proposed various methods of point and interval estimation [18]. Panahi H. performed statistical inference on the unknown parameters of the EPD under the progressive first-failure censoring (PFIF-C) scheme using cumin essential oil data [19]. Al-Omari I. A. et al. studied system reliability estimation when the distributions of stress and strength are independent and both follow the EPD, under simple random sampling, ranked set sampling, and median ranked set sampling methods [20]. Nawaz A. C. et al. presented a Bayesian analysis of a three-component mixture model of the EPD under the right Type-I censoring scheme [21]. Gül F. A. investigated the classical and Bayesian estimation of reliability under the assumption that both stress and strength variables follow the EPD [22]. Jabbari H. K. et al. explored the use of ranked set sampling for parameter estimation of the exponentiated Pareto distribution [23]. Kexin W. et al. employed both classical and Bayesian estimation methods to estimate the parameters, reliability function, and hazard function of the EPD under an adaptive Type-II progressive censoring scheme, and compared the performance of these methods through MCMC simulations and real data analysis [24]. Under the assumption of the EPD, Saini S. investigated the estimation of multicomponent stress–strength reliability and evaluated the performance of MLE and Bayesian estimation via simulation studies and real data [25]. Alsadat N. et al. investigated the inference of multicomponent stress–strength reliability under the PFIF-C scheme based on the EPD, assuming that both stress and strength follow EPDs with a common second shape parameter. Parameter and reliability estimates were obtained using MLE and Bayesian methods, and both asymptotic confidence intervals and HPD credible intervals were constructed [26]. Saini S. investigated the classical and Bayesian estimation of multicomponent stress–strength reliability based on progressively censored data under the assumption that stress and strength are independently and identically distributed as EPDs. The effectiveness of the proposed methods was validated through MCMC simulations and two real datasets [27].

From the studies reviewed above, it is evident that current research on the EPD in stress–strength reliability models primarily proceeds along two directions. The first involves assuming that both stress and strength follow the EPD, and subsequently comparing the performance of various point and interval estimators for the EPD parameters and the stress–strength reliability. The second direction integrates the EPD with progressive censoring schemes. By removing unfailed units in stages, progressive censoring eliminates the need to wait indefinitely for extreme samples that may never fail, while still retaining sufficient observations to capture the tail behavior of the distribution, which aligns naturally with the heavy-tailed structure of the EPD. Moreover, the EPD involves multiple shape parameters, and its likelihood function lacks an explicit analytical solution, rendering parameter estimation computationally demanding. Progressive censoring alleviates this data burden by reducing sample sizes without substantially compromising estimation precision.

Motivated by these considerations, the primary contributions of this paper are twofold.

(1) Assuming that each component of the system is subjected to two different stresses with two corresponding strengths, and that both the stress and strength variables follow the EPD, the stress–strength reliability model for a multi-state system composed of multi-state components is derived based on the generalized survival signature theory.

(2) Under the PFIF-C scheme, the performance of three point estimation methods (MLE, empirical Bayesian estimation, and weakly informative prior for Bayesian estimation) and two interval estimation methods (asymptotic confidence intervals and Bayesian HPD credible intervals) is evaluated.

The remainder of this paper is organized as follows. Section 2 presents the model assumptions and relevant theoretical foundations. Section 3 derives the stress–strength reliability model for multi-state systems. Section 4 develops the MLE and Bayesian estimation with both informative and non-informative priors. Section 5 provides the computational methods for asymptotic confidence intervals and HPD credible intervals. Section 6 conducts simulation experiments based on a consecutive three-state 3-out-of-5: G system. Furthermore, a real dataset is used to illustrate the proposed method and model. Section 7 is the conclusion of this paper.

2. Assumptions, Definitions and PFIF-C Design Scheme

This section presents some assumptions and definitions related to the stress–strength model, and the basic steps of the PFIF-C scheme are provided.

2.1. Assumptions

Assumption 1. *Assume that the multi-state system consists of n independent and identically distributed multi-state components, each with two different types of strength (X_1, X_2) and subject to two different types of stress (Y_1, Y_2), where all these variables are mutually independent.*

Assumption 2. *Both the stress and strength variables follow the EPD, where $X_i \sim \mathrm{EPD}(v_i, w)$ and $Y_i \sim \mathrm{EPD}(u_i, w)$, $i = 1, 2$. $F_i(x_i)$ and $G_i(y_i)$ are CDFs of the variables X_i and Y_i, respectively, while $f_i(x_i)$ and $g_i(y_i)$ are their PDFs, for $i = 1, 2$.*

2.2. Definitions

Definition 1. *Let τ denote the state of the multi-state component, then it can be defined as*

$$\tau = \begin{cases} s_0, & \text{if } X_1 \le Y_1\ ,\ X_2 \le Y_2, \\ s_1, & \text{if } X_1 > Y_1, X_2 \le Y_2 \text{ or } X_1 \le Y_1, X_2 > Y_2, \\ s_2, & \text{if } X_1 > Y_1\ ,\ X_2 > Y_2. \end{cases} \tag{2}$$

τ_s is the state of the multi-state system.

Definition 2. *For a discrete multi-state system with $m+1$ states, let $r_{n_1,n_2,\ldots,n_j}(n)$, where $n_1 \ge n_2 \ge \ldots \ge n_j$ $(j \le m)$, and denote the number of path sets in which there are exactly n_k components in state s_k or above, for $k = 1, 2, \ldots, j$. The generalized survival signature $\overline{p}_{n_1,n_2,\ldots,n_j}(n)$ of this system is then defined as*

$$\overline{p}_{n_1,n_2,\ldots,n_j}(n) = \frac{r_{n_1,n_2,\ldots,n_j}(n)}{\binom{n}{n_j}\binom{n-n_j}{n_{j-1}-n_j}\cdots\binom{n-n_2}{n_1-n_2}}, \tag{3}$$

where $\overline{p}_{n_1,n_2,\ldots,n_j}(n)$ denotes the probability that the system is in an operating state, given that exactly n_k components are in state s_k or above, for $k = 1, 2, \ldots, j$ [28].

2.3. PFIF-C Design Scheme

Progressive censoring refers to a scheme in which, each time a failure occurs, a certain number of surviving units are randomly removed from the test. PFIF-C is a censoring scheme based on progressive censoring that incorporates the concept of first failure. The experimental design of PFIF-C is as follows:

(1) The samples are divided into M independent groups, with h samples in each group. m is the predetermined number of failures.
(2) When the first failure occurs at time $X_{1:m}$, the group in which the failure occurs is removed, along with Q_1 surviving groups. When the second failure occurs at time $X_{2:m}$, the group containing the second failure is removed, along with Q_2 surviving groups, and so on. Finally, when the m-th failure occurs at time $X_{m:m}$, the group in which the m-th failure occurs is removed, together with the remaining Q_m surviving groups.
(3) Consequently, the PFIF-C data $X_{1:m}, X_{2:m}, \ldots, X_{m:m}$ can be obtained, with the prefixed censoring scheme $Q = (Q_1, Q_2, \ldots, Q_m)$, where $M = m + Q_1 + Q_2 + \ldots + Q_m$.

3. Multi-State System Stress–Strength Reliability

For the multi-state system described by the assumptions in Section 2.1, let $R_c^{(s_k)}(y_1, y_2)$ ($k = 0,\ 1,\ 2$) denote the component reliability in state s_k.

$$\begin{aligned} R_c^{(s_0)}(y_1, y_2) &= P\{\tau = s_0 | Y_1 = y_1, Y_2 = y_2\} \\ &= P\{X_1 \le y_1, X_2 \le y_2\} \\ &= \left[1 - (1+y_1)^{-w}\right]^{v_1} \left[1 - (1+y_2)^{-w}\right]^{v_2}, \end{aligned} \tag{4}$$

$$\begin{aligned} R_c^{(s_1)}(y_1, y_2) &= P\{\tau = s_1 | Y_1 = y_1, Y_2 = y_2\} \\ = \quad & P\{X_1 > y_1, X_2 \le y_2\} + P\{X_1 \le y_1, X_2 > y_2\} \\ &= \left\{1 - \left[1 - (1+y_1)^{-w}\right]^{v_1}\right\} \left[1 - (1+y_2)^{-w}\right]^{v_2} \\ &+ \left[1 - (1+y_1)^{-w}\right]^{v_1} \left\{1 - \left[1 - (1+y_2)^{-w}\right]^{v_2}\right\}, \end{aligned} \tag{5}$$

$$\begin{aligned} R_c^{(s_2)}(y_1, y_2) &= P\{\tau = s_2 | Y_1 = y_1, Y_2 = y_2\} \\ &= P\{X_1 > y_1, X_2 > y_2\} \\ &= \left\{1 - \left[1 - (1+y_1)^{-w}\right]^{v_1}\right\} \left\{1 - \left[1 - (1+y_2)^{-w}\right]^{v_2}\right\} \end{aligned} \tag{6}$$

Based on Definition 2 and Equations (4)–(6), let $R_s^{(s_2)}(y_1, y_2)$ and $R_s^{(\ge s_1)}(y_1, y_2)$ denote the system reliability in state s_2 and in state s_1 or above, respectively, conditional on $Y_1 = y_1$ and $Y_2 = y_2$. They can be expressed as

$$\begin{aligned} & R_s^{(s_2)}(y_1, y_2) = P\{\tau_s = s_2 | Y_1 = y_1, Y_2 = y_2\} \\ & = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \left\{ \begin{array}{c} \overline{p}_{n_1,n_2}(n) \binom{n}{n_1} \binom{n_1}{n_2} \\ \times \left[R_c^{(s_0)}(y_1, y_2)\right]^{n-n_1} \left[R_c^{(s_1)}(y_1, y_2)\right]^{n_1-n_2} \left[R_c^{(s_2)}(y_1, y_2)\right]^{n_2} \end{array} \right\} \\ & = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \left\{ \begin{array}{c} \overline{p}_{n_1,n_2}(n) \binom{n}{n_1} \binom{n_1}{n_2} \left\{\left[1 - (1+y_1)^{-w}\right]^{v_1} \left[1 - (1+y_2)^{-w}\right]^{v_2}\right\}^{n-n_1} \\ \times \left\{ \begin{array}{c} \left\{1 - \left[1 - (1+y_1)^{-w}\right]^{v_1}\right\} \left[1 - (1+y_2)^{-w}\right]^{v_2} \\ + \left[1 - (1+y_1)^{-w}\right]^{v_1} \left\{1 - \left[1 - (1+y_2)^{-w}\right]^{v_2}\right\} \end{array} \right\}^{n_1-n_2} \\ \times \left\{\left\{1 - \left[1 - (1+y_1)^{-w}\right]^{v_1}\right\} \left\{1 - \left[1 - (1+y_2)^{-w}\right]^{v_2}\right\}\right\}^{n_2} \end{array} \right\} \\ & = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \left\{ \begin{array}{c} \overline{p}_{n_1,n_2}(n) \binom{n}{n_1} \binom{n_1}{n_2} \\ \times \left[1 - (1+y_1)^{-w}\right]^{(n-n_1)v_1} \left[1 - (1+y_2)^{-w}\right]^{(n-n_1)v_2} \\ \times \sum_{k=0}^{n_1-n_2} \left\{ \begin{array}{c} \binom{n_1-n_2}{k} \left[1 - (1+y_2)^{-w}\right]^{kv_2} \\ \times \sum_{k_1=0}^{k} \binom{k}{k_1} (-1)^{k_1} \left[1 - (1+y_1)^{-w}\right]^{k_1 v_1} \\ \times \left[1 - (1+y_1)^{-w}\right]^{(n_1-n_2-k)v_1} \\ \times \sum_{k_2=0}^{n_1-n_2-k} \binom{n_1-n_2-k}{k_2} (-1)^{k_2} \left[1 - (1+y_2)^{-w}\right]^{k_2 v_2} \end{array} \right\} \\ \times \sum_{l_1=0}^{n_2} \sum_{l_2=0}^{n_2} \left\{ \begin{array}{c} \binom{n_2}{l_1} \binom{n_2}{l_2} (-1)^{l_1+l_2} \\ \left[1 - (1+y_1)^{-w}\right]^{v_1 l_1} \left[1 - (1+y_2)^{-w}\right]^{v_2 l_2} \end{array} \right\} \end{array} \right\} \\ & = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \left\{ \begin{array}{c} \overline{p}_{n_1,n_2}(n) \binom{n}{n_1} \binom{n_1}{n_2} \sum_{k=0}^{n_1-n_2} \sum_{k_1=0}^{k} \sum_{k_2=0}^{n_1-n_2-k} \sum_{l_1=0}^{n_2} \sum_{l_2=0}^{n_2} \binom{n_1-n_2}{k} \binom{k}{k_1} \\ \times \binom{n_1-n_2-k}{k_2} \binom{n_2}{l_1} \binom{n_2}{l_2} (-1)^{k_1+k_2+l_1+l_2} \\ \times \left[1 - (1+y_1)^{-w}\right]^{(n-n_2-k+k_1+l_1)v_1} \left[1 - (1+y_2)^{-w}\right]^{(n-n_1+k+k_2+l_2)v_2} \end{array} \right\}, \end{aligned} \tag{7}$$

$$
\begin{aligned}
&R_s^{(\geq s_1)}(y_1,y_2) = P\{\tau_s \geq s_1 | Y_1 = y_1, Y_2 = y_2\} \\
&= \sum_{n_1=0}^{n} \overline{p}_{n_1}(n) \binom{n}{n_1} \left[R_c^{(s_0)}(y_1,y_2)\right]^{n-n_1} \left[1 - R_c^{(s_0)}(y_1,y_2)\right]^{n_1} \\
&= \sum_{n_1=0}^{n} \overline{p}_{n_1}(n) \binom{n}{n_1} \sum_{l=0}^{n_1} \binom{n_1}{l} (-1)^l \left[R_c^{(s_0)}(y_1,y_2)\right]^{n-n_1+l} \\
&= \sum_{n_1=0}^{n} \overline{p}_{n_1}(n) \binom{n}{n_1} \sum_{l=0}^{n_1} \binom{n_1}{l} (-1)^l \left\{\left[1-(1+y_1)^{-w}\right]^{v_1} \left[1-(1+y_2)^{-w}\right]^{v_2}\right\}^{n-n_1+l}.
\end{aligned}
\tag{8}
$$

$$
R_s^{(s_0)}(y_1,y_2) = P\{\tau = s_0 | Y_1 = y_1, Y_2 = y_2\} = 1 - R_s^{(\geq s_1)}(y_1,y_2), \tag{9}
$$

$$
R_s^{(s_1)}(y_1,y_2) = P\{\tau = s_1 | Y_1 = y_1, Y_2 = y_2\} = R_s^{(\geq s_1)}(y_1,y_2) - R_s^{(s_2)}(y_1,y_2). \tag{10}
$$

