

Geometry and Constants in Finite Ring Continuum

Yosef Akhtman ^{1,2} 
¹ Gamma Earth Sàrl, 1162 St-Prex, Switzerland; ya@gamma.earth

² Faculty of Space Technologies, AGH University of Krakow, 30-059 Krakow, Poland

Abstract

The current paper studies the global shell layer of the Finite Ring Continuum framework in the symmetry-complete regime realized here by framed finite fields, $\mathbb{F}_p(t; 0, 1, e_t)$, with $p = 4t + 1$. We show that a single symmetry-complete shell carries a unified finite Euclidean datum for which its continuum comparison interpretation reproduces the familiar structural roles of e , π , and i of a one-phase step with an exponential kernel, a half-period, and a quarter-turn, respectively. In the same shell, the orbital geometry is generated by additive meridian action and multiplicative phase action from that same frame datum. The resulting orbital shell has a canonical spherical completion, combinatorially equivalent to the two-sphere, with labels depending on the chosen frame, but the shell type fixed up to isomorphism. Arbitrary finite-precision approximation on this external spherical comparison object is then obtained within every fixed symmetry-complete shell by the scale-periodic framed-rational refinement generated by the same frame datum. The Fourier formalism is developed strictly as a discrete Fourier transform over the shell ring, with conventional continuum Fourier language becoming a continuum large- p comparison case of that shell formalism.

Keywords: symmetry-complete shells; framed involutions; finite geometry; combinatorial two-sphere; operational approximation; discrete Fourier transform over a ring; finitist harmonic analysis; Finite Ring Continuum

1. Introduction

The finite ring continuum is used to develop arithmetic and geometry inside finite algebraic constructions rather than by taking an actual continuum as primitive. The resultant symmetry-complete shells over finite fields $\mathbb{F}_p$ with $p = 4t + 1$ were established in [1], where we demonstrated that these constructs provide a finite setting in which geometric orientation, reciprocal duality, and harmonic phase structure are all available without leaving the arithmetic of a finite field. Furthermore, the symmetry-complete prime case is the first regime in which these Euclidean symmetries appear in exact combinatorial form inside a strictly finitist formalism.

The present paper continues to work with the symmetry-complete shell layer studied here: Fields of radius t are denoted as $\mathbb{F}_p(t; 0, 1, e_t)$ when the full frame is explicit. The orbital two-dimensional geometry, the primitive frame datum $(0, 1, e_t)$, the induced Euclidean values (i_t, π_t) , and the finite Fourier duality are naturally carried by these objects. In the present reading, the shell is first a symmetry space: Things exist because their structure is preserved under the available shell actions. The orbital geometry then appears as the finite representation of those internal symmetries, generated in a chosen frame by additive meridian advance and multiplicative phase advance. The orbital shell is studied in that



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chosen frame, so its longitude labels depend on the frame, while the shell fixes only the resulting isomorphism class and its invariants. The central claim is not merely that finite analogues of i , π , and e can be written down. It is that once the primitive frame datum $(0, 1, e_t)$ is fixed, the symmetry-complete shell studied here determines the half-period π_t and the oriented quarter-turn i_t so that one framed shell carries the full Euclidean datum used in this paper. Section 6 then shows that this framed datum is transported exactly to the sampled classical roles of a one-phase step, a half-period, a quarter-turn, and an exponential kernel. In that sense, this paper proposes a finite structural origin for the same roles that continuum mathematics usually treats as primitive.

The classical sphere S^2 enters here only as an external global comparison object built from the shell coordinates. The finite orbital shell S_p remains the global shell, for which its terminal latitude is collapsed to an antipodal point only in the associated global spherical quotient. The comparison theorem is stated at the level of finitely many marked points together with finite families of Lipschitz observables. The unrefined vertex grid of S_p becomes denser as p grows, but the stronger point proved here is different: arbitrary prescribed precision is already available inside every fixed symmetry-complete shell, including the small ones, through the scale-periodic framed-rational refinement generated by the same frame datum.

This external spherical comparison is not yet the observer-local spatial chart in which familiar metric or differential geometry would be written explicitly. Observer-local development is deferred to future work. Accordingly, the term “Euclidean” in the present paper refers not to the global shell S_p itself but to the later observer-local space-like reading for which the shell and its external comparison structure provide the present prerequisite layer.

The present paper is one explicit step in a broader programme, outlined in Figure 1, which aims to show that a strictly finite formalism can carry the full operational load of scientific description, leaving the continuum as an idealized degenerative description rather than a fundamental ontology [2–4]. The empirical success of continuum mathematics is consistent with that position, since physical confirmation is always mediated by finite measurements, finite resolutions, and finite agreement criteria. The present paper establishes one concrete shell layer of that programme: a single symmetry-complete shell already carries the bounded geometric and harmonic structures studied here, the fixed-shell refinement mechanism is already present even for a small p , and the classical continuum appears as the external degenerate comparison case of that finite formalism [3].

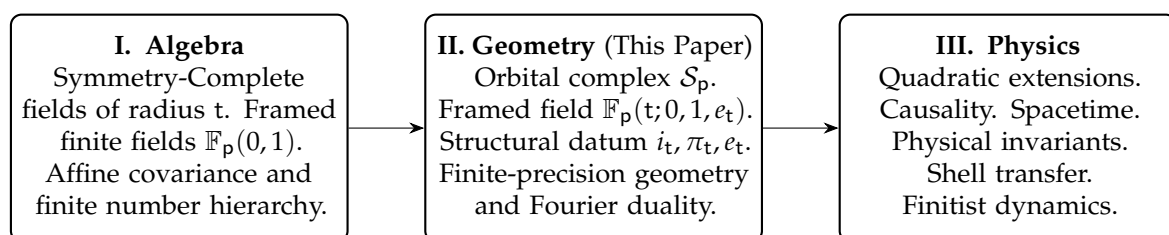


Figure 1. FRC programme placement of the present work. Paper I [1] provides the algebraic foundations, while the current Paper II isolates the Euclidean shell layer in the symmetry-complete regime and leaves extension, causality, and dynamics to later work.

This paper pursues four objectives:

- (i) Construct a strictly finitist orbital shell S_p as an action-generated symmetry space of a symmetry-complete field of radius t , together with its canonical spherical completion.
- (ii) Identify the core involutive symmetry axes of the shell regime studied here together with their minimal framing depth: additively framed negation $R(x) = -x$, affinely

framed reciprocal inversion $I(x) = x^{-1}$ on $\mathbb{F}_p^\times$, and extended-frame Euclidean conjugation C_{e_t} .

- (iii) Prove an operational finite-precision theorem showing that each fixed symmetry-complete shell, through its scale-periodic framed-rational refinement, models every bounded experiment consisting of finitely many marked points and a finite family of Lipschitz comparison observables to arbitrary prescribed accuracy.
- (iv) Formulate the shell's harmonic duality strictly as a discrete Fourier transform over the finite ring $\mathbb{F}_p$.

This paper is situated in a clear scholarly setting. On the foundational side, it is closer to feasibility and bounded-arithmetic viewpoints, where the distinction between formal existence and operational accessibility is explicit [5,6], and to finitist or non-Cantorian size programmes that seek mathematically coherent alternatives to unrestricted actual infinity [7]. On the geometric side, the orbital shell and its external degenerate continuum reading belong to the broader tradition of finite or combinatorial models approximating continuum geometric outputs, including Regge's coordinate-free discretization of geometry, Dodziuk's combinatorial Hodge theory, and modern discrete differential geometry [8–10]. On the harmonic side, the Fourier formalism used here is not ad hoc: Discrete Fourier transforms over finite fields, finite groups, and related finite rings are classical tools in number-theoretic and coding-theoretic analysis [11–14]. These comparisons do not identify FRC with any one prior framework, but they do locate the present claims inside recognizable mathematical terrain.

Relative to earlier work in the FRC series, the present paper isolates the first stage at which the symmetry-complete shell studied here can be treated as a self-contained object. The published algebra paper [1] establishes the symmetry-complete shell regime used here, including the quarter-turn condition $p \equiv 1 \pmod{4}$, the framed-complex layer, and the basic Klein-type orbit structure. The present article builds on that algebraic base and contributes three specifically geometric and harmonic steps: an action-generated orbital shell with canonical spherical completion combinatorially equivalent to S^2 , an explicit bounded-experiment approximation theorem for the resulting shell grids, and a strictly finite Fourier formalism written in the same setting.

