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On Quotients of Gamma and Beta Random Variables and Related Hypergeometric Identities

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Abstract

This study presents the development of a comprehensive framework wherein probabilistic and analytical techniques collaboratively produce identities pertinent to special functions. The fundamental premise is that ratios of random variables, particularly within the family of Gamma and Beta distributions, inherently generate integral representations and hypergeometric structures that can be harnessed to derive non-trivial summation formulas. A pivotal outcome of this investigation is that, in the context of independent and identically distributed random variables, symmetry plays a crucial role. A key property of ratios of i.i.d. random variables translates into straightforward probabilistic assertions that directly lead to precise analytic identities, thereby enabling the recovery of classical results such as Kummer's and Watson's summation theorems as consequences of distributional symmetry. When the assumption of identical distribution is relaxed, hypergeometric representations continue to exist; however, the absence of symmetry impedes closed-form evaluations of the same nature. This contrast underscores symmetry as the fundamental mechanism underlying exact summation formulas and elucidates why such identities are contingent upon specific parameter configurations. More generally, this methodology offers a probabilistic interpretation of hypergeometric identities and establishes a conceptual connection between probability theory and the theory of special functions. Furthermore, it suggests that more expansive constructions, based on non-identically distributed variables or iterated processes, could be fruitfully explored. In particular, examining ratios may lead to new identities or alternative derivations of classical results.

Keywords: continuous random variables; gamma and beta distributions; hypergeometric functions; appel function; summation theorems

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1. Introduction

The present paper continues our line of research devoted to the rediscovery of identities for special functions by means of probabilistic methods involving independent and identically distributed random variables. In particular, we turn our attention to Gamma and Beta prime random variables, whose densities already belong to the realm of special functions. In this sense, the present analysis represents a further step with respect to previous contributions, where the probabilistic framework was restricted to more elementary distributions. The study of quotients of independent and identically distributed (i.i.d.) random variables in connection with problems of mathematical analysis can be traced

back to the contribution of Bourgade et al. [1]. In that work, the authors exploit specific properties of independent Cauchy random variables to provide a probabilistic proof of Euler's formula for even values of the Riemann zeta function, namely $\zeta(2n)$, as well as of the Dirichlet beta function at odd arguments, i.e., $\beta(2n+1)$. A key ingredient of their approach is the observation that the associated probability density integrates to 1, which enables the derivation of non-trivial integral identities. In the same contribution, it is also suggested that analogous results may be obtained by considering quotients rather than products of Cauchy random variables. This perspective is further elaborated upon in the scholarly work of Pace [2], whose inspiration is derived from the elementary proof delineated in [3]. Within this framework, probability theory assumes a dual function. Firstly, it motivates the initial analytical constructions by framing them as instances of a more general probabilistic problem. Secondly, it facilitates a seamless progression of subsequent arguments by leveraging a straightforward symmetry. This symmetry is inherent in the utilisation of quotients derived from independent and identically distributed (i.i.d.) non-negative random variables.

From this perspective, the geometric division of the probability space into two equally probable regions—the unit interval $[0, 1]$ and its complement $[1, \infty)$—serves as a fundamental constraint. The requirement that the cumulative distribution function (CDF) of the quotient attain precisely $\frac{1}{2}$ at the critical point $y = 1$ establishes a foundational anchor point. This point enables the derivation of an analytical identity, whereby the normalisation constant, representing the global information in the density, is constrained to conform to the local behaviour dictated by exchangeability. This symmetry-driven methodology effectively transforms a probabilistic necessity into a formal summation formula.

A further interesting contribution along these lines can be found in [4], where the author uses specific features of the hyperbolic secant probability distribution. In particular, by considering the sum of two independent hyperbolic secant random variables and exploiting its distribution, an alternative probabilistic proof of the Basel problem is obtained. Moreover, following the approach of [1], general formulas for $\zeta(2n)$ and $\beta(2n+1)$ are derived within the same probabilistic framework. The remarkable versatility of this methodology has led to numerous identities involving the values of $\zeta(2n)$. Parallel to this probabilistic line of research, several analytic approaches to the Basel problem and related evaluations of $\zeta(2n)$ have been developed, notably through the use of multiple integrals; see, for instance, ref. [3] and related works. These developments have continued in more recent contributions. In particular, in [5], both probabilistic and multiple-integration techniques are combined. Along similar probabilistic lines, we also mention [6]. Specifically, that work studies alternating and zeta-type series for positive integers k through a multiple-integration framework. Starting from classical double integrals leading to Euler's evaluation of $\zeta(2)$, the constructions are extended to arbitrary dimensions. The first generalisation produces the density of the quotient of k independent non-negative Cauchy random variables and involves a logarithmic integral connected to the alternating series. The second approach, after an appropriate change in variables, reduces the problem to the volume of a convex polytope, which can be interpreted probabilistically as the distribution of pairwise sums of independent uniform random variables; combinatorial arguments then yield closed formulas for both series. The third construction involves integration over a hyperbolic polytope associated with k Cauchy variables and admits a probabilistic interpretation in terms of pairwise products. This formulation is shown to coincide with the second approach, revealing a shared underlying combinatorial structure.

More recently, in [5], the probabilistic approach based on non-negative Cauchy random variables has been employed to derive further values of the Riemann zeta function at even arguments and of the Dirichlet beta function at odd arguments, in parallel with the multiple-

integral method, demonstrating a faster computational performance of the probabilistic framework as the order increases. In the same work, by considering a generalised version of Cauchy random variables with index $n > 1$, the authors also obtained probabilistic proofs of reflection formulas for the digamma and trigamma functions. The present paper fits naturally within this line of research.

The main focus of this contribution is on other classes of probability distributions, particularly the Gamma and Beta prime random variables. This choice leads to probabilistic derivations closely related to Kummer's theorem [7], reported also in [8], and Watson's Theorem [9]. Specifically, the quotient of two Gamma random variables yields a Beta distribution whose parameters naturally connect with Kummer's theorem. Proceeding further, either by taking the quotient of four Gamma random variables or, equivalently, by combining the two Beta variables obtained in the previous step, one arrives at expressions related to Watson's theorem. Additionally, in the present work we have also investigated the case in which the random variables are independent but not identically distributed, considering two different scenarios: half-broken symmetry and fully broken symmetry. It is interesting to observe how, in these two cases, the absence of the symmetry induced by i.i.d. random variables prevents us from recovering the results related to Kummer's and Watson's theorems.

In the last part of this section, we present some useful definitions and results that will be used in the rest of the paper. We begin by recalling a basic probabilistic property that underlies the present work. Let X and Y be independent and identically distributed (i.i.d.) non-negative random variables, and define $\mathrm{T} := \mathrm{X}/\mathrm{Y}$. By exchangeability, one immediately obtains

$$\mathcal{P}(\mathrm{T} \leq 1) = \mathcal{P}(\mathrm{X} \leq \mathrm{Y}) = \frac{1}{2}, \tag{1}$$

a simple symmetry that plays a central role in probabilistic approaches to identities involving special functions.

A key technical ingredient of the present work is the use of Gamma and Beta distributed building blocks. Recall that the Gamma distribution is defined by

$$\Gamma(x; \alpha, \lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \qquad \alpha, \lambda > 0, \tag{2}$$

and is supported on the positive half-axis.