Based on Equations (7) and (8), let $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ denote the stress–strength reliability of the system in state s_2 and in state s_1 or above, respectively. The expressions are as follows

$$
\begin{aligned}
R_s^{(s_2)} &= \int_0^{+\infty}\int_0^{+\infty} R_s^{(s_2)}(y_1,y_2)\mathrm{d}G_1(y_1)\mathrm{d}G_2(y_2) \\
&= \sum_{n_1=0}^{n}\sum_{n_2=0}^{n_1} \left\{
\begin{array}{c}
\overline{p}_{n_1,n_2}(n)\binom{n}{n_1}\binom{n_1}{n_2}\sum_{k=0}^{n_1-n_2}\sum_{k_1=0}^{k}\sum_{k_2=0}^{n_1-n_2-k}\sum_{l_1=0}^{n_2}\sum_{l_2=0}^{n_2}\binom{n_1-n_2}{k}\binom{k}{k_1} \\
\times\binom{n_1-n_2-k}{k_2}\binom{n_2}{l_1}\binom{n_2}{l_2}(-1)^{k_1+k_2+l_1+l_2} \\
\times\int_0^{+\infty}\left\{\begin{array}{c}\left[1-(1+y_1)^{-w}\right]^{(n-n_2-k+k_1+l_1)v_1} \\ \times wu_1(1+y_1)^{-w-1}\left[1-(1+y_1)^{-w}\right]^{u_1-1}\end{array}\right\}\mathrm{d}y_1 \\
\times\int_0^{+\infty}\left\{\begin{array}{c}\left[1-(1+y_2)^{-w}\right]^{(n-n_1+k+k_2+l_2)v_2} \\ \times wu_2(1+y_2)^{-w-1}\left[1-(1+y_2)^{-w}\right]^{u_2-1}\end{array}\right\}\mathrm{d}y_2
\end{array}
\right\} \\
&= \sum_{n_1=0}^{n}\sum_{n_2=0}^{n_1} \left\{
\begin{array}{c}
\overline{p}_{n_1,n_2}(n)\binom{n}{n_1}\binom{n_1}{n_2}\sum_{k=0}^{n_1-n_2}\sum_{k_1=0}^{k}\sum_{k_2=0}^{n_1-n_2-k}\sum_{l_1=0}^{n_2}\sum_{l_2=0}^{n_2}\binom{n_1-n_2}{k}\binom{k}{k_1} \\
\times\binom{n_1-n_2-k}{k_2}\binom{n_2}{l_1}\binom{n_2}{l_2}(-1)^{k_1+k_2+l_1+l_2} \\
\times\dfrac{u_1u_2}{[(n-n_2-k+k_1+l_1)v_1+u_1]\cdot[(n-n_1+k+k_2+l_2)v_2+u_2]}
\end{array}
\right\},
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
R_s^{(\geq s_1)} &= \int_0^{+\infty}\int_0^{+\infty} R_s^{(\geq s_1)}(y_1,y_2)\mathrm{d}G_1(y_1)\mathrm{d}G_2(y_2) \\
&= \sum_{n_1=0}^{n} \left\{
\begin{array}{c}
\overline{p}_{n_1}(n)\binom{n}{n_1}\sum_{l=0}^{n_1}\binom{n_1}{l}(-1)^l \\
\times\int_0^{+\infty}\left\{\begin{array}{c}\left[1-(1+y_1)^{-w}\right]^{(n-n_1+l)v_1} \\ \times wu_1(1+y_1)^{-w-1}\left[1-(1+y_1)^{-w}\right]^{u_1-1}\end{array}\right\}\mathrm{d}y_1 \\
\times\int_0^{+\infty}\left\{\begin{array}{c}\left[1-(1+y_2)^{-w}\right]^{(n-n_1+l)v_2} \\ \times wu_2(1+y_2)^{-w-1}\left[1-(1+y_2)^{-w}\right]^{u_2-1}\end{array}\right\}\mathrm{d}y_2
\end{array}
\right\} \\
&= \sum_{n_1=0}^{n} \overline{p}_{n_1}(n)\binom{n}{n_1}\sum_{l=0}^{n_1}\binom{n_1}{l}(-1)^l\frac{u_1u_2}{[(n-n_1+l)v_1+u_1][(n-n_1+l)v_2+u_2]}.
\end{aligned}
\tag{12}
$$

This is precisely the advantage of the signature theory, as it allows the system structure to be separated from the component lifetime distributions. The three-state nature of the components does not enter the censoring scheme or the likelihood function of the continuous parameters. Instead, it is incorporated into the reliability stage, where the state of a component is defined by comparing its strength variables against the corresponding stresses.

4. Point Estimation

In order to estimate the parameters in the EPD and the stress–strength reliabilities $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$, m independent PFIF-C experiments are first performed, with s ordered variable observations obtained from each experiment. In the data simulation process, the censored observations of the strength variables X_1, X_2 and stress variables Y_1, Y_2 are assumed to be independently generated. Specifically, the censoring process (including censoring time and censoring indicator) for each variable is simulated independently of the other variables. This setup corresponds to practical scenarios where strength and stress data are collected separately and independently—for example, conducting independent experiments on the same batch of components under different stress levels, or obtaining strength and stress data from different experimental platforms. Under this assumption, the likelihood function for each variable factorizes into a product of independent components, thereby simplifying the parameter estimation procedure. The observed values of the strength and stress random variables are expressed as follows

$$\begin{pmatrix} X_{11}^{(i)} & X_{12}^{(i)} & \dots & X_{1s}^{(i)} \\ X_{21}^{(i)} & X_{22}^{(i)} & \dots & X_{2s}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m1}^{(i)} & X_{m2}^{(i)} & \dots & X_{ms}^{(i)} \end{pmatrix}, \begin{pmatrix} Y_{11}^{(i)} & Y_{12}^{(i)} & \dots & Y_{1s}^{(i)} \\ Y_{21}^{(i)} & Y_{22}^{(i)} & \dots & Y_{2s}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{m1}^{(i)} & Y_{m2}^{(i)} & \dots & Y_{ms}^{(i)} \end{pmatrix}$$

where $X_{ij}^{(1)}, X_{ij}^{(2)}, Y_{ij}^{(1)}, Y_{ij}^{(2)}$ denote the j-th ordered observations of the strength variables $(X_1,\ X_2)$ and the stress variables (Y_1, Y_2), respectively, in the i-th PFIF-C experiment, where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, s$.

4.1. Maximum Likelihood Estimator (MLE)

The likelihood function is given by

$$\begin{aligned} L(v_1, v_2, u_1, u_2, w) &= L_{X_1} L_{X_2} L_{Y_1} L_{Y_2} \\ &\propto (wv_1)^{ms} \prod_{i=1}^{m} \prod_{j=1}^{s} \left\{ \begin{array}{l} \left(1+X_{ij}^{(1)}\right)^{-w-1} \left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1-1} \\ \times \left\{1-\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1}\right\}^{h(Q_j+1)-1} \end{array} \right\} \\ &\times (wv_2)^{ms} \prod_{i=1}^{m} \prod_{j=1}^{s} \left\{ \begin{array}{l} \left(1+X_{ij}^{(2)}\right)^{-w-1} \left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2-1} \\ \times \left\{1-\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2}\right\}^{h(Q_j+1)-1} \end{array} \right\} \\ &\times (wu_1)^{ms} \prod_{i=1}^{m} \prod_{j=1}^{s} \left\{ \begin{array}{l} \left(1+Y_{ij}^{(1)}\right)^{-w-1} \left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1-1} \\ \times \left\{1-\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1}\right\}^{h(Q_j+1)-1} \end{array} \right\} \\ &\times (wu_2)^{ms} \prod_{i=1}^{m} \prod_{j=1}^{s} \left\{ \begin{array}{l} \left(1+Y_{ij}^{(2)}\right)^{-w-1} \left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2-1} \\ \times \left\{1-\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2}\right\}^{h(Q_j+1)-1} \end{array} \right\}. \end{aligned} \tag{13}$$

Hence the log-likelihood function can be expressed as

$$
\begin{aligned}
&\ln L(v_1,v_2,u_1,u_2,w) \propto ms[\ln(wv_1)+\ln(wv_2)+\ln(wu_1)+\ln(wu_2)] \\
&-(w+1)\sum_{i=1}^{m}\sum_{j=1}^{s}\left[\ln\left(1+X_{ij}^{(1)}\right)+\ln\left(1+X_{ij}^{(2)}\right)+\ln\left(1+Y_{ij}^{(1)}\right)+\ln\left(1+Y_{ij}^{(2)}\right)\right] \\
&+\sum_{i=1}^{m}\sum_{j=1}^{s}\left\{\begin{array}{l}(v_1-1)\ln\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]+(v_2-1)\ln\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right] \\ +(u_1-1)\ln\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]+(u_2-1)\ln\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]\end{array}\right\} \\
&+\sum_{i=1}^{m}\sum_{j=1}^{s}\left\{\begin{array}{c}\left[h\left(Q_j+1\right)-1\right] \\ \times\left\{\begin{array}{l}\ln\left\{1-\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1}\right\}+\ln\left\{1-\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2}\right\} \\ +\ln\left\{1-\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1}\right\}+\ln\left\{1-\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2}\right\}\end{array}\right\}\end{array}\right\}.
\end{aligned}
\tag{14}
$$

Since the MLEs of the parameters $\hat{v}_1,\ \hat{v}_2,\ \hat{u}_1,\ \hat{u}_2,\ \hat{w}$ cannot be obtained analytically from the equations of first-order partial derivatives of the log-likelihood function in the Appendix A, the Nelder–Mead simplex algorithm is employed to solve them, where the relative convergence tolerance is set to 1×10^{-8} and the maximum number of iterations is 2000. By the invariance property of MLE, substituting the MLEs of the parameters into (11) and (12) yields the MLEs of $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$.

4.2. Bayesian Estimator

In this section, the Bayesian method is used to estimate the parameters v_1, v_2, u_1, u_2, w. Statistical inference for the parameters is performed using both the empirical Bayesian estimation and the weakly informative prior Bayesian estimation.

4.2.1. Empirical Bayesian Estimation

Empirical Bayesian estimation is a parameter estimation method that lies between frequentist and classical Bayesian approaches, addressing the question of how to estimate hyperparameters from the available data. In this framework, the Gamma distribution is frequently employed as a conjugate prior for the parameters of exponential family distributions, owing to its desirable properties. Therefore, in this section, the gamma distribution is chosen as the prior distribution for each parameter. Assuming $v_1, v_2, u_1, u_2, w \sim Gamma(a_i, b_i),\ i = 1,2,3,4,5$, and that the prior distributions of the parameters are independent, the joint prior density is given as follows

$$
\begin{aligned}
\pi(v_1,v_2,u_1,u_2,w) \propto {} & {v_1}^{a_1-1}{v_2}^{a_2-1}{u_1}^{a_3-1}{u_2}^{a_4-1}w^{a_5-1} \\
& \times\exp\{-(b_1v_1+b_2v_2+b_3u_1+b_4u_2+b_5w)\},
\end{aligned}
\tag{15}
$$

where $a_i, b_i, i = 1,2,3,4,5$ are hyperparameters. According to the method proposed by Dey et al. [29] for specifying the hyperparameters of gamma priors, the estimates of the parameters are first obtained via MLE, and then the mean and variance of the gamma priors are set equal to the mean and variance of these MLEs.

Based on the likelihood function (13) and the joint prior distribution (15), the joint posterior distribution and the conditional posterior densities of the parameters can be expressed as

$$
\begin{aligned}
&\pi(v_1,v_2,u_1,u_2,w|X_1,X_2,Y_1,Y_2) \\
&= \frac{L(X_1,X_2,Y_1,Y_2|v_1,v_2,u_1,u_2,w)\pi(v_1,v_2,u_1,u_2,w)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(v_1,v_2,u_1,u_2,w)\pi(v_1,v_2,u_1,u_2,w)\mathrm{d}v_1\mathrm{d}v_2\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}w}
\end{aligned}
\tag{16}
$$

$$\pi_1(v_1|w) \propto {v_1}^{ms+a_1-1}e^{-b_1v_1}\prod_{i=1}^{m}\prod_{j=1}^{s}\left\{\begin{array}{c}\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1-1} \\ \times\left\{1-\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1}\right\}^{h(Q_j+1)-1}\end{array}\right\} \tag{17}$$

$$\pi_2(v_2|w) \propto {v_2}^{ms+a_2-1}e^{-b_2v_2}\prod_{i=1}^{m}\prod_{j=1}^{s}\left\{\begin{array}{c}\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2-1} \\ \times\left\{1-\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2}\right\}^{h(Q_j+1)-1}\end{array}\right\} \tag{18}$$

$$\pi_3(u_1|w) \propto {u_1}^{ms+a_3-1}e^{-b_3u_1}\prod_{i=1}^{m}\prod_{j=1}^{s}\left\{\begin{array}{c}\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1-1} \\ \times\left\{1-\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1}\right\}^{h(Q_j+1)-1}\end{array}\right\} \tag{19}$$

$$\pi_4(u_2|w) \propto {u_2}^{ms+a_4-1}e^{-b_4u_2}\prod_{i=1}^{m}\prod_{j=1}^{s}\left\{\begin{array}{c}\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2-1} \\ \times\left\{1-\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2}\right\}^{h(Q_j+1)-1}\end{array}\right\} \tag{20}$$

$$\pi_5(w|v_1,v_2,u_1,u_2) \propto w^{4ms+a_5-1}e^{-b_5w}\times\prod_{i=1}^{m}\prod_{j=1}^{s}\left\{\begin{array}{l}\left\{\begin{array}{c}\left(1+X_{ij}^{(1)}\right)^{-w-1}\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1-1} \\ \times\left\{1-\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1}\right\}^{h(Q_j+1)-1}\end{array}\right\} \\ \times\left\{\begin{array}{c}\left(1+X_{ij}^{(2)}\right)^{-w-1}\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2-1} \\ \times\left\{1-\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2}\right\}^{h(Q_j+1)-1}\end{array}\right\} \\ \times\left\{\begin{array}{c}\left(1+Y_{ij}^{(1)}\right)^{-w-1}\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1-1} \\ \times\left\{1-\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1}\right\}^{h(Q_j+1)-1}\end{array}\right\} \\ \times\left\{\begin{array}{c}\left(1+Y_{ij}^{(2)}\right)^{-w-1}\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2-1} \\ \times\left\{1-\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2}\right\}^{h(Q_j+1)-1}\end{array}\right\}\end{array}\right\} \tag{21}$$

Under the squared loss function, the Bayesian estimates of the stress–strength reliability $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ are the means of the posterior distribution and can be represented as follows

$$\begin{aligned}\hat{R}_s^{(s_2)} &= E\left(R_s^{(s_2)}|X_1,X_2,Y_1,Y_2\right) \\ &= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} R_s^{(s_2)}\pi(v_1,v_2,u_1,u_2,w|X_1,X_2,Y_1,Y_2)\mathrm{d}v_1\mathrm{d}v_2\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}w\end{aligned} \tag{22}$$

$$\begin{aligned}\hat{R}_s^{(\geq s_1)} &= E\left(R_s^{(\geq s_1)}|X_1,X_2,Y_1,Y_2\right) \\ &= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} R_s^{(\geq s_1)}\pi(v_1,v_2,u_1,u_2,w|X_1,X_2,Y_1,Y_2)\mathrm{d}v_1\mathrm{d}v_2\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}w\end{aligned} \tag{23}$$

Bayesian estimation of each parameter and the stress–strength reliability is performed using the MCMC algorithm. However, when the conditional distribution cannot be directly sampled, a Metropolis–Hastings (MH) step is executed individually for each parameter. Therefore, a block-update Metropolis-within-Gibbs algorithm is developed by combining Gibbs sampling with the MH algorithm. The specific steps are shown in Algorithm 1.