The contribution is that these structures coexist and interact inside one symmetry-complete shell package rather than appearing as disconnected finite analogies. The point is not just that each ingredient has a finite counterpart in isolation but that one symmetry-complete shell simultaneously carries the frame datum $(0, 1, e_t)$, the induced Euclidean values (i_t, π_t) , the involution package, an action-generated orbital shell together with its spherical completion, the external degenerate continuum reading, and the harmonic transform, with exact external transport of the shell datum to the sampled classical phase roles. This unified shell architecture is what this paper contributes to the mathematical and foundational discussion.

Importantly, the congruence $p \equiv 1 \pmod{4}$ ensures that -1 is a square in $\mathbb{F}_p$, so the shell carries a quarter-turn subgroup of order 4. This is the minimal algebraic condition required for the Euclidean phase structure studied here. This paper therefore works in the symmetry-complete prime case because it makes the relevant symmetry pattern exact and combinatorial, especially at small p . This is a choice of the exact symmetry regime and not a claim that the system itself is organized by an internal prime/composite distinction. By Dirichlet's theorem, there are infinitely many such primes [15] (Chapter 13), but the operational approximation theorem of Section 5 does not rely on varying p : its refinement mechanism is already present in each fixed symmetry-complete shell.

The rest of this paper is organized as follows. Sections 2–4 define the shell and its structural data, with the shell built from the additive meridian action and the multiplicative

phase action of the framed finite field. The same sections isolate the core involutive symmetry axes already present in the shell regime studied here and organize them by minimal framing depth. Section 5 proves the operational approximation theorem for the resulting shell grids. Section 6 develops the discrete Fourier transform formalism needed in longitude cycles of length $p - 1$, specialized to the shell setting and separated from external continuum-comparison language. Section 7 discusses the significance and limits of the resulting symmetry architecture, and Section 8 summarizes the main conclusions.

2. Framework: Symmetry-Complete Shells and Frames

We work throughout with the symmetry-complete regime realized here by odd primes of the form $p = 4t + 1$, so that t is the radius/chronon parameter, and $\mathbb{F}_p$ is a finite field with $\text{card}(\mathbb{F}_p) = p$. In the present paper, t is used only in its radial shell role. Its broader temporal reading belongs to the later causality layer of the FRC programme [16].

Definition 1 (Symmetry-Complete Field of Radius t). *A symmetry-complete field of radius t is a pair $(t, \mathbb{F}_p)$ with $p = 4t + 1$ prime. Equivalently, $\mathbb{F}_p^\times$ is cyclic of order $4t$ and has a unique subgroup of order 4, and in any framed realization $\mathbb{F}_p(0, 1)$, the remaining nonzero elements split into $t - 1$ orbital classes of the form*

$$\{x, -x, x^{-1}, -x^{-1}\}.$$

These are the generic orbits of the commuting involution pair generated by negation and reciprocal inversion, as established already in a published algebra paper [1]. When emphasizing the geometric role of the same object, we also refer to it simply as a symmetry-complete shell.

Remark 1 (Inherited algebraic layer used in this paper). *Only a small part of the earlier algebraic development is used here. The present article needs four inherited facts: (i) $\mathbb{F}_p^\times$ is cyclic of order $4t$; (ii) it therefore has a unique subgroup of order 4, written as $\mathcal{Q}_p$; (iii) the commuting involutions generated by framed negation and inversion produce generic four orbits $\{x, -x, x^{-1}, -x^{-1}\}$; (iv) once the oriented quarter-turn i_t is fixed, one may write framed-complex expressions $a + bi_t$ over $\mathbb{F}_p$. Everything else used in this paper is developed explicitly below.*

Definition 2 (Extended affine reference frame). *An extended affine reference frame on a symmetry-complete shell is the finite field*

$$\mathbb{F}_p(t; 0, 1, e_t),$$

or equivalently the quadruple $(t; 0, 1, e_t)$, consisting of the radius/chronon together with three chosen frame elements, written in that frame as the origin $0 \in \mathbb{F}_p$, the unit $1 \in \mathbb{F}_p$, and the exponential unit $e_t \in \mathbb{F}_p^\times$. Externally, e_t is represented by a primitive generator of $\mathbb{F}_p^\times$. This extends the framed finite field $\mathbb{F}_p(0, 1)$ of the algebra paper by adjoining a phase chart.

Remark 2. *The shell $(t, \mathbb{F}_p)$ and the frame $\mathbb{F}_p(t; 0, 1, e_t)$ are different kinds of data. The shell determines the finite field and the structural possibilities available at that grade. The frame is specified by the primitive datum $(0, 1, e_t)$, which fixes the affine chart and the phase chart. Explicit symmetry formulas appear only after that datum has been fixed.*

Definition 3 (Half-Period and Phase Cycle). *For a symmetry-complete shell in the regime studied here, set*

$$\pi_t := \frac{p-1}{2} = 2t.$$

The meridian-step interval is

$$I_p := \{0, 1, \dots, \pi_t\},$$

and the phase cycle is the cyclic group

$$\Phi_p := \mathbb{Z}/(p-1)\mathbb{Z}.$$

Operationally, I_p counts discrete steps along a fixed meridian from the observer origin toward the terminal shell latitude L_{π_t} , while Φ_p indexes the $p-1$ distinct meridians or longitude phases via the chosen exponential unit e_t . A phase label $m \in \Phi_p$ corresponds to the field element $e_t^m \in \mathbb{F}_p^\times$. Thus, the shell carries two primitive actions used throughout the geometry below: additive meridian advance generated by 1, and multiplicative phase advance generated by e_t .

Remark 3 (Action-Generated Shell Points). In a chosen frame $\mathbb{F}_p(t; 0, 1, e_t)$, the shell points are generated from the observer origin by the two action coordinates $a \in I_p$ and $m \in \Phi_p$. We write the generated vertex in phase direction m and meridian step a as

$$\mathcal{V}_p(a, m) = ae_t^m.$$

Here, a records the additive depth along the chosen meridian, while m records multiplicative phase advance and hence selects one of the $p-1$ meridians, as depicted in Figure 2. The value $a = 0$ gives the observer origin for every m , and the value $a = \pi_t$ gives the terminal shell latitude. The antipode appears only after that terminal latitude is collapsed in the associated global spherical quotient.

Proposition 1 (Primitive Generators Form One Orbit). If $g \in \mathbb{F}_p^\times$ is primitive, then every primitive generator of $\mathbb{F}_p^\times$ has the form

$$g^u, \quad u \in (\mathbb{Z}/(p-1)\mathbb{Z})^\times.$$

Hence, the set of primitive generators is a torsor for the automorphism group of the cyclic group $\mathbb{F}_p^\times$.

Proof. Since $\mathbb{F}_p^\times$ is cyclic of order $p-1$, its generators are exactly the power g^u with $\gcd(u, p-1) = 1$ [17] (Chapter 5). $\square$

Proposition 2 (Phase-Frame Change). Let $g' = g^u$ with $u \in (\mathbb{Z}/(p-1)\mathbb{Z})^\times$. Then, the map

$$\rho_u: \Phi_p \rightarrow \Phi_p, \quad m \mapsto um$$

is a cyclic reindexing of longitude phases. Reversing the meridian orientation acts by

$$\sigma: I_p \rightarrow I_p, \quad a \mapsto \pi_t - a.$$

Both reindexings preserve the shell incidence structure defined below.

Proof. The first statement is immediate because u is invertible modulo $p-1$. The second is the reflection of the interval I_p . Both maps are bijections; hence, they preserve adjacency once the shell is defined by step relations in $I_p \times \Phi_p$. $\square$

3. Orbital Shell and Spherical Completion

The central finite geometry object of this paper is the orbital shell $\mathcal{S}_p$. It should be read first as an action-generated shell and only then as a CW complex. In a chosen frame $\mathbb{F}_p(t; 0, 1, e_t)$, the shell is generated by additive meridian advance and multiplicative

phase advance. The finite shell vertices beyond the observer's origin are written in action notation as

$$\mathcal{V}_p(a, m) = ae_t^m, \quad 1 \leq a \leq \pi_t, \quad m \in \Phi_p,$$

where a and m remain the two independent shell coordinates: a is the additive step count along a fixed meridian, while m is the multiplicative phase index selecting one of the $p - 1$ meridians. In this notation, 1 is the additive action generator, and e_t is the multiplicative phase generator. At $a = 0$, all phase labels agree:

$$0e_t^m = 0, \quad m \in \Phi_p,$$

so the north pole belongs internally to the shell as the observer origin. At the opposite boundary, $a = \pi_t$ gives the terminal latitude L_{π_t} . In the associated global spherical quotient, the whole latitude is collapsed to a single antipodal point S . Thus, $\mathcal{S}_p$ is the finite action-generated shell itself, while its spherical completion is obtained by collapsing L_{π_t} to S .