A random variable X is said to follow $(\sim)$ a Beta prime distribution [10–13] with parameters $a > 0$ and $b > 0$, denoted by $X \sim \mathrm{BetaPrime}(x; a, b)$, if it has density

$$\mathrm{BetaPrime}(x; a, b) = \frac{1}{\mathrm{Beta}\,(a, b)} \frac{x^{a-1}}{(1+x)^{a+b}}, \qquad x > 0. \tag{3}$$

Another important building block of our work is the following result on computing the density of the ratio of two i.i.d. random variables. Let X and Y be i.i.d. non-negative random variables with density functions f_{X} and f_{Y}, respectively. Their quotient $\mathrm{T} = \mathrm{Y}/\mathrm{X}$ admits the density representation

$$f_{\mathrm{T}}(t) = \int_0^{\infty} x\, f_{\mathrm{Y}}(tx)\, f_{\mathrm{X}}(x)\, \mathrm{d}x. \tag{4}$$

Our approach is also connected with a broader line of research on the algebra of Gamma e Beta distributions and probabilistic representations of special functions, where identities for products, quotients and transforms of beta and gamma random variables are related to Mellin transforms, Barnes lemmas and hypergeometric-type structures; see, among others [14–19].

Notation. For the reader's convenience, we list below the main notations used throughout the paper.

Symbol	Meaning
$\mathrm{P}(\cdot)$	Probability of an event
$\Gamma(\cdot)$	Euler Gamma function
$\mathrm{Beta}(\cdot,\cdot)$	Beta function
$(a)_n$	Pochhammer symbol
${}_2\mathrm{F}_1$	Gaussian hypergeometric function
${}_3\mathrm{F}_2$	Generalised hypergeometric function
F_1	Appell hypergeometric series of two variables
$q_{\mathrm{T}}(t)$	PDF of the quotient of two Gamma i.i.d. random variables
$q_{\mathrm{S}}(t)$	PDF of the quotient of two Beta Prime i.i.d. random variables
$f_{\mathrm{T}}(t)$	PDF of the quotient of two Gamma independent random variables
$f_{\mathrm{R}}(t)$	PDF of the quotient of two Beta Prime independent random variables

The paper is organised as follows. Section 2 is devoted to the core of our analysis. After introducing the notation and the preliminary results needed to properly frame the problem, we first focus on the quotient of two Gamma random variables and subsequently on the quotient of four Gamma random variables. The main objective of the present work is to provide a partial proof, restricted to a specific set of parameters, of Kummer-type summation results by means of the ratio of two Gamma random variables, and of Watson-type results by adopting a probabilistic approach based on four Gamma variables, or equivalently on two Beta prime distributions. Further developments are naturally envisaged by increasing the number of quotients of random variables involved. However, as already observed in [5], the computational complexity grows rapidly with the number of variables, eventually necessitating numerical approximation methods. We also investigate cases in which the assumption of identical distribution is relaxed, obtaining results that, due to the absence of symmetry, no longer yield closed-form expressions analogous to those of the symmetric setting; in particular, in one instance, it becomes necessary to resort to numerical approximation techniques. Finally, motivated by the recurring structural patterns identified both in [5] and in the present contribution, we suggest that similar summation results may be obtained by considering a larger number of quotients of random variables. In particular, the underlying probabilistic structure and the role of symmetry appear to persist, indicating that this approach could be extended to more general settings involving additional variables.

In the end, Section 3 contains the conclusions and outlines possible directions for future research.

2. Gamma Distribution and Hypergeometric Identities

2.1. Notation and Preliminary Results

To facilitate our discussion, we establish the notation and terminology to be employed. The Gaussian hypergeometric function, which can be traced to Euler's early contributions, owes its current form primarily to Gauss's seminal paper [20] in relation to the power series.

$$ {}_2\mathrm{F}_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|\, x\right) := \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\,\frac{x^n}{n!}, \tag{5}$$

where $(a)_n, (b)_n, (c)_n$ are the Pochhammer symbols which are defined, among others in [8], as, for $x > 0$ and $n \in \mathbb{N}$,

$$(x)_n := \frac{\Gamma(x+n)}{\Gamma(x)} = (x+n-1)(x+n-2)\cdots(x+1)x.$$

This function solves the ordinary differential equation:

$$x\,(1-x)\,y'' + [c - (a+b+1)\,x]\,y' - a\,b\,y = 0, \tag{6}$$

where a, b, c are parameters (real or complex) not depending on x, which stands for the independent variable of the equation, and fulfil the initial conditions $y(0) = 1$ as well as $y'(0) = (ab)/c$. Observe that the radius of convergence of series (5) is one unless a or b are negative integers (see [8] (Chap. 9) for details). At the boundary of the convergence circle $|x| = 1$, series (5) converges for $\mathrm{Re}\,(a+b) < \mathrm{Re}\,c$ and diverges for $\mathrm{Re}\,(a+b) \geq \mathrm{Re}\,c$. We are not concerned here with the exhaustive study of the solutions of the differential Equation (6), which Kummer completed [7] and which is reported in the more recent classic monograph of Slater and Patra [21,22]. What is crucial for the following is the integral representation theorem for the function ${}_2\mathrm{F}_1$, for which we refer to [8] (Chap. 9) and [21] (Chap. 1).

Theorem 1. *If* $\mathrm{Re}\,c > \mathrm{Re}\,a > 0$ *and if* $|x| < 1$, *then:*

$${}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| x\right) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 \frac{u^{a-1}\,(1-u)^{c-a-1}}{(1-x\,u)^b}\,\mathrm{d}u. \tag{7}$$

The fundamental fact for us here is that the representation of Formula (7) is the analytic continuation of ${}_2\mathrm{F}_1$ when $\mathrm{Re}\,c > \mathrm{Re}\,a > 0$, to $\mathbb{C} \setminus [1, \infty)$. In [23] (pages 239–240), the continuation argument is provided rigorously.

We will also deal with the generalised hypergeometric ${}_3\mathrm{F}_2$ power series

$${}_3\mathrm{F}_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}\middle| x\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n\,(a_2)_n\,(a_3)_n}{(b_1)_n\,(b_2)_n}\,\frac{x^n}{n!}, \tag{8}$$

which converges for $|x| < 1$ where a_1, a_2, a_3, b_1, b_2 are real or complex parameters. In the case where the parameters a_1, a_2 and a_3 take on negative integer values, there are definitional problems, illustrated, for example, on the Wolfram Functions site [21,24,25], which, however, are not relevant to our discussion. In the case where $|x| = 1$, the series (8) converges if $\mathrm{Re}(b_1 + b_2) > \mathrm{Re}(a_1 + a_2 + a_3)$.

The subsequent lemma, which involves evaluating an integral containing the hypergeometric function, is deliberately not formulated under the most general assumptions on the hypergeometric coefficients. Instead, it is formulated under conditions relevant to our discussion. The reduction in generality is compensated by the ability to apply the monotone convergence theorem, thereby permitting the interchange of the hypergeometric series and the integral.

The following Lemma is an instrumental result that will be useful in the sequel in order to explicitly compute the resulting PDF (Probability Density Function) using the quotient rule (4).