Algorithm 1. Block-update Metropolis-within-Gibbs algorithm

Step 1. Input initial parameters $v_1^{(0)}, v_2^{(0)}, u_1^{(0)}, u_2^{(0)}, w^{(0)}$ (with the MLE values of each parameter as the initial values).
Step 2. At each iteration t, new candidate parameter values $v_1^*, v_2^*, u_1^*, u_2^*$ and w^* are generated by sampling from the proposal distributions

$$q\left(v_1^*|v_1^{(t-1)}\right),\ q\left(v_2^*|v_2^{(t-1)}\right),\ q\left(u_1^*|u_1^{(t-1)}\right),\ q\left(u_2^*|u_2^{(t-1)}\right),\ q\left(w^*|w^{(t-1)}\right).$$

In the random walk MH algorithm, a symmetric distribution is typically used as the proposal distribution, such as the normal distribution or the uniform distribution.
Step 3. Compute the acceptance probability for each parameter based on the ratio of the target conditional distributions. Since symmetric proposal distributions are used, the proposal ratio cancels out.

$$A_{v_1} = \min\left\{1, \frac{\pi_1\left(v_1^*|w^{(t-1)}\right)}{\pi_1\left(v_1^{(t-1)}|w^{(t-1)}\right)}\right\}, A_{v_2} = \min\left\{1, \frac{\pi_2\left(v_2^*|w^{(t-1)}\right)}{\pi_2\left(v_2^{(t-1)}|w^{(t-1)}\right)}\right\},$$
$$A_{u_1} = \min\left\{1, \frac{\pi_3\left(u_1^*|w^{(t-1)}\right)}{\pi_3\left(u_1^{(t-1)}|w^{(t-1)}\right)}\right\}, A_{u_2} = \min\left\{1, \frac{\pi_4\left(u_2^*|w^{(t-1)}\right)}{\pi_4\left(u_4^{(t-1)}|w^{(t-1)}\right)}\right\},$$
$$A_w = \min\left\{1, \frac{\pi_5\left(w^*|v_1^{(t-1)}, v_2^{(t-1)}, u_1^{(t-1)}, u_2^{(t-1)}\right)}{\pi_5\left(w^{(t-1)}|v_1^{(t-1)}, v_2^{(t-1)}, u_1^{(t-1)}, u_2^{(t-1)}\right)}\right\},$$

Step 4. Generate random numbers $u_{v_1}, u_{v_2}, u_{u_1}, u_{u_2}$ and u_w from a uniform distribution $U(0,1)$.

(1) If $u_{v_1} \le A_{v_1}$, accept v_1^* and let $v_1^{(t)} = v_1^*$. Otherwise, let $v_1^{(t)} = v_1^{(t-1)}$.
(2) If $u_{v_2} \le A_{v_2}$, accept v_2^* and let $v_2^{(t)} = v_2^*$. Otherwise, let $v_2^{(t)} = v_2^{(t-1)}$.
(3) If $u_{u_1} \le A_{u_1}$, accept u_1^* and let $u_1^{(t)} = u_1^*$. Otherwise, let $u_1^{(t)} = u_1^{(t-1)}$.
(4) If $u_{u_2} \le A_{u_2}$, accept u_2^* and let $u_2^{(t)} = u_2^*$. Otherwise, let $u_2^{(t)} = u_2^{(t-1)}$.
(5) If $u_w \le A_w$, accept w^* and let $w^{(t)} = w^*$. Otherwise, let $w^{(t)} = w^{(t-1)}$.

Step 5. Set the accepted or current parameter values as the initial values for the next iteration, and repeat Steps 2–4 until convergence or the maximum number of iterations is attained.
Step 6. The posterior expectation is approximated by Monte Carlo integration using MCMC samples

$$\hat{R}_s^{(s_2)} = \frac{1}{N}\sum_{i=1}^{N} R_s^{(s_2)}\left(v_1^{(i)}, v_2^{(i)}, u_1^{(i)}, u_2^{(i)}, w^{(t)}\right),$$
$$\hat{R}_s^{(\ge s_1)} = \frac{1}{N}\sum_{i=1}^{N} R_s^{(\ge s_1)}\left(v_1^{(i)}, v_2^{(i)}, u_1^{(i)}, u_2^{(i)}, w^{(t)}\right),$$

where N is the number of MCMC posterior samples generated during the iteration process.
Step 7. Output the Bayesian estimates of each parameter and the stress–strength reliability.

4.2.2. Weakly Informative Prior Bayesian Estimation

This section adopts a weakly informative prior for Bayesian estimation and continues to use the Gamma prior distribution. The main advantage of the Gamma prior is that, by adjusting its shape and rate parameters, it can flexibly achieve a smooth transition from a non-informative prior to an informative prior. Specifically, when both parameters approach zero, the Gamma prior reduces to a weakly informative prior, also known as a diffuse prior. Based on this property, the hyperparameters of the Gamma prior are uniformly set to $a_i = 0.001$, $b_i = 0.001$, $i = 1, 2, \ldots, 5$. Under this specification, the Gamma density becomes extremely flat over the positive real line, approximating a uniform distribution and imposing almost no prior beliefs. Consequently, the posterior distribution is primarily determined by the data. This specification is suitable for situations where prior information is lacking and effectively ensures the objectivity of the inference results.

5. Interval Estimation

5.1. Asymptotic Confidence Interval

Let $\boldsymbol{\theta} = (v_1, v_2, u_1, u_2, w)$ be the parameter vector, $H(\boldsymbol{\theta})$ is the matrix of second-order partial derivatives of the log-likelihood function with respect to all parameters (i.e., the Hessian matrix), and $I(\boldsymbol{\theta})$ is the Fisher information matrix. The results of the second derivative calculations are detailed in the Appendix A.

$$H(\boldsymbol{\theta}) = \begin{pmatrix} \frac{\partial^2 \ln L}{\partial v_1^2} & \frac{\partial^2 \ln L}{\partial v_1 \partial v_2} & \frac{\partial^2 \ln L}{\partial v_1 \partial u_1} & \frac{\partial^2 \ln L}{\partial v_1 \partial u_2} & \frac{\partial^2 \ln L}{\partial v_1 \partial w} \\ \frac{\partial^2 \ln L}{\partial v_2 \partial v_1} & \frac{\partial^2 \ln L}{\partial v_2^2} & \frac{\partial^2 \ln L}{\partial v_2 \partial u_1} & \frac{\partial^2 \ln L}{\partial v_2 \partial u_2} & \frac{\partial^2 \ln L}{\partial v_2 \partial w} \\ \frac{\partial^2 \ln L}{\partial u_1 \partial v_1} & \frac{\partial^2 \ln L}{\partial u_1 \partial v_2} & \frac{\partial^2 \ln L}{\partial u_1^2} & \frac{\partial^2 \ln L}{\partial u_1 \partial u_2} & \frac{\partial^2 \ln L}{\partial u_1 \partial w} \\ \frac{\partial^2 \ln L}{\partial u_2 \partial v_1} & \frac{\partial^2 \ln L}{\partial u_2 \partial v_2} & \frac{\partial^2 \ln L}{\partial u_2 \partial u_1} & \frac{\partial^2 \ln L}{\partial u_2^2} & \frac{\partial^2 \ln L}{\partial u_2 \partial w} \\ \frac{\partial^2 \ln L}{\partial w \partial v_1} & \frac{\partial^2 \ln L}{\partial w \partial v_2} & \frac{\partial^2 \ln L}{\partial w \partial u_1} & \frac{\partial^2 \ln L}{\partial w \partial u_2} & \frac{\partial^2 \ln L}{\partial w^2} \end{pmatrix} \tag{24}$$

$$\boldsymbol{I}(\boldsymbol{\theta}) = -E[H(\boldsymbol{\theta})]$$

Obviously, the exact result of the expectation is difficult to obtain. Since the observed Fisher information matrix is a consistent estimator of the expected Fisher information matrix under regularity conditions, the observed information matrix can be used in place of the expected information matrix to construct confidence intervals or perform hypothesis testing. Therefore, the expectation operator E is omitted, and the parameter $\boldsymbol{\theta}$ is directly replaced by its MLE $\hat{\boldsymbol{\theta}}$ to obtain the observed Fisher information matrix, which is used to approximate the asymptotic variance–covariance matrix.

The inverse matrix of $\boldsymbol{I}(\hat{\boldsymbol{\theta}})$ is denoted as $\boldsymbol{I}^{-1}(\hat{\boldsymbol{\theta}})$, and the diagonal elements of $\boldsymbol{I}^{-1}(\hat{\boldsymbol{\theta}})$ are the asymptotic variances of the estimates $\hat{v}_1, \hat{v}_2, \hat{u}_1, \hat{u}_2, \hat{w}$, respectively. Then, the $(1-\alpha)100\%$ confidence interval for $\boldsymbol{\theta}$ is

$$\left(\hat{\boldsymbol{\theta}} - z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\boldsymbol{\theta}})}, \hat{\boldsymbol{\theta}} + z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\boldsymbol{\theta}})}\right),$$

where $z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ quantile of the standard normal distribution.

In the PDF of EPD, with v, $w > 0$, for any $x > 0$, it holds that $wv > 0$, $(1+x)^{-w-1} > 0$, $0 < (1+x)^{-w} < 1$, and $\left[1-(1+x)^{-w}\right]^{v-1} > 0$. Therefore, regardless of the values of v and w, the support of this distribution is always $(0, \infty)$, which does not depend on the specific values of the parameters v and w. All second-order partial derivatives of the log-likelihood function have been derived in the Appendix A. Since $v, w > 0$, and in practice, X_1, X_2, Y_1, Y_2 all have positive lower bounds, these second-order partial derivatives are continuous everywhere in their domain. The analytical expression of $I(\boldsymbol{\theta})$ is extremely complex, and its expectation involves high-dimensional integrals, making a closed-form symbolic proof difficult. Therefore, a widely accepted approach is used to verify that the matrix is nonsingular: numerical scanning over a reasonable parameter space. Based on the parameter vector $\hat{\boldsymbol{\theta}} = (\hat{v}_1, \hat{v}_2, \hat{u}_1, \hat{u}_2, \hat{w})$, a broad parameter space Θ is specified (for example, $\left[0.9 \times \hat{\theta}_i, 1.1 \times \hat{\theta}_i\right]$ for each parameter). $N = 1000$ parameter combinations are uniformly sampled within Θ, and for each combination, the determinant of the observed information matrix is numerically computed. The determinant is always greater than zero, and all eigenvalues of the matrix are positive. The experimental results are shown in Figures A1 and A2 in the Appendix B. Figure A1 shows the distribution of the natural logarithm of the determinant of the observed information matrix, and Figure A2 shows the distribution of the five eigenvalues. From these figures, it can be concluded that the observed information matrix is positive definite. For joint MLE based on independent samples, classical theory guarantees joint asymptotic normality. Under an independent censoring scheme, as long as the censoring scheme is predetermined and does not depend on unknown parameters, the asymptotic normality of the MLE remains valid [30]. The accuracy of the first-order and second-order partial derivatives of the log-likelihood function was validated through numerical differentiation, as detailed in Table A1 in the Appendix B.

Therefore, the MLE of the parameter vector $\boldsymbol{\theta}$ is asymptotically normal. Furthermore, since the reliability model is a differentiable function of these parameters (with first-order partial derivatives provided in the Appendix A), the Delta method implies that its estimator is also asymptotically normal [31,32]. According to the multivariate Delta method, if the estimator of $\boldsymbol{\theta}$ is asymptotically normal and $\boldsymbol{\varphi}(\cdot)$ is a function that is differentiable at $\boldsymbol{\theta}$, then the estimator of this function is also asymptotically normal. Define

$$\nabla\boldsymbol{\varphi}^T_{(s_2)} = \left(\frac{\partial R_s^{(s_2)}}{\partial v_1}, \frac{\partial R_s^{(s_2)}}{\partial v_2}, \frac{\partial R_s^{(s_2)}}{\partial u_1}, \frac{\partial R_s^{(s_2)}}{\partial u_2}, \frac{\partial R_s^{(s_2)}}{\partial w}\right),$$

$$\nabla\boldsymbol{\varphi}^T_{(\geq s_1)} = \left(\frac{\partial R_s^{(\geq s_1)}}{\partial v_1}, \frac{\partial R_s^{(\geq s_1)}}{\partial v_2}, \frac{\partial R_s^{(\geq s_1)}}{\partial u_1}, \frac{\partial R_s^{(\geq s_1)}}{\partial u_2}, \frac{\partial R_s^{(\geq s_1)}}{\partial w}\right).$$

where the first derivatives of $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ with respect to v_1, v_2, u_1, u_2, w are detailed in the Appendix A.

