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A Pair of Hermitian and Anti-Hermitian Solutions of a Generalized Sylvester Matrix Equation over Commutative Quaternion Algebra

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Abstract

In this paper, we investigate a pair of Hermitian and anti-Hermitian solutions of a generalized Sylvester matrix equation $AXB + C\overline{X}D + EYF = G$ over commutative quaternion algebra by using a complex representation of commutative quaternion matrices, the Kronecker product, vec-operation, and Moore–Penrose-generalized inverse. We establish the necessary and sufficient conditions for the existence of solutions. Moreover, we derive explicit expressions when they are solvable. We also provide two numerical examples to illustrate the main results.

Keywords: matrix equation; Hermitian; anti-Hermitian; least square solution

1. Introduction

In 1843, William Rowan Hamilton discovered real quaternion algebra $\mathbb{H}$, which marked a pivotal advancement in hypercomplex algebras. $\mathbb{H}$ is a four-dimensional noncommutative associative algebra over the real field $\mathbb{R}$, which is defined in [1] by the set

$$\mathbb{H} = \left\{ a = a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = -1, ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Its utility extends beyond pure mathematics into diverse domains, such as the statistical analysis of quaternion-valued stochastic processes [2], color image representations exploiting rotational symmetries [3], and feature extraction in facial recognition systems [4]. The noncommutative multiplication inherent in $\mathbb{H}$ imposes significant analytical constraints and computational complexities. Seeking commutative alternatives, in 1892, Corrado Segre [5] proposed the commutative quaternion, which shares the same vector space structure

$$b = b_0 + b_1 i + b_2 j + b_3 k$$

over $\mathbb{R}$, and its multiplicative structure is defined by $i^2 = k^2 = -1, j^2 = 1, ij = ji = k$, $jk = kj = i, ki = ik = -j, ijk = -1$, and $b_0, b_1, b_2, b_3 \in \mathbb{R}$. The set of commutative quaternions $\mathbb{H}_c$ is four-dimensional commutative Clifford algebra and forms a ring, which contains non-trivial idempotents, zero divisors, and nilpotent elements. These structural properties underpin significant applications, including the stable state modeling in Hopfield neural networks, hypercomplex signal processing frameworks, and multi-channel image

Check for updates

Academic Editor: Calogero Vetro

Received: 8 March 2026
Revised: 1 April 2026
Accepted: 2 April 2026
Published: 6 April 2026

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transformations (see [6–10]). Commutative quaternion algebra remains an active domain of theoretical investigation and applied research. Several related properties were investigated in the context of commutative quaternion matrices (see [11]).

Matrix equations occur frequently throughout the mathematical sciences, arising in both theoretical and applied contexts, from stability theory to the design of control systems. In particular, Hermitian and anti-Hermitian solutions to such matrix equations over real, complex, quaternion, split quaternion, commutative quaternion and dual quaternion algebra—by using the tools of matrix decomposition, generalized inverse, Kronecker product, iterative algorithms, real representation, complex representation, etc.—have been an active area of research in matrix algebra. For instance, the classic matrix equation

$$AXB = C \tag{1}$$

is essential in system control, which is extensively studied (see [12–14]). Specifically, Chen et al. [15] derived the conditions for solvability and presented the explicit general solution to (1) over dual quaternion algebra, and provided the expression of $\phi-$Hermitian solutions when they exist. The Sylvester matrix equation $AX - XB = C$ arises in diverse fields, including singular system control [16], perturbation theory [17], sensitivity analysis [18], and control theory [19]. In [20], Yuan et al. investigated Hermitian solutions to the split quaternion matrix equation $AXB + CXD = E$ by using the complex representation of quaternion matrices. In [21], Yu et al. considered Hermitian solutions to the generalized quaternion matrix equation $AXB + CX * D = E$ by the real representation of quaternion matrices. In [22], Kösal et al. deduced explicit analytical solutions for the commutative quaternion matrix equations $X - A\overline{\widetilde{X}}B = C$, $X - A\overline{\widetilde{X}}B = C$, and $X - A\widetilde{\overline{X}}B = C$ by real representation of matrices, which are conventionally termed Kalman–Yakubovich-conjugate matrix equations. Building upon this work, Kösal et al. [23] derived a general formulation for solving the matrix equation $AX = B$ over the commutative quaternion ring. The review of the well-known generalized Sylvester matrix equation $AX - YB = C$ can be found in [24]. Other forms of generalized Sylvester matrix equations, such as $AXB + CYD = E$, have also been extensively studied (see [25,26]). For systems of matrix equations over quaternion algebra, explicit formulas and determinant representations of η-skew-Hermitian solutions were presented in [27]. In addition to direct solution methods, iterative algorithms and optimization-based approaches have been widely developed for large-scale matrix equations, with applications ranging from image reconstruction to inverse problems.

While the above-mentioned works cover a wide range of matrix equations over various quaternion algebras, the problem of Hermitian and anti-Hermitian solutions to the generalized Sylvester matrix equation

$$AXB + C\overline{X}D + EYF = G \tag{2}$$

over the commutative quaternion algebra has not yet been studied. The main difficulty arises from the algebraic structure of commutative quaternions, which contains zero divisors and idempotents, preserving of Hermitian and anti-Hermitian constraints. At the same time, the commutativity of multiplication simplifies computation but leads to challenges in algebraic transformations of structure matrices.

Motivated by the continued interest in Hermitian and anti-Hermitian solutions and the broad applications of matrix equations over commutative quaternion algebras, we aim to study the solvability conditions for a pair of Hermitian and anti-Hermitian solutions to the matrix Equation (2) over commutative quaternion algebra, where X and Y are unknown Hermitian and anti-Hermitian matrices, respectively.

https://doi.org/10.3390/sym18040619

The paper is organized as follows. In Section 2, we review the relevant properties and show the structure of the associated operator $\mathrm{vec}(AXB)$, $\mathrm{vec}(C\overline{X}D)$, and $\mathrm{vec}(EYF)$, where X and Y are unknown Hermitian and anti-Hermitian matrices, respectively. In Section 3, we establish the necessary and sufficient conditions for the existence of a pair of Hermitian and anti-Hermitian solutions to (2) and provide its solution set. Algorithms and numerical examples are presented in Section 4.

Throughout, let $\mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}$, $\mathbb{SR}^{n\times n}$, $\mathbb{ASR}^{n\times n}$, $\mathbb{H}_c$, and $\mathbb{H}_c^{m\times n}$ denote the set of all complex $m\times n$ matrices, real $m\times n$ matrices, $n\times n$ real symmