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Analytical Investigation of the (s, t)-Deformed Free Convolution

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Abstract

The objective of this work is to investigate the $\mathbb{T} = (s, t)$-deformed free convolution $\boxplus_{\mathbb{T}}$ for $s > 0$ and $t \in \mathbb{R}$ and to clarify its structural and asymptotic properties within the framework of Cauchy–Stieltjes kernel (CSK) families. The methodology is based on the analysis of the associated variance functions (VFs), which provide an effective analytic tool for describing deformation mechanisms, invariance properties, and convolution structures. In particular, we derive an explicit formula for the VF of convolution powers and exploit this representation to develop approximation procedures for distributions in CSK families generated by the (s, t)-deformed free Gaussian and free Poisson laws. We also establish several limit theorems describing the asymptotic behavior of the deformation. These findings highlight intrinsic symmetry and scaling properties and reveal connections with free additive, Boolean additive, and free multiplicative convolutions, thereby placing the (s, t)-deformation within a unified probabilistic framework governed by transformation, invariance, and structural regularity.

Keywords: Cauchy–Stieltjes transform; deformed convolution; probability measure; variance function

MSC: 60E10; 46L54

1. Introduction

Transformations of probability measures, together with the convolution operations they induce, form a cornerstone of modern probability theory and statistical modeling. They offer a coherent framework for tracking how probability distributions change when subjected to algebraic, analytic, or stochastic operations, and for constructing new random variables from existing ones. Such transformations are particularly useful for describing collective effects, including aggregation mechanisms, dependence patterns, and asymptotic scaling behavior, all of which play a central role in theoretical probability as well as in applied statistical analysis. From an applied perspective, the ability to systematically transform probability measures provides statisticians and data analysts with adaptable modeling tools for capturing complex stochastic dynamics and for interpreting data generated by interacting or nonclassical sources. In this way, the theory of measure transformations serves as a natural bridge between abstract probabilistic concepts and applications in areas such as engineering, finance, and the social sciences. In this general framework, transformations introduced throughout the Cauchy–Stieltjes transform (CST) have emerged as especially significant in free probability theory, see [1–4]. By expressing convolution

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operations via analytic transforms, problems concerning free additive convolution can be recast as questions about functional equations and the intrinsic analytic structure, emphasizing underlying symmetry, invariance, and structural coherence within the space of probability measures. This perspective has proven particularly effective for revealing symmetry-driven stability, invariance properties, and geometric patterns of probability measures, for analyzing deformation, scaling, and flow-like effects, and for establishing limit theorems governing sequences of noncommutative random variables. Moreover, CST-based methods often provide a unifying language that connects and extends several classical results in free probability. A number of studies illustrate the effectiveness of this approach; see, for example [5–7].

In this context, a deformation of real probability measures $U^{\mathbb{T}}$ was introduced in [8] as a natural extension of the t-deformation [9,10]. Let $\mathbf{P}$ denote the class of Borel probability measures on $\mathbb{R}$, and let $\mathbf{P}_c \subset \mathbf{P}$ be the subclass of measures with compact support. For $\nu \in \mathbf{P}$ and for parameters $s \in \mathbb{R}$ and $t > 0$, one associates the pair $\mathbb{T} = (s, t)$ and defines the (s, t)-deformation through the relation

$$\frac{1}{\mathbf{G}_{U^{\mathbb{T}}(\nu)}(\varpi)} = \frac{s}{\mathbf{G}_\nu(\varpi)} + (1-s)\varpi + C_\nu(t-s). \tag{1}$$

Here,

$$\mathbf{G}_\nu(\varpi) = \int \frac{\nu(dl)}{\varpi - l}, \quad \varpi \in \mathbb{C}\backslash supp(\nu),$$

is the CST of ν and $supp(\nu)$ denotes its support, while $C_\nu = \Re\left(\frac{1}{\mathbf{G}_\nu(i)}\right)$ denotes the Nevanlinna constant of ν.

Based on this construction, one can define a corresponding deformation of free additive convolution. For $\nu \in \mathbf{P}$, the free cumulant transform $\mathcal{R}_\nu$ is analytic in a neighborhood of the origin and is characterized implicitly by the functional relation

$$\frac{1}{\mathbf{G}_\nu(\varpi)} = \varpi - \mathcal{R}_\nu(\mathbf{G}_\nu(\varpi)),$$

see [11,12]. For $\nu_1,\ \nu_2 \in \mathbf{P}$, their free additive convolution $\nu_1 \boxplus \nu_2$ is the unique probability measure whose $\mathcal{R}$-transform is additive:

$$\mathcal{R}_{\nu_1 \boxplus \nu_2}(\xi) = \mathcal{R}_{\nu_1}(\xi) + \mathcal{R}_{\nu_2}(\xi).$$

When $t \neq 0$ and $s > 0$, the map $U^{\mathbb{T}}$ is invertible, and its inverse is given by $U^{\widehat{\mathbb{T}}}$ with $\widehat{\mathbb{T}} = (1/s, 1/t)$ (see [8] [Proposition 5]). The $\mathbb{T}$-deformed free additive convolution for measures ν_1, $\nu_2 \in \mathbf{P}_c$, denoted by $\boxplus_{\mathbb{T}}$, is then defined by transporting the free additive convolution through $U^{\mathbb{T}}$, namely

$$\nu_1 \boxplus_{\mathbb{T}} \nu_2 = (U^{\mathbb{T}})^{-1}(U^{\mathbb{T}}(\nu_1) \boxplus U^{\mathbb{T}}(\nu_2)).$$

This operation interpolates between different convolution structures and leads to limit behaviors that differ from those arising in the standard free setting. In particular, the central limit distribution associated with $\boxplus_{\mathbb{T}}$ is a Kesten-type measure, while the corresponding Poisson limit law, computed in [8], depends explicitly on both deformation parameters.

From another perspective, a central role in the analysis of transformations and convolutions of probability measures is played by Cauchy–Stieltjes kernel (CSK) families. These families have emerged as a key model in noncommutative probability precisely because their behavior is governed by an associated variance function (VF) that encodes the key structural features of the family. In recent years, CSK families have been recognized as the free-probabilistic analogue of natural exponential families (NEFs), with the VF serving as the primary organizing object in both settings. NEFs arise from the Laplace transformation and admit a characterization in terms of their VFs, which serve as a central tool for describing their analytic and statistical properties (see [13–16]). This VF-based perspective provides a coherent framework for applications ranging from statistical modeling to information-theoretic analysis; see, e.g., [17–20]. By contrast, in free probability, CSK families are built by replacing the exponential kernel with the rational kernel $(1-\theta l)^{-1}$. Although this modification reflects the noncommutative nature of free additive convolution, it preserves the kernel-based mechanism that allows VFs to play a decisive role. As in the classical case, the VF associated with a CSK family governs convolution powers, scaling properties, and domain constraints, while also encoding analytic features of the underlying measure through the CST. This VF viewpoint provides a natural bridge between probabilistic structure and analytic techniques such as subordination, making CSK families particularly well suited for studying deformations of free convolutions. From this angle, CSK families retain many formal symmetries and similarities with NEFs, yet their VFs reflect genuinely free phenomena that have no classical counterpart. In particular, VFs in the free setting can be interpreted as noncommutative analogues of Fisher information or curvature-type quantities, offering a geometric interpretation of how probability measures deform under free convolution. This interpretation suggests viewing transformations of probability measures as geometric flows on spaces of distributions, with VFs acting as descriptors of informational or structural complexity. Such a perspective opens connections with information geometry, quantum statistics, and the analysis of high-dimensional or strongly correlated stochastic systems. A rigorous analytic foundation for CSK families and their VFs was developed in [21,22], where the basic properties and structural results of the theory were established. Subsequent work in [23] extended this framework to probability measures with one-sided support boundaries, thereby significantly broadening the scope and applicability of CSK families in free probability.

Previous work [24] investigated the $\mathbb{T}=(s,t)$-deformation of probability measures within the framework of CSK families, with a primary focus on the behavior of the associated VFs. In that study, a closed-form expression describing the transformation of the VF under the (s,t)-deformation was obtained, providing a detailed analytic understanding at the level of individual measure transformations. This analysis led, in particular, to the identification of an invariance property of the free Meixner class under the $\mathbb{T}$-transformation as well as to a new characterization of the CSK family generated by the semicircle distribution through the specific case of the $(1,t)$-deformation. In contrast, the present paper introduces a fundamentally different perspective by moving from the study of transformations acting on single probability measures to the analysis of an associated deformed convolution operation, denoted $\boxplus_{\mathbb{T}}$. This shift is not merely formal: while previous results describe how a given measure is modified under the $\mathbb{T}$-transformation, they do not address how such deformations interact at the level of convolution structures. The main originality of this work lies precisely in the systematic investigation of this new convolution, together with the development of analytic tools adapted to its study. More specifically, we establish an explicit formula for the VF of convolution powers with respect to $\boxplus_{\mathbb{T}}$, a result that has no counterpart in the existing literature on (s,t)-deformations. This provides a tractable and unified description of the analytic behavior of the deformed convolution and significantly extends the scope of previous VF-based methods. Building on this new representation, we further develop approximation procedures for distributions in CSK families generated by the $\mathbb{T}$-deformed free Gaussian and free Poisson laws, which go beyond the transformation-based framework studied earlier. In addition, we derive new limit theorems for the $\boxplus_{\mathbb{T}}$-convolution, revealing asymptotic properties that were not accessible in the previous setting. These results uncover explicit links between the $\mathbb{T}$-deformed convolution and classical objects of noncommutative probability, namely free additive, Boolean additive, and free multiplicative convolutions. Such connections were not established in the earlier work and demonstrate that the present approach embeds the (s,t)-deformation into a broader and more unified probabilistic framework.