The approximate $(1-\alpha)100\%$ confidence intervals for $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ are

$$\left(\hat{R}_s^{(s_2)} - z_{\frac{\alpha}{2}}\sqrt{Var\left(\hat{R}_s^{(s_2)}\right)}, \hat{R}_s^{(s_2)} + z_{\frac{\alpha}{2}}\sqrt{Var\left(\hat{R}_s^{(s_2)}\right)}\right),$$

$$\left(\hat{R}_s^{(\geq s_1)} - z_{\frac{\alpha}{2}}\sqrt{Var\left(\hat{R}_s^{(\geq s_1)}\right)}, \hat{R}_s^{(\geq s_1)} + z_{\frac{\alpha}{2}}\sqrt{Var\left(\hat{R}_s^{(\geq s_1)}\right)}\right),$$

where $Var\left(\hat{R}_s^{(s_2)}\right) = \nabla\boldsymbol{\varphi}^T_{(s_2)}\boldsymbol{I}^{-1}(\hat{\boldsymbol{\theta}})\nabla\boldsymbol{\varphi}_{(s_2)}$, $Var\left(\hat{R}_s^{(\geq s_1)}\right) = \nabla\boldsymbol{\varphi}^T_{(\geq s_1)}\boldsymbol{I}^{-1}(\hat{\boldsymbol{\theta}})\nabla\boldsymbol{\varphi}_{(\geq s_1)}$.

5.2. Highest Posterior Density (HPD) Interval

The HPD interval is the shortest credible interval in Bayesian statistics. This is due to the fact that, for a given credible level $1-\alpha$, the HPD interval selects the region with the highest posterior density, thereby minimizing the interval length. Algorithmically, for MCMC posterior samples, the shortest interval covering the $1-\alpha$ probability is found via a sliding window approach. This can also be implemented in R using the coda package. In this paper, the HPD interval is computed using the sliding window algorithm based on sorted samples. The HPDinterval( ) function from the coda package is used as the primary method. When the coda package is unavailable, the sliding window algorithm is implemented manually.

6. Simulation

6.1. System Description

The multi-state consecutive k-out-of-n: G system has wide applications in fields such as power systems, communication networks, missile systems, and sensor networks. Taking the three-state consecutive 3-out-of-5: G system as an example, let the component states be s_0, s_1 and s_2 (s_0 represents complete failure, s_1 represents partial degradation, and s_2 represents perfect functioning). For a given threshold level s_k (where $k = 1$ or 2), the system is in state s_k or above if and only if there exists at least one set of three consecutive components whose states are all not lower than s_k.

Tables 1 and 2 are the generalized survival signatures of $\overline{p}_{n_1,n_2}(n)$ and $\overline{p}_{n_1}(n)$, respectively. Table 1 is used to compute $R_s^{(s_2)}$, and Table 2 is used to compute $R_s^{(\geq s_1)}$. Taking $\overline{p}_{4,4}(5)$ in Table 1 as an example, according to Formula (3), the numerator $r_{4,4}(5)$ indicates that among four components that are in state s_1 or above, a consecutive segment of length 4 must be included. Let the component numbers be 1, 2, 3, 4 and 5. There are two cases for the path set: $\{1,2,3,4\}$ and $\{2,3,4,5\}$. The denominator is $\binom{5}{4}\binom{1}{0} = 5$. Thus, $\overline{p}_{4,4}(5) = 0.4$.

Table 1. The values of the generalized survival signature $\overline{p}_{n_1,n_2}(n)$.

n_1	n_2	$r_{n_1,n_2}(n)$	$\overline{p}_{n_1,n_2}(n)$
3	3	3	0.3
4	3	6	0.3
4	4	2	0.4
5	3	3	0.3
5	4	4	0.8
5	5	1	1

Table 2. The values of the generalized survival signature $\overline{p}_{n_1}(n)$.

n_1	$r_{n_1}(n)$	$\overline{p}_{n_1}(n)$
3	3	0.3
4	2	0.4
5	1	1

6.2. Parameters Estimation

The PFIF-C data for the random variables X_1, X_2, Y_1, Y_2 are generated using the inverse transform method. The specific Algorithm 2 procedure is as follows.

Algorithm 2. Numerical simulation

Step 1. Input the true parameter vector $\boldsymbol{\theta} = (v_1, v_2, u_1, u_2, \omega) = (3, 2.5, 1.8, 1.5, 2.5)$.
Step 2. $X_i \sim \text{EPD}(v_i, w)$ and $Y_i \sim \text{EPD}(u_i, w)$, $i = 1, 2$. For each random variable, M independent random numbers are generated from the uniform distribution $U(0, 1)$. Then, a complete set of failure sample data is generated using the inverse transform method.
Step 3. PFIF-C samples are generated from the complete failure sample data. The $M = 150$ observations are randomly divided into 30 groups, with five observations in each group. The minimum value in each group is taken as the "first failure time" of that group. In this way, 30 "group first failure time samples" are obtained.
Step 4. Assume $s = 15$ is the observed number of failures. The censoring scheme is set as $Q = (0, 0, 2, 0, 0, 3, 0, 3, 0, 4, 0, 1, 0, 0, 2)$, where Q_i $(i = 1, 2, \ldots, 15)$ is the number of additional groups removed at each failure. PFIF-C samples are generated using the progressive censoring scheme.

Algorithm 2. *Cont.*

(1) The first failure times corresponding to the 30 groups are sorted in ascending order.
(2) For the sorted groups, upon the occurrence of the first failure group, that group is removed, followed by the random removal of Q_1 additional groups, and the corresponding failure observation data are recorded.
(3) When the second failure group occurs, remove this group, then randomly remove Q_2 additional groups, and record the corresponding failure observation data. This procedure is repeated successively until the s-th failure group occurs, whereupon this group and all remaining groups are removed, and the corresponding failure observation data are recorded.
(4) Finally, s failure observation data are obtained, which constitute the PFIF-C sample.

Step 5. Repeat Steps 2–4 m times to obtain several groups of observations corresponding to the stress variables and strength variables.

6.2.1. Parameter Estimation for Stress and Strength Variables

In this section, simulation experiments are conducted under two different sample sizes to compare the performance of three methods: MLE, empirical Bayesian estimation, and weakly informative Bayesian estimation. First, 100 sets of PFIF-C data are generated using Algorithm 2, from which 10 or 30 sets are randomly drawn each time to compute the estimates. A total of 1000 simulation runs are performed. Figures 2 and 3 show the cumulative averages of the three algorithms under these two sample sizes, respectively.

It can be observed from Figures 2 and 3 that as the number of simulations increases, the cumulative averages of the parameters gradually converge and stabilize. Comparing Figures 2 and 3, under small sample sizes, the convergence rates of the cumulative averages of the MLE, empirical Bayesian estimation, and weakly informative Bayesian estimation are similar, with no significant differences in fluctuation patterns. As the sample size gradually increases, empirical Bayesian estimation achieves the fastest convergence rate by adaptively updating the prior hyperparameters using the marginal information from the samples. Relying on asymptotic normality and the law of large numbers, MLE exhibits the second fastest convergence rate. Due to the lack of prior information constraints and consequently a more dispersed posterior distribution, weakly informative prior Bayesian estimation shows the slowest convergence rate.

Tables 3–5 present the estimation results and error analysis for MLE, empirical Bayesian estimation, and weakly informative prior Bayesian estimation based on 1000 simulation experiments. When the sample size is 10, i.e., under small sample conditions, empirical Bayesian estimation is the most accurate based on bias and RMSE analysis. The biases of the parameters range from 0.0191 to 0.0905, and the RMSE ranges from 0.0398 to 0.1139. Weakly informative prior Bayesian estimation ranks second, with biases ranging from 0.0196 to 0.0937 and RMSE ranging from 0.0849 to 0.1971. MLE performs the worst, with biases ranging from 0.0202 to 0.0941 and RMSE ranging from 0.0850 to 0.1967. This is primarily because, under small samples, MLE is purely data-driven without regularization. The weakly informative prior introduces vague regularization and moderately shrinks the estimates, resulting in lower bias than MLE; however, the prior is fixed and has limited adaptability. Empirical Bayesian estimation automatically learns the prior hyperparameters from the data, achieving adaptive shrinkage, the smallest bias, and the optimal stability.

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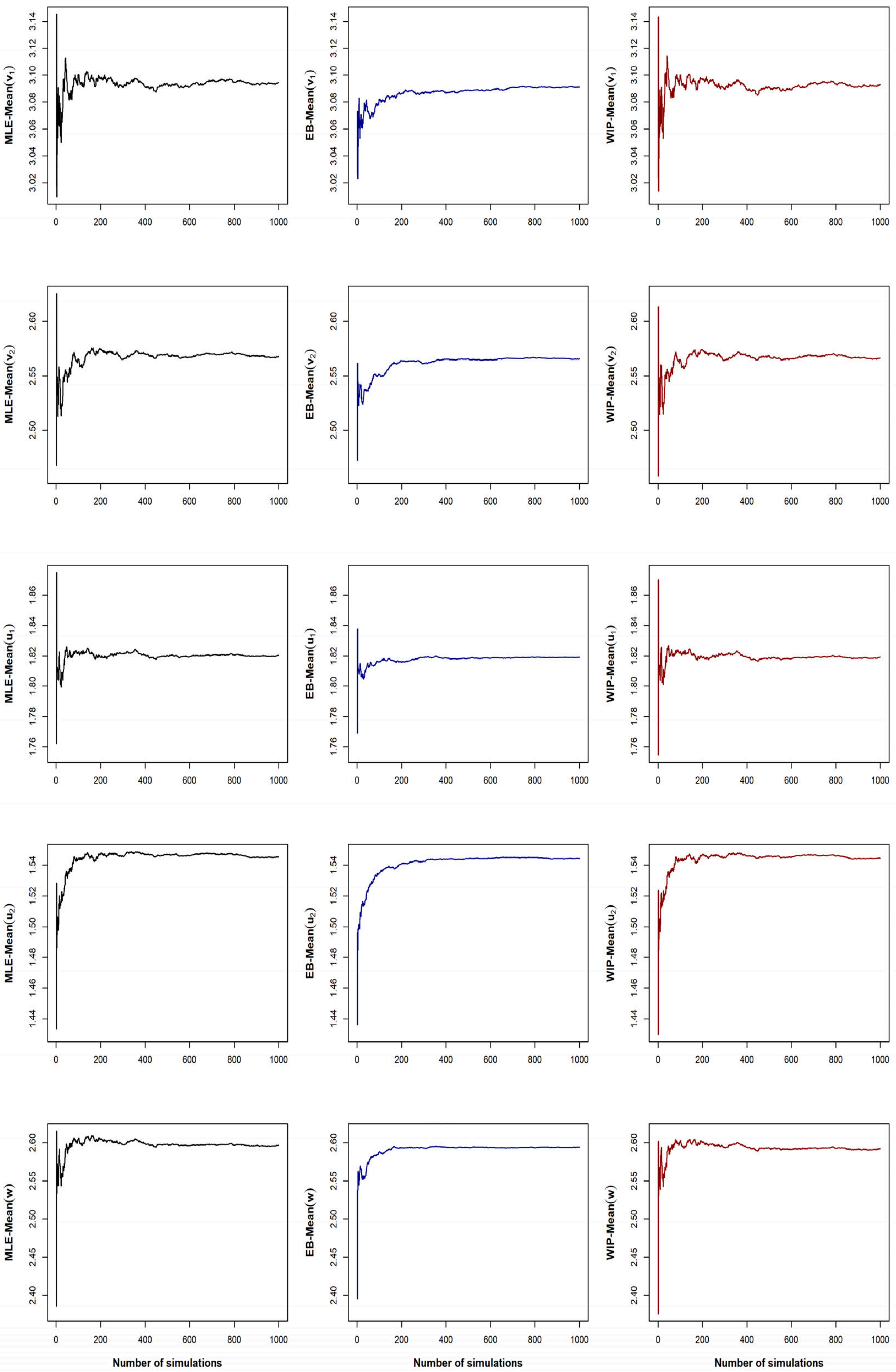

Figure 2. Cumulative mean convergence trajectories for v_1, v_2, u_1, u_2, w based on 1000 simulations with sample size $n = 10$.

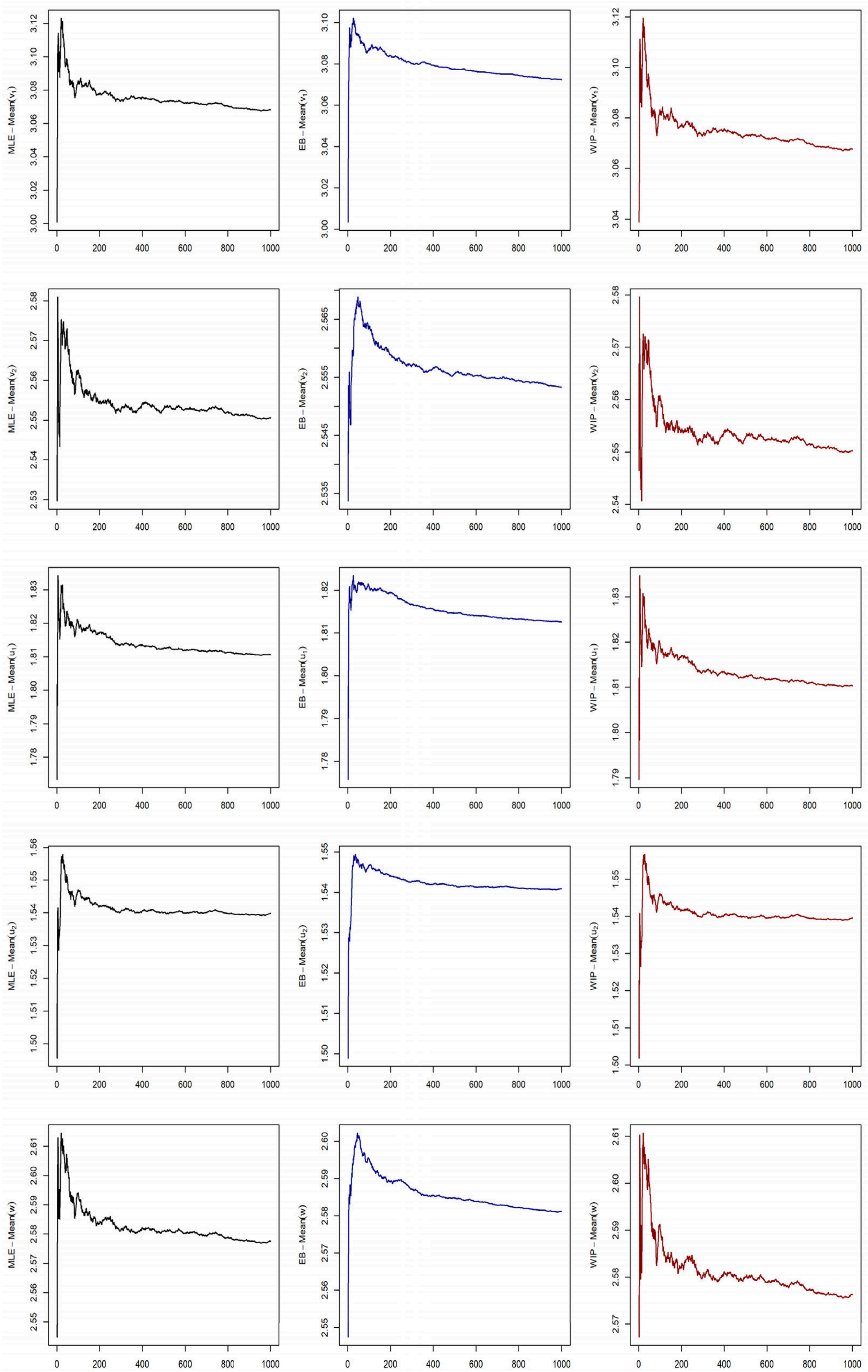

Figure 3. Cumulative mean convergence trajectories for v_1, v_2, u_1, u_2, w based on 1000 simulations with sample size $n = 30$.