Lemma 1. *Let* $\alpha, a, b, c > 0$ *such that*

$$\alpha + c - a - b > 0. \tag{9}$$

Then,

$$\int_0^1 t^{\alpha-1}\,{}_2\mathrm{F}_1\left(\begin{matrix}a,\,b\\c\end{matrix}\,\middle|\,1-t\right)\mathrm{d}t=\frac{1}{\alpha}\,{}_3\mathrm{F}_2\left(\begin{matrix}a,\,b,\,1\\c,\,1+\alpha\end{matrix}\,\middle|\,1\right). \tag{10}$$

Proof. Recall that $t\in(0,\,1)\implies 1-t\in(0,\,1)$ and that

$${}_2\mathrm{F}_1\left(\begin{matrix}a,b\\c\end{matrix}\,\middle|\,1-t\right)=\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\,\frac{(1-t)^n}{n!}. \tag{11}$$

Series (11) has positive addends; hence, we can integrate the left-hand side of (10) term by term in virtue of the monotone convergence theorem, yielding

$$\begin{aligned}\int_0^1 t^{\alpha-1}\,{}_2\mathrm{F}_1\left(\begin{matrix}a,\,b\\c\end{matrix}\,\middle|\,1-t\right)\mathrm{d}t&=\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\,\frac{1}{n!}\int_0^1 t^{\alpha-1}\,(1-t)^n\,\mathrm{d}t\\&=\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\,\frac{1}{n!}\,\frac{\Gamma(\alpha)\,\Gamma(n+1)}{\Gamma(\alpha+n+1)}\\&=\frac{1}{\alpha}\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\,\frac{1}{n!}\,\frac{\Gamma(\alpha+1)\,\Gamma(n+1)}{\Gamma(\alpha+n+1)}\\&=\frac{1}{\alpha}\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\,\frac{1}{n!}\,\frac{(1)_n}{(\alpha+1)_n}\\&=\frac{1}{\alpha}\,{}_3\mathrm{F}_2\left(\begin{matrix}a,\,b,\,1\\c,\,1+\alpha\end{matrix}\,\middle|\,1\right).\end{aligned}$$

Recalling (9), we ensure the convergence of the ${}_3\mathrm{F}_2$ evaluated at 1. □

In our article, we present a probabilistic proof of two famous hypergeometric summation results due to Kummer and Watson.

Theorem 2 (Kummer summation). *If* $1+a-b\neq 0$ *is not a negative integer and if* $\operatorname{Re} b<1$*, then:*

$${}_2\mathrm{F}_1\left(\begin{matrix}a;b\\1+a-b\end{matrix}\,\middle|\,-1\right)=\frac{\Gamma(1+a-b)\,\Gamma(1+\frac{a}{2})}{\Gamma(1+a)\,\Gamma(1+\frac{a}{2}-b)}$$

The proof of Theorem 2 can be found [21] (Chap. 2). A simplified proof, but with additional hypotheses, is provided in [26], while the source is [7] (p. 53).

Theorem 3 (Watson summation). *If* $1+a-b\neq 0$ *is not a negative integer and if* $\operatorname{Re} b<1$*, then:*

$${}_3F_2\left(\begin{matrix}a,b,c\\\frac{1}{2}(a+b+1),2c\end{matrix}\,\middle|\,1\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}.$$

The proof of Theorem 3 can be found in [21] (Chap. 2) and in [25] (Chap. 3).

Research in this field has not stopped at these classic results; there are several recent contributions, including [27–30].

While the classical proofs of Kummer and Watson, like their later counterparts, are grounded exclusively in the framework of hypergeometric analysis, our probabilistic derivation posits that these identities transcend their analytical significance and instead reveal profound structural properties associated with the quotients of independent and identically distributed random variables. This approach elucidates the intrinsic connection between series summability and the preservation of specific distributional symmetries, offering a statistical perspective that conventional algebraic manipulation often obscures.

In addition to their significance in pure mathematics, these summation formulas have a range of applications: Kummer's formula, for instance, is employed in physical and mathematical contexts to ascertain closed-form solutions to differential equations describing wave propagation problems [31]. In quantum mechanics, Kummer's identities are utilised to normalize wave functions; in stochastic models [32]; in analytic number theory related to Ramanujan's series [33]; in statistics [34], where Kummer's summation facilitates the simplification of integrals encountered in multivariate probability distributions; and in combinatorics [35], where Kummer's formula is applied to convert binomial sums into hypergeometric series.

Similarly, Watson's theorem plays a fundamental role in quantum mechanics for calculating the Clebsch–Gordan coefficients and Wigner symbols. These coefficients, which govern angular-momentum coupling, are often expressed as hypergeometric series. Watson's theorem allows them to be reduced to products of functions, simplifying the calculation of transition amplitudes. The connection between Wigner's symbols and ${}_3F_2$ is profound and not merely a formal coincidence: in fact, the parameter c is related to the intermediate angular momentum, while a and b are related to the sums and differences of the angular momenta of the individual subsystems. For further references in the field, we recommend two fundamental texts on the theory of angular momentum: ref. [36], where in Chapter 8.2.1 (page 248), the exact table relating each hypergeometric transformation (Watson, Whipple, Dixon) to the specific restriction on the quantum numbers is presented, and [37]. Connections of Watson's theorem with quantum gravity are exposed in [38], where it is examined how the Wigner $6j$ symbols [39], define the transition amplitude in quantum spacetime, while in [40] in the field of quantum chemistry, (3) is used in the calculation of multi-centre superposition integrals and in [41] Formula (3) is used in quantum information theory. Queuing and stochastic process theory employ birth-and-death processes and their representation in terms of discrete orthogonal polynomials. Karlin and McGregor [42] showed that the transition probabilities of a birth–death process can be expressed as integrals of orthogonal polynomials. Many of these (such as the Meixner, Charlier, or Hahn polynomials) are hypergeometric functions, where Watson's formula allows the moments of the distribution to be calculated by using special functions to solve transition equations in stochastic processes and [43] deals with explicit solutions for transient queue states where hypergeometric functions and summation formulas are used to simplify the results.

2.2. Ratios of Independent and Identically Distributed Gamma and Beta Random Variables

The analysis developed in this subsection highlights a key structural feature of ratios of random variables. When considering quotients of independent and identically distributed Gamma random variables, a natural symmetry emerges: no variable is privileged over the other, and this invariance is reflected in the identity (1). As we will show, this seemingly elementary probabilistic symmetry is sufficiently strong to generate nontrivial analytic identities. In particular, it allows us to recover special cases of classical summation theorems for hypergeometric functions, such as Kummer's and Watson's Theorems 2 and 3. This provides evidence that these identities are not merely isolated analytical facts, but rather manifestations of deeper structural properties associated with ratios of independent and identically distributed random variables. While the classical proofs of Kummer's and Watson's theorems are formulated entirely within the framework of hypergeometric analysis, our probabilistic derivation suggests a different perspective. These identities can be interpreted as consequences of distributional symmetries, revealing a connection between summability of series and invariance properties of random variables. In this sense, probabilistic arguments uncover a structural layer that is often hidden by

purely algebraic manipulations. In the next subsection, we will investigate what happens when the assumption of identical distribution is relaxed. In that setting, hypergeometric series representations still arise, but the symmetry is no longer sufficient to produce closed-form evaluations such as those of Kummer or Watson. This contrast further emphasises the fundamental role played by symmetry in the emergence of exact identities.