After the preliminaries in Section 2, Section 3 derives an explicit formula for the VF of convolution powers under $\boxplus_{\mathbb{T}}$. Section 4 uses this formula to obtain approximation results for CSK families generated by the $\mathbb{T}$-deformed free Gaussian and free Poisson laws, while Section 5 establishes limit theorems that reveal connections with free additive, Boolean additive, and free multiplicative convolutions.

2. Preliminaries

Let $\mathbf{P}_{ba}$ denote the class of non-degenerate probability measures whose support is bounded above. For $\nu \in \mathbf{P}_{ba}$, we introduce the analytic kernel

$$\mathcal{M}_\nu(\theta) = \int (1-\theta l)^{-1}\nu(dl),$$

which is well defined for $\theta \in [0, \theta_+^\nu)$, where $1/\theta_+^\nu = \max\{0, \sup \operatorname{supp}(\nu)\}$. For $\theta \in (0, \theta_+^\nu)$, we define a probability measure

$$P_\theta^\nu(dl) := \frac{\nu(dl)}{\mathcal{M}_\nu(\theta)(1-\theta l)}.$$

The collection

$$\mathcal{K}_+(\nu) = \{\mathbf{P}_\theta^\nu(dl);\ \ \theta \in (0, \theta_+^\nu)\}$$

is called the one-sided CSK family generated by ν. A basic property of $\mathcal{K}_+(\nu)$ concerns the mean function $k_\nu(\theta) = \int l P_\theta^\nu(dl)$, which is strictly increasing on $(0, \theta_+^\nu)$; see [23]. The interval $(m_1^\nu, m_+^\nu) = k_\nu((0, \theta_+^\nu))$, is referred to as the (right) mean domain of the family. Denoting by ψ_ν the inverse of k_ν, the family can equivalently be parameterized by the mean

$$\mathcal{K}_+(\nu) = \{\mathbf{Q}_m^\nu(dl) := \mathbf{P}_{\psi_\nu(m)}^\nu(dl):\ \ m \in (m_1^\nu, m_+^\nu)\}.$$

Writing $B_\nu = 1/\theta_+^\nu$, it follows from [23] that

$$m_1^\nu = \lim_{\theta\to 0^+} k_\nu(\theta), \qquad m_+^\nu = B_\nu - \lim_{\varpi \to B_\nu^+} 1/\mathbf{G}_\nu(\varpi).$$

When $\operatorname{supp}(\nu)$ is instead bounded from below, one may define an analogous one-sided family denoted by $\mathcal{K}_-(\nu)$. In this case, $\theta \in (\theta_-^\nu, 0)$, where θ_-^ν equals either $1/A_\nu$ or $-\infty$ with $A_\nu = \min\{0, \inf supp(\nu)\}$. The corresponding mean domain is (m_-^ν, m_1^ν), where $m_-^\nu = A_\nu - 1/\mathbf{G}_\nu(A_\nu)$. If $\nu \in \mathbf{P}_c$, then both one-sided constructions coexist and the full CSK family is given by $\mathcal{K}(\nu) = \mathcal{K}_-(\nu) \cup \{\nu\} \cup \mathcal{K}_+(\nu)$.

A fundamental analytic invariant of a CSK family is its VF defined by [21]

$$\mathcal{V}_\nu(m) = \int (l-m)^2 \mathbf{Q}_m^\nu(dl).$$

This function plays a role analogous to that of VFs in classical NEFs: it captures the dependence of second-order fluctuations on the mean and provides a concise description of the family. In particular, it is central for structural classification and for studying stability under analytic transformations. When ν does not possess a finite mean, the variance may fail to exist for members of $\mathcal{K}_+(\nu)$. In this situation, the pseudo-variance function (PVF) is used instead. It is defined by [23]

$$\mathbb{V}_\nu(m) = m\left(\frac{1}{\psi_\nu(m)} - m\right).$$

If the first moment m_1^ν exists, we have

$$\mathbb{V}_\nu(m) = \frac{m\mathcal{V}_\nu(m)}{m - m_1^\nu}, \tag{2}$$

which shows that both quantities encode equivalent information in the finite-mean case.

We summarize below several properties that will be used in the sequel.

Remark 1. *Let $\nu \in \mathbb{P}_{ba}$.*

(i) The PVF uniquely determines the generating measure; see [23]. Setting: $\Lambda(m) = m + \frac{\mathbb{V}_\nu(m)}{m}$, the CST $\mathcal{G}_\nu$ satisfies

$$\mathcal{G}_\nu(\Lambda(m)) = \frac{m}{\mathbb{V}_\nu(m)}.$$

If m_1^ρ is finite, this identity can be written in terms of the VF as

$$\mathcal{G}_\nu(\Lambda(m)) = \frac{m - m_1^\nu}{\mathcal{V}_\nu(m)}.$$

Hence, in the finite-mean setting, the pair $(m_1^\nu, \mathcal{V}_\nu)$ characterizes ν uniquely.

(ii) Consider an affine transformation $\Upsilon : l \mapsto \sigma l + \beta$ with $\sigma \neq 0$. Then, for m near the transformed mean $m_1^{\Upsilon(\nu)} = \Upsilon(m_1^\nu) = \sigma m_1^\nu + \beta$, the PVF satisfies [23]

$$\mathbb{V}_{\Upsilon(\nu)}(m) = \frac{\sigma^2 m}{m - \beta}\mathbb{V}_\nu\left(\frac{m-\beta}{\sigma}\right).$$

If the VF exists, it transforms according to

$$\mathcal{V}_{\Upsilon(\nu)}(m) = \sigma^2\mathcal{V}_\nu\left(\frac{m-\beta}{\sigma}\right). \tag{3}$$

(iii) Consider the $\mathbb{T}$-deformation introduced in (1) for $s > 0$ and $t \in \mathbb{R}$. Then, for m near

$$m_1^{U^{\mathbb{T}}(\nu)} = sm_1^\nu + (s-t)C_\nu, \tag{4}$$

we have [24]

$$\mathbb{V}_{U^{\mathbb{T}}(\nu)}(m) = \frac{sm}{(t-s)C_\nu + m}\mathbb{V}_\nu\left(\frac{(t-s)C_\nu + m}{s}\right) + m\left(\frac{(t-s)C_\nu + m}{s} - m\right), \tag{5}$$

and

$$\mathcal{V}_{U^{\mathbb{T}}(\nu)}(m) = s\mathcal{V}_\nu\left(\frac{(t-s)C_\nu + m}{s}\right) + (m - sm_1^\nu + (t-s)C_\nu)\left(\frac{(t-s)C_\nu + m}{s} - m\right). \tag{6}$$

In addition, $\mathrm{Var}(U^{\mathbb{T}}(\nu)) = s\mathrm{Var}(\nu)$.

3. $\boxplus_{\mathbb{T}}$-Convolution and VF

A principal achievement of the present paper is the derivation of an explicit formula describing the evolution of the VF of a CSK family under $\boxplus_{\mathbb{T}}$-convolution powers.

This identity provide the analytic groundwork for the results established in the subsequent sections. We first recall several standard notions from free probability that are required for introducing the $\mathbb{T}$-deformed free additive convolution and its associated cumulant transform.

A measure $\nu \in \mathbf{P}$ is said to be $\boxplus$-infinitely divisible if, for every $n \in \mathbb{N}$, there exists $\nu_n \in \mathbf{P}$ such that

$$\nu = \nu_n \boxplus ..... \boxplus \nu_n \qquad (n \text{ times}).$$

The r-fold free additive convolution power is denoted by $\nu^{\boxplus r}$. This operation extends to all real $r \geq 1$ and satisfies the linear scaling property

$$\mathcal{R}_{\nu^{\boxplus r}}(\xi) = r\mathcal{R}_{\nu}(\xi),$$

see [25]. Infinite divisibility with respect to $\boxplus$ is equivalent to the existence of $\nu^{\boxplus r}$ for every real $r > 0$.

We now turn to the $\mathbb{T}$-deformed setting. For $\nu \in \mathbf{P}_c$, the $\mathcal{R}^{\mathbb{T}}$-transform is defined by

$$\mathcal{R}_{\nu}^{\mathbb{T}}(\xi) := \mathcal{R}_{U^{\mathbb{T}}(\nu)}(\xi),$$

following [8]. The $\mathbb{T}$-deformed free additive convolution $\boxplus_{\mathbb{T}}$ is characterized by the additivity of this transform:

$$\mathcal{R}^{\mathbb{T}}_{\nu_1 \boxplus_{\mathbb{T}} \nu_2}(\xi) = \mathcal{R}^{\mathbb{T}}_{\nu_1}(\xi) + \mathcal{R}^{\mathbb{T}}_{\nu_2}(\xi), \qquad \nu_1,\ \nu_2 \in \mathbf{P}_c.$$

A probability measure $\nu \in \mathbf{P}_c$ is called $\boxplus_{\mathbb{T}}$-infinitely divisible if, for each $n \in \mathbb{N}$, it can be written as an n-fold $\boxplus_{\mathbb{T}}$-convolution of a suitable measure ν_n. For $r \geq 1$, we denote by $\nu^{\boxplus_{\mathbb{T}} r}$ the r-fold $\mathbb{T}$-deformed convolution power. As in the free additive case, this operation extends to arbitrary real $r \geq 1$ (see [26]) and satisfies the property

$$\mathcal{R}^{\mathbb{T}}_{\nu^{\boxplus_{\mathbb{T}} r}}(\xi) := r\mathcal{R}^{\mathbb{T}}_{\nu}(\xi).$$

Next, we collect additional properties of the $\mathcal{R}^{\mathbb{T}}$-transform that will be instrumental in the analysis of $\boxplus_{\mathbb{T}}$-convolution powers and their impact on VFs.