This global shell and its global spherical quotient should therefore not be identified with the later observer-local space-like chart. The familiar Euclidean space belongs to that derived chart, whereas $\mathcal{S}_p$ is the finite prerequisite symmetry object constructed in the present paper.

The arithmetic source of the construction is the finite orbit cylinder

$$C_p^{\text{orb}} := I_p \times \Phi_p,$$

equipped with the two shell motions

$$A(a, m) = (a + 1, m), \quad P(a, m) = (a, m + 1),$$

whenever the right-hand side is defined. Here, A records one additive increment in the direction e_t^m , and P records one multiplicative phase step by e_t . The lower boundary $a = 0$ already represents the observer origin, while the opposite boundary $a = \pi_t$ is the terminal latitude later collapsed to the antipode in the spherical completion.

This construction combines the two primitive shell actions in one object. Meridians are generated by additive increments of the phase ray e_t^m , while latitudes are generated by multiplication by e_t . The CW complex below records that interaction in a finite combinatorial form.

Definition 4 (Action-generated orbital shell). *Fix an extended affine reference frame $\mathbb{F}_p(t; 0, 1, e_t)$ on a symmetry-complete prime shell. The orbital shell $\mathcal{S}_p$ is the finite CW complex with*

(i) *Vertices*

$$V(\mathcal{S}_p) = \{N\} \cup \{v_{a,m} \mid 1 \leq a \leq \pi_t, m \in \Phi_p\},$$

(ii) *Action interpretation*

$$N = 0, \quad v_{a,m} \equiv ae_t^m \quad (1 \leq a \leq \pi_t),$$

(iii) *Meridian edges joining N to $v_{1,m}$ and $v_{a,m}$ to $v_{a+1,m}$ for $1 \leq a \leq \pi_t - 1$;*

(iv) *Latitude edges joining $v_{a,m}$ to $v_{a,m+1}$ for $1 \leq a \leq \pi_t$;*

(v) *North triangular faces $[N, v_{1,m}, v_{1,m+1}]$;*

(vi) *Quadrilateral faces*

$$[v_{a,m}, v_{a,m+1}, v_{a+1,m+1}, v_{a+1,m}]$$

for $1 \leq a \leq \pi_t - 1$.

All longitude indices are taken as modulo $p - 1$.

The meridian indexed by $m \in \Phi_p$ is the path

$$M_m := \{N\} \cup \{v_{a,m} : 1 \leq a \leq \pi_t\},$$

and the latitude indexed by a is the cycle

$$L_a := \{v_{a,m} : m \in \Phi_p\}.$$

In action notation, M_m records the additive sequence

$$0, e_t^m, 2e_t^m, \dots, \pi_t e_t^m$$

toward the terminal latitude L_{π_t} , while L_a records the multiplicative orbit of fixed meridian step a under repeated multiplication by e_t .

Remark 4 (Observer Origin and Spherical Antipode). *In the chosen frame, N is the observer's internal origin. The opposite shell boundary is the terminal latitude L_{π_t} . In the associated global spherical quotient, the whole latitude is collapsed to the single antipodal point S . Under frame change, the origin moves with the chosen frame, while the terminal latitude remains the opposite shell boundary.*

Figure 2 shows a concrete shell visualization in the case $p = 13$. The gray meridians and orange latitudes display the longitude–latitude cellulation induced by the orbit cylinder, while N marks the observer's origin, and the top orange latitude records the terminal shell boundary for which its image in the external spherical comparison is the antipode.

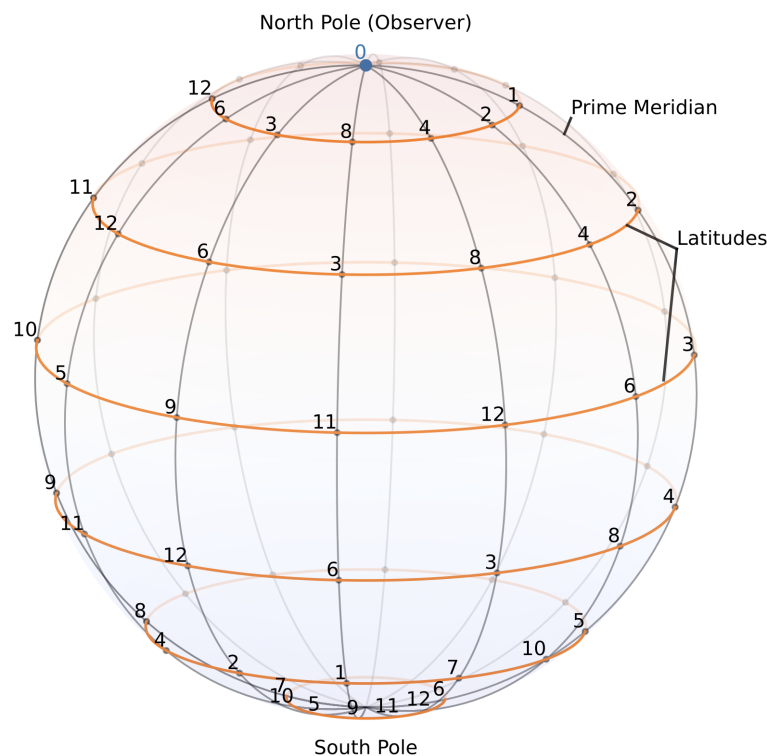


Figure 2. Observer-framed visualization of the orbital shell for the symmetry-complete prime shell $\mathbb{F}_{13}(3;0,1,2)$. Gray curves are additive meridians, orange curves are multiplicative latitudes, N is the observer's origin, and the upper orange latitude is the terminal shell boundary for which its image in the external spherical comparison is the antipode.

Theorem 1 (Canonical spherical completion). *For every symmetry-complete prime shell and every chosen extended affine reference frame, collapsing the terminal latitude $L_{\pi_t} \subset \mathcal{S}_p$ to a single point S yields a finite regular CW complex combinatorially isomorphic to S^2 .*

Proof. Consider the orbit cylinder

$$C_p := [0, \pi_t] \times (\mathbb{R}/(p-1)\mathbb{Z}),$$

subdivided by the unit lattice in the a - and m -directions. This is the topological realization of the discrete orbit cylinder C_p^{orb} . The lower boundary circle already represents the observer's origin because $0e_t^m = 0$ for every phase label m . The shell $\mathcal{S}_p$ records the cell structure before the upper boundary's collapse. Now, collapse the boundary circles

$$\{0\} \times (\mathbb{R}/(p-1)\mathbb{Z}) \quad \text{and} \quad \{\pi_t\} \times (\mathbb{R}/(p-1)\mathbb{Z})$$

to single points N and S , respectively. The quotient is homeomorphic to the suspension of a circle and hence homeomorphic to S^2 [18] (Example 0.14). The vertices, edges, and faces listed above induce exactly the corresponding quotient cellulation, so the spherical completion is a finite regular CW complex with the required combinatorial type. $\square$

Remark 5 (Shell and Completion Counts). *The orbital shell has*

$$\text{card } V(\mathcal{S}_p) = \pi_t(p-1) + 1, \quad \text{card } E(\mathcal{S}_p) = 2\pi_t(p-1), \quad \text{card } F(\mathcal{S}_p) = \pi_t(p-1).$$

Hence,

$$\chi(\mathcal{S}_p) = \text{card } V - \text{card } E + \text{card } F = 1,$$

as expected for a disk with one boundary latitude. After collapsing L_{π_t} to S , the spherical completion has

$$(\pi_t - 1)(p-1) + 2$$

vertices and

$$(2\pi_t - 1)(p-1)$$

edges, so its Euler characteristic is 2, as expected for a sphere.