Let $X_1, X_2 \sim \text{Gamma}(\alpha, \lambda)$ be two i.i.d. Gamma random variables, following the definition in (2). Consider the quotient

$$\mathrm{T} = \frac{\mathrm{X}_1}{\mathrm{X}_2}.$$

Using the quotient distribution of Formula (4), we compute the density of T as

$$q_{\mathrm{T}}(t) = \int_0^\infty x \, \frac{\lambda^\alpha (t\,x)^{\alpha-1} e^{-\lambda t x}}{\Gamma(\alpha)} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \, \mathrm{d}\, x.$$

Simplifying, we get

$$q_{\mathrm{T}}(t) = \frac{\lambda^{2\alpha} t^{\alpha-1}}{\Gamma^2(\alpha)} \int_0^\infty x^{2\alpha-1} e^{-(1+t)\lambda x} \, \mathrm{d}\, x.$$

Evaluating the integral,

$$\int_0^\infty x^{2\alpha-1} e^{-(1+t)\lambda x} \, \mathrm{d}\, x = \frac{\Gamma(2\alpha)}{[(1+t)\lambda]^{2\alpha}},$$

so that the PDF of the quotient is

$$q_{\mathrm{T}}(t) = \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} \frac{t^{\alpha-1}}{(1+t)^{2\alpha}}. \tag{12}$$

Hence, combining (3) and (12), we can say that $\mathrm{T} \sim \text{BetaPrime}(\alpha, \alpha)$. This result is not merely a computational convenience: it reveals a structural link between Gamma- and Beta-type distributions. In particular, the emergence of the Beta prime law reflects the invariance of the Gamma family under scaling and normalisation. It highlights how ratio transformations naturally lead to distributions supported on the positive half-line that decay polynomially. This interplay will be crucial in what follows, as it allows us to translate probabilistic symmetries into analytic identities.

By [2], since T is the quotient of two i.i.d. random variables, we can invoke the symmetry in (1), i.e.,

$$\frac{1}{2} = \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} \int_0^1 \frac{t^{\alpha-1}}{(1+t)^{2\alpha}} \, \mathrm{d}t. \tag{13}$$

We observe that the integral on the right-hand side of (13) is a well-known result, re-obtained through probabilistic arguments.

Expanding $(1+t)^{-2\alpha}$ into a geometric series,

$$(1+t)^{-2\alpha} = \sum_{n=0}^{\infty} \binom{2\alpha+n-1}{n} (-t)^n,$$

and using the monotone convergence theorem of Beppo Levi (see [44], pp. 82–84), we obtain

$$\begin{aligned} \frac{1}{2} &= \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} \int_0^1 \sum_{n=0}^{\infty} \binom{2\alpha+n-1}{n} (-1)^n t^{\alpha+n-1} \mathrm{d}\, t \\ &= \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} \sum_{n=0}^{\infty} \binom{2\alpha+n-1}{n} (-1)^n \int_0^1 t^{\alpha+n-1} \mathrm{d}\, t. \end{aligned}$$

Evaluating the integral, we get that

$$\frac{1}{2} = \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} \sum_{n=0}^{\infty} \binom{2\alpha+n-1}{n} \frac{(-1)^n}{\alpha+n}. \tag{14}$$

Equation (14) can be written in hypergeometric form as, via (5),

$$\frac{1}{2} = \frac{\Gamma(2\alpha)}{\alpha\,\Gamma^2(\alpha)}\, {}_2\mathrm{F}_1\left(\begin{matrix} \alpha; 2\alpha \\ \alpha+1 \end{matrix} \middle| -1 \right),$$

or equivalently

$${}_2\mathrm{F}_1\left(\begin{matrix} \alpha; 2\alpha \\ \alpha+1 \end{matrix} \middle| -1 \right) = \frac{\alpha\,\Gamma^2(\alpha)}{2\,\Gamma(2\alpha)}. \tag{15}$$

Remark 1. *We notice that* (15) *is a particular case of Theorem 2 due to Kummer, where* $a = 2\alpha$ *and* $b = \alpha$*. This identity can be interpreted as a probabilistic derivation of a classical hypergeometric evaluation. What is remarkable here is that the special value of the* ${}_2\mathrm{F}_1$ *arises naturally from a symmetry argument, rather than from transformation formulas or contour integration. This reinforces the idea that probabilistic constructions can serve as a unifying framework for special function theory.*

We proceed similarly by comparing the quotient of four i.i.d. Gamma distributions. In particular, since we know by [10–13] that the ratio of two i.i.d. $\Gamma(\alpha, \lambda)$ is a BetaPrime(α, α), we consider

$$\mathrm{T}_1, \mathrm{T}_2 \sim \mathrm{BetaPrime}(\alpha, \alpha),$$

and define

$$\mathrm{S} = \frac{\mathrm{T}_1}{\mathrm{T}_2}.$$

Before continuing, we record the following elementary but useful change in variable, which will be instrumental in the sequel.

Lemma 2. *Let* $f : (0,1) \to \mathbb{R}$ *be an integrable function. Then the following identity holds:*

$$\int_0^{\infty} f(x)\,\mathrm{d}x = \int_0^1 f\left(\frac{x}{1-x}\right) \frac{\mathrm{d}x}{(1-x)^2}.$$

Proof. The result follows from the change in variable

$$x = \frac{t}{1-t}, \qquad t \in (0,1),$$

for which

$$\mathrm{d}x = \frac{\mathrm{d}t}{(1-t)^2}.$$

Substituting into the integral over $(0,\infty)$ yields the desired identity. □

Using the quotient formula again,

$$q_{\mathrm{S}}(t) = \int_0^{\infty} x\, q_{\mathrm{T}_1}(tx) q_{\mathrm{T}_2}(x)\,\mathrm{d}x$$

and substituting the densities, we obtain

$$\begin{aligned} q_S(t) &= \frac{\Gamma^2(2\alpha)}{\Gamma^4(\alpha)} t^{\alpha-1} \int_0^\infty \frac{x^{2\alpha-1}}{(1+x)^{2\alpha}(1+tx)^{2\alpha}} \, \mathrm{d}\, x \\ &\overset{\text{Lemma 2}}{=} \frac{\Gamma^2(2\alpha)}{\Gamma^4(\alpha)} t^{\alpha-1} \int_0^1 \frac{x^{2\alpha-1}(1-x)^{2\alpha-1}}{(1-(1-t)x)^{2\alpha}} \, \mathrm{d}\, x \\ &\overset{\text{Theorem 1}}{=} \frac{\Gamma^4(2\alpha)}{\Gamma(4\alpha)\,\Gamma^4(\alpha)} t^{\alpha-1} \, {}_2\mathrm{F}_1\left(\begin{matrix} 2\alpha; 2\alpha \\ 4\alpha \end{matrix} \middle| 1-t \right). \end{aligned} \tag{16}$$

The density above already exhibits a significantly richer structure, involving a non-trivial coupling between two rational terms. The appearance of the hypergeometric function ${}_2\mathrm{F}_1$ is therefore not accidental, but rather reflects the intrinsic combinatorial complexity of the ratio of Beta prime variables. Since (16) is a PDF, we can invoke the symmetry (1), so that

$$\begin{aligned} \frac{1}{2} &= \frac{\Gamma^4(2\alpha)}{\Gamma(4\alpha)\,\Gamma^4(\alpha)} \int_0^1 t^{\alpha-1} \, {}_2\mathrm{F}_1\left(\begin{matrix} 2\alpha; 2\alpha \\ 4\alpha \end{matrix} \middle| 1-t \right) \mathrm{d}\, t \\ &\overset{\text{Lemma 1}}{=} \frac{\Gamma^4(2\alpha)}{\alpha\,\Gamma(4\alpha)\,\Gamma^4(\alpha)} \, {}_3\mathrm{F}_2\left(\begin{matrix} 2\alpha; 2\alpha; 1 \\ 4\alpha; \alpha+1 \end{matrix} \middle| 1 \right) \end{aligned} \tag{17}$$

Equivalently,

$${}_3\mathrm{F}_2\left(\begin{matrix} 2\alpha; 2\alpha; 1 \\ 4\alpha; \alpha+1 \end{matrix} \middle| 1 \right) = \frac{\alpha\,\Gamma(4\alpha)\,\Gamma^4(\alpha)}{2\,\Gamma^4(2\alpha)}. \tag{18}$$