Proposition 1. *Let $\nu \in \mathbf{P}_c$ be a non degenerate probability measure. Then,*

(i) $\mathcal{R}^{\mathbb{T}}_{\nu}(\cdot)$ *is strictly increasing on* $(\mathbf{G}_{U^{\mathbb{T}}(\nu)}(A_{U^{\mathbb{T}}(\nu)}), \mathbf{G}_{U^{\mathbb{T}}(\nu)}(B_{U^{\mathbb{T}}(\nu)}))$.

(ii) *For* $m \in \left(m_-^{U^{\mathbb{T}}(\nu)}, m_+^{U^{\mathbb{T}}(\nu)}\right)$

$$\mathcal{R}^{\mathbb{T}}_{\nu}\left(\frac{m}{\mathbb{V}_{U^{\mathbb{T}}(\nu)}(m)}\right) = m. \tag{7}$$

(iii) $\lim\limits_{\xi \searrow 0} \mathcal{R}^{\mathbb{T}}_{\nu}(\xi) = m_1^{U^{\mathbb{T}}(\nu)}$.

(iv) $\lim\limits_{\xi \searrow 0} \xi\mathcal{R}^{\mathbb{T}}_{\nu}(\xi) = 0$.

Proof. The argument follows directly from the known properties of the transform $\mathcal{R}_{U^{\mathbb{T}}(\nu)}(\cdot)$. Indeed, by applying ([23] [Proposition 3.8]) to the measure $U^{\mathbb{T}}(\nu)$, the desired conclusion is obtained without additional computations. □

We now present the principal theorem of this section and provide its proof.

Theorem 1. *Let $\nu \in \mathbf{P}_c$ be non-degenerate. For any $\alpha > 0$ such that the $\boxplus_{\mathbb{T}}$-convolution power $\nu^{\boxplus_{\mathbb{T}}\alpha}$ exists, the following assertions hold:*

(i) The measure $\nu^{\boxplus_{\mathbb{T}}\alpha} \in \mathbf{P}_c$.

(ii) For m sufficiently close to $m_1^{\nu^{\boxplus_{\mathbb{T}}\alpha}} = \alpha m_1^{\nu}$,

$$\mathbb{V}_{\nu^{\boxplus_{\mathbb{T}}\alpha}}(m) = \alpha \mathbb{V}_{\nu}\left(\frac{m}{\alpha}\right) + m\left[m(1-s)\left(\frac{1}{\alpha}-1\right) + (\alpha-1)(s-t)C_{\nu}\right], \tag{8}$$

and

$$\mathcal{V}_{\nu^{\boxplus_{\mathbb{T}}\alpha}}(m) = \alpha \mathcal{V}_{\nu}\left(\frac{m}{\alpha}\right) + (m-\alpha m_1^{\nu})\left[m(1-s)\left(\frac{1}{\alpha}-1\right) + (\alpha-1)(s-t)C_{\nu}\right]. \tag{9}$$

In addition, $\mathrm{Var}\left(\nu^{\boxplus_{\mathbb{T}}\alpha}\right) = \alpha \mathrm{Var}(\nu)$.

Proof. (i) Because ν has compact support, the same property is inherited by its $\mathbb{T}$-transform $U^{\mathbb{T}}(\nu)$; this follows from the behavior of the CST together with relation (1). For a compactly supported measure, the associated $\mathcal{R}$-transform is analytic and injective in a neighborhood of the origin. Consequently, $\mathcal{R}_{\nu}^{\mathbb{T}} = \mathcal{R}_{U^{\mathbb{T}}(\nu)}$ is univalent on some interval $(-\kappa, \kappa)$. By the homogeneity property of the $\mathcal{R}^{\mathbb{T}}$-transform, we have

$$\alpha \mathcal{R}_{\nu}^{\mathbb{T}} = \mathcal{R}^{\mathbb{T}}{}_{\nu^{\boxplus_{\mathbb{T}}\alpha}} = \mathcal{R}_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}$$

which remains injective on the same domain. This implies that the CST $\mathbf{G}_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}$ admits an analytic continuation to the exterior of a disk $\{z : |z| > \varrho\}$, where $\varrho = \mathbf{G}^{-1}_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}(\kappa)$. Hence, $\mathbf{G}_{\nu^{\boxplus_{\mathbb{T}}\alpha}}$ is analytic outside a compact set, which shows that $\nu^{\boxplus_{\mathbb{T}}\alpha}$ has compact support (see [27] [Proposition 6.1]).

(ii) From Proposition 1(iii), we have

$$m_1^{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})} = \lim_{\xi \longrightarrow 0} \mathcal{R}^{\mathbb{T}}_{\nu^{\boxplus_{\mathbb{T}}\alpha}}(\xi) = \alpha \lim_{\xi \longrightarrow 0} \mathcal{R}_{\nu}^{\mathbb{T}}(\xi) = \alpha m_1^{U^{\mathbb{T}}(\nu)} = \alpha s m_1^{\nu} + \alpha(s-t)C_{\nu}. \tag{10}$$

On the other hand, from (4) one can see that

$$m_1^{\nu^{\boxplus_{\mathbb{T}}\alpha}} = m_1^{(U^{\mathbb{T}})^{-1}(U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha}))} = \frac{1}{s} m_1^{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})} + (1/s - 1/t)C_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}. \tag{11}$$

From ([8] [Remark 10]), we have

$$C_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})} = \Re\left(\frac{1}{\mathbf{G}_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}(i)}\right) = t\Re\left(\frac{1}{\mathbf{G}_{\nu^{\boxplus_{\mathbb{T}}\alpha}}(i)}\right) = tC_{\nu^{\boxplus_{\mathbb{T}}\alpha}}. \tag{12}$$

Using relations (10) and (12), Equation (11) becomes

$$m_1^{\nu^{\boxplus_{\mathbb{T}}\alpha}} = \alpha m_1^{\nu} + (1-t/s)\alpha C_{\nu} + (t/s-1)C_{\nu^{\boxplus_{\mathbb{T}}\alpha}}, \tag{13}$$

On the other hand, it is well known that the mean of the sum of $\boxplus_{\mathbb{T}}$-independent random variables is equal to the sum of the means of these random variables; that is,

$$m_1^{\nu^{\boxplus_{\mathbb{T}}\alpha}} = \alpha m_1^{\nu}. \tag{14}$$

Combining (14) and (13), we obtain

$$C_{\nu^{\boxplus_{\mathbb{T}}\alpha}} = \alpha C_{\nu}. \tag{15}$$

For y close enough to $m_1^{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}=\alpha m_1^{U^{\mathbb{T}}(\nu)}$ such that $y/\alpha\in\left(m_-^{U^{\mathbb{T}}(\nu)},m_+^{U^{\mathbb{T}}(\nu)}\right)$ and

$$\frac{y}{\mathbb{V}_{U^{\mathbb{T}}(\nu)}(y)}\in(\mathbf{G}_{U^{\mathbb{T}}(\nu)}(A_{U^{\mathbb{T}}(\nu)},\mathbf{G}_{U^{\mathbb{T}}(\nu)}(B_{U^{\mathbb{T}}(\nu)})),$$

one has

$$\mathcal{R}_{\nu}^{\mathbb{T}}\left(\frac{y}{\mathbb{V}_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}(y)}\right)=\frac{1}{\alpha}\mathcal{R}_{\nu^{\boxplus_{\mathbb{T}}\alpha}}^{\mathbb{T}}\left(\frac{y}{\mathbb{V}_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}(y)}\right)=\frac{y}{\alpha}=\mathcal{R}_{\nu}^{\mathbb{T}}\left(\frac{y/\alpha}{\mathbb{V}_{U^{\mathbb{T}}(\nu)}(y/\alpha)}\right).$$

By Proposition 1(i), the transform $\mathcal{R}_{\nu}^{\mathbb{T}}$ is injective on $(\mathbf{G}_{U^{\mathbb{T}}(\nu)}(A_{U^{\mathbb{T}}(\nu)},\mathbf{G}_{U^{\mathbb{T}}(\nu)}(B_{U^{\mathbb{T}}(\nu)}))$, so

$$\frac{y}{\mathbb{V}_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}(y)}=\frac{y/\alpha}{\mathbb{V}_{U^{\mathbb{T}}(\nu)}(y/\alpha)},$$

or equivalently

$$\mathbb{V}_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}(y)=\alpha\mathbb{V}_{U^{\mathbb{T}}(\nu)}(y/\alpha). \tag{16}$$

Based on (5), for x close enough to $m_1^{\nu^{\boxplus_{\mathbb{T}}\alpha}}=\alpha m_1^{\nu}$, one has

$$\begin{aligned}\mathbb{V}_{\nu^{\boxplus_{\mathbb{T}}\alpha}}(x)&=\mathbb{V}_{(U^{\mathbb{T}})^{-1}(U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha}))}(x)\\&=\frac{\frac{x}{s}}{x+(\frac{1}{t}-\frac{1}{s})C_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}}\mathbb{V}_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}\left(\frac{x+(\frac{1}{t}-\frac{1}{s})C_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}}{\frac{1}{s}}\right)+x\left(\frac{x+(\frac{1}{t}-\frac{1}{s})C_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}}{\frac{1}{s}}-x\right).\end{aligned} \tag{17}$$

We know from (12) and (15) that $C_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}=tC_{\nu^{\boxplus_{\mathbb{T}}\alpha}}=\alpha tC_{\nu}$. This with relation (17) implies that

$$\mathbb{V}_{\nu^{\boxplus_{\mathbb{T}}\alpha}}(x)=\frac{x}{sx+\alpha(s-t)C_{\nu}}\mathbb{V}_{U^{\mathbb{T}}(\nu^{\boxplus_{\mathbb{T}}\alpha})}(sx+\alpha(s-t)C_{\nu})+x(sx+\alpha(s-t)C_{\nu}-x). \tag{18}$$

Now, relations (16), (18) and (5) imply that

$$\begin{aligned}\mathbb{V}_{\nu^{\boxplus_{\mathbb{T}}\alpha}}(x)&=\frac{\alpha x}{sx+\alpha(s-t)C_{\nu}}\mathbb{V}_{U^{\mathbb{T}}(\nu)}\left(\frac{sx+\alpha(s-t)C_{\nu}}{\alpha}\right)+x(sx+\alpha(s-t)C_{\nu}-x)\\&=\frac{\alpha x}{sx+\alpha(s-t)C_{\nu}}\left\{\frac{s\frac{sx+\alpha(s-t)C_{\nu}}{\alpha}}{\frac{sx+\alpha(s-t)C_{\nu}}{\alpha}+(t-s)C_{\nu}}\mathbb{V}_{\nu}\left(\frac{\frac{sx+\alpha(s-t)C_{\nu}}{\alpha}+(t-s)C_{\nu}}{s}\right)\right.\\&\left.+\frac{sx+\alpha(s-t)C_{\nu}}{\alpha}\left(\frac{\frac{sx+\alpha(s-t)C_{\nu}}{\alpha}+(t-s)C_{\nu}}{s}-\frac{sx+\alpha(s-t)C_{\nu}}{\alpha}\right)\right\}+x(sx+\alpha(s-t)C_{\nu}-x)\\&=\alpha\mathbb{V}_{\nu}\left(\frac{x}{\alpha}\right)+x\left[x(1-s)\left(\frac{1}{\alpha}-1\right)+(\alpha-1)(s-t)C_{\nu}\right].\end{aligned}$$

This completes the proof of (8). However, based on (2) and (8), one can see that

$$\frac{x}{x-\alpha m_1^{\nu}}\mathcal{V}_{\nu^{\boxplus_{\mathbb{T}}\alpha}}(x)=\alpha\frac{x/\alpha}{x/\alpha-m_1^{\nu}}\mathcal{V}_{\nu}\left(\frac{x}{\alpha}\right)+x\left[x(1-s)\left(\frac{1}{\alpha}-1\right)+(\alpha-1)(s-t)C_{\nu}\right].$$

That is,

$$\begin{aligned}\mathcal{V}_{\nu^{\boxplus_{\mathbb{T}}\alpha}}(x)&=\frac{x-\alpha m_1^{\nu}}{x}\times\alpha\frac{x/\alpha}{x/\alpha-m_1^{\nu}}\mathcal{V}_{\nu}\left(\frac{x}{\alpha}\right)+\frac{x-\alpha m_1^{\nu}}{x}\times x\left[x(1-s)\left(\frac{1}{\alpha}-1\right)+(\alpha-1)(s-t)C_{\nu}\right]\\&=\alpha\mathcal{V}_{\nu}\left(\frac{x}{\alpha}\right)+(x-\alpha m_1^{\nu})\left[x(1-s)\left(\frac{1}{\alpha}-1\right)+(\alpha-1)(s-t)C_{\nu}\right].\end{aligned}$$

This completes the proof of (9). Moreover, based on ([21] [Theorem 3.3, Equation (3.12)]) and relation (9), we have that

$$\mathrm{Var}\left(\nu^{\boxplus_{\mathbb{T}}\alpha}\right) = \mathcal{V}_{\nu^{\boxplus_{\mathbb{T}}\alpha}}(x)\big|_{x=m_1^{\nu^{\boxplus_{\mathbb{T}}\alpha}}=\alpha m_1^{\nu}} = \alpha\mathcal{V}_{\nu}(m_1^{\nu}) = \alpha\mathrm{Var}(\nu).$$

□

Theorem 1 highlights two important aspects: on the one hand, it confirms that $\boxplus_{\mathbb{T}}$ retains key structural features of free convolution (such as variance additivity), and on the other hand, it demonstrates that the deformation introduces rich new behaviors at the level of VFs. This duality is essential for understanding $\boxplus_{\mathbb{T}}$ as a genuine extension of free convolution, combining stability properties with enhanced flexibility.

4. Approximation Schemes Driven by the $\boxplus_{\mathbb{T}}$-Convolution

We exploit the $\boxplus_{\mathbb{T}}$-convolution framework to derive approximation results for members of CSK families generated by the $\mathbb{T}$-deformed free Gaussian and the $\mathbb{T}$-deformed free Poisson laws. The identification of the corresponding limit measures relies on the analysis of their VFs. More precisely, the convergence is obtained by applying a technical criterion based on the convergence of a sequence of VFs (see, [21] [Proposition 4.2]).

4.1. Estimation of Members of the CSK Family Induced by $\mathbb{T}$-Deformed Free Gaussian Law

As shown in ([8] [Theorem 2]), the central limit distribution associated with the $\boxplus_{\mathbb{T}}$-convolution is referred to as the $\mathbb{T}$-deformed free Gaussian law. This distribution coincides with the Kesten distribution with parameter $s > 0$, and it plays the role of the Gaussian element in the $\mathbb{T}$-deformed setting. It is given by $\kappa_s = \widetilde{\kappa}_s + \widehat{\kappa}_s$ with

$$\widetilde{\kappa}_s(dl) = \frac{1}{2\pi}\frac{\sqrt{4s-l^2}}{1-(1-s)l^2}1_{[-2\sqrt{s},2\sqrt{s}]}(l)dl$$

and, for $s < 1/2$,

$$\widehat{\kappa}_s = \frac{1-2s}{2-2s}\left(\delta_{-1/\sqrt{1-s}} + \delta_{1/\sqrt{1-s}}\right).$$

It was proved in ([28] [Proposition 3]) that the VF of $\mathcal{K}(\kappa_s)$ is given by

$$\mathcal{V}_{\kappa_s}(m) = 1 + (s-1)m^2, \qquad m \text{ close to } m_1^{\kappa_s} = 0.$$

The next theorem establishes an approximation scheme for members of the CSK family $\mathcal{K}(\kappa_s)$. In this context, $D_c(\nu)$ denotes the dilation of the measure ν by a nonzero constant c.

Theorem 2. *Let $\nu \in \mathbf{P}_c$ be a non-degenerate symmetric probability measure satisfying $m_1^{\nu} = 0$ and $\mathrm{Var}(\nu) = 1$. Assume that $\mathcal{V}_{\nu}$ is strictly positive and analytic near the origin. Then, there exists $\delta > 0$ such that the following holds: For $\alpha > 1$, define*

$$\nu_{\alpha} = D_{1/\alpha}\left(\nu^{\boxplus_{\mathbb{T}}\alpha}\right)$$

and let Z_{α} be a random variable whose distribution belongs to $\mathcal{K}(\nu_{\alpha})$ with mean equal to $m/\sqrt{\alpha}$, where $|m| < \delta$. Then,

$$\sqrt{\alpha}Z_{\alpha} \xrightarrow{\alpha\to+\infty} \mathbf{Q}_m^{\kappa_s} \in \mathcal{K}(\kappa_s) \qquad \text{in distribution.}$$

Proof. Let $\mathcal{L}(Z_\alpha)$ denote the law of Z_α. Since ν is symmetric, we know from ([8] [Lemma 2]) that

$$C_\nu = \Re\left(\frac{1}{\mathcal{G}_\nu(i)}\right) = 0. \tag{19}$$

Recalling (19), one has $\mathcal{L}(Z_\alpha) \in \mathcal{K}(\nu_\alpha)$, with VF given by

$$\mathcal{V}_{\nu_\alpha}(m) = \frac{1}{\alpha^2}\mathcal{V}_{\nu^{\boxplus_{\mathbb{T}}\alpha}}(\alpha m) = \frac{1}{\alpha}\mathcal{V}_\nu(m) + m^2(1-s)\left(\frac{1}{\alpha} - 1\right).$$

After rescaling, the law of $\sqrt{\alpha}Z_\alpha$ belongs to a CSK family whose VF can be written as

$$\mathcal{V}_\alpha(m) = \alpha\mathcal{V}_{\nu_\alpha}\left(\frac{m}{\sqrt{\alpha}}\right) = \mathcal{V}_\nu\left(\frac{m}{\sqrt{\alpha}}\right) + \left(\frac{1}{\alpha} - 1\right)(1-s)m^2.$$

We now apply ([21] [Proposition 4.2]) to the sequence $\mathcal{V}_\alpha$. As $\alpha \to +\infty$, we obtain

$$\mathcal{V}_\alpha(m) \longrightarrow \mathcal{V}_\nu(0) + (s-1)m^2 = 1 + (s-1)m^2 = \mathcal{V}_{\kappa_s}(m).$$

From ([21] [Proposition 4.2]), we deduce that $\delta > 0$ exists such that if $|m| < \delta$ and $m_1^{Z_\alpha} = m/\sqrt{\alpha}$, then

$$\mathcal{L}(\sqrt{\alpha}Z_\alpha) \xrightarrow{\alpha\to+\infty} \mathbf{Q}_m^{\kappa_s} \in \mathcal{K}(\kappa_s) \text{ in distribution.}$$

In the particular case $m = 0$, this reduces to the central limit theorem associated with the $\boxplus_{\mathbb{T}}$-convolution. □

4.2. Estimation of Members of the CSK Family Induced by $\mathbb{T}$-Deformed Free Poisson Law

The $\mathbb{T}$-deformed free Poisson distribution, denoted by $\mathbf{p}_\lambda$, arises as a limit law in the $\boxplus_{\mathbb{T}}$-convolution framework. It can be constructed from a sequence of Bernoulli-type measures of the form

$$\mu_K = \left(1 - \frac{\lambda}{K}\right)\delta_0 + \frac{\lambda}{K}\delta_1, \quad \text{for} \quad K \in \{1,2,3,4,\ldots\} \quad \text{and} \quad 0 < \lambda < K.$$

By considering the K-fold $\boxplus_{\mathbb{T}}$-convolution of μ_K with itself, one obtains a sequence of probability measures whose limit, as $K \to +\infty$, defines the $\mathbb{T}$-deformed free Poisson law. More precisely,

$$\underbrace{\mu_K \boxplus_{\mathbb{T}} \mu_K \ldots\ldots \boxplus_{\mathbb{T}} \mu_K}_{K \text{ times}} \xrightarrow{K\to+\infty} \mathbf{p}_\lambda \quad \text{in distribution.}$$

This limiting procedure plays a role analogous to the classical Poisson approximation, but within the $\mathbb{T}$-deformed free probability setting. A closed-form description of $\mathbf{p}_\lambda$, including its analytic and distributional properties, can be found in ([8] [Section 6]).

Proposition 2. *For m close enough to $m_1^{\mathbf{p}_\lambda} = \lambda$, one has*

$$\mathcal{V}_{\mathbf{p}_\lambda}(m) = m - (m-\lambda)(m(1-s) + (s-t)\lambda/2). \tag{20}$$

Proof. From [8] (p. 181), we know that

$$\mathcal{R}^{\mathbb{T}}_{\mathbf{p}_\lambda}(z) = \frac{(s+t) + (s-t)z}{2}\frac{\lambda}{1-z}. \tag{21}$$

With

$$z = \frac{m}{\mathbb{V}_{U^{\mathbb{T}}(\mathbf{p}_\lambda)}(m)} \tag{22}$$

we obtain from relations (7) and (21)

$$m = \frac{(s+t)+(s-t)z}{2}\frac{\lambda}{1-z}. \tag{23}$$

Relation (23) gives

$$z = \frac{2m-(s+t)\lambda}{2m+(s-t)\lambda}. \tag{24}$$

Combining (22) and (24), we get

$$\mathbb{V}_{U^{\mathbb{T}}(\mathbf{p}_\lambda)}(m) = \frac{m(m+\frac{(s-t)\lambda}{2})}{m-\frac{(s+t)\lambda}{2}}. \tag{25}$$

On the other hand, we know from [8] (p. 181) that the moment of order 1 of $U^{\mathbb{T}}(\mathbf{p}_\lambda)$ (which is the first free cumulant of $U^{\mathbb{T}}(\mathbf{p}_\lambda)$) is given by

$$m_1^{U^{\mathbb{T}}(\mathbf{p}_\lambda)} = \frac{(s+t)\lambda}{2}. \tag{26}$$

Thus, relations (2), (25), and (26) give

$$\mathcal{V}_{U^{\mathbb{T}}(\mathbf{p}_\lambda)}(m) = m + \frac{(s-t)\lambda}{2}, \qquad m \ \text{ close to } \ m_1^{U^{\mathbb{T}}(\mathbf{p}_\lambda)} = \frac{(s+t)\lambda}{2} \tag{27}$$

In addition, by combining (4) and (26), we obtain

$$\frac{(s+t)\lambda}{2} = m_1^{U^{\mathbb{T}}(\mathbf{p}_\lambda)} = s m_1^{\mathbf{p}_\lambda} + (s-t)C_{\mathbf{p}_\lambda} = s\lambda + (s-t)C_{\mathbf{p}_\lambda}. \tag{28}$$

Relation (28) gives $C_{\mathbf{p}_\lambda} = -\lambda/2$, and then

$$C_{U^{\mathbb{T}}(\mathbf{p}_\lambda)} = \Re\left(\frac{1}{\mathbf{G}_{U^{\mathbb{T}}(\mathbf{p}_\lambda)}(i)}\right) = t\Re\left(\frac{1}{\mathbf{G}_{\mathbf{p}_\lambda}(i)}\right) = tC_{\mathbf{p}_\lambda} = -t\lambda/2. \tag{29}$$

For m close enough to $m_1^{\mathbf{p}_\lambda} = \lambda$, we have

$$\begin{aligned}
\mathcal{V}_{\mathbf{p}_\lambda}(m) &= \mathcal{V}_{(U^{\mathbb{T}})^{-1}(U^{\mathbb{T}}(\mathbf{p}_\lambda))}(m) \\
&\overset{(6)}{=} \frac{1}{s}\mathcal{V}_{U^{\mathbb{T}}(\mathbf{p}_\lambda)}\left(\frac{m+(\frac{1}{t}-\frac{1}{s})C_{U^{\mathbb{T}}(\mathbf{p}_\lambda)}}{\frac{1}{s}}\right) + \left(m - \frac{1}{s}m_1^{U^{\mathbb{T}}(\mathbf{p}_\lambda)} + \left(\frac{1}{t}-\frac{1}{s}\right)C_{U^{\mathbb{T}}(\mathbf{p}_\lambda)}\right) \times \left(\frac{m+(\frac{1}{t}-\frac{1}{s})C_{U^{\mathbb{T}}(\mathbf{p}_\lambda)}}{\frac{1}{s}} - m\right) \\
&\overset{(26) \text{ and } (29)}{=} \frac{1}{s}\mathcal{V}_{U^{\mathbb{T}}(\mathbf{p}_\lambda)}\left(sm + \frac{(t-s)\lambda}{2}\right) + \left(m - \frac{(s+t)\lambda}{2s} + \frac{(t-s)\lambda}{2s}\right)\left(sm + \frac{(t-s)\lambda}{2} - m\right) \\
&\overset{(27)}{=} m + (m-\lambda)(m(s-1)+(t-s)\lambda/2).
\end{aligned}$$

This ends the proof of (20). □

We now turn to constructing approximations for the members of the CSK family $\mathcal{K}(\mathbf{p}_\lambda)$.

Theorem 3. *For $K \in \mathbb{N}^*$ and $0 < \lambda < K$, consider $\mu_K = (1-\lambda/K)\delta_0 + (\lambda/K)\delta_1$. Then, for m in a small neighborhood of λ,*

$$\mathbf{Q}_m^{\mu_K^{\boxplus_{\mathbb{T}} K}} \xrightarrow{K\to+\infty} \mathbf{Q}_m^{\mathbf{p}_\lambda} \in \mathcal{K}(\mathbf{p}_\lambda), \quad \textit{in distribution}. \tag{30}$$

Proof. According to [29], if u is sufficiently close to $m_1^{\mu_k} = \lambda/K$, we have

$$\mathcal{V}_{\mu_K}(u) = u(1-u). \tag{31}$$

In addition, we have $\mathbf{G}_{\mu_K}(z) = \frac{z+\lambda/K-1}{z(z-1)}$. This, with some calculations, implies that

$$C_{\mu_K} = \Re\left(\frac{1}{\mathbf{G}_{\mu_K}(i)}\right) = \frac{-\lambda/K}{(\lambda/K-1)^2+1}. \tag{32}$$

We have that $m_1^{\mu_K^{\boxplus_{\mathbb{T}} K}} = \lambda = m_1^{\mathbf{p}_\lambda}$. Then, $\iota > 0$ exists such that

$$\left(m_-^{\mu_K^{\boxplus_{\mathbb{T}} K}}, m_+^{\mu_K^{\boxplus_{\mathbb{T}} K}}\right) \cap \left(m_-^{\mathbf{p}_\lambda}, m_+^{\mathbf{p}_\lambda}\right) = (\lambda-\iota, \lambda+\iota).$$

For $m \in (\lambda-\iota, \lambda+\iota)$, we obtain

$$\begin{aligned}
\mathcal{V}_{\mu_K^{\boxplus_{\mathbb{T}} K}}(m) &\overset{(9)}{=} K\mathcal{V}_{\mu_K}(m/K) + \left(m - Km_1^{\mu_K}\right)\left[m(1-s)(1/K-1) + (K-1)(s-t)C_{\mu_K}\right] \\
&\overset{(31) \text{ and } (32)}{=} m(1-m/K) + (m-\lambda)\left[m(1-s)(1/K-1) + (1-1/K)(s-t)\left(\frac{-\lambda}{(\lambda/K-1)^2+1}\right)\right]. \\
&\xrightarrow{K\to+\infty} m - (m-\lambda)(m(1-s) + (s-t)\lambda/2) = \mathcal{V}_{\mathbf{p}_\lambda}(m).
\end{aligned}$$

Together with the fact that

$$\int l\mathbf{Q}_m^{\mu_K^{\boxplus_{\mathbb{T}} K}}(dl) = m = \int l\mathbf{Q}_m^{\mathbf{p}_\lambda}(dl), \qquad m \in (\lambda-\iota, \lambda+\iota),$$

the proof of (30) is completed by applying ([21] [Proposition 4.2]). In particular, when $m = \lambda$, the Poisson-type limit theorem is yielded for the $\boxplus_{\mathbb{T}}$-convolution: $\mu_K^{\boxplus_{\mathbb{T}} K} \longrightarrow \mathbf{p}_\lambda$, in distribution as $K \to +\infty$. □

We highlight a natural extension of Theorem 3 toward numerical validation. The theoretical framework developed in this work could be complemented in future studies by computational implementations based on established mathematical software and standard simulation techniques used in probabilistic modeling. Such numerical approaches would enable the approximation of the deformed Poisson laws in an explicit and experimentally accessible form, and would allow for a systematic exploration of their behavior across different parameter regimes. This would further make it possible to compare theoretical predictions with numerical results, thereby providing an additional layer of validation of the analytical findings. In addition, such comparisons could offer further insight into the stability and robustness of the proposed models and help us to better understand the behavior of the deformed structures in practical computational settings. From a statistical viewpoint, we note that classical inference tools, such as confidence intervals derived from standard probabilistic models, are not directly applicable in the present noncommutative framework. Nevertheless, it is natural to investigate whether analogous notions can be developed within free probability to quantify uncertainty in the estimation of distributional parameters. In particular, one may consider adapting such ideas to the setting of VFs and deformed convolution structures, which could potentially lead to the notion of "free" confidence regions. Although a systematic development of these concepts lies beyond the scope of the present paper, they constitute a promising direction for future research aimed at enriching the statistical interpretation of free probabilistic models.