Proposition 3 (Reference-Frame Covariance of the Orbital Shell). *Let $e_t' = e_t^u$ with $u \in (\mathbb{Z}/(p-1)\mathbb{Z})^\times$. Then,*

$$\varphi_u(v_{a,m}) := v'_{a,um}, \quad \varphi_u(N) = N'$$

extends to a cellular isomorphism $\mathcal{S}_p \rightarrow \mathcal{S}_p'$. It preserves the terminal latitude and therefore descends to spherical completions. Upon spherical completion, meridian reversal $a \mapsto \pi_t - a$ gives a cellular involution, exchanging the poles.

Proof. The map $m \mapsto um$ is a permutation of Φ_p , so it carries latitude and meridian adjacency in $\mathcal{S}_p$ to the corresponding adjacency in $\mathcal{S}_p'$. Since it preserves the latitude index a , it also preserves the terminal latitude L_{π_t} and therefore descends to spherical completions after the latitude is collapsed. In the spherical completion, the quotient identifies the terminal latitude with S , so the reflection $a \mapsto \pi_t - a$ becomes a well-defined cellular involution exchanging N and S . $\square$

4. Symmetry Axes and Structural Data

The structural roles of the classical constants are organized here through a framed Euclidean datum. In a chosen frame $\mathbb{F}_p(t; 0, 1, e_t)$, the primitive datum is $(0, 1, e_t)$, and from this, the shell determines the Euclidean values π_t and i_t :

- (i) The frame is given by the three chosen values 0, 1, and e_t .
- (ii) The same datum generates the two shell actions: additive meridian advance by 1 and multiplicative phase advance by e_t .
- (iii) The shell then determines the half-period π_t .
- (iv) The same frame and shell determine the quarter-turn subgroup $\mathcal{Q}_p$ and the oriented value i_t .
- (v) Explicit involutions are written in that frame.

Proposition 4 (Shell Half-Period). *Let $\pi_t = (p - 1)/2$. For every primitive generator $g \in \mathbb{F}_p^\times$,*

$$g^{\pi_t} = -1.$$

Consequently, π_t depends only on p and not on the choice of g .

Proof. $\mathbb{F}_p^\times$ is cyclic of even order $p - 1$. A cyclic group has a unique element of order 2, namely -1 , and the $(p - 1)/2$ -th power of any generator is that element. $\square$

Definition 5 (Quarter-Turn Subgroup). *The quarter-turn subgroup of the shell is*

$$\mathcal{Q}_p := \{x \in \mathbb{F}_p^\times : x^4 = 1\}.$$

Proposition 5 (Quarter-Turn Structure). *$\mathcal{Q}_p$ is the unique subgroup of $\mathbb{F}_p^\times$ of order 4. If g is a chosen primitive generator, then*

$$g^{(p-1)/4} = g^t$$

has order 4, belongs to $\mathcal{Q}_p$, and satisfies

$$(g^{(p-1)/4})^2 = -1.$$

Consequently, $\mathcal{Q}_p$ contains exactly two square roots of -1 . We write the distinguished oriented one as i_t so that

$$i_t^2 = -1 \quad \text{and} \quad \mathcal{Q}_p = \{\pm 1, \pm i_t\}.$$

Proof. Because $\mathbb{F}_p^\times$ is cyclic of order $4t$, it has a unique subgroup of order 4. The element $g^{(p-1)/4}$ has order 4; hence, it generates that subgroup. Squaring gives

$$(g^{(p-1)/4})^2 = g^{(p-1)/2} = -1$$

by Proposition 4. $\square$

Remark 6 (Structural Set in a Chosen Framed Field). *In the notation of the algebra paper, once the framed finite field $\mathbb{F}_p(t; 0, 1, e_t)$ is fixed, the corresponding structural set is*

$$\{1, i_t, -1, -i_t\} = \{i_t^k : 0 \leq k < 4\}.$$

Thus the quarter-turn subgroup $\mathcal{Q}_p$ appears in the chosen frame as the structural set of fourth roots of unity.

Proposition 6 (Exponential Unit in a Chosen Frame). *A choice of exponential unit e_t in the frame is equivalent to a choice of a primitive generator of $\mathbb{F}_p^\times$. More precisely, if g is primitive, then the primitive generators are exactly*

$$\{g^u : u \in (\mathbb{Z}/(p-1)\mathbb{Z})^\times\},$$

so different frames may use different representatives for the same phase role.

Proof. This is Proposition 1. The shell determines the cyclic group $\mathbb{F}_p^\times$, and a primitive generator of that cyclic group determines the phase chart used in the chosen frame. $\square$

Remark 7 (Primitive and Induced Euclidean Values). *Once the primitive frame datum $(0, 1, e_t)$ has been fixed, the Euclidean values are determined by the shell relations*

$$\pi_t = \frac{p-1}{2} = 2t, \quad i_t = e_t^\dagger, \quad i_t^2 = -1.$$

Under frame changes, the external representatives may change, but the new frame again writes its own datum in the same form.

Figure 3 shows these data on the phase cycle for the shell $p = 13, t = 3$. The primitive frame value e_t , the induced half-period $\pi_t = 6$, and the induced quarter-turns $\pm i_t$ all appear directly on the same finite cycle.

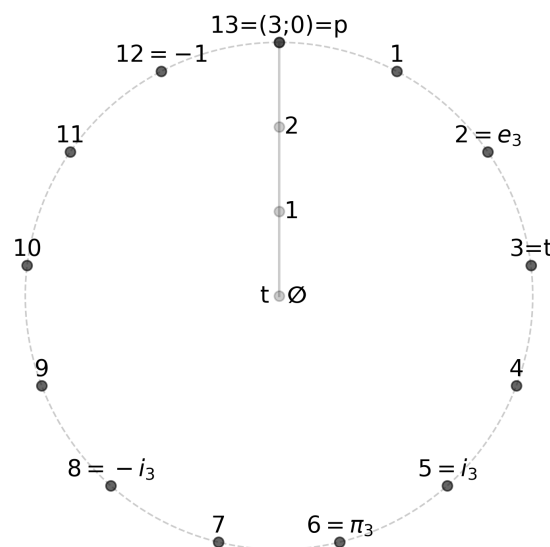


Figure 3. Phase-cycle illustration of the framed Euclidean datum in the symmetry-complete shell $\mathbb{F}_p(3;0,1,e_t)$ with $p = 13$. The primitive frame value is $e_t = 2$, and the same cycle determines the induced values $\pi_t = 6$ and $\pm i_t$.

Framed Involutive Symmetries of the Symmetry-Complete Shell

At the symmetry-complete shell stage studied here, the natural symmetry axes are involutions. Their explicit formulas depend on the chosen frame. The shell determines the corresponding symmetry classes and orbit structure.

Proposition 7 (Additively Framed Negation). *The map*

$$R: \mathbb{F}_p \rightarrow \mathbb{F}_p, \quad R(x) := 0 - x = -x,$$

is an involution relative to the chosen additive origin 0. Restricted to the additive cycle underlying reversible enumeration, it is the framed realization of shell reversal.

Proof. Clearly, $R^2(x) = x$ for every $x \in \mathbb{F}_p$. On the additive structure of $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$, the map $x \mapsto -x$ reverses the direction of the cyclic order. $\square$

Proposition 8 (Affinely Framed Inversion and Four-Orbit Symmetry). *The map*

$$I: \mathbb{F}_p^\times \rightarrow \mathbb{F}_p^\times, \quad I(x) := 1/x = x^{-1},$$

is an involution of the unit cycle relative to the chosen affine pair $(0, 1)$. Moreover, R and I commute on $\mathbb{F}_p^\times$, so they generate a four-element involution action

$$G_K := \langle R, I \rangle = \{\text{id}, R, I, R \circ I\}.$$

For every $x \in \mathbb{F}_p^\times \setminus \mathcal{Q}_p$, the G_K -orbit of x is

$$\{x, -x, x^{-1}, -x^{-1}\}.$$

Proof. Since $(x^{-1})^{-1} = x$, the map I is an involution on $\mathbb{F}_p^\times$. For $x \in \mathbb{F}_p^\times$,

$$I(R(x)) = (-x)^{-1} = -x^{-1} = R(I(x)),$$

so R and I commute. The orbit description is then exactly the orbit of x under the four commuting maps id , R , I , and $R \circ I$. If $x \notin \mathcal{Q}_p$, these four points are distinct; otherwise, degeneracies occur only on the quarter-turn subgroup. $\square$

Remark 8 (Framed Orbital Classes). *Proposition 8 recovers the orbital classes already identified in the published algebra paper [1]. In the present geometry paper, these same involutions supply the symmetry backbone of the shell regime studied here.*

Remark 9 (Transport Under Reference-Frame Change). *If $g' = g^u$ with $u \in (\mathbb{Z}/(p-1)\mathbb{Z})^\times$, then*

$$g'^{(p-1)/4} = (g^{(p-1)/4})^u.$$

Since u is odd, $g'^{(p-1)/4}$ is again one of the two square roots of -1 in $\mathcal{Q}_p$. The distinguished shell value i_t is not defined by generator choice; it remains fixed once the shell orientation is fixed. Thus, changing the generator only reparametrizes the phase chart in which the quarter-turn data are written.