Remark 2. *Notice that* (18) *is a particular case of Theorem 3 due to Watson and Whipple with* $a = b = 2\alpha$ *and* $c = 1$. *This identity represents a higher-order analogue of* (15), *and can be viewed as a probabilistic counterpart of Watson-type summation formulas for* ${}_3\mathrm{F}_2$. *We want to emphasise that Mathematica is incapable of evaluating the integral we computed in Formula* (17). *For this reason, Lemma 1 has been introduced. Moreover, Mathematica is not able to symbolically simplify the left-hand side of Formula* (18), *where the* ${}_3\mathrm{F}_2$ *appears. It can only evaluate it when a specific numerical value is assigned to the parameter* α, *without, however, capturing the right-hand side of Formula* (18). *The striking feature is that such a non-trivial evaluation emerges from a simple symmetry argument applied to ratios of independent random variables. This perspective suggests that probabilistic methods may provide a systematic way to generate and interpret hypergeometric identities of increasing complexity.*

The results obtained in this subsection show that classical hypergeometric summation formulas can be naturally interpreted through a probabilistic lens. In particular, the emergence of special values such as (15) and (18) is directly linked to the symmetry induced by ratios of independent and identically distributed random variables. This highlights the essential role of symmetry as the underlying mechanism responsible for the exact evaluation of these series. From this perspective, Kummer's and Watson's theorems are not only analytic statements, but also reflect intrinsic invariance properties of probabilistic models. The fact that such identities arise from simple considerations of ratios of random variables suggests that probabilistic methods may provide a unifying framework for understanding a wide class of special-function identities. Beyond their theoretical interest, these summation formulas have numerous applications across different areas of mathematics and physics. Kummer's formula, for instance, appears in the study of differential equations, particularly in wave propagation problems [31], in stochastic models [32], in analytic number theory in

connection with Ramanujan's series [33], in multivariate statistics [34], and even in combinatorics [35], where it is used to transform binomial sums into hypergeometric series. Similarly, Watson's theorem plays a fundamental role in quantum mechanics, especially in computing Clebsch–Gordan coefficients and Wigner symbols, which describe angular momentum coupling. These quantities are often expressed in terms of hypergeometric series, and Watson's theorem allows them to be reduced to products of simpler functions. The connection between Wigner $3j$ symbols [39], and ${}_3\mathrm{F}_2$ functions is not merely formal, but reflects the underlying structure of angular momentum composition. For further references, see [36,37]. Moreover, Watson-type identities appear in quantum gravity [38], quantum chemistry [40], and quantum information theory [41]. In probability theory and stochastic processes, hypergeometric functions arise in the study of birth-and-death processes through orthogonal polynomials [42], while summation formulas such as Watson's play a role in simplifying transition probabilities and moment calculations [43]. These connections reinforce the concept that the identities elucidated in this subsection are not isolated outcomes but are components of a more extensive mathematical framework wherein symmetry, probability, and special functions are profoundly interconnected.

2.3. Ratios of Independent Gamma and Beta Random Variables

In the probabilistic approach developed in this work, identities involving hypergeometric functions naturally arise from the study of ratios of independent random variables. In several classical situations, an additional symmetry appears when the variables are not only independent but also identically distributed. In particular, if X and Y are i.i.d. positive random variables, then the simple symmetry (1) often leads to remarkably compact closed forms. In our previous investigations, this symmetry allowed us to recover classical summation theorems for hypergeometric series, including well-known identities due to Kummer and Watson.

In the present section, we deliberately relax the assumption of identical distributions and consider ratios of Gamma random variables that are independent but not identically distributed. The purpose of this analysis is to highlight the role played by the symmetry of the i.i.d. case and to show how its absence affects the structure of the resulting identities.

We begin by studying the ratio of two independent Gamma random variables having different shape parameters but a common rate parameter. This case already breaks the symmetry of the i.i.d. setting while preserving a relatively simple analytic structure. We then extend the analysis to ratios involving four independent Gamma variables with different shape parameters.

It is worth noting that even if one allows both the shape and the rate parameters to differ, certain structural properties remain unchanged. In particular, as we will see, if $X_1 \sim \mathrm{Gamma}(\alpha_1, \lambda_1)$ and $X_2 \sim \mathrm{Gamma}(\alpha_2, \lambda_2)$ are independent, the ratio

$$\frac{X_1}{X_2} \sim \mathrm{BetaPrime}(\alpha_1, \alpha_2)$$

as in the case where $X_1 \sim \mathrm{Gamma}(\alpha_1, \lambda)$ and $X_2 \sim \mathrm{Gamma}(\alpha_2, \lambda)$ are independent.

The analysis that follows shows that, once the symmetry of the i.i.d. case is lost, the resulting identities no longer collapse to a single hypergeometric summation formula. Instead, the normalisation condition leads to a combination of two ${}_3\mathrm{F}_2$ series, reflecting the absence of the probabilistic symmetry that in the i.i.d. case produces the classical summation formulas of Kummer and Watson.

2.3.1. Half-Broken Symmetry

Let X_1 and X_2 be independent Gamma random variables with a common rate $\lambda > 0$ and different shape parameters $\alpha_1, \alpha_2 > 0$, namely $\mathrm{X}_i \sim \mathrm{Gamma}(\alpha_i, \lambda)$ for $i = 1, 2$, using the definition in (2). Define the ratio

$$\mathrm{T} := \frac{\mathrm{X}_1}{\mathrm{X}_2}.$$

To compute the density of T, we need to apply (4), i.e.,

$$f_{\mathrm{T}}(t) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} t^{\alpha_1-1} \int_0^\infty x^{\alpha_1+\alpha_2-1} e^{-\lambda(1+t)x}\, \mathrm{d}x = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} t^{\alpha_1-1} \frac{\Gamma(\alpha_1+\alpha_2)}{(\lambda(1+t))^{\alpha_1+\alpha_2}}$$
$$= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{t^{\alpha_1-1}}{(1+t)^{\alpha_1+\alpha_2}}, \qquad t > 0.$$

First of all, we notice that

$$\int_0^\infty f_{\mathrm{T}}(t)\, \mathrm{d}t = 1.$$

Hence by (3), T follows a BetaPrime distribution with parameters (α_1, α_2). Indeed, the normalising constant that allows the integral to be equal to 1 is exactly $\mathrm{Beta}(\alpha_1, \alpha_2)^{-1}$. Notice that in the case $\alpha_1 = \alpha_2$, we recover exactly Kummer theorem already found in (15).