Table 3. MLE means for parameters of stress and strength variables under different sample sizes.

Sample Size	MLE	Estimate	Bias	RMSE	ACI_L	ACI_U	Width	CP
$(n_{X_1}, n_{X_2}, n_{Y_1}, n_{Y_2}) = (10, 10, 10, 10)$	$\hat{v}_1$	3.0941	0.0941	0.1967	2.7261	3.4620	0.7359	95.4%
	$\hat{v}_2$	2.5674	0.0674	0.1513	2.2868	2.8479	0.5611	95.6%
	$\hat{u}_1$	1.8202	0.0202	0.0850	1.6480	1.9925	0.3445	96.0%
	$\hat{u}_2$	1.5454	0.0454	0.0858	1.4078	1.6829	0.2751	91.0%
	$\hat{\omega}$	2.5966	0.0966	0.1786	2.2968	2.8964	0.5996	91.7%
$(n_{X_1}, n_{X_2}, n_{Y_1}, n_{Y_2}) = (30, 30, 30, 30)$	$\hat{v}_1$	3.0681	0.0681	0.1096	2.8580	3.2782	0.4203	95.6%
	$\hat{v}_2$	2.5507	0.0507	0.0845	2.3900	2.7113	0.3213	95.5%
	$\hat{u}_1$	1.8106	0.0106	0.0427	1.7118	1.9094	0.1976	97.9%
	$\hat{u}_2$	1.5398	0.0398	0.0551	1.4607	1.6189	0.1582	90.6%
	$\hat{\omega}$	2.5777	0.0777	0.1078	2.4054	2.7500	0.3446	91.8%

Table 4. Empirical Bayesian posterior means for parameters of stress and strength variables under different sample sizes.

Sample Size	EB	Estimate	Bias	RMSE	HPD_L	HPD_U	Width	CP
$(n_{X_1}, n_{X_2}, n_{Y_1}, n_{Y_2}) = (10, 10, 10, 10)$	$\hat{v}_1$	3.0905	0.0905	0.1139	2.8806	3.3031	0.4225	96.6%
	$\hat{v}_2$	2.5650	0.0650	0.0868	2.4015	2.7307	0.3293	95.4%
	$\hat{u}_1$	1.8191	0.0191	0.0398	1.7141	1.9249	0.2107	99.6%
	$\hat{u}_2$	1.5443	0.0443	0.0560	1.4590	1.6305	0.1714	90.0%
	$\hat{\omega}$	2.5937	0.0937	0.1075	2.4407	2.7472	0.3065	88.1%
$(n_{X_1}, n_{X_2}, n_{Y_1}, n_{Y_2}) = (30, 30, 30, 30)$	$\hat{v}_1$	3.0723	0.0723	0.0803	2.9517	3.1939	0.2422	91.4%
	$\hat{v}_2$	2.5533	0.0533	0.0606	2.4592	2.6482	0.1890	92.4%
	$\hat{u}_1$	1.8126	0.0126	0.0220	1.7521	1.8732	0.1210	99.3%
	$\hat{u}_2$	1.5409	0.0409	0.0445	1.4919	1.5903	0.0984	88.4%
	$\hat{\omega}$	2.5811	0.0811	0.0852	2.4929	2.6691	0.1762	85.2%

Table 5. Weakly informative prior Bayesian posterior means for parameters of stress and strength variables under different sample sizes.

Sample Size	WIP	Estimate	Bias	RMSE	HPD_L	HPD_U	Width	CP
$(n_{X_1}, n_{X_2}, n_{Y_1}, n_{Y_2}) = (10, 10, 10, 10)$	$\hat{v}_1$	3.0937	0.0937	0.1971	2.7409	3.4544	0.7135	94.6%
	$\hat{v}_2$	2.5666	0.0666	0.1521	2.2978	2.8423	0.5445	94.7%
	$\hat{u}_1$	1.8196	0.0196	0.0849	1.6536	1.9885	0.3349	94.8%
	$\hat{u}_2$	1.5449	0.0449	0.0856	1.4118	1.6801	0.2682	90.2%
	$\hat{\omega}$	2.5929	0.0929	0.1774	2.3052	2.883	0.5777	89.1%
$(n_{X_1}, n_{X_2}, n_{Y_1}, n_{Y_2}) = (30, 30, 30, 30)$	$\hat{v}_1$	3.0678	0.0678	0.1109	2.8663	3.2732	0.4069	93.5%
	$\hat{v}_2$	2.5503	0.0503	0.0857	2.3958	2.7069	0.3111	93.1%
	$\hat{u}_1$	1.8104	0.0104	0.0433	1.715	1.9069	0.1919	97.3%
	$\hat{u}_2$	1.5396	0.0396	0.0552	1.4628	1.6171	0.1542	86.2%
	$\hat{\omega}$	2.5763	0.0763	0.1083	2.4114	2.7435	0.3321	86.6%

From the perspective of interval estimation, the empirical Bayesian estimation yields the shortest HPD credible intervals for all parameters, with lengths ranging from 0.1714 to 0.4225. The weakly informative prior Bayesian estimation ranks second, with HPD credible interval lengths ranging from 0.2682 to 0.7135. The asymptotic confidence intervals of MLE are the longest, with lengths ranging from 0.2751 to 0.7359. However, in terms of coverage probability (CP), MLE performs slightly better than the two Bayesian methods. This is primarily due to the degree of shrinkage inherent in each estimation method. Empirical Bayesian estimation adaptively learns the prior hyperparameters from the data, resulting in the strongest shrinkage, a highly concentrated posterior distribution, and the shortest HPD intervals. Weakly informative prior Bayesian estimation provides moderate shrinkage, but its adaptive capability is inferior to that of empirical Bayesian estimation, leading to interval estimates of moderate length. The MLE asymptotic confidence interval involves no prior shrinkage and is entirely determined by the sample information. Under small sample conditions, sampling fluctuations are large, resulting in greater variance and the widest intervals. From the perspective of coverage probability, the reason why MLE has a slightly higher coverage rate than the two Bayesian methods lies in the different logic of interval construction. MLE constructs confidence intervals based on the classical frequentist approach, resulting in very wide intervals under small sample conditions. In contrast, Bayesian HPD credible intervals are posterior probability intervals that trade off a slight loss in coverage probability for reduced bias and RMSE in parameter estimation, with inherent shrinkage leading to shorter intervals. When the intervals become narrower, the range that contains the true parameter becomes smaller, resulting in a slightly lower coverage probability.

As the sample size increases, the statistical information contained in the data becomes richer, and the biases of all three estimation methods gradually decrease. However, a horizontal comparison reveals that the bias of empirical Bayesian estimation ranges from 0.0126 to 0.0811, that of MLE ranges from 0.0106 to 0.0777, and that of weakly informative prior Bayesian estimation ranges from 0.0104 to 0.0763. Empirical Bayesian estimation introduces hyperparameter estimation error, resulting in larger bias compared to the other two methods. Furthermore, the coverage probabilities of the HPD credible intervals for both Bayesian methods show a slight decrease. This is because as the sample size increases, the posterior distribution becomes tighter, and the interval estimates shrink, leading to a slightly lower probability that the true value falls within the credible interval. In contrast, the asymptotic confidence intervals of MLE gradually improve in terms of asymptotic properties, and their coverage probability increases as the sample size grows.

6.2.2. Estimation for Stress–Strength Reliabilities $\boldsymbol{R_s^{(s_2)}}$ and $\boldsymbol{R_s^{(\geq s_1)}}$

Figures 4 and 5 present the trends of the cumulative averages of stress–strength reliabilities $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ under different sample sizes across 1000 simulation runs. It can be observed from the figures that as the number of experiments increases, the cumulative averages gradually stabilize. A comparison between Figures 4 and 5 reveals that when the sample size increases, stress–strength reliabilities $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ stabilize more rapidly. In terms of the magnitude of curve fluctuation, the stability of empirical Bayesian estimation is slightly better than that of MLE, and MLE is slightly better than that of weakly informative prior Bayesian estimation.

Tables 6–8 present the estimation results of stress–strength reliabilities $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ under MLE, empirical Bayesian estimation, and weakly informative prior Bayesian estimation for different sample sizes. The estimation biases of the reliabilities decrease as the sample size increases. A horizontal comparison reveals that the overall biases of the models are almost identical, with very small differences. The main reason for this is, on the one hand, that complex models have a smoothing and diluting effect on the errors of individual parameters, making the final model insensitive to estimation errors of single parameters. Only when the parameter estimates exhibit systematic large biases do the model outputs show significant differences. On the other hand, these parameter estimation methods themselves are asymptotically consistent, so the estimated values fluctuate within a small range around the true values, resulting in negligible differences in bias at the model level.

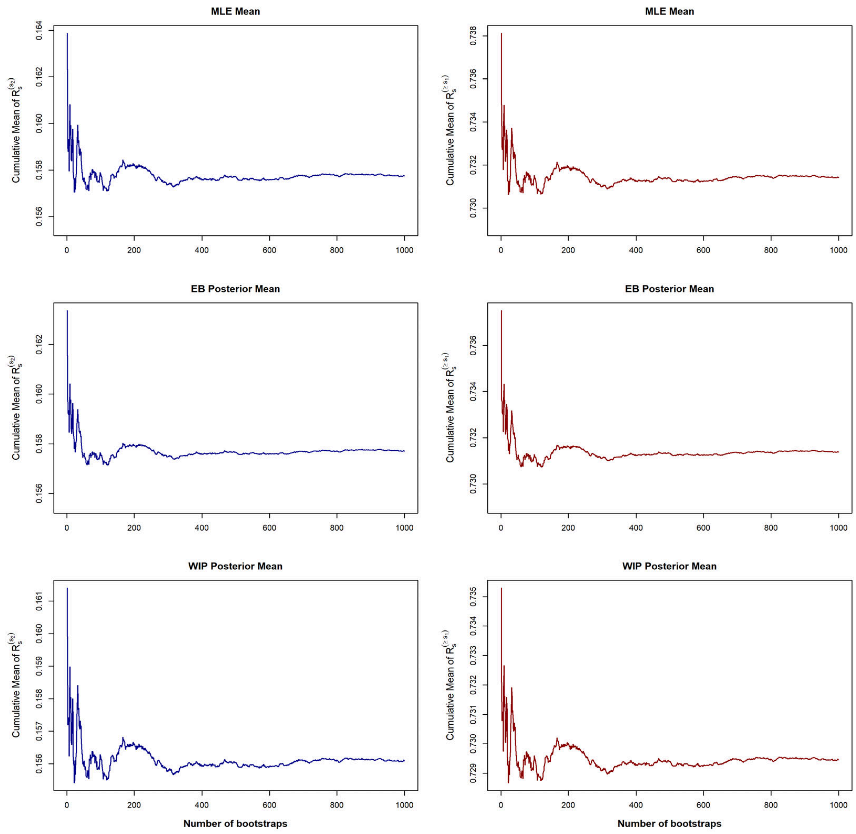

Figure 4. Cumulative means of stress–strength reliabilities when sample size is equal to 10.

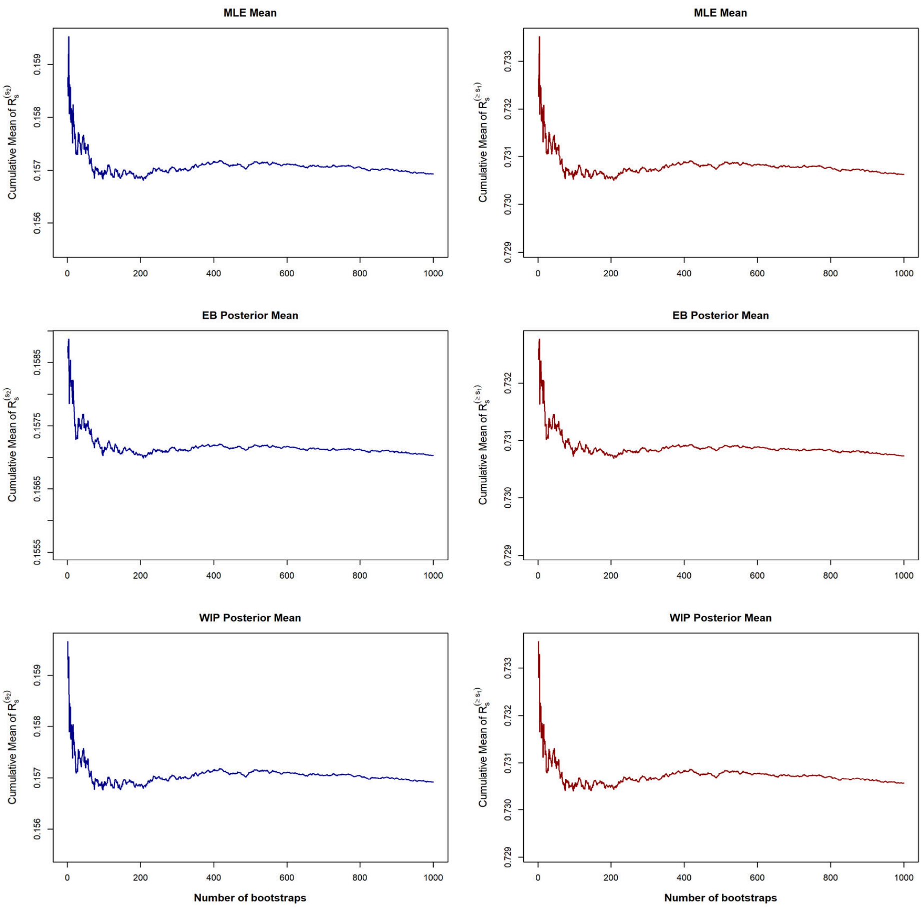

Figure 5. Cumulative means of stress–strength reliabilities when sample size is equal to 30.