Remark 10 (Framed-Complex Layer). *Once an extended frame $\mathbb{F}_p(t; 0, 1, e_t)$ is fixed, the shell's oriented quarter-turn i_t gives a framed-complex presentation in the sense of the algebra paper [1]. In particular, expressions of the form*

$$a + bi_t, \quad a, b \in \mathbb{F}_p,$$

may be used internally as finitist framed-complex quantities. Thus, complex-style elements are available inside the theory after a reference frame has been chosen, even though the phase chart in which they are written still depends on the chosen exponential unit e_t .

Proposition 9 (Extended-Frame Euclidean Conjugation). *In the same framed-complex presentation, one has the involution on coordinate expressions*

$$C_{e_t}(a + bi_t) := a - bi_t.$$

This map is an involution of the framed-complex presentation determined by e_t . It is entirely internal to the chosen framed shell presentation and should be read as a static orientation-reversing duality of that presentation.

Proof. Applying C_{e_t} twice gives

$$C_{e_t}(C_{e_t}(a + bi_t)) = C_{e_t}(a - bi_t) = a + bi_t,$$

so $C_{e_t}^2 = \text{id}$. The definition depends on the chosen exponential unit e_t through the framed-complex coordinate system, so it depends on the chosen frame. $\square$

Remark 11 (Hierarchy Boundary). *The map C_{e_t} is a Euclidean orientation-reversing involution of the chosen framed-complex presentation. No hierarchy dynamics is assumed here, and the temporal involution studied elsewhere is distinct from this Euclidean conjugation [1].*

Tables 1 and 2 summarizes the involutive symmetries and the Euclidean datum used in the present formulation. The frame is fixed by $(0, 1, e_t)$; the same datum generates additive meridian action and multiplicative phase action; and the shell then determines π_t , $\mathcal{Q}_p$, and the oriented quarter-turn i_t . Together, Tables 1 and 2 give the symmetry-centered reading used in the remainder of this paper: the frame supplies the primitive datum, and the shell determines the Euclidean values and involutive structure written in that frame.

Table 1. Core involutive symmetry axes of the symmetry-complete shell studied here.

Symmetry Axis	Action	Status
Additive negation	$R(x) = 0 - x$	requires origin 0
Reciprocal inversion	$I(x) = 1/x$ on $\mathbb{F}_p^\times$	requires affine pair $(0, 1)$
Euclidean conjugation	$C_{e_t}(a + bi_t) = a - bi_t$	requires extended frame $(0, 1, e_t)$

Table 2. Primitive frame datum and induced Euclidean values in the symmetry-complete formulation used here.

Continuum-Theory Role	FRC Object	Framed Meaning
t time index [16]	t	radial parameter of the shell
0 as distinguished zero	0	relative origin of the observer
1 as distinguished unit	1	relative scale and additive meridian generator
e as exponential base	e_t	phase scale and multiplicative latitude generator
π as half-period	π_t	1st cardinal phase position; $\pi_t = 2t = (p - 1)/2$
i as oriented quarter-turn	i_t	2nd cardinal phase position; $i_t = e_t^t$ and $i_t^2 = -1$

5. Operational Finite Precision on the External Spherical Comparison

The orbital shell $\mathcal{S}_p$ of the framed finite field $\mathbb{F}_p(t; 0, 1, e_t)$ is the global finite geometry object of FRC. The classical sphere enters here only as an external global comparison reading of the shell coordinates. The finite vertex set $V(\mathcal{S}_p)$ is the zeroth shell grid determined directly by the action-generated shell: the observer origin $N = 0$ together with the shell points ae_t^m for $1 \leq a \leq \pi_t$. In that external spherical comparison, the terminal latitude L_{π_t} collapses to the antipodal image. Its spacing improves as p grows, but arbitrary precision in the present framework does not rely on sending $p \rightarrow \infty$. The stronger mechanism

is the scale-periodic framed-rational refinement already available in every fixed frame $\mathbb{F}_p(t; 0, 1, e_t)$, including the small symmetry-complete shells, as established in [1]. The present section therefore separates the coarse base grid from the fixed-shell refinement grids derived from the same shell.

Definition 6 (Base External Comparison Map). *Fix the framed finite field $\mathbb{F}_p(t; 0, 1, e_t)$. For $0 \leq a \leq \pi_t$ and $m \in \Phi_p$, set*

$$\theta_a := \frac{\pi a}{\pi_t}, \quad \phi_m := \frac{2\pi m}{p-1}.$$

Define

$$\Xi_{p,e_t}(N) = (0, 0, 1),$$

and for $1 \leq a \leq \pi_t$,

$$\Xi_{p,e_t}(v_{a,m}) := (\sin \theta_a \cos \phi_m, \sin \theta_a \sin \phi_m, \cos \theta_a) \in S^2 \subset \mathbb{R}^3.$$

The formula for Ξ_{p,e_t} uses only the shell chart coordinates (a, m) ; its dependence on e_t enters through the meaning of the longitude label m , namely the identification of shell phase with the field element e_t^m . At $a = \pi_t$, the whole terminal latitude L_{π_t} maps to the same external comparison antipodal image because $\theta_{\pi_t} = \pi$.

The map Ξ_{p,e_t} does not endow the shell with a continuum ontology. It records the external spherical comparison used for the finite-precision statements of this paper.

Proposition 10 (Base-Shell Grid Estimate). *Let d_{S^2} be the geodesic distance on the unit sphere. Every point $x \in S^2$ lies within distance*

$$\delta_p := \frac{\sqrt{2}\pi}{p-1}$$

of some point in the finite set $\Xi_{p,e_t}(V(\mathcal{S}_p))$.

Proof. Write x in spherical coordinates (θ, ϕ) with $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Choose $a \in \{0, \dots, \pi_t\}$ and $m \in \Phi_p$ so that

$$|\theta - \theta_a| \leq \frac{\pi}{p-1}, \quad |\phi - \phi_m| \leq \frac{\pi}{p-1}.$$

If $a = 0$, choose the north pole N . If $a = \pi_t$, choose any terminal-latitude vertex $v_{\pi_t,m}$. The same estimate holds because the terminal latitude collapses to a single antipodal image:

$$d_{S^2}(x, \Xi_{p,e_t}(N)) \leq \frac{\pi}{p-1} \quad \text{or} \quad d_{S^2}(x, \Xi_{p,e_t}(v_{\pi_t,m})) \leq \frac{\pi}{p-1},$$

Assume now that $1 \leq a \leq \pi_t - 1$. Consider the linear path

$$\gamma(s) := ((1-s)\theta + s\theta_a, (1-s)\phi + s\phi_m), \quad 0 \leq s \leq 1.$$

Since the sphere metric in these coordinates is

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 \leq d\theta^2 + d\phi^2,$$

the length of γ is bounded by the Euclidean length of the corresponding straight segment in the (θ, ϕ) -plane:

$$\text{length}(\gamma) \leq \sqrt{|\theta - \theta_a|^2 + |\phi - \phi_m|^2} \leq \frac{\sqrt{2}\pi}{p-1}.$$

Therefore,

$$d_{S^2}(x, \Xi_{p,e_t}(v_{a,m})) \leq \delta_p.$$

□

Remark 12 (Role of the Base Grid). *Proposition 10 describes only the coarse vertex grid of the finite orbital shell itself. It is useful as the exact combinatorial starting layer, but it is not the source of arbitrary precision in the present framework.*

Definition 7 (Scale-Periodic Shell Refinement). *Fix the framed finite field $\mathbb{F}_p(\mathfrak{t}; 0, 1, e_t)$. Let $\mathcal{Q}_{e_t}^{\leq N} \subset \mathbb{R}$ denote the truncated framed-rational grid at zoom depth N generated by the primitive frame datum $(0, 1, e_t)$ in the sense of [1]. Define the refined meridian and phase coordinate sets*

$$I_{p,e_t}^{\leq N} := \pi_t(\mathcal{Q}_{e_t}^{\leq N} \cap [0, 1]), \quad \Phi_{p,e_t}^{\leq N} := (p-1)(\mathcal{Q}_{e_t}^{\leq N} \cap [0, 1]).$$

The external spherical comparison grid at depth N is the set

$$\Gamma_{p,e_t}^{\leq N} \subset S^2$$

consisting of the poles $(0, 0, \pm 1)$ together with all points

$$(\sin(\pi u / \pi_t) \cos(2\pi v / (p-1)), \sin(\pi u / \pi_t) \sin(2\pi v / (p-1)), \cos(\pi u / \pi_t))$$

for $u \in I_{p,e_t}^{\leq N} \cap (0, \pi_t)$ and $v \in \Phi_{p,e_t}^{\leq N}$.