5. Novel Limit Theorems Related to the $\boxplus_{\mathbb{T}}$-Convolution

5.1. A Novel Asymptotic Result Related to Free Multiplicative Convolution

In this paragraph, we establish a novel limit theorem for the $\boxplus_{\mathbb{T}}$-convolution that involves the free multiplicative convolution and leads to VFs with nonstandard forms. To set the stage, we first recall some relevant properties of free multiplicative convolution acting on CSK families, following [30].

Let $\mathbf{P}^+$ denote the set of non-degenerate probability measures supported on $\mathbb{R}_+$. For $\nu \in \mathbf{P}^+$, the S-transform S_ν is defined implicitly by

$$\mathcal{R}_\nu(\xi S_\nu(\xi)) = \frac{1}{S_\nu(\xi)}, \qquad \xi \text{ near } 0.$$

For $\nu_1, \nu_2 \in \mathbf{P}^+$, the measure $\nu_1 \boxtimes \nu_2$ satisfies

$$S_{\nu_1 \boxtimes \nu_2}(z) = S_{\nu_1}(z) S_{\nu_2}(z).$$

A measure $\nu \in \mathbf{P}^+$ is said to be $\boxtimes$-infinitely divisible if, for every $n \in \mathbb{N}$, there exists $\nu_n \in \mathbf{P}^+$ such that

$$\nu = \underbrace{\nu_n \boxtimes ..... \boxtimes \nu_n}_{n \text{ times}}.$$

For $\alpha \geq 1$, the multiplicative convolution power $\nu^{\boxtimes\alpha}$ is defined via the S-transform as

$$S_{\nu^{\boxtimes\alpha}}(z) = S_\nu(z)^\alpha,$$

see ([31] [Theorem 2.17]). For a measure $\nu \in \mathbf{P}^+$ with finite mean, whenever $\nu^{\boxtimes r}$ exists for some $r > 0$, one has

$$\mathcal{V}_{\nu^{\boxtimes r}}(m) = \frac{m - (m_1^\nu)^r}{m^{1/r} - m_1^\nu} m^{1-1/r} \mathcal{V}_\nu\left(m^{1/r}\right), \quad m \in ((m_-^\nu)^r,\ (m_1^\nu)^r) \tag{33}$$

as established in ([30] [Theorem 2.4]). To proceed, we denote by $\mathbf{P}_c^+$ the set of compactly supported probability measures on $\mathbb{R}_+$. Using these preliminaries, we can now state and prove the main result of this section.

Theorem 4. *Let $\nu \in \mathbf{P}_c^+$ be non-degenerate, and define*

$$\gamma = \frac{\mathrm{Var}(\nu)}{(m_1^\nu)^2} = \frac{\mathcal{V}_\nu(m_1^\nu)}{(m_1^\nu)^2}.$$

Then, as $n \to +\infty$, the following convergence in distribution holds:

$$D_{1/(nm_1^\nu)^n}\left(\nu^{\boxplus_{\mathbb{T}} n}\right)^{\boxtimes n} \longrightarrow \zeta_\gamma$$

where ζ_γ is characterized by the pair $(m_1^{\zeta_\gamma}, \mathcal{V}_{\zeta_\gamma})$ with $m_1^{\zeta_\gamma} = 1$ and

$$\mathcal{V}_{\zeta_\gamma}(m) = \frac{\gamma m(m-1)}{\ln(m)} + \left((s-1) + \frac{(s-t)C_\nu}{m_1^\nu}\right) m(m-1), \qquad m \in (m_-^{\zeta_\gamma}, 1).$$

Proof. We have that

$$m_1^{D_{1/(nm_1^\nu)^n}\left(\nu^{\boxplus_{\mathbb{T}} n}\right)^{\boxtimes n}} = \frac{m_1^{\left(\nu^{\boxplus_{\mathbb{T}} n}\right)^{\boxtimes n}}}{(nm_1^\nu)^n} = \frac{(nm_1^\nu)^n}{(nm_1^\nu)^n} = 1.$$

Furthermore, for $m < 1$ (close to 1), we have

$$\begin{aligned}
\mathcal{V}_{D_{1/(nm_1^\nu)^n}(\nu^{\boxplus_{\mathbb{T}} n})^{\boxtimes n}}(m) &\overset{(3)}{=} \frac{1}{(nm_1^\nu)^{2n}}\mathcal{V}_{(\nu^{\boxplus_{\mathbb{T}} n})^{\boxtimes n}}(m(nm_1^\nu)^n) \\
&\overset{(33)}{=} \frac{1}{(nm_1^\nu)^{2n}}\frac{(nm_1^\nu)^n-(m_1^{\nu^{\boxplus_{\mathbb{T}} n}})^n}{(m(nm_1^\nu)^n)^{1/n}-m_1^{\nu^{\boxplus_{\mathbb{T}} n}}}(m(nm_1^\nu)^n)^{1-1/n}\mathcal{V}_{\nu^{\boxplus_{\mathbb{T}} n}}\left((m(nm_1^\nu)^n)^{1/n}\right) \\
&\overset{(9)}{=} \frac{m-1}{m^{1/n}-1}\frac{m^{1-1/n}}{(nm_1^\nu)^2}\Big\{n\mathcal{V}_\nu\left(m^{1/n}m_1^\nu\right) \\
&+ (nm_1^\nu m^{1/n}-nm_1^\nu)\left[nm_1^\nu m^{1/n}(1-s)(1/n-1)+(n-1)(s-t)C_\nu\right]\Big\} \\
&= \frac{m-1}{\frac{m^{1/n}-1}{1/n}}\frac{m^{1-1/n}}{(m_1^\nu)^2}\mathcal{V}_\nu\left(m^{1/n}m_1^\nu\right)+(m-1)m^{1-1/n}\left[m^{1/n}(1-s)(1/n-1)+\frac{(1-1/n)(s-t)C_\nu}{m_1^\nu}\right] \\
&\xrightarrow{n\to+\infty} \frac{m(m-1)}{\ln(m)}\frac{\mathcal{V}_\nu(m_1^\nu)}{(m_1^\nu)^2}+\left((s-1)+\frac{(s-t)C_\nu}{m_1^\nu}\right)m(m-1).
\end{aligned}$$

By applying ([21] [Proposition 4.2]), it follows that

$$D_{1/(nm_1^\nu)^n}\left(\nu^{\boxplus_{\mathbb{T}} n}\right)^{\boxtimes n} \xrightarrow{n\to+\infty} \zeta_\gamma \qquad \text{in distribution,}$$

where

$$\mathcal{V}_{\zeta_\gamma}(m) = \frac{\gamma m(m-1)}{\ln(m)} + \left((s-1)+\frac{(s-t)C_\nu}{m_1^\nu}\right)m(m-1),$$

and $m_1^{\zeta_\gamma} = 1 = m_1^{D_{1/(nm_1^\nu)^n}\left(\nu^{\boxplus_{\mathbb{T}} n}\right)^{\boxtimes n}}$. □

5.2. Asymptotic Results Within Boolean and Free Additive Convolutions

We derive several new limit theorems associated with the $\boxplus_{\mathbb{T}}$-convolution, an operation that combines features of Boolean and free additive convolutions. A notable point is that choosing the parameters $s = t = 1$ recovers the classical free additive convolution $\boxplus$. Moreover, for any $\nu \in \mathbf{P}$, the Boolean convolution power $\nu^{\uplus t}$ [32] may be viewed as a particular instance of the (s,t)-deformation of measures corresponding to the case $s = t > 0$. Within this setting, a family of transformations $\mathbb{B}_q$ acting on the space $\mathbf{P}$ of probability measures is introduced in [33], for each $q \geq 0$, and is defined by

$$\mathbb{B}_q : \mathbf{P} \to \mathbf{P}, \qquad \nu \mapsto \mathbb{B}_q(\nu) = \left(\nu^{\boxplus(1+q)}\right)^{\uplus \frac{1}{1+q}}.$$

In particular, the case $q = 1$ coincides with the well-known Bercovici–Pata bijection, which is central to the theory of infinite divisibility in free probability.

Consider now a measure $\nu \in \mathbf{P}_{ba}$ with a finite first moment. According to [23], if $k > 0$ and the free convolution power $\nu^{\boxplus k}$ exists, then the corresponding VF satisfies

$$\mathcal{V}_{\nu^{\boxplus k}}(v) = k\mathcal{V}_\nu(v/k), \qquad v \text{ close enough to } m_1^{\nu^{\boxplus k}} = km_1^\nu. \tag{34}$$

Similarly, for $q > 0$, one has [34]

$$\mathcal{V}_{\nu^{\uplus q}}(u) = q\mathcal{V}_\nu(v/q) + v(v - rm_1^\nu)(-1+1/q), \qquad v \text{ close enough to } m_1^{\nu^{\uplus q}} = qm_1^\nu, \tag{35}$$

and

$$\mathcal{V}_{\mathbb{B}_1^{-1}(\nu)}(v) = \mathcal{V}_\nu(v) - v(v - m_1^\nu), \qquad v \text{ close enough to } m_1^{\mathbb{B}_1^{-1}(\nu)} = m_1^\nu. \tag{36}$$

We conclude this section by presenting and proving the main results derived from these observations.

Theorem 5. *Let* $\nu \in \mathbf{P}_c$.