Table 6. Means of stress–strength reliabilities $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ under MLE.

Sample Size	SSR	Estimate	Bias	RMSE	CI_L	CI_U	Width	CP
$(n_{X_1}, n_{X_2}, n_{Y_1}, n_{Y_2}) = (10, 10, 10, 10)$	$\hat{R}_s^{(S_2)}$	0.1577	0.0022	0.0094	0.1398	0.1757	0.0200	92.2%
	$\hat{R}_s^{(\geq S_1)}$	0.7314	0.0023	0.0102	0.7119	0.7510	0.0307	95.5%
$(n_{X_1}, n_{X_2}, n_{Y_1}, n_{Y_2}) = (30, 30, 30, 30)$	$\hat{R}_s^{(S_2)}$	0.1569	0.0014	0.0046	0.1484	0.1655	0.0117	96.0%
	$\hat{R}_s^{(\geq S_1)}$	0.7306	0.0016	0.0050	0.7213	0.7400	0.0178	99.1%

Table 7. Posterior means of stress–strength reliabilities $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ under empirical Bayes estimation.

Sample Size	SSR	Estimate	Bias	RMSE	HPD_L	HPD_U	Width	CP
$(n_{X_1}, n_{X_2}, n_{Y_1}, n_{Y_2}) = (10, 10, 10, 10)$	$\hat{R}_s^{(S_2)}$	0.1577	0.0022	0.0065	0.1423	0.1734	0.0311	98.4%
	$\hat{R}_s^{(\geq S_1)}$	0.7314	0.0023	0.0071	0.7143	0.7483	0.0340	98.0%
$(n_{X_1}, n_{X_2}, n_{Y_1}, n_{Y_2}) = (30, 30, 30, 30)$	$\hat{R}_s^{(S_2)}$	0.1570	0.0015	0.0033	0.1481	0.1660	0.0179	99.2%
	$\hat{R}_s^{(\geq S_1)}$	0.7307	0.0017	0.0036	0.7209	0.7406	0.0197	99.0%

Table 8. Posterior means of stress–strength reliabilities $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ under weakly informative priors.

Sample Size	SSR	Estimate	Bias	RMSE	HPD_L	HPD_U	Width	CP
$(n_{X_1}, n_{X_2}, n_{Y_1}, n_{Y_2}) = (10, 10, 10, 10)$	$\hat{R}_s^{(S_2)}$	0.1577	0.0022	0.0095	0.1384	0.1774	0.0390	96.0%
	$\hat{R}_s^{(\geq S_1)}$	0.7312	0.0021	0.0103	0.7097	0.7525	0.0428	95.3%
$(n_{X_1}, n_{X_2}, n_{Y_1}, n_{Y_2}) = (30, 30, 30, 30)$	$\hat{R}_s^{(S_2)}$	0.1569	0.0014	0.0046	0.1457	0.1682	0.0225	98.4%
	$\hat{R}_s^{(\geq S_1)}$	0.7306	0.0015	0.0050	0.7182	0.7429	0.0247	98.3%

Under small sample conditions, in terms of interval estimation width, the MLE asymptotic confidence intervals for the parameters are the widest. However, when mapped to the stress–strength reliability model, the interval width becomes the shortest. This is because error propagation in nonlinear models is not linear additive, and the uncertainty in parameter estimation undergoes nonlinear shrinkage after being mapped through the complex model. Empirical Bayesian estimation achieves the smallest parameter bias, but the uncertainty in the prior hyperparameters is amplified by the model. Consequently, the corresponding HPD interval length for the reliability is longer than that of MLE but shorter than that of weakly informative prior Bayesian estimation. Weakly informative prior Bayesian estimation has the most sources of uncertainty. Under the combined effects of fixed weak prior information, fluctuations in the likelihood function, and diffuse posterior tails, this method produces the longest interval estimates for the reliability. However, coverage probability is decoupled from interval length. MLE asymptotic confidence intervals have the shortest length but the lowest coverage probability. Empirical Bayesian estimation yields HPD credible intervals of moderate length with extremely high coverage probability exceeding 98%. Weakly informative prior HPD credible intervals are the widest, with intermediate coverage probability. That is to say, empirical Bayesian estimation achieves the highest coverage probability with an appropriate interval width, attaining an optimal balance in estimation accuracy. As the sample size increases, the observed information becomes richer, and the interval lengths for reliabilities $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ gradually shorten, while the coverage probabilities gradually increase.

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6.3. Sensitivity Analysis

6.3.1. Sensitivity Analysis of Hyperparameters in Weakly Informative Prior Bayesian Estimation

In this section, a sensitivity analysis is conducted on the hyperparameters of the Gamma prior used as the weakly informative prior. Three sets of comparative experiments are designed, where a_i and b_i ($i = 1, 2, ..., 5$) are set to 0.01, 0.1, and 1, respectively, to analyze the impact of hyperparameter values on the estimation results. Table 9 presents the estimation results for the parameters v_1, v_2, u_1, u_2 and w of the stress and strength variables, and Table 10 presents the estimation results for stress–strength reliabilities $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$. Overall, as a_i and b_i increase, the estimation bias and RMSE of the parameters and stress–strength reliabilities gradually decrease, the length of the HPD credible intervals gradually shortens, and the CP coverage probability gradually improves. It is worth noting that, based on the analysis of Tables 5 and 8, even with the weakest prior $(0.001, 0.001)$, the conclusions regarding the parameters remain consistent with those obtained under the stronger prior $(1, 1)$. This indicates that the amount of information contained in the data of this study is sufficient to overcome the influence of prior specification, and the estimation results exhibit satisfactory robustness.

Table 9. Weakly informative priors for stress and strength parameters under different hyperparameter specifications.

Hyperparameters	WIP	Estimate	Bias	RMSE	HPD_L	HPD_U	Width	CP
$a_i = b_i = 0.01$, $i = 1, 2, \ldots, 5.$	$\hat{v}_1$	3.0926	0.0926	0.1978	2.7398	3.453	0.7132	94.8%
	$\hat{v}_2$	2.5658	0.0658	0.1523	2.2973	2.8405	0.5432	94.1%
	$\hat{u}_1$	1.8192	0.0192	0.0851	1.6534	1.9882	0.3348	95.1%
	$\hat{u}_2$	1.5445	0.0445	0.0858	1.4122	1.6799	0.2678	89.2%
	$\hat{\omega}$	2.5920	0.0920	0.1774	2.3039	2.8823	0.5784	90.5%
$a_i = b_i = 0.1$, $i = 1, 2, \ldots, 5.$	$\hat{v}_1$	3.0883	0.0883	0.1940	2.7366	3.4486	0.7121	94.4%
	$\hat{v}_2$	2.5628	0.0628	0.1497	2.2943	2.8375	0.5432	94.3%
	$\hat{u}_1$	1.8177	0.0177	0.0847	1.6527	1.9870	0.3343	95.6%
	$\hat{u}_2$	1.5433	0.0433	0.0849	1.4109	1.6786	0.2677	89.9%
	$\hat{\omega}$	2.5883	0.0883	0.1744	2.3007	2.8788	0.5781	91.1%
$a_i = b_i = 1$, $i = 1, 2, \ldots, 5.$	$\hat{v}_1$	3.0386	0.0386	0.1715	2.6962	3.3892	0.6931	96.7%
	$\hat{v}_2$	2.5272	0.0272	0.1337	2.2649	2.7956	0.5308	96.0%
	$\hat{u}_1$	1.7985	−0.0015	0.0803	1.6356	1.9649	0.3293	95.3%
	$\hat{u}_2$	1.5295	0.0295	0.0768	1.3985	1.6624	0.2639	92.0%
	$\hat{\omega}$	2.5461	0.0461	0.1526	2.2627	2.8311	0.5683	94.0%

Table 10. Weakly informative priors for stress–strength reliabilities under different hyperparameter specifications.

Hyperparameters	SSR	Estimate	Bias	RMSE	HPD_L	HPD_U	Width	CP
$a_i = b_i = 0.01$, $i = 1, 2, \ldots, 5.$	$\hat{R}_s^{(S_2)}$	0.1577	0.0022	0.0094	0.1383	0.1773	0.0390	95.8%
	$\hat{R}_s^{(\geq S_1)}$	0.7312	0.0021	0.0102	0.7097	0.7525	0.0427	95.3%
$a_i = b_i = 0.1$, $i = 1, 2, \ldots, 5.$	$\hat{R}_s^{(S_2)}$	0.1575	0.0020	0.0093	0.1382	0.1772	0.0390	95.7%
	$\hat{R}_s^{(\geq S_1)}$	0.7311	0.0020	0.0101	0.7096	0.7524	0.0428	95.7%
$a_i = b_i = 1$, $i = 1, 2, \ldots, 5.$	$\hat{R}_s^{(S_2)}$	0.1561	0.0006	0.0090	0.1370	0.1756	0.0386	96.7%
	$\hat{R}_s^{(\geq S_1)}$	0.7295	0.0004	0.0099	0.7080	0.7507	0.0427	96.4%

6.3.2. Sensitivity Analysis of the Common Shape Parameter w

A sensitivity analysis for parameter w was conducted using the profile likelihood method, with the experimental results presented in Table 11. As shown by the relative log-likelihood values in the last column of Table 11, as w increases, the relative log-likelihood initially increases, reaches its maximum at the MLE $\hat{w} = 2.58$, and then decreases. This indicates that w is well-identifiable. The estimates of the other parameters increase with w, suggesting a stable linear relationship between them and w. This finding is consistent with the first-order partial derivatives of the log-likelihood function, which analytically demonstrates that the estimates of v_1, v_2, u_1, u_2 all depend on w. In stress–strength reliability models $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$, the parameter w cancels out; therefore, theoretically, the reliability models are independent of w. Based on the likelihood ratio test theory, the 95% confidence interval for w is obtained, and within this interval, the relative changes in reliability for $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ are both less than 5%, which are very small. This demonstrates that both $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ are insensitive to w, consistent with the theoretical derivation. However, since v_1, v_2, u_1, u_2 increase as w increases, and the reliability models depend on these four parameters, the overall reliability tends to increase with w.

Table 11. Profile likelihood sensitivity analysis for parameter w.

w	Ln L	v_1	v_2	u_1	u_2	$\hat{R}_s^{(S_2)}$	$\hat{R}_s^{(\geq S_1)}$	ΔLn L
1.50	−7804.8196	2.0524	1.7807	1.3614	1.1943	0.1280	0.6957	−305.3615
1.58	−7754.1006	2.1264	1.8383	1.3968	1.2221	0.1303	0.6987	−254.6425
1.67	−7709.0172	2.2010	1.8960	1.4320	1.2496	0.1326	0.7017	−209.5591
1.75	−7669.1708	2.2763	1.9539	1.4670	1.2768	0.1348	0.7046	−169.7127
1.83	−7634.2152	2.3522	2.0122	1.5017	1.3038	0.1371	0.7074	−134.7570
1.92	−7603.8470	2.4290	2.0706	1.5365	1.3307	0.1393	0.7102	−104.3889
2.00	−7577.7996	2.5065	2.1294	1.5711	1.3573	0.1416	0.7129	−78.3415
2.08	−7555.8366	2.5849	2.1886	1.6056	1.3838	0.1438	0.7156	−56.3785
2.17	−7537.7477	2.6642	2.2481	1.6400	1.4101	0.1460	0.7183	−38.2896
2.25	−7523.3450	2.7444	2.3081	1.6745	1.4363	0.1483	0.7209	−23.8869
2.33	−7512.4598	2.8256	2.3685	1.7089	1.4624	0.1505	0.7235	−13.0017
2.42	−7504.9399	2.9077	2.4294	1.7433	1.4885	0.1527	0.7260	−5.4818
2.50	−7500.6477	2.9910	2.4908	1.7777	1.5144	0.1550	0.7285	−1.1895
2.58	−7499.4581	3.0753	2.5527	1.8121	1.5403	0.1572	0.7310	0.0000
2.67	−7501.2575	3.1607	2.6152	1.8466	1.5661	0.1595	0.7334	−1.7994
2.75	−7505.9418	3.2473	2.6782	1.8812	1.5919	0.1617	0.7358	−6.4836
2.83	−7513.4157	3.3350	2.7418	1.9158	1.6176	0.1639	0.7382	−13.9575
2.92	−7523.5917	3.4240	2.8059	1.9505	1.6434	0.1662	0.7406	−24.1336
3.00	−7536.3893	3.5142	2.8707	1.9852	1.6691	0.1684	0.7429	−36.9312
3.08	−7551.7341	3.6056	2.9361	2.0201	1.6947	0.1707	0.7452	−52.2760
3.17	−7569.5573	3.6982	3.0021	2.0549	1.7204	0.1729	0.7474	−70.0991
3.25	−7589.7950	3.7922	3.0688	2.0900	1.7460	0.1752	0.7497	−90.3368
3.33	−7612.3880	3.8876	3.1362	2.1251	1.7717	0.1774	0.7519	−112.9299
3.42	−7637.2811	3.9843	3.2042	2.1604	1.7974	0.1797	0.7541	−137.8230
3.50	−7664.4230	4.0824	3.2729	2.1958	1.8231	0.1820	0.7563	−164.9648

6.4. Real Data Analysis

A dataset discussed by Maurya and Tripathi [33] is selected for real data analysis, where the data represents the failure times of a specific type of software. The original data values vary over a wide range, so each value is divided by 50,000 for analysis. Following the data selection method in [26], starting from the 120th observation, that reference describes $X_{11} - X_{15}$ as the failure times of the 120th to 124th units, Y_1 as the failure time of the 125th unit, $X_{21} - X_{25}$ as the failure times of the 126th to 130th units, and Y_2 as the failure time of the 131st unit. The simulation continues until the failure of the 167th unit. A total of six variables are selected, each containing eight failure times. To adapt to the model in this paper, the above variables are transformed. First, the Kolmogorov–Smirnov (K-S) test is used to determine whether each variable follows the EPD. Then, four variables with similar second shape parameters are selected as the two strength variables X_1, X_2 and the two stress variables Y_1, Y_2 for this study. The test results are shown in Table 12. All p-values for the variables are greater than 0.05, indicating that all four variables follow the EPD.

$$(X_1, X_2) = \begin{pmatrix} 0.0096 & 0.0036 \\ 5.4600 & 1.1976 \\ 4.7568 & 0.0912 \\ 14.7636 & 2.9400 \\ 1.1004 & 0.0024 \\ 2.1240 & 0.0096 \\ 7.9716 & 7.8276 \\ 6.7380 & 5.2896 \end{pmatrix} (Y_1, Y_2) = \begin{pmatrix} 0.0120 & 1.6488 \\ 0.0168 & 1.5264 \\ 0.0384 & 1.5408 \\ 1.5336 & 1.3236 \\ 5.93592 & 14.4912 \\ 2.3472 & 0.0012 \\ 0.0036 & 0.0048 \\ 16.9416 & 6.9864 \end{pmatrix}$$

Table 12. Kolmogorov–Smirnov (K-S) test.