The zeroth shell grid comes from the actual vertices of S_p . The refined grids $\Gamma_{p,e_t}^{\leq N}$ are external comparison refinements built over the same additive and multiplicative shell coordinates by means of the framed-rational scale structure already available in the fixed frame.

Proposition 11 (Fixed-Shell Density of the Refined Shell Grids). *Fix a symmetry-complete shell $(\mathfrak{t}, \mathbb{F}_p)$ and a frame $\mathbb{F}_p(\mathfrak{t}; 0, 1, e_t)$. For every $\eta > 0$, there exists $N \in \mathbb{N}$ such that every point $x \in S^2$ lies within distance η of some point in $\Gamma_{p,e_t}^{\leq N}$.*

Proof. The algebra paper proves that for every fixed frame $\mathbb{F}_p(\mathfrak{t}; 0, 1, e_t)$, the framed-rational grids generated by e_t become dense under increasing zoom depth, and this refinement is scale-periodic rather than dependent on changing the underlying field [1]. Hence, for any $\rho > 0$, there exists N such that every number in $[0, 1]$ is within ρ of some point of $\mathcal{Q}_{e_t}^{\leq N} \cap [0, 1]$, and every number in $[0, 1)$ is within ρ of some point of $\mathcal{Q}_{e_t}^{\leq N} \cap [0, 1)$.

Choose N so that this holds with

$$\rho := \frac{\eta}{\sqrt{2}\pi}.$$

Let $x \in S^2$ have spherical coordinates (θ, ϕ) with $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Choose

$$a \in \mathcal{Q}_{e_t}^{\leq N} \cap [0, 1], \quad b \in \mathcal{Q}_{e_t}^{\leq N} \cap [0, 1)$$

such that

$$\left| \frac{\theta}{\pi} - a \right| < \rho, \quad \left| \frac{\phi}{2\pi} - b \right| < \rho.$$

Set

$$u := \pi_{\mathfrak{t}} a \in I_{\mathfrak{p}, e_{\mathfrak{t}}}^{\leq N}, \quad v := (\mathfrak{p} - 1)b \in \Phi_{\mathfrak{p}, e_{\mathfrak{t}}}^{\leq N}.$$

Then,

$$\left| \theta - \frac{\pi u}{\pi_{\mathfrak{t}}} \right| < \frac{\eta}{\sqrt{2}}, \quad \left| \phi - \frac{2\pi v}{\mathfrak{p} - 1} \right| < \frac{\eta}{\sqrt{2}}.$$

Using the same path-length estimate as in the proof of Proposition 10, the corresponding point of $\Gamma_{\mathfrak{p}, e_{\mathfrak{t}}}^{\leq N}$ lies within distance η of x . $\square$

Definition 8 (*L*-Bounded Observable Family). Fix $L \in \mathbb{N}$. An *L*-bounded observable family on r marked points, with $r \leq L$, is a finite set $\mathcal{O}$ of functions

$$O: (S^2)^r \rightarrow \mathbb{R}$$

such that each $O \in \mathcal{O}$ is *L*-Lipschitz with respect to the product metric

$$d^{(r)}((x_1, \dots, x_r), (y_1, \dots, y_r)) := \sum_{k=1}^r d_{S^2}(x_k, y_k).$$

This definition captures bounded finite experiments: a finite number of marked points, a finite list of measurements, and a uniform upper bound on experimental sensitivity.

Theorem 2 (Operational Finite-Precision Theorem for a Fixed Symmetry-Complete Shell). Fix a symmetry-complete shell $(\mathfrak{t}, \mathbb{F}_{\mathfrak{p}})$ and a framed finite field $\mathbb{F}_{\mathfrak{p}}(\mathfrak{t}; 0, 1, e_{\mathfrak{t}})$. Let $L \in \mathbb{N}$, let $r \leq L$, let $\mathcal{O}$ be an *L*-bounded observable family on r marked points, and let $\epsilon > 0$. For every configuration

$$X = (x_1, \dots, x_r) \in (S^2)^r$$

there exists a refinement depth N and points

$$y_1, \dots, y_r \in \Gamma_{\mathfrak{p}, e_{\mathfrak{t}}}^{\leq N}$$

such that

$$|O(X) - O(y_1, \dots, y_r)| < \epsilon$$

for every $O \in \mathcal{O}$.

Proof. Choose $\eta > 0$ such that

$$L^2 \eta < \epsilon.$$

By Proposition 11, there exists N such that, for each k , one can choose $y_k \in \Gamma_{\mathfrak{p}, e_{\mathfrak{t}}}^{\leq N}$ with

$$d_{S^2}(x_k, y_k) \leq \eta.$$

Hence,

$$d^{(r)}(X, (y_1, \dots, y_r)) \leq r\eta \leq L\eta.$$

Each observable $O \in \mathcal{O}$ is *L*-Lipschitz, so

$$|O(X) - O(y_1, \dots, y_r)| \leq L \cdot L\eta < \epsilon.$$

$\square$

6. Finite Fourier Duality in a Chosen Reference Frame

In this paper, Fourier formalism is strictly finitist in the sense of a discrete Fourier transform over a finite ring. The algebraic statements collected here are the basic DFT identities needed on shell phase cycles, written explicitly because later constructions use them in framed form. We work in the framed finite field $\mathbb{F}_p(t; 0, 1, e_t)$. The relevant ring is $\mathbb{F}_p$, the relevant transform length is

$$n := p - 1,$$

and the relevant root of unity is the chosen exponential unit $e_t \in \mathbb{F}_p^\times$. Thus, the shell transform lives entirely on the finite phase cycle carried by each latitude. This places the section inside the finite-field and finite-group Fourier tradition [11–14], now interpreted in the present setting.

Proposition 12 (Root-of-Unity Facts). *Let $n = p - 1$. The chosen exponential unit $e_t \in \mathbb{F}_p^\times$ is a principal n -th root of unity in $\mathbb{F}_p$, i.e.,*

$$e_t^n = 1 \quad \text{and} \quad \sum_{j=0}^{n-1} e_t^{jk} = 0 \quad (1 \leq k < n).$$

Moreover, n is invertible in $\mathbb{F}_p$, with the field equality

$$n^{-1} = -1 \quad \text{in } \mathbb{F}_p.$$

Proof. Because e_t has order $p - 1 = n$, we have $e_t^n = 1$ and $e_t^k \neq 1$ for $1 \leq k < n$. Since $\mathbb{F}_p$ is an integral domain, the geometric-series identity gives

$$(e_t^k - 1) \sum_{j=0}^{n-1} e_t^{jk} = e_t^{nk} - 1 = 0,$$

and therefore the sum is zero for $1 \leq k < n$. Finally, $n = p - 1 \equiv -1 \pmod{p}$, so $n^{-1} = -1$ in $\mathbb{F}_p$. $\square$

Definition 9 (Ring Fourier Transform on the Phase Cycle). *Let $V_{e_t} := \mathbb{F}_p^n$. For $v = (v_0, \dots, v_{n-1}) \in V_{e_t}$, define its ring Fourier transform*

$$\mathcal{F}_{e_t}(v) = (f_0, \dots, f_{n-1}) \in V_{e_t}$$

by

$$f_k := \sum_{j=0}^{n-1} v_j e_t^{jk}, \quad 0 \leq k < n.$$

Proposition 13 (Inverse Formula). *For every $v \in V_{e_t}$,*

$$v_j = - \sum_{k=0}^{n-1} (\mathcal{F}_{e_t}(v))_k e_t^{-jk}, \quad 0 \leq j < n.$$