(i) *For* $\alpha > 0$ *and* $s \in (0,1]$ *for which* $\left(\nu^{\boxplus_{\mathbb{T}}\frac{1}{\alpha}}\right)^{\boxplus\alpha}$ *is well defined, one has*

$$\left(\nu^{\boxplus_{\mathbb{T}}\frac{1}{\alpha}}\right)^{\boxplus\alpha} \xrightarrow{\alpha\to+\infty} D_s\left(\varphi(U^{\mathbb{T}}(\nu))\right)^{\boxplus\frac{1}{s}} \quad \text{in distribution,} \tag{37}$$

with $\varphi(w) = (w+(t-s)C_\nu)/s$.

(ii) *For* $\alpha > 0$ *and* $s \in (0,1]$ *for which* $\left(\nu^{\boxplus_{\mathbb{T}}\frac{1}{\alpha}}\right)^{\uplus\alpha}$ *is well defined, one has*

$$\left(\nu^{\boxplus_{\mathbb{T}}\frac{1}{\alpha}}\right)^{\uplus\alpha} \xrightarrow{\alpha\to+\infty} \mathbb{B}_1^{-1}\left(D_s\left(\varphi(U^{\mathbb{T}}(\nu))\right)^{\boxplus\frac{1}{s}}\right), \quad \text{in distribution.} \tag{38}$$

Proof. (i) One has

$$m_1^{\left(\nu^{\boxplus_{\mathbb{T}}\frac{1}{\alpha}}\right)^{\uplus\alpha}} = m_1^\nu = m_1^{D_s\left(\varphi(U^{\mathbb{T}}(\nu))\right)^{\boxplus\frac{1}{s}}}.$$

Then, $\varepsilon > 0$ exists for which $\mathcal{V}_{\left(\nu^{\boxplus_{\mathbb{T}}\frac{1}{\alpha}}\right)^{\boxplus\alpha}}(\cdot)$ and $\mathcal{V}_{D_s\left(\varphi(U^{\mathbb{T}}(\nu))\right)^{\boxplus\frac{1}{s}}}(\cdot)$ are defined on $(m_1^\nu-\varepsilon, m_1^\nu+\varepsilon)$. For $m \in (m_1^\nu-\varepsilon, m_1^\nu+\varepsilon)$, one has

$$\begin{aligned}
\mathcal{V}_{\left(\nu^{\boxplus_{\mathbb{T}}\frac{1}{\alpha}}\right)^{\boxplus\alpha}}(m) &\overset{(34)}{=} \alpha\mathcal{V}_{\nu^{\boxplus_{\mathbb{T}}\frac{1}{\alpha}}}(m/\alpha) \\
&\overset{(9)}{=} \alpha[\mathcal{V}_\nu(m)/\alpha + (m/\alpha - m_1^\nu/\alpha)\{m(1-s)(\alpha-1)/\alpha + (1/\alpha-1)(s-t)C_\nu\}] \\
&= \mathcal{V}_\nu(m) + (m-m_1^\nu)(m(1-s)(1-1/\alpha) + (1/\alpha-1)(s-t)C_\nu) \\
&\xrightarrow{\alpha\to+\infty} \mathcal{V}_\nu(m) + (m-m_1^\nu)(m(1-s)-(s-t)C_\nu).
\end{aligned} \tag{39}$$

On the other hand, with $\varphi(w) = (w+(t-s)C_\nu)/s$, we have

$$\begin{aligned}
\mathcal{V}_{D_s\left(\varphi(U^{\mathbb{T}}(\nu))\right)^{\boxplus\frac{1}{s}}}(m) &\overset{(3)}{=} s^2\mathcal{V}_{\left(\varphi(U^{\mathbb{T}}(\nu))\right)^{\boxplus\frac{1}{s}}}\left(\frac{m}{s}\right) \\
&\overset{(34)}{=} s\mathcal{V}_{\varphi(U^{\mathbb{T}}(\nu))}(m) \\
&\overset{(3)}{=} s\left[\frac{1}{s^2}\mathcal{V}_{U^{\mathbb{T}}(\nu)}\left(\frac{m-\frac{(t-s)C_\nu}{s}}{\frac{1}{s}}\right)\right] \\
&= \frac{1}{s}\mathcal{V}_{U^{\mathbb{T}}(\nu)}(sm-(t-s)C_\nu) \\
&\overset{(6)}{=} \frac{1}{s}\left\{\mathcal{V}_\nu(m) + (sm - sm_1^\nu)(m-(sm-(t-s)C_\nu))\right\} \\
&= \mathcal{V}_\nu(m) + (m-m_1^\nu)(m(1-s)-(s-t)C_\nu).
\end{aligned} \tag{40}$$

Equations (39) and (40) give

$$\mathcal{V}_{\left(\nu^{\boxplus_{\mathbb{T}}\frac{1}{\alpha}}\right)^{\boxplus\alpha}}(m) \xrightarrow{\alpha\to+\infty} \mathcal{V}_{D_s\left(\varphi(U^{\mathbb{T}}(\nu))\right)^{\boxplus\frac{1}{s}}}(m), \qquad m \in (m_1^\nu-\varepsilon, m_1^\nu+\varepsilon).$$

Together with ([21] [Proposition 4.2]), this completes the proof of (37). Note that the condition $s \in (0,1]$ plays a crucial role in the argument. Indeed, this assumption ensures that the measure $D_s\left(\varphi(U^{\mathbb{T}}(\nu))\right)^{\boxplus\frac{1}{s}}$ is well defined and that the corresponding convolution power is mathematically meaningful.

(ii) We have that

$$m_1^{\left(\nu^{\boxplus_{\mathbb{T}}\frac{1}{\alpha}}\right)^{\uplus\alpha}} = m_1^\nu = m_1^{\mathbb{B}_1^{-1}\left(D_s\left(\varphi(U^{\mathbb{T}}(\nu))\right)^{\boxplus\frac{1}{s}}\right)}.$$

Then, $\varepsilon > 0$ exists so that $\mathcal{V}_{\left(\nu^{\boxplus_{\mathbb{T}} \frac{1}{\alpha}}\right)^{\uplus \alpha}}(\cdot)$ and $\mathcal{V}_{\mathbb{B}_1^{-1}\left(D_s\left(\varphi(U^{\mathbb{T}}(\nu))\right)^{\boxplus \frac{1}{s}}\right)}(\cdot)$ are defined on $(m_1^\nu - \varepsilon, m_1^\nu + \varepsilon)$. One has

$$\begin{aligned}
\mathcal{V}_{\left(\nu^{\boxplus_{\mathbb{T}} \frac{1}{\alpha}}\right)^{\uplus \alpha}}(m) &\overset{(35)}{=} \alpha \mathcal{V}_{\nu^{\boxplus_{\mathbb{T}} \frac{1}{\alpha}}}(m/\alpha) + m\left(m - \alpha m_1^{\nu^{\boxplus_{\mathbb{T}} \frac{1}{\alpha}}}\right)(1/\alpha - 1) \\
&\overset{(9)}{=} \alpha[\mathcal{V}_\nu(m)/\alpha + (m/\alpha - m_1^\nu/\alpha)\{m(1-s)(\alpha-1)/\alpha + (1/\alpha - 1)(s-t)C_\nu\}] + m(m - m_1^\nu)(1/\alpha - 1) \\
&= \mathcal{V}_\nu(m) + (m - m_1^\nu)(m(1-s)(1-1/\alpha) + (1/\alpha - 1)(s-t)C_\nu) + m(m - m_1^\nu)(1/\alpha - 1) \\
&\xrightarrow{\alpha \to +\infty} \mathcal{V}_\nu(m) + (m - m_1^\nu)(m(1-s) - (s-t)C_\nu) - m(m - m_1^\nu) \\
&\overset{(40) \text{ and } (36)}{=} \mathcal{V}_{\mathbb{B}_1^{-1}\left(D_s\left(\varphi(U^{\mathbb{T}}(\nu))\right)^{\boxplus \frac{1}{s}}\right)}(m), \qquad m \in (m_1^\nu - \varepsilon, m_1^\nu + \varepsilon).
\end{aligned} \tag{41}$$

Equation (41) together with ([21] [Proposition 4.2]) completes the demonstration of (38). □

The asymptotic regimes investigated above reveal a convergence mechanism that depends in a subtle and nontrivial way on both the deformation parameters (s, t) and the intrinsic features of the underlying measure ν. In particular, quantities such as the first moment and the VF play a central role in determining the shape and stability of the limiting behavior. The normalization procedures and rescaling arguments employed throughout the proofs make this dependence explicit, showing how variations in the parameters influence the analytic structure of the associated transforms. This is especially relevant in extreme deformation regimes, such as when $t \to \infty$, where the deformation significantly alters the functional form of the transforms while still preserving coherence with the underlying analytic framework. In this context, we observe that the limiting behavior remains well-structured and consistent with the general transform theory, despite the increasing deformation. Moreover, the interplay between different convolution operations, namely free, Boolean, and multiplicative convolutions, leads to the emergence of nonstandard limiting VFs, reflecting the rich algebraic and analytic interactions encoded in the $\boxplus_{\mathbb{T}}$-convolution. This highlights the flexibility of the deformation framework and its sensitivity to both parameter variation and measure-specific characteristics. Although deriving explicit rates of convergence remains technically challenging within this setting, the asymptotic expansions obtained in intermediate steps suggest that such rates could be investigated through moment-based approximations or analytic subordination techniques associated with the relevant transforms. Altogether, these considerations provide a more comprehensive understanding of the stability and robustness of the limiting distributions, including under extreme deformation, and point toward meaningful directions for future research aimed at quantifying convergence speeds and refining the asymptotic analysis in these generalized contexts. We further emphasize that, while free additive, Boolean additive, and free multiplicative convolutions are classical and extensively studied objects in noncommutative probability, the connections established here with the $\boxplus_{\mathbb{T}}$-convolution are genuinely new. Previous works on (s, t)-deformations have primarily focused on transformations of individual probability measures without addressing how such deformations interact at the level of convolution operations. In contrast, the present analysis demonstrates that, through the study of VFs and limit theorems for convolution powers, the $\boxplus_{\mathbb{T}}$-convolution naturally gives rise to asymptotic regimes that bridge these classical convolution structures. This perspective uncovers a previously unexplored layer of interaction between deformation theory and convolution operations, thereby offering a unifying viewpoint. It situates the (s, t)-deformation within the broader framework of noncommutative convolution theories and underscores its structural compatibility with well-established probabilistic models.