Random Variable	MLEs	K-S	p-Value
X_1	$\hat{v}_1 = 0.9494,\ \hat{w} = 0.6199$	0.3011	0.3857
X_2	$\hat{v}_2 = 0.3476,\ \hat{w} = 0.6079$	0.2146	0.7844
Y_1	$\hat{u}_1 = 0.3630,\ \hat{w} = 0.5578$	0.2548	0.5907
Y_2	$\hat{u}_2 = 0.4562,\ \hat{w} = 0.5561$	0.3887	0.1345

Tables 13 and 14 present the estimation results for each parameter as well as for stress–strength reliabilities. From Table 13, it can be seen that the parameter estimation results are highly robust, with the three methods yielding consistent results, indicating satisfactory model identifiability and stable, reliable estimates. Furthermore, the empirical Bayes estimates and the MLE estimates are almost identical, with differences not exceeding 1%. Under weakly informative priors, a slight deviation in w is expected, which can be attributed to the limited sample size and reflects a slight shrinkage of the estimates toward the prior mean. Table 14 presents the estimation results for the stress–strength reliability. The estimates obtained by different methods are highly consistent. The 95% credible intervals from the empirical Bayes method are the narrowest, reflecting the additional information provided by the informative prior. The intervals under MLE and the weakly informative prior are wider, which is primarily determined by the sample size.

Table 13. Point estimation results of each parameter under three different estimation methods.

Parameters	MLE	BE	WIP
$\hat{v}_1$	0.9217	0.9214	0.9074
$\hat{v}_2$	0.3446	0.3447	0.3419
$\hat{u}_1$	0.3688	0.3690	0.3668
$\hat{u}_2$	0.4654	0.4650	0.4536
$\hat{u}_2$	0.5891	0.5886	0.5649

Table 14. Estimation results of stress–strength reliability models $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$.

Method	Reliabilities	Estimate	ACI (HPD)	Width
MLE	$\hat{R}_s^{(S_2)}$	0.0826	[0.0318, 0.1971]	0.1653
	$\hat{R}_s^{(\geq S_1)}$	0.6937	[0.4685, 0.9193]	0.4508
EB	$\hat{R}_s^{(S_2)}$	0.0827	[0.0609, 0.1067]	0.0458
	$\hat{R}_s^{(\geq S_1)}$	0.6920	[0.6353, 0.7483]	0.1130
WIP	$\hat{R}_s^{(S_2)}$	0.1075	[0.0105, 0.3254]	0.3149
	$\hat{R}_s^{(\geq S_1)}$	0.6919	[0.4713, 0.8935]	0.4222

7. Conclusions

This paper analyzes the stress–strength reliability of a multi-state system composed of multi-state components, assuming that each component has two strengths and is subjected to two different stresses. The strength variables and stress variables are assumed to be mutually independent and follow the exponentiated Pareto distribution. First, for each individual component, the stress–strength reliability under its different states is derived. Then, based on the generalized survival signature theory, the stress–strength reliabilities $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ of the multi-state system are derived. Second, under the PFIF-C scheme, multiple point estimation methods, namely MLE, Empirical Bayesian estimation and weakly informative prior Bayesian estimation, are employed to estimate the parameters of the stress and strength variables under large samples, and the point estimates of reliabilities $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$ are obtained. Subsequently, for the MLE, asymptotic confidence intervals for each parameter and the reliability models are constructed based on the Fisher information matrix. Meanwhile, for the two Bayesian estimation methods, HPD credible intervals are provided based on MCMC sampling. Finally, the performance of various parameter estimation methods was analyzed under different sample sizes. Under small samples, the empirical Bayes estimation achieved the smallest bias and the shortest interval estimation length, followed by Bayesian estimation with weakly informative priors, and finally MLE. As the sample size increased, MLE and Bayesian estimation with weakly informative priors gradually outperformed empirical Bayes estimation, with their estimation accuracy improving slightly and progressively. For stress–strength reliabilities $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$, the estimation results gradually stabilized as the sample size increased. Additionally, a sensitivity analysis was conducted on the hyperparameters of the weakly informative priors. Under small samples, estimation accuracy gradually increased as the strength of the prior information grew. The Profile likelihood method was used to perform a sensitivity analysis on the value of w, and the results indicated that the model could identify w reasonably well. Finally, the feasibility of the model was validated using a real dataset.

Future research can be pursued along the following directions. On the one hand, this paper treats the stress and strength variables as mutually independent. As a natural extension of the model, future research may consider possible dependence between them and employ a copula function to analyze its impact on the reliability model. On the other hand, this paper focus on a discrete multi-state system, and the model could be extended to a continuous multi-state system in future research. Furthermore, for complex systems, the calculation of the generalized survival signature is also a challenging problem.

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Data Availability Statement: The dataset analyzed in this study is derived from a publicly available online resource, as detailed in reference [33]. The data are directly accessible via the following URL: https://www.cse.cuhk.edu.hk/~lyu/book/reliability/DATA/CH4/SS3.DAT (accessed on accessed on 11 February 2026).

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Calculation of Partial Derivatives

(1) The first-order partial derivatives of the log-likelihood function.

$$\begin{aligned}\frac{\partial \ln L}{\partial v_1} &= \frac{ms}{v_1} + \sum_{i=1}^{m}\sum_{j=1}^{s} \ln\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right] \\ &- \sum_{i=1}^{m}\sum_{j=1}^{s} \left[h\left(Q_j+1\right)-1\right] \frac{\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1} \ln\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]}{1-\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1}}\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial v_2} &= \frac{ms}{v_2} + \sum_{i=1}^{m}\sum_{j=1}^{s} \ln\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right] \\ &- \sum_{i=1}^{m}\sum_{j=1}^{s} \left[h\left(Q_j+1\right)-1\right] \frac{\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2} \ln\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]}{1-\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2}}\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial u_1} &= \frac{ms}{u_1} + \sum_{i=1}^{m}\sum_{j=1}^{s} \ln\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right] \\ &- \sum_{i=1}^{m}\sum_{j=1}^{s} \left[h\left(Q_j+1\right)-1\right] \frac{\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1} \ln\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]}{1-\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1}}\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial u_2} &= \frac{ms}{u_2} + \sum_{i=1}^{m}\sum_{j=1}^{s} \ln\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right] \\ &- \sum_{i=1}^{m}\sum_{j=1}^{s} \left[h\left(Q_j+1\right)-1\right] \frac{\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2} \ln\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]}{1-\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2}}\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \ln L}{\partial w} &= \frac{4ms}{w} - \sum_{i=1}^{m}\sum_{j=1}^{s}\left[\ln\left(1+X_{ij}^{(1)}\right)+\ln\left(1+X_{ij}^{(2)}\right)+\ln\left(1+Y_{ij}^{(1)}\right)+\ln\left(1+Y_{ij}^{(2)}\right)\right] \\
&+\sum_{i=1}^{m}\sum_{j=1}^{s}\left\{\begin{array}{c}
\dfrac{(v_1-1)\left(1+X_{ij}^{(1)}\right)^{-w}\ln\left(1+X_{ij}^{(1)}\right)}{1-\left(1+X_{ij}^{(1)}\right)^{-w}} \\
+\dfrac{(v_2-1)\left(1+X_{ij}^{(2)}\right)^{-w}\ln\left(1+X_{ij}^{(2)}\right)}{1-\left(1+X_{ij}^{(2)}\right)^{-w}} \\
+\dfrac{(u_1-1)\left(1+Y_{ij}^{(1)}\right)^{-w}\ln\left(1+Y_{ij}^{(1)}\right)}{1-\left(1+Y_{ij}^{(1)}\right)^{-w}} \\
+\dfrac{(u_2-1)\left(1+Y_{ij}^{(2)}\right)^{-w}\ln\left(1+Y_{ij}^{(2)}\right)}{1-\left(1+Y_{ij}^{(2)}\right)^{-w}}
\end{array}\right\} \\
&-\sum_{i=1}^{m}\sum_{j=1}^{s}\left\{\begin{array}{c}
\left[h\left(Q_j+1\right)-1\right] \\
\times\left\{\begin{array}{c}
\dfrac{v_1\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1-1}\left(1+X_{ij}^{(1)}\right)^{-w}\ln\left(1+X_{ij}^{(1)}\right)}{1-\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1}} \\
+\dfrac{v_2\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2-1}\left(1+X_{ij}^{(2)}\right)^{-w}\ln\left(1+X_{ij}^{(2)}\right)}{1-\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2}} \\
+\dfrac{u_1\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1-1}\left(1+Y_{ij}^{(1)}\right)^{-w}\ln\left(1+Y_{ij}^{(1)}\right)}{1-\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1}} \\
+\dfrac{u_2\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2-1}\left(1+Y_{ij}^{(2)}\right)^{-w}\ln\left(1+Y_{ij}^{(2)}\right)}{1-\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2}}
\end{array}\right\}
\end{array}\right\}
\end{aligned}
$$

(2) The second-order partial derivatives of the log-likelihood function.

$$
\frac{\partial^2 \ln L}{\partial v_1 \partial v_2} = \frac{\partial^2 \ln L}{\partial v_2 \partial v_1} = 0, \frac{\partial^2 \ln L}{\partial v_1 \partial u_1} = \frac{\partial^2 \ln L}{\partial u_1 \partial v_1} =, 0 \frac{\partial^2 \ln L}{\partial v_1 \partial u_2} = \frac{\partial^2 \ln L}{\partial u_2 \partial v_1} = 0
$$

$$
\frac{\partial^2 \ln L}{\partial v_2 \partial u_1} = \frac{\partial^2 \ln L}{\partial u_1 \partial v_2} = 0, \frac{\partial^2 \ln L}{\partial v_2 \partial u_2} = \frac{\partial^2 \ln L}{\partial u_2 \partial v_2} = 0, \frac{\partial^2 \ln L}{\partial u_1 \partial u_2} = \frac{\partial^2 \ln L}{\partial u_2 \partial u_1} = 0
$$

$$
\frac{\partial^2 \ln L}{\partial v_1^2} = -\frac{ms}{v_1^2} - \sum_{i=1}^{m}\sum_{j=1}^{s}\left\{\begin{array}{c}
\left[h\left(Q_j+1\right)-1\right] \\
\times\dfrac{\left\{\ln\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]\right\}^2\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1}}{\left\{1-\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1}\right\}^2}
\end{array}\right\}
$$

$$
\frac{\partial^2 \ln L}{\partial v_2^2} = -\frac{ms}{v_2^2} - \sum_{i=1}^{m}\sum_{j=1}^{s}\left\{\begin{array}{c}
\left[h\left(Q_j+1\right)-1\right] \\
\times\dfrac{\left\{\ln\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]\right\}^2\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2}}{\left\{1-\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2}\right\}^2}
\end{array}\right\}
$$

$$\frac{\partial^2 \ln L}{\partial u_1^2} = -\frac{ms}{u_1^2} - \sum_{i=1}^{m}\sum_{j=1}^{s}\left\{ \begin{array}{c} [h(Q_j+1)-1] \\ \times \dfrac{\left\{\ln\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]\right\}^2 \left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1}}{\left\{1-\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1}\right\}^2} \end{array} \right\}$$

$$\frac{\partial^2 \ln L}{\partial u_2^2} = -\frac{ms}{u_2^2} - \sum_{i=1}^{m}\sum_{j=1}^{s}\left\{ \begin{array}{c} [h(Q_j+1)-1] \\ \times \dfrac{\left\{\ln\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]\right\}^2 \left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2}}{\left\{1-\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2}\right\}^2} \end{array} \right\}$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial w^2} = -\frac{4ms}{w^2} - \textstyle\sum_{i=1}^{m}\sum_{j=1}^{s} & \left\{ \begin{array}{c} \dfrac{(v_1-1)\left(1+X_{ij}^{(1)}\right)^{-w}\left[\ln\left(1+X_{ij}^{(1)}\right)\right]^2}{\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^2} \\ +\dfrac{(v_2-1)\left(1+X_{ij}^{(2)}\right)^{-w}\left[\ln\left(1+X_{ij}^{(2)}\right)\right]^2}{\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^2} \\ +\dfrac{(u_1-1)\left(1+Y_{ij}^{(1)}\right)^{-w}\left[\ln\left(1+Y_{ij}^{(1)}\right)\right]^2}{\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^2} \\ +\dfrac{(u_2-1)\left(1+Y_{ij}^{(2)}\right)^{-w}\left[\ln\left(1+Y_{ij}^{(2)}\right)\right]^2}{\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^2} \end{array} \right\} \\ -\textstyle\sum_{i=1}^{m}\sum_{j=1}^{s} & \left\{ \begin{array}{c} [h(Q_j+1)-1] \\ \times\left\{ \begin{array}{c} \dfrac{\left\{ \begin{array}{c} v_1\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1-2}\left(1+X_{ij}^{(1)}\right)^{-w}\left[\ln\left(1+X_{ij}^{(1)}\right)\right]^2 \\ \times\left\{v_1\left(1+X_{ij}^{(1)}\right)^{-w}-\left\{1-\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1}\right\}\right\} \end{array} \right\}}{\left\{1-\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1}\right\}^2} \\ +\dfrac{\left\{ \begin{array}{c} v_2\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2-2}\left(1+X_{ij}^{(2)}\right)^{-w}\left[\ln\left(1+X_{ij}^{(2)}\right)\right]^2 \\ \times\left\{v_2\left(1+X_{ij}^{(2)}\right)^{-w}-\left\{1-\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2}\right\}\right\} \end{array} \right\}}{\left\{1-\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2}\right\}^2} \\ +\dfrac{\left\{ \begin{array}{c} u_1\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1-2}\left(1+Y_{ij}^{(1)}\right)^{-w}\left[\ln\left(1+Y_{ij}^{(1)}\right)\right]^2 \\ \times\left\{u_1\left(1+Y_{ij}^{(1)}\right)^{-w}-\left\{1-\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1}\right\}\right\} \end{array} \right\}}{\left\{1-\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1}\right\}^2} \\ +\dfrac{\left\{ \begin{array}{c} u_2\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2-2}\left(1+Y_{ij}^{(2)}\right)^{-w}\left[\ln\left(1+Y_{ij}^{(2)}\right)\right]^2 \\ \times\left\{u_2\left(1+Y_{ij}^{(2)}\right)^{-w}-\left\{1-\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2}\right\}\right\} \end{array} \right\}}{\left\{1-\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2}\right\}^2} \end{array} \right\} \end{array} \right\} \end{aligned}$$