Equivalently,

$$\mathcal{F}_{e_t}^{-1}(f)_j = n^{-1} \sum_{k=0}^{n-1} f_k e_t^{-jk}.$$

Proof. Let $f = \mathcal{F}_{e_t}(v)$. Then,

$$n^{-1} \sum_{k=0}^{n-1} f_k e_t^{-jk} = n^{-1} \sum_{k=0}^{n-1} \sum_{j'=0}^{n-1} v_{j'} e_t^{j'k} e_t^{-jk} = n^{-1} \sum_{j'=0}^{n-1} v_{j'} \sum_{k=0}^{n-1} e_t^{(j'-j)k}.$$

By Proposition 12, the inner sum is 0 when $j' \neq j$ and n when $j' = j$. Hence, the right-hand side equals v_j . $\square$

Remark 13 (Matrix Formulation). If W_{e_t} denotes the $n \times n$ matrix with entries

$$(W_{e_t})_{k,j} := e_t^{jk},$$

then $\mathcal{F}_{e_t}(v) = W_{e_t}v$ and

$$W_{e_t}^{-1} = n^{-1} (e_t^{-jk})_{j,k} = -(e_t^{-jk})_{j,k}.$$

The same transform also has a direct polynomial interpretation on the finite phase cycle.

Proposition 14 (Polynomial Reading). For $v = (v_0, \dots, v_{n-1}) \in V_{e_t}$, define the polynomial

$$P_v(X) := \sum_{j=0}^{n-1} v_j X^j \in \mathbb{F}_p[X].$$

Then,

$$(\mathcal{F}_{e_t}(v))_k = P_v(e_t^k), \quad 0 \leq k < n.$$

Thus, the ring Fourier transform exchanges coefficient data with values on the finite phase cycle $\{1, e_t, e_t^2, \dots, e_t^{n-1}\} \subset \mathbb{F}_p^\times$.

Proof. This is immediate from the definition

$$P_v(e_t^k) = \sum_{j=0}^{n-1} v_j (e_t^k)^j = \sum_{j=0}^{n-1} v_j e_t^{jk} = (\mathcal{F}_{e_t}(v))_k.$$

$\square$

Every latitude L_j of the shell is a copy of the phase cycle $\Phi_p \cong \mathbb{Z}/n\mathbb{Z}$, so Definition 9 applies verbatim to any shell quantity recorded around a latitude.

Remark 14 (Framed-Complex Coefficient Layer and External Transport). Our earlier paper constructs finitist framed complex numbers over symmetry-complete prime shells [1]. Therefore, the same coefficient vectors and spectra may be viewed inside that framed-complex layer, and framed-complex expressions $a + bi_t$ are already available internally once i_t is fixed. The transform formulas themselves already live over the base ring $\mathbb{F}_p$. The map χ_{p,e_t} introduced below has a narrower role: it is an external transport from that already available finite phase structure to the classical complex n -th roots of unity, used here only for comparison with sampled continuum phase roles.

The remaining structural point is covariance under change of the chosen phase generator.

Proposition 15 (Covariance Under Generator Change). Let $e_t' = e_t^u$ with $u \in (\mathbb{Z}/n\mathbb{Z})^\times$. For a shell phase quantity $q: \mathbb{F}_p^\times \rightarrow \mathbb{F}_p$, let

$$v_j^{(e_t)} := q(e_t^j), \quad v_j^{(e_t')} := q(e_t'^j).$$

Then,

$$v_j^{(e'_t)} = v_{uj}^{(e_t)}$$

and the corresponding spectra agree:

$$\mathcal{F}_{e'_t}(v^{(e'_t)}) = \mathcal{F}_{e_t}(v^{(e_t)}).$$

Proof. The coefficient relation is immediate from $e_t'^j = e_t^{uj}$. Then,

$$(\mathcal{F}_{e'_t}(v^{(e'_t)}))_k = \sum_{j=0}^{n-1} v_j^{(e'_t)} e_t'^{jk} = \sum_{j=0}^{n-1} v_{uj}^{(e_t)} e_t^{ujk}.$$

Since u is invertible modulo n , the substitution $m = uj$ permutes the index set $\{0, \dots, n-1\}$. Hence,

$$(\mathcal{F}_{e'_t}(v^{(e'_t)}))_k = \sum_{m=0}^{n-1} v_m^{(e_t)} e_t^{mk} = (\mathcal{F}_{e_t}(v^{(e_t)}))_k.$$

□

Proposition 16 (Exact External Transport of the Framed Datum). *Fix the framed finite field $\mathbb{F}_p(t; 0, 1, e_t)$ and set $n = p - 1$. Define*

$$\chi_{p,e_t}: \mathbb{F}_p^\times \rightarrow \mathbb{C}^\times, \quad \chi_{p,e_t}(e_t^m) := \exp\left(\frac{2\pi i m}{n}\right), \quad m \in \mathbb{Z}/n\mathbb{Z}.$$

Then, χ_{p,e_t} is a cyclic group isomorphism from $\mathbb{F}_p^\times$ onto the complex n -th roots of unity, and for every $m \in \mathbb{Z}/n\mathbb{Z}$,

$$\begin{aligned} \chi_{p,e_t}(e_t^{m+1}) &= e^{2\pi i/n} \chi_{p,e_t}(e_t^m), \\ \chi_{p,e_t}(e_t^{m+\pi_t}) &= -\chi_{p,e_t}(e_t^m), \quad \chi_{p,e_t}(i_t e_t^m) = i \chi_{p,e_t}(e_t^m). \end{aligned}$$

Moreover, for all $j, r \in \mathbb{Z}/n\mathbb{Z}$,

$$\chi_{p,e_t}(e_t^{jr}) = e^{2\pi i jr/n}.$$

Proof. Because e_t generates the cyclic group $\mathbb{F}_p^\times$ of order n , the map χ_{p,e_t} is the unique group isomorphism sending e_t to $e^{2\pi i/n}$. The first identity is then immediate. Since $\pi_t = n/2$, we obtain

$$\chi_{p,e_t}(e_t^{m+\pi_t}) = \exp\left(\frac{2\pi i(m+n/2)}{n}\right) = -\exp\left(\frac{2\pi i m}{n}\right).$$

Likewise, $i_t = e_t^\dagger$ and $n = 4t$, so

$$\chi_{p,e_t}(i_t e_t^m) = \chi_{p,e_t}(e_t^{m+t}) = \exp\left(\frac{2\pi i(m+t)}{n}\right) = i \exp\left(\frac{2\pi i m}{n}\right).$$

The kernel identity is the definition of χ_{p,e_t} applied to e_t^{jr} . □

Remark 15 (Status of the Fourier identities). *This section records the finite Fourier facts used in this paper. They are entirely ring-internal: The signal space, the transform kernel, the inverse transform, and the covariance statement all live inside the finite ring $\mathbb{F}_p$. What matters here is that the longitude cycles of the shell studied here carry this harmonic structure natively, with the chosen unit e_t providing the phase chart and with the external degenerate continuum comparison introduced afterward.*

Remark 16 (Shell DFT is Not Just a Generic Cyclic-Group Transform). *A generic cyclic group of order n supports a discrete Fourier transform once a generator has been chosen. The symmetry-complete shell studied here carries more than that. Its phase cycle is already integrated with the shell half-period π_t , the quarter-turn structure $\mathcal{Q}_p = \{\pm 1, \pm i_t\}$, the framed involution package, the orbital shell $\mathcal{S}_p$, and the frame-transport laws developed in the preceding sections. Thus, the point of the present Fourier section is not the existence of a transform on a cyclic set, but the fact that the same symmetry-complete shell supports geometry, involutions, constants, and harmonic analysis in one coordinated arithmetic object. The extra structure used here consists of the exact compatibilities between the transform and the shell half-period, quarter-turn subgroup, involution package, orbital shell geometry, generator-change covariance, and external transport laws established in the surrounding propositions.*