Let $(\mathrm{X}_1, \mathrm{X}_2)$ and $(\mathrm{Y}_1, \mathrm{Y}_2)$ be independent pairs of independent Gamma random variables with common rate $\lambda > 0$ and shapes $\alpha_1, \alpha_2 > 0$ and $\beta_1, \beta_2 > 0$, respectively:

$$\mathrm{X}_i \sim \mathrm{Gamma}(\alpha_i, \lambda), \quad i = 1, 2, \qquad \mathrm{Y}_j \sim \mathrm{Gamma}(\beta_j, \lambda), \quad j = 1, 2,$$

and all four variables are mutually independent. Define the two Beta prime ratios

$$\mathrm{T}_1 := \frac{\mathrm{X}_1}{\mathrm{X}_2}, \qquad \mathrm{T}_2 := \frac{\mathrm{Y}_1}{\mathrm{Y}_2},$$

and consider their quotient

$$\mathrm{R} := \frac{\mathrm{T}_1}{\mathrm{T}_2} = \frac{\mathrm{X}_1\mathrm{Y}_2}{\mathrm{X}_2\mathrm{Y}_1}, \qquad \mathrm{R} > 0.$$

Therefore,

$$f_{\mathrm{R}}(t) = \frac{1}{\mathrm{Beta}(\alpha_1, \alpha_2)\mathrm{Beta}(\beta_1, \beta_2)} \int_0^\infty x^{\alpha_1+\beta_1-1}(x+1)^{-\beta_1-\beta_2}(1+tx)^{-\alpha_1-\alpha_2}\, \mathrm{d}x$$
$$\overset{\text{Lemma 2}}{=} \frac{t^{\alpha_1-1}}{\mathrm{Beta}(\alpha_1, \alpha_2)\mathrm{Beta}(\beta_1, \beta_2)} \int_0^1 \frac{(1-u)^{\alpha_2+\beta_2-1} u^{\alpha_1+\beta_1-1}}{(1-(1-t)\,u)^{\alpha_1+\alpha_2}}\, \mathrm{d}u$$
$$\overset{\text{Theorem 1}}{=} \frac{\mathrm{Beta}(\alpha_1+\beta_1, \alpha_2+\beta_2)}{\mathrm{Beta}(\alpha_1, \alpha_2)\mathrm{Beta}(\beta_1, \beta_2)} t^{\alpha_1-1}\, {}_2\mathrm{F}_1\left(\begin{matrix} \alpha_1+\alpha_2; \alpha_1+\beta_1 \\ \alpha_1+\alpha_2+\beta_1+\beta_2 \end{matrix} \middle| 1-t \right), \qquad t > 0.$$

Since R is composed of a ratio of two independent but not identically distributed random variables, we have that:

$$\int_0^\infty f_{\mathrm{R}}(t)\, \mathrm{d}t = 1.$$

Equivalently,

$$\int_0^\infty t^{\alpha_1-1}\, {}_2\mathrm{F}_1\left(\begin{matrix} \alpha_1+\alpha_2; \alpha_1+\beta_1 \\ \alpha_1+\alpha_2+\beta_1+\beta_2 \end{matrix} \middle| 1-t \right) \mathrm{d}t = \frac{\mathrm{Beta}(\alpha_1, \alpha_2)\mathrm{Beta}(\beta_1, \beta_2)}{\mathrm{Beta}(\alpha_1+\beta_1, \alpha_2+\beta_2)}. \tag{19}$$

In order to compute the left-hand side of (19), we split the domain of integration into $[0,1]$ and $(1,\infty)$. Therefore, by Lemma 1, we have

$$\int_0^1 t^{\alpha_1-1}\,{}_2\mathrm{F}_1\left(\begin{matrix}\alpha_1+\alpha_2;\alpha_1+\beta_1\\ \alpha_1+\alpha_2+\beta_1+\beta_2\end{matrix}\middle|1-t\right)dt=\frac{1}{\alpha_1}\,{}_3\mathrm{F}_2\left(\begin{matrix}\alpha_1+\alpha_2;\alpha_1+\beta_1;1\\ \alpha_1+\alpha_2+\beta_1+\beta_2;\alpha_1+1\end{matrix}\middle|1\right).$$

We are left with computing

$$\int_1^\infty t^{\alpha_1-1}\,{}_2\mathrm{F}_1\left(\begin{matrix}\alpha_1+\alpha_2;\alpha_1+\beta_1\\ \alpha_1+\alpha_2+\beta_1+\beta_2\end{matrix}\middle|1-t\right)\mathrm{d}t. \tag{20}$$

For $t>1$, Pfaff's transformation [8] is employed with $x=1-t$. It is important to note that for $t>1$, the transformation $1-t$ within Pfaff's formula is interpreted through the analytic continuation of the ${}_2\mathrm{F}_1$.

$${}_2\mathrm{F}_1\left(\begin{matrix}\alpha_1+\alpha_2;\alpha_1+\beta_1\\ \alpha_1+\alpha_2+\beta_1+\beta_2\end{matrix}\middle|1-t\right)=t^{-\alpha_1-\alpha_2}\,{}_2\mathrm{F}_1\left(\begin{matrix}\alpha_1+\alpha_2;\alpha_2+\beta_2\\ \alpha_1+\alpha_2+\beta_1+\beta_2\end{matrix}\middle|1-\frac{1}{t}\right),$$

and, setting $u=t^{-1}$, (20) becomes

$$\int_1^\infty t^{-\alpha_2-1}\,{}_2\mathrm{F}_1\left(\begin{matrix}\alpha_1+\alpha_2;\alpha_2+\beta_2\\ \alpha_1+\alpha_2+\beta_1+\beta_2\end{matrix}\middle|1-\frac{1}{t}\right)\mathrm{d}t=\int_0^1 u^{\alpha_2-1}\,{}_2\mathrm{F}_1\left(\begin{matrix}\alpha_1+\alpha_2;\alpha_2+\beta_2\\ \alpha_1+\alpha_2+\beta_1+\beta_2\end{matrix}\middle|1-u\right)\mathrm{d}u.$$

Therefore, by Lemma 1, we have

$$\int_0^1 u^{\alpha_2-1}\,{}_2\mathrm{F}_1\left(\begin{matrix}\alpha_1+\alpha_2;\alpha_2+\beta_2\\ \alpha_1+\alpha_2+\beta_1+\beta_2\end{matrix}\middle|1-u\right)du=\frac{1}{\alpha_2}\,{}_3\mathrm{F}_2\left(\begin{matrix}\alpha_1+\alpha_2;\alpha_2+\beta_2;1\\ \alpha_1+\alpha_2+\beta_1+\beta_2;\alpha_2+1\end{matrix}\middle|1\right).$$

Notice that sufficient conditions for convergence are summarised below:

$$\mathrm{Re}(\alpha_1)>0,\mathrm{Re}(\alpha_2)>0,\mathrm{Re}(\beta_1)>0,\mathrm{Re}(\beta_2)>0,\ \beta_1,\beta_2\notin\{-1,-2,\dots\}.$$

Finally, recalling (19), we have

$$\begin{aligned}\frac{1}{\alpha_1}\,{}_3\mathrm{F}_2\left(\begin{matrix}\alpha_1+\alpha_2;\alpha_1+\beta_1;1\\ \alpha_1+\alpha_2+\beta_1+\beta_2;\alpha_1+1\end{matrix}\middle|1\right)+\frac{1}{\alpha_2}\,{}_3\mathrm{F}_2\left(\begin{matrix}\alpha_1+\alpha_2;\alpha_2+\beta_2;1\\ \alpha_1+\alpha_2+\beta_1+\beta_2;\alpha_2+1\end{matrix}\middle|1\right)\\ =\frac{\mathrm{Beta}(\alpha_1,\alpha_2)\,\mathrm{Beta}(\beta_1,\beta_2)}{\mathrm{Beta}(\alpha_1+\beta_1,\alpha_2+\beta_2)}.\end{aligned} \tag{21}$$

Notice that if $\alpha_1=\alpha_2=\beta_1=\beta_2=\alpha$, we recover Formula (18), or equivalently the case of complete symmetry. Observe also that the symmetry can be identified within the parameters of the Beta functions on the right-hand side of (21). Furthermore, it is noteworthy that the breaking of this symmetry manifests in one of the numerator parameters of the ${}_3\mathrm{F}_2$, which, in our case, corresponds to the second parameter. Indeed, it is evident that precisely in the second parameter of the ${}_3\mathrm{F}_2$ in (21) we observe the parameters of the Beta Prime distributions, namely α_1,β_1 and α_2,β_2. It is important to note that in the present context, the symmetry observed for identically distributed random variables is lost. Specifically, since the shape parameters of the Gamma variables are not equal, the probabilistic symmetry that, in the i.i.d. case, implies (1) no longer holds. Consequently, the resulting identity does not simplify to a single closed hypergeometric expression.