$$\frac{\partial^2 \ln L}{\partial v_1 \partial w} = \frac{\partial^2 \ln L}{\partial w \partial v_1} = \sum_{i=1}^{m}\sum_{j=1}^{s} \frac{\left(1+X_{ij}^{(1)}\right)^{-w} \ln\left(1+X_{ij}^{(1)}\right)}{1-\left(1+X_{ij}^{(1)}\right)^{-w}}$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{s}\left\{\begin{array}{c}[h(Q_j+1)-1] \\ \times\dfrac{\left\{\begin{array}{c}\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1-1}\left(1+X_{ij}^{(1)}\right)^{-w}\ln\left(1+X_{ij}^{(1)}\right) \\ \times\left\{v_1\ln\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]+\left\{1-\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1}\right\}\right\}\end{array}\right\}}{\left\{1-\left[1-\left(1+X_{ij}^{(1)}\right)^{-w}\right]^{v_1}\right\}^2}\end{array}\right\}$$

$$\frac{\partial^2 \ln L}{\partial v_2 \partial w} = \frac{\partial^2 \ln L}{\partial w \partial v_2} = \sum_{i=1}^{m}\sum_{j=1}^{s} \frac{\left(1+X_{ij}^{(2)}\right)^{-w} \ln\left(1+X_{ij}^{(2)}\right)}{1-\left(1+X_{ij}^{(2)}\right)^{-w}}$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{s}\left\{\begin{array}{c}[h(Q_j+1)-1] \\ \times\dfrac{\left\{\begin{array}{c}\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2-1}\left(1+X_{ij}^{(2)}\right)^{-w}\ln\left(1+X_{ij}^{(2)}\right) \\ \times\left\{v_2\ln\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]+\left\{1-\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2}\right\}\right\}\end{array}\right\}}{\left\{1-\left[1-\left(1+X_{ij}^{(2)}\right)^{-w}\right]^{v_2}\right\}^2}\end{array}\right\}$$

$$\frac{\partial^2 \ln L}{\partial u_1 \partial w} = \frac{\partial^2 \ln L}{\partial w \partial u_1} = \sum_{i=1}^{m}\sum_{j=1}^{s} \frac{\left(1+Y_{ij}^{(1)}\right)^{-w} \ln\left(1+Y_{ij}^{(1)}\right)}{1-\left(1+Y_{ij}^{(1)}\right)^{-w}}$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{s}\left\{\begin{array}{c}[h(Q_j+1)-1] \\ \times\dfrac{\left\{\begin{array}{c}\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1-1}\left(1+Y_{ij}^{(1)}\right)^{-w}\ln\left(1+Y_{ij}^{(1)}\right) \\ \times\left\{u_1\ln\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]+\left\{1-\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1}\right\}\right\}\end{array}\right\}}{\left\{1-\left[1-\left(1+Y_{ij}^{(1)}\right)^{-w}\right]^{u_1}\right\}^2}\end{array}\right\}$$

$$\frac{\partial^2 \ln L}{\partial u_2 \partial w} = \frac{\partial^2 \ln L}{\partial w \partial u_2} = \sum_{i=1}^{m}\sum_{j=1}^{s} \frac{\left(1+Y_{ij}^{(2)}\right)^{-w} \ln\left(1+Y_{ij}^{(2)}\right)}{1-\left(1+Y_{ij}^{(2)}\right)^{-w}}$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{s}\left\{\begin{array}{c}[h(Q_j+1)-1] \\ \times\dfrac{\left\{\begin{array}{c}\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2-1}\left(1+Y_{ij}^{(2)}\right)^{-w}\ln\left(1+Y_{ij}^{(2)}\right) \\ \times\left\{u_2\ln\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]+\left\{1-\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2}\right\}\right\}\end{array}\right\}}{\left\{1-\left[1-\left(1+Y_{ij}^{(2)}\right)^{-w}\right]^{u_2}\right\}^2}\end{array}\right\}$$

(3) Partial derivatives of stress–strength reliabilities $R_s^{(s_2)}$ and $R_s^{(\geq s_1)}$.

$$\frac{\partial R_s^{(s_2)}}{\partial v_1} = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n_1}\left\{\begin{array}{c}\overline{p}_{n_1,n_2}(n)\binom{n}{n_1}\binom{n_1}{n_2}\sum_{k=0}^{n_1-n_2}\sum_{k_1=0}^{k}\sum_{k_2=0}^{n_1-n_2-k}\sum_{l_1=0}^{n_2}\sum_{l_2=0}^{n_2}\binom{n_1-n_2}{k} \\ \times\binom{k}{k_1}\binom{n_1-n_2-k}{k_2}\binom{n_2}{l_1}\binom{n_2}{l_2}(-1)^{k_1+k_2+l_1+l_2+1} \\ \times\dfrac{u_1u_2(n-n_2-k+k_1+l_1)}{\left\{\begin{array}{c}[(n-n_2-k+k_1+l_1)v_1+u_1]^2 \\ \times[(n-n_1+k+k_2+l_2)v_2+u_2]\end{array}\right\}}\end{array}\right\}$$

$$\frac{\partial R_s^{(s_2)}}{\partial v_2} = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n_1}\left\{\begin{array}{c} \overline{p}_{n_1,n_2}(n)\binom{n}{n_1}\binom{n_1}{n_2}\sum_{k=0}^{n_1-n_2}\sum_{k_1=0}^{k}\sum_{k_2=0}^{n_1-n_2-k}\sum_{l_1=0}^{n_2}\sum_{l_2=0}^{n_2}\binom{n_1-n_2}{k} \\ \times\binom{k}{k_1}\binom{n_1-n_2-k}{k_2}\binom{n_2}{l_1}\binom{n_2}{l_2}(-1)^{k_1+k_2+l_1+l_2+1} \\ \times\dfrac{u_1u_2(n-n_1+k+k_2+l_2)}{\left\{\begin{array}{c}[(n-n_2-k+k_1+l_1)v_1+u_1] \\ \times[(n-n_1+k+k_2+l_2)v_2+u_2]^2\end{array}\right\}} \end{array}\right\}$$

$$\frac{\partial R_s^{(s_2)}}{\partial u_1} = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n_1}\left\{\begin{array}{c} \overline{p}_{n_1,n_2}(n)\binom{n}{n_1}\binom{n_1}{n_2}\sum_{k=0}^{n_1-n_2}\sum_{k_1=0}^{k}\sum_{k_2=0}^{n_1-n_2-k}\sum_{l_1=0}^{n_2}\sum_{l_2=0}^{n_2}\binom{n_1-n_2}{k} \\ \times\binom{k}{k_1}\binom{n_1-n_2-k}{k_2}\binom{n_2}{l_1}\binom{n_2}{l_2}(-1)^{k_1+k_2+l_1+l_2} \\ \times\dfrac{v_1u_2(n-n_2-k+k_1+l_1)}{\left\{\begin{array}{c}[(n-n_2-k+k_1+l_1)v_1+u_1]^2 \\ \times[(n-n_1+k+k_2+l_2)v_2+u_2]\end{array}\right\}} \end{array}\right\}$$

$$\frac{\partial R_s^{(s_2)}}{\partial u_2} = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n_1}\left\{\begin{array}{c} \overline{p}_{n_1,n_2}(n)\binom{n}{n_1}\binom{n_1}{n_2}\sum_{k=0}^{n_1-n_2}\sum_{k_1=0}^{k}\sum_{k_2=0}^{n_1-n_2-k}\sum_{l_1=0}^{n_2}\sum_{l_2=0}^{n_2}\binom{n_1-n_2}{k} \\ \times\binom{k}{k_1}\binom{n_1-n_2-k}{k_2}\binom{n_2}{l_1}\binom{n_2}{l_2}(-1)^{k_1+k_2+l_1+l_2} \\ \times\dfrac{v_2u_1(n-n_1+k+k_2+l_2)}{\left\{\begin{array}{c}[(n-n_2-k+k_1+l_1)v_1+u_1] \\ \times[(n-n_1+k+k_2+l_2)v_2+u_2]^2\end{array}\right\}} \end{array}\right\}$$

$$\frac{\partial R_s^{(s_2)}}{\partial w} = 0$$

$$\frac{\partial R_s^{(\geq s_1)}}{\partial v_1} = \sum_{n_1=0}^{n}\left\{\overline{p}_{n_1}(n)\binom{n}{n_1}\sum_{l=0}^{n_1}\binom{n_1}{l}(-1)^{l+1}\frac{u_1u_2(n-n_1+l)}{\left\{\begin{array}{c}[(n-n_1+l)v_1+u_1]^2 \\ \times[(n-n_1+l)v_2+u_2]\end{array}\right\}}\right\}$$

$$\frac{\partial R_s^{(\geq s_1)}}{\partial v_2} = \sum_{n_1=0}^{n}\left\{\overline{p}_{n_1}(n)\binom{n}{n_1}\sum_{l=0}^{n_1}\binom{n_1}{l}(-1)^{l+1}\frac{u_1u_2(n-n_1+l)}{\left\{\begin{array}{c}[(n-n_1+l)v_1+u_1] \\ \times[(n-n_1+l)v_2+u_2]^2\end{array}\right\}}\right\}$$

$$\frac{\partial R_s^{(\geq s_1)}}{\partial u_1} = \sum_{n_1=0}^{n}\left\{\overline{p}_{n_1}(n)\binom{n}{n_1}\sum_{l=0}^{n_1}\binom{n_1}{l}(-1)^{l}\frac{v_1u_2(n-n_1+l)}{\left\{\begin{array}{c}[(n-n_1+l)v_1+u_1]^2 \\ \times[(n-n_1+l)v_2+u_2]\end{array}\right\}}\right\}$$

$$\frac{\partial R_s^{(\geq s_1)}}{\partial u_2} = \sum_{n_1=0}^{n}\left\{\overline{p}_{n_1}(n)\binom{n}{n_1}\sum_{l=0}^{n_1}\binom{n_1}{l}(-1)^{l}\frac{u_1v_2(n-n_1+l)}{\left\{\begin{array}{c}[(n-n_1+l)v_1+u_1] \\ \times[(n-n_1+l)v_2+u_2]^2\end{array}\right\}}\right\}$$

$$\frac{\partial R_s^{(\geq s_1)}}{\partial w} = 0$$

Appendix B. Validation of Results

Table A1. Numerical differentiation verifies the accuracy of partial derivatives.

Parameter	Analytic	Numeric	Absolute Error
v_1	1.39269017	1.39268955	6.18×10^{-7}
v_2	−2.99006193	−2.99006195	2.89×10^{-8}
u_1	−2.15444899	−2.15444864	3.51×10^{-7}
u_2	−4.74915589	−4.74915614	2.52×10^{-7}
w	1.73164186	1.73164142	4.33×10^{-7}
v_1, v_1	−951.65387682	−951.65387683	1.54×10^{-8}
v_2, v_2	−1383.18574019	−1383.18574019	3.26×10^{-9}
$u_1, u_1,$	−2737.30543019	−2737.30543028	9.02×10^{-8}
u_2, u_2	−3775.49823531	−3775.49823535	3.86×10^{-8}
w, w	−3019.77951576	−3019.77951581	5.37×10^{-8}
v_1, w	967.33560417	967.33560416	2.29×10^{-9}
v_2, w	1033.09375004	1033.09375002	2.87×10^{-8}
u_1, w	1132.95370881	1132.95370885	4.27×10^{-8}
u_2, w	1173.27834877	1173.27834880	3.36×10^{-8}

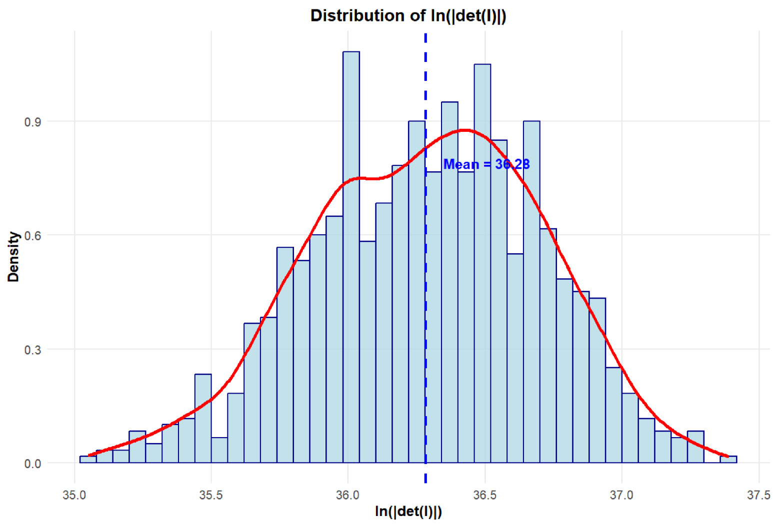

Figure A1. Distribution of $\ln(|det(I)|)$ of the fisher information matrix based on 1000 random samples (MLE $\pm$ 10%).

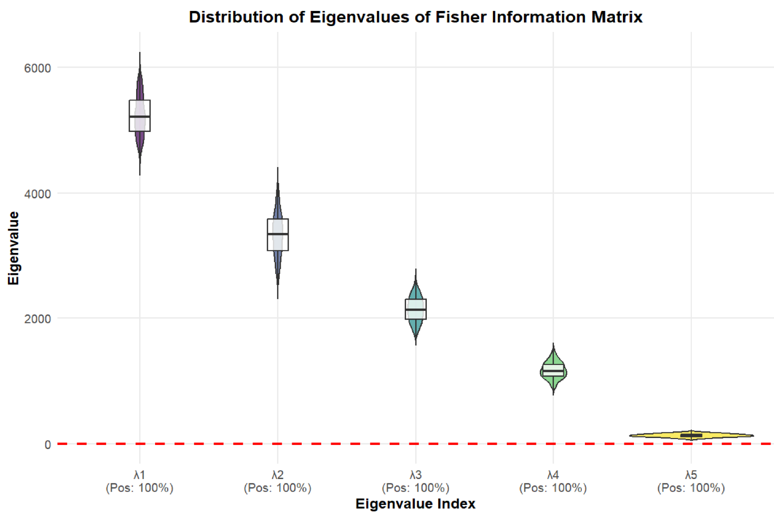

Figure A2. Distribution of eigenvalues for fisher information matrix based on 1000 random samples (MLE $\pm$ 10%).

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