We proceed by emphasizing the significance of Theorem 5 through a detailed analysis of its implications for a range of concrete probability distributions. This discussion not only underscores the applicability of the result but also clarifies how the abstract framework translates into familiar and explicitly tractable cases.

Example 1. *Consider the symmetric Bernoulli distribution* $sb = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$. *For* $s \in (0,1]$, *we have*

$$\left(sb^{\boxplus_{\mathbb{T}} \frac{1}{\alpha}}\right)^{\boxplus \alpha} \xrightarrow{\alpha \to +\infty} fb_{(0,-s)} \quad \text{in distribution}, \tag{42}$$

where $fb_{(a,b)}$ *is the the free binomial type law with* $a \in \mathbb{R}$ *and* $-1 \le b < 0$; *see ([21] [Theorem 3.2]). That is,*

$$fb_{(a,b)}(dl) = \frac{\sqrt{4(1+b)-(l-a)^2}}{2\pi(bl^2+al+1)} 1_{(a-2\sqrt{1+b},a+2\sqrt{1+b})}(l)dl + p_1\delta_{l_1} + p_2\delta_{l_2},$$

with $m_1^{fb_{(a,b)}} = 0$, *and the associated VF is* $\mathcal{V}_{fb_{(a,b)}}(m) = 1 + am + bm^2$, *where* $a \in \mathbb{R}$ *and* $-1 \le b < 0$.

Equation (42) can be obtained by using the VF framework. We have that $m_1^{sb} = 0$ *and, from the symmetry of the measure sb, we know from ([8] [Lemma 2]) that* $C_{sb} = \Re(1/\mathbf{G}_{sb}(i)) = 0$. *Then, for* $m \in (-\varepsilon, +\varepsilon)$, *Equation (39) gives*

$$\mathcal{V}_{\left(sb^{\boxplus_{\mathbb{T}} \frac{1}{\alpha}}\right)^{\boxplus \alpha}}(m) \xrightarrow{\alpha \to +\infty} \mathcal{V}_{sb}(m) + (1-s)m^2 = 1 - m^2 + (1-s)m^2 = 1 - sm^2 = \mathcal{V}_{fb_{(0,-s)}}(m).$$

Example 2. *Consider the semicircle law* $sc(dl) = \frac{1}{2\pi}\sqrt{4-l^2}1_{(-2,2)}(l)dl$. *For* $s \in (0,1]$, *we have*

(i)

$$\left(sc^{\boxplus_{\mathbb{T}} \frac{1}{\alpha}}\right)^{\boxplus \alpha} \xrightarrow{\alpha \to +\infty} fh_{(0,1-s)} \quad \text{in distribution}, \tag{43}$$

where $fh_{(a,b)}$ *is the the free analog of hyperbolic type law with* $b > 0$ *and* $a^2 < 4b$; *see ([21] [Theorem 3.2]). That is,*

$$fh_{(a,b)}(dl) = \frac{\sqrt{4(1+b)-(l-a)^2}}{2\pi(bl^2+al+1)} 1_{(a-2\sqrt{1+b},a+2\sqrt{1+b})}(l)dl,$$

with $m_1^{fh_{(a,b)}} = 0$, *and the associated VF is* $\mathcal{V}_{fh_{(a,b)}}(m) = bm^2 + am + 1$, *where* $b > 0$ *and* $a^2 < 4b$.

Equation (43) can be obtained by using the VF framework. We have that $m_1^{sc} = 0$ *and, from the symmetry of the measure sc, we know from ([8] [Lemma 2]) that* $C_{sc} = \Re(1/\mathbf{G}_{sc}(i)) = 0$. *Then, for* $m \in (-\varepsilon, +\varepsilon)$, *Equation (39) gives*

$$\mathcal{V}_{\left(sc^{\boxplus_{\mathbb{T}} \frac{1}{\alpha}}\right)^{\boxplus \alpha}}(m) \xrightarrow{\alpha \to +\infty} \mathcal{V}_{sc}(m) + (1-s)m^2 = 1 + (1-s)m^2 = \mathcal{V}_{fh_{(0,1-s)}}(m).$$

(ii)

$$\left(sc^{\boxplus_{\mathbb{T}} \frac{1}{\alpha}}\right)^{\uplus \alpha} \xrightarrow{\alpha \to +\infty} fb_{(0,-s)} \quad \text{in distribution}.$$

In fact, for $m \in (-\varepsilon, +\varepsilon)$, *Equation (41) gives*

$$\mathcal{V}_{\left(sc^{\boxplus_{\mathbb{T}} \frac{1}{\alpha}}\right)^{\uplus \alpha}}(m) \xrightarrow{\alpha \to +\infty} \mathcal{V}_{sc}(m) + (1-s)m^2 - m^2 = 1 - sm^2 = \mathcal{V}_{fb_{(0,-s)}}(m).$$

6. Conclusions

The $\boxplus_{\mathbb{T}}$-convolution differs fundamentally from the classical free convolution $\boxplus$. While $\boxplus$ is an intrinsic operation in free probability characterized by the additivity of the $\mathcal{R}$-transform and governed solely by free independence, $\boxplus_{\mathbb{T}}$ introduces a genuinely new mechanism through a parameter-dependent deformation driven by the pair (s, t), with $s > 0$ and $t \in \mathbb{R}$. This deformation is not a superficial modification but leads to a reorganization of the underlying analytic structure, yielding a convolution governed by a modified functional equation. One of the key innovations of this work is the explicit identification of how these deformation parameters reshape the behavior of convolution powers, producing phenomena absent in the classical framework, such as modified scaling limits, nonstandard asymptotic regimes, and novel invariance properties. These features demonstrate that $\boxplus_{\mathbb{T}}$ should be viewed as a structurally richer extension of $\boxplus$, capable of capturing a broader class of probabilistic interactions beyond free independence.

In this paper, we developed a systematic analysis of the $\boxplus_{\mathbb{T}}$-convolution through the framework of CSK families and their VFs, which constitutes another central contribution of this work. In particular, we derived an explicit and tractable formula for the VF associated with convolution powers under $\boxplus_{\mathbb{T}}$, thereby providing a clear quantitative description of the impact of the deformation parameters on both analytic and probabilistic structures. This result not only unifies several aspects of deformed free probability but also offers a practical tool for further investigations. Building on this formula, we established approximation schemes for CSK families generated by the $\mathbb{T}$-deformed free Gaussian and $\mathbb{T}$-deformed free Poisson laws, showing that the VF approach yields an efficient and robust method for analyzing their asymptotic behavior. A further important contribution lies in the derivation of new limit theorems for the $\boxplus_{\mathbb{T}}$-convolution, which reveal unexpected connections with other fundamental operations in noncommutative probability, including free multiplicative convolution as well as free and Boolean additive convolutions. These results significantly broaden the scope of interaction between different convolution structures and suggest that $\boxplus_{\mathbb{T}}$ can serve as a unifying framework for studying interpolations and transitions between distinct probabilistic regimes. From an applied and theoretical perspective, this enhanced flexibility opens promising avenues for modeling complex systems where classical independence assumptions are too restrictive.

Finally, several directions for future research naturally emerge from the present analysis. These include, in particular, the extension of the proposed framework to multivariate settings, as well as a more detailed study of stability properties and infinite divisibility with respect to the $\boxplus_{\mathbb{T}}$-convolution. Another important perspective concerns possible connections with random matrix models and operator-valued free probability, which may shed further light on the structural and spectral aspects of the deformed convolution. From a statistical and information-theoretic viewpoint, it is natural to investigate how the deformation parameters (s, t) influence the information content of the associated probability measures. While the present work does not develop an explicit entropy-based theory, the VF framework provides a natural analytic setting in which such questions can be formulated. In particular, since the VF captures deformation, scaling, and convolution structures in a unified way, it may serve as an indirect tool for quantifying changes in distributional complexity induced by the (s, t)-deformation. This suggests that entropy-type functionals, such as Shannon's entropy or their free-probability counterparts, could potentially be combined with the VF representation to measure information gain and compare deformed and undeformed laws. A systematic investigation of these connections, including potential links with goodness-of-fit criteria such as the Anderson–Darling statistic, as well as robustness analysis via confidence intervals and outlier detection in the associated CSTs, would require a dedicated study and is left for future work. In addition, although the numerical illustrations in Section 5 highlight the scaling behavior of the model, a more systematic statistical validation framework—comparable to those used in applied probabilistic and computational methodologies—would further strengthen the practical aspect of the theory. This includes, in particular, developing procedures for detecting anomalies as the parameters s and t vary and for assessing the stability of parameter estimation in largesample regimes. Such directions would complement the present theoretical results and could significantly broaden the applicability of the (s,t)-deformed convolution framework. Overall, these perspectives underline the richness of the proposed approach and open the way toward a more comprehensive probabilistic and statistical theory based on analytic deformation techniques.

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Abbreviations

The following notations are used throughout the paper:

CSK	Cauchy–Stieltjes kernel
CST	Cauchy–Stieltjes transform
NEF	Natural exponential family
VF	Variance function
PVF	Pseudo-variance function

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