Conversely, in the independent and identically distributed (i.i.d.) scenario, this symmetry produces more straightforward closed-form solutions that accurately correspond to well-known summation theorems for hypergeometric series, such as those established by Kummer and Watson. To underscore this point, consider the case where the quotient of independent Gamma distributions with varying parameters for the shape and rate is examined.

2.3.2. Fully Broken Symmetry

Let us now examine the scenario in which the symmetry is entirely disrupted by using Gamma distributions with distinct shape and rate parameters. It is noteworthy that, by entirely losing symmetry in the rate parameter λ, the ${}_3\mathrm{F}_2$ function evaluated at 1 is no longer obtained. In fact, the outcome results from the negative ratio of the rate parameters. Let $\mathrm{X}_1 \sim \mathrm{Gamma}(\alpha_1, \lambda_1)$ and $\mathrm{X}_2 \sim \mathrm{Gamma}(\alpha_2, \lambda_2)$ be two independent random variables, then the ratio

$$\mathrm{T} = \frac{\mathrm{X}_1}{\mathrm{X}_2} \sim \mathrm{BetaPrime}(\alpha_1, \alpha_2),$$

as in the case where $\mathrm{X}_1 \sim \mathrm{Gamma}(\alpha_1, \lambda)$ and $\mathrm{X}_2 \sim \mathrm{Gamma}(\alpha_2, \lambda)$ are independent. Indeed, applying the quotient rule (4), we recover that

$$\begin{aligned} f_{\mathrm{T}}(t) &= \int_0^\infty \frac{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2} t^{\alpha_1-1} x^{\alpha_1+\alpha_2-1} e^{-x(\lambda_2+\lambda_1 t)}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \mathrm{d}\, x = \frac{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2} t^{\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty x^{\alpha_1+\alpha_2-1} e^{-x(\lambda_2+\lambda_1 t)} \mathrm{d}\, x \\ &= \frac{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\Gamma(\alpha_1+\alpha_2) t^{\alpha_1-1}(\lambda_2+\lambda_1 t)^{-\alpha_1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} = \left(\frac{\lambda_1}{\lambda_2}\right)^{\alpha_1} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{t^{\alpha_1-1}}{\left(1+\frac{\lambda_1}{\lambda_2} t\right)^{\alpha_1+\alpha_2}}. \end{aligned} \tag{22}$$

Notice that

$$1 = \int_0^\infty f_{\mathrm{T}}(t)\, \mathrm{d}\, t = \left(\frac{\lambda_1}{\lambda_2}\right)^{\alpha_1} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty \frac{t^{\alpha_1-1}}{\left(1+\frac{\lambda_1}{\lambda_2} t\right)^{\alpha_1+\alpha_2}} \mathrm{d}\, t,$$

i.e., $\mathrm{T} \sim \mathrm{BetaPrime}(\alpha_1, \alpha_2)$ multiplied by the normalising constant $\left(\frac{\lambda_1}{\lambda_2}\right)^{\alpha_1}$. Again, we observe that if $\alpha_1 = \alpha_2$ and $\lambda_1 = \lambda_2$, we recover (15).

Now, consider the last equation in (22) as the new probability density function to which we apply the quotient rule. Formally, let $(\mathrm{X}_1, \mathrm{X}_2)$ and $(\mathrm{Y}_1, \mathrm{Y}_2)$ be independent pairs of independent Gamma random variables with rates $\lambda_j > 0$ for $j = 1, \ldots, 4$ and shapes $\alpha_1, \alpha_2 > 0$ and $\beta_1, \beta_2 > 0$, respectively:

$$\mathrm{X}_i \sim \mathrm{Gamma}(\alpha_i, \lambda_j), \qquad \mathrm{Y}_j \sim \mathrm{Gamma}(\beta_i, \lambda_j), \quad i = 1, 2 \quad \text{and} \quad j = 1, \ldots, 4$$

and all four variables are mutually independent. Define the two Beta prime ratios

$$\mathrm{T}_1 := \frac{\mathrm{X}_1}{\mathrm{X}_2}, \qquad \mathrm{T}_2 := \frac{\mathrm{Y}_1}{\mathrm{Y}_2},$$

and consider their quotient

$$\mathrm{R} := \frac{\mathrm{T}_1}{\mathrm{T}_2} = \frac{\mathrm{X}_1\mathrm{Y}_2}{\mathrm{X}_2\mathrm{Y}_1}, \qquad \mathrm{R} > 0.$$

Therefore, we have that

$$
\begin{aligned}
f_{\mathrm{R}}(t) &= \frac{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\beta_1}\lambda_4^{\beta_2}t^{\alpha_1-1}}{\mathrm{Beta}(\alpha_1,\alpha_2)\,\mathrm{Beta}(\beta_1,\beta_2)}\int_0^\infty \frac{x^{\alpha_1+\beta_1-1}}{(\lambda_4+\lambda_3 x)^{\beta_1+\beta_2}(\lambda_2+\lambda_1 t x)^{\alpha_1+\alpha_2}}\,\mathrm{d}x \\
&\overset{\text{Computations}}{=} \frac{\lambda_1^{\alpha_1}\lambda_2^{-\alpha_1}\lambda_3^{\beta_1}\lambda_4^{-\beta_1}t^{\alpha_1-1}}{\mathrm{Beta}(\alpha_1,\alpha_2)\,\mathrm{Beta}(\beta_1,\beta_2)}\int_0^\infty \frac{x^{\alpha_1+\beta_1-1}}{\left(1+\frac{\lambda_3}{\lambda_4}x\right)^{\beta_1+\beta_2}\left(1+\frac{\lambda_1}{\lambda_2}tx\right)^{\alpha_1+\alpha_2}}\,\mathrm{d}x \\
&\overset{\text{Lemma 2}}{=} \frac{\lambda_1^{\alpha_1}\lambda_2^{-\alpha_1}\lambda_3^{\beta_1}\lambda_4^{-\beta_1}t^{\alpha_1-1}}{\mathrm{Beta}(\alpha_1,\alpha_2)\,\mathrm{Beta}(\beta_1,\beta_2)}\int_0^1 \frac{x^{\alpha_1+\beta_1-1}(1-x)^{\alpha_2+\beta_2-1}}{\left(1-\left(1-\frac{\lambda_3}{\lambda_4}\right)x\right)^{\beta_1+\beta_2}\left(1-\left(1-\frac{\lambda_1}{\lambda_2}t\right)x\right)^{\alpha_1+\alpha_2}}\,\mathrm{d}x,
\end{aligned}
\tag{23}
$$

for $t>0$. Note that the last integral in (23) is an Appell F_1 series expressed in its integral representation (see [21]), i.e., letting

$$
K_2 = \frac{\lambda_1^{\alpha_1}\lambda_2^{-\alpha_1}\lambda_3^{\beta_1}\lambda_4^{-\beta_1}}{\mathrm{Beta}(\alpha_1,\alpha_2)\,\mathrm{Beta}(\beta_1,\beta_2)}\,\frac{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\alpha_2+\beta_1+\beta_2)} = \frac{\lambda_1^{\alpha_1}\lambda_2^{-\alpha_1}\lambda_3^{\beta_1}\lambda_4^{-\beta_1}\mathrm{Beta}(\alpha_1+\alpha_2,\beta_1+\beta_2)}{\mathrm{Beta}(\alpha_1,\beta_1)\mathrm{Beta}(\alpha_2,\beta_2)},
$$

so that, for $\alpha_1+\beta_1>0$ and $\alpha_2+\beta_2>0$,

$$
f_{\mathrm{R}}(t) = K_2\, t^{\alpha_1-1}\,\mathrm{F}_1\left(\begin{matrix}\alpha_1+\beta_1,\beta_1+\beta_2\\ \alpha_1+\alpha_2,\alpha_1+\alpha_2+\beta_1+\beta_2\end{matrix}\,\middle|\, 1-\frac{\lambda_3}{\lambda_4},1-\frac{\lambda_1}{\lambda_2}\,t\right). \tag{24}
$$

Since (24) is again a probability density function, we can say that the integral in the range $[0,\infty)$ is equal to 1, i.e.,

$$
\begin{aligned}
\frac{1}{K_2} &= \int_0^\infty f_{\mathrm{R}}(t)\,\mathrm{d}\,t \\
&= \int_0^\infty t^{\alpha_1-1}\,\mathrm{F}_1\left(\begin{matrix}\alpha_1+\beta_1,\beta_1+\beta_2\\ \alpha_1+\alpha_2,\alpha_1+\alpha_2+\beta_1+\beta_2\end{matrix}\,\middle|\, 1-\frac{\lambda_3}{\lambda_4},1-\frac{\lambda_1}{\lambda_2}\,t\right)\mathrm{d}\,t.
\end{aligned}
\tag{25}
$$

or equivalently

$$
\int_0^\infty t^{\alpha_1-1}\mathrm{F}_1\left(\begin{matrix}\alpha_1+\beta_1,\beta_1+\beta_2\\ \alpha_1+\alpha_2,\alpha_1+\alpha_2+\beta_1+\beta_2\end{matrix}\,\middle|\, 1-\frac{\lambda_3}{\lambda_4},1-\frac{\lambda_1}{\lambda_2}\,t\right)\mathrm{d}\,t = \frac{\lambda_2^{\alpha_1}\lambda_4^{\beta_1}}{\lambda_1^{\alpha_1}\lambda_3^{\beta_1}}\,\frac{\mathrm{Beta}(\alpha_1,\beta_1)\,\mathrm{Beta}(\alpha_2,\beta_2)}{\mathrm{Beta}(\alpha_1+\alpha_2,\beta_1+\beta_2)}.
$$

We note that the evaluation of the integral on the left-hand side could in principle be carried out by combining Lemma 1 with Lemma 2, which would lead to a representation in terms of a sum of generalised hypergeometric series of type ${}_3\mathrm{F}_2$. However, we have not included these calculations, since, due to the lack of symmetry, such an approach does not yield a single closed-form expression. In particular, the loss of symmetry prevents the simplifications that occur in the i.i.d. setting. As a consequence, a purely symbolic validation of Formula (25), or of its equivalent compact form given above, would lead to a highly cumbersome expression, involving several terms each containing special functions of increasing complexity. Such a representation would not provide a clearer interpretation of the result, nor would it simplify its verification. For this reason, we preferred to validate the identity numerically, sacrificing a closed symbolic expression in favour of a more transparent and reliable assessment of the accuracy of the obtained result. It is, however, evident that the absence of symmetry conditions on the parameters, namely the non-i.i.d. case, which translates into the impossibility of exploiting the symmetry described in (1), prevents us from recovering, as a special case, the result given by Watson's theorem given in (18). However, in Table 1 below, we report numerical evaluations obtained for three different parameter choices that confirm the validity of (25).

Parameter set 1:

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{3}, \quad \beta_1 = \frac{1}{5}, \quad \beta_2 = \frac{1}{10}, \quad \lambda_1 = \frac{1}{4}, \quad \lambda_2 = \frac{1}{12}, \quad \lambda_3 = \frac{1}{6}, \quad \lambda_4 = \frac{1}{8}.$$

Parameter set 2:

$$\alpha_1 = 1, \quad \alpha_2 = \frac{1}{4}, \quad \beta_1 = \frac{1}{10}, \quad \beta_2 = \frac{1}{4}, \quad \lambda_1 = \frac{1}{7}, \quad \lambda_2 = \frac{1}{100}, \quad \lambda_3 = \frac{1}{200}, \quad \lambda_4 = \frac{1}{1000}.$$

Parameter set 3:

$$\alpha_1 = 1, \quad \alpha_2 = \frac{1}{4}, \quad \beta_1 = \frac{1}{10}, \quad \beta_2 = \frac{1}{4}, \quad \lambda_1 = \frac{1}{7}, \quad \lambda_2 = 300, \quad \lambda_3 = \frac{1}{200}, \quad \lambda_4 = \frac{1}{1000}.$$

Table 1 compares the numerical evaluation of the right-hand side of (25), obtained by `NIntegrate` of Mathematica®, Version 14.3 [24], with the corresponding value of $1/K_2$ for three different parameter sets. The relative error shows an excellent agreement between the two quantities.

Table 1. Comparison between the numerical evaluation of the left-hand side of (25) and the corresponding value of $1/K_2$ for three different choices of parameters.

Parameter Set	RHS of (25)	$1/K_2$	Relative Error
1	11.840732012856772215	11.840732012856772215	0
2	1.7111441725347451600	1.7111406841317187133	2.038642×10^{-6}
3	51334.466351918946776	51334.220523951561400	4.788774×10^{-6}

We stress that, although the absolute difference is larger in the third parameter set, the relative error remains of order 10^{-6}, thus still providing strong numerical support for (25). The relative error is computed as

$$\frac{|\text{RHS of (25)} - 1/K_2|}{|1/K_2|}.$$

3. Conclusions

In this study, we have established a unified framework in which probabilistic and analytical methods collaborate to derive identities for special functions. The fundamental concept is that ratios of random variables, especially within the Gamma distribution, inherently produce integral representations and hypergeometric structures that can be utilised to deduce non-trivial summation formulas. A key outcome of our analysis is that, for independent and identically distributed random variables, symmetry plays a decisive role. The invariance under exchange of the variables translates into simple probabilistic statements, such as (1), which in turn yield exact analytic identities. In this way, classical results such as Kummer's and Watson's summation theorems emerge as direct consequences of distributional symmetries, rather than purely analytic constructions. This perspective suggests that such identities reflect deeper structural properties of the underlying probabilistic models. In particular, the summability of certain hypergeometric series can be interpreted as a manifestation of the preservation of symmetry under ratio transformations. This provides an alternative viewpoint in which special functions are not only analytic objects, but also encode intrinsic probabilistic invariances. When the assumption of identical distribution is relaxed, the situation changes substantially. Although hypergeometric representations still arise, the loss of symmetry prevents the emergence of closed-form evaluations analogous to those of Kummer or Watson. This contrast highlights the fundamental role of symmetry as the mechanism enabling exact summation formulas and clarifies why such identities appear only in specific parameter

configurations. More broadly, the approach developed here opens the way to a systematic probabilistic interpretation of hypergeometric identities. It suggests that higher-order constructions, based on iterated ratios or more general distributional settings, may lead to new classes of identities or to alternative derivations of known results. At the same time, it provides a conceptual bridge between probability theory, special functions, and applications in areas such as mathematical physics, statistics, and stochastic processes. Future research may explore these connections further, both by extending the class of distributions under consideration and by investigating the role of symmetry in more general non-autonomous or non-identically distributed settings. In this direction, probabilistic methods may offer new insights into the structure and interpretation of special functions beyond the classical analytic framework.

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