Remark 17 (Preview of a Larger Representational Reading). *The present section uses only the local shell transform needed in this paper: At fixed meridian step a , the phase-position coordinate m is replaced by its dual spectral coordinate k . In a larger shell interpretation not developed here, the working chart is (t, a, m, k) , and Fourier transform is read not merely as a local latitude transform but as a representational quarter-turn between primal and dual shell domains. In that reading, one Fourier step exchanges the geometric roles of a and m , double Fourier produces the minus structure, and four steps close the cycle. The shell quarter-turn datum i_t is then expected to preserve its role across the whole representational cycle. These stronger claims are deferred to subsequent work.*

Remark 18 (External Degenerate Large- p Reading). *Fix the framed finite field $\mathbb{F}_p(t; 0, 1, e_t)$ and set $n = p - 1$. Externally, the phase index $m \in \Phi_p$ may be read by an observer as the angle*

$$\phi_m := \frac{2\pi m}{n}, \quad 0 \leq m < n,$$

so the phase cycle becomes a uniform n -point sampling grid on S^1 . In that external comparison reading, the ring DFT of Definition 9 is the finite shell form for which its degenerate large- p case is the classical n -point complex DFT on sampled trigonometric data [3], although the internal coefficient ring and normalization remain those of $\mathbb{F}_p$. Consequently, for any fixed trigonometric polynomial,

$$F(\phi) = \sum_{|k| \leq N} c_k e^{ik\phi},$$

external exact recovery by the classical sample DFT holds whenever $n > 2N$. More generally, for periodic analytic $F: S^1 \rightarrow \mathbb{C}$, the external trapezoidal or sample coefficient approximations on the grid ϕ_m converge rapidly to the classical Fourier coefficients as $p \rightarrow \infty$ along primes $p \equiv 1 \pmod{4}$ [19–21]. These are external sampling facts about the large- p phase grid: they describe the degenerate continuum case of the shell formalism, not internal theorems of the shell itself.

7. Discussion

This article occupies a precise place both within the FRC programme and within the broader literature on symmetry in finite mathematical structures. Specifically, it gives the programme a coherent symmetry-complete shell layer carrying the framed Euclidean datum:

- (i) A finite orbital shell $\mathcal{S}_p$ together with a canonical spherical completion;
- (ii) An action-generated symmetry space in which additive meridian motion and multiplicative phase motion are carried by the same frame datum;
- (iii) A finite structural account of the framed Euclidean datum $(0, 1, e_t, i_t, \pi_t)$;
- (iv) A symmetry backbone realized through additively framed negation, affinely framed reciprocal inversion, and extended-frame Euclidean conjugation;

- (v) A strict finitist meaning of “geometry to arbitrary precision”;
- (vi) Finite Fourier duality on the shell.

Euclidean Datum in Finite Field. The most distinctive contribution of this paper is the way the Euclidean datum is organized. Continuum mathematics usually treats π , i , and e as basic ingredients belonging to different chapters of the theory: geometry, complex structure, and exponential phase. Here, the frame is fixed first by $(0, 1, e_t)$, and then, the same shell determines the half-period π_t and the quarter-turn i_t . Section 6 sharpens that point by showing exact external transport from (e_t, π_t, i_t) to the sampled classical roles of a one-phase step, a half-period, a quarter-turn, and a Fourier kernel. The result is not a list of disconnected finite analogues but one framed shell datum carrying the same roles inside a single shell.

Emergent Symmetry-Space. The shell is not introduced here as a neutral grid on which symmetry is later imposed. It is already a symmetry space: The shell points are the stable action positions produced by additive meridian advance and multiplicative phase advance in the chosen frame. In that sense, the orbital shell is the finite representation of the internal symmetries available in the shell arithmetic itself, while the spherical completion records the associated global spherical quotient of its terminal boundary.

Scope of the Euclidean Result. The approximation theorem concerns fixed bounded experiments with finitely many marked points and finitely many Lipschitz observables. Its key mechanism is the scale-periodic framed-rational refinement available inside each fixed shell frame, while the finite vertex set of S_p serves only as the coarse combinatorial grid layer. This is true already for the small symmetry-complete shells; increasing p is not what creates arbitrary precision. The proof deliberately uses classical analysis only at the level of the external spherical comparison, because the theorem is about that comparison reading of the shell rather than internal continuum reconstruction.

Boundary of the Approximation Claim. No claim of Gromov-Hausdorff convergence of the shell family is being made here. The point established in this paper is narrower: The global shell S_p , together with fixed-shell framed-rational refinement and the comparison map Ξ_{p,e_t} , supports bounded finite-precision approximation on the external spherical comparison object. The observer-local spatial chart in which familiar local metric or differential geometry would be developed is not constructed here and is deferred to future work.

Potential Applications. The present article is theoretical, but the structures it isolates are operationally relevant. The orbital shell provides a finitist model for finite-resolution spherical sampling; the phase-cycle transform provides a native shell Fourier tool for periodic shell data; and the framed symmetry package gives a controlled language for reversible finite geometry and phase processing. Within the FRC programme, this symmetry-complete shell layer also serves as the prerequisite for the quadratic-extension causality analysis developed in [16]. These are not yet empirical applications in the narrow sense, but they do indicate where the formalism can carry practical computational or modeling load: bounded geometric approximation, shell-based signal analysis, finite comparison tasks, and staged extension to algebraic spacetime models.

8. Conclusions

In the symmetry-complete shell regime studied here, realized by the finite field $\mathbb{F}_p(t; 0, 1, e_t)$ with $p = 4t + 1$, the shell supports an action-generated orbital shell with canonical spherical completion combinatorially equivalent to the two-sphere, a coherent family of involutive symmetries, a finite structural account of the framed Euclidean datum, and a ring-internal Fourier duality. More fundamentally, the shell is presented as a

symmetry space generated by additive meridian motion and multiplicative phase motion from the same frame datum, with the observer origin internal to the shell and the terminal latitude collapsing to the antipode only in the associated global spherical quotient. The labels depend on the chosen frame, while the shell type and its spherical completion are the same up to isomorphism. The framed Euclidean datum (e_t, π_t, i_t) also transports exactly to the sampled classical roles of one phase step and exponential kernel, a half-period, and a quarter-turn, respectively.

Equally important, the approximation theorem is a comparison-level statement about bounded experiments and prescribed tolerances and not an internal smooth-manifold claim, and this is proven by combining the coarse shell geometry with the scale-periodic framed-rational refinement already available in each fixed frame, including the small symmetry-complete shells. Likewise, the Fourier section develops the ring-internal transform identities naturally carried by the shell phase cycle. In that form, the result gives the FRC programme a technically coherent account of the shell geometry, its framed Euclidean datum, and harmonic duality.

This paper does not prove that the classical continuum is impossible or inconsistent. It does show, however, that in the shell setting treated here, the continuum is not operationally forced: a strictly finite shell already carries the bounded geometric and harmonic load under study, supplies a finite structural origin for the framed Euclidean datum used in the theory, and admits the conventional continuum formalism as an external degenerate comparison case. This is the foundational point established here. The broader displacement thesis remains programme-level, but the present shell analysis provides one explicit finite-first layer on which that broader thesis would have to build.

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Notations

t	Fundamental radius/chronon parameter in the symmetry-complete regime studied here, $p = 4t + 1$. In this paper, it is used in its radial shell role; its broader temporal reading belongs to the later causality layer.
$\mathbb{F}_p$	Symmetry-complete field of radius t in the regime studied here, meaning a prime field with $p = 4t + 1$. When the full frame is explicit, we write $\mathbb{F}_p(t; 0, 1, e_t)$.
S_p	Orbital shell of the symmetry-complete field, generated by additive meridian action and multiplicative phase action in a chosen frame; its terminal latitude has a canonical spherical completion obtained by collapsing it to the antipode.
I_p	Meridian-step interval $\{0, 1, \dots, \pi_t\}$, recording additive shell depth along a fixed meridian from the observer origin to the terminal latitude.
Φ_p	Phase cycle $\mathbb{Z}/(p-1)\mathbb{Z}$, indexing the $p-1$ distinct meridians or longitude labels in the chosen frame.
$\mathcal{Q}_p$	Quarter-turn subgroup of $\mathbb{F}_p^\times$, namely the unique subgroup of order 4; in a chosen frame, this is $\{1, i_t, -1, -i_t\}$.
e_t	Exponential unit of the chosen frame.
π_t	Induced half-period in that frame, defined by $\pi_t = (p-1)/2 = 2t$.

i_t Induced oriented quarter-turn in that frame, with $i_t = e_t^\dagger$ and $i_t^2 = -1$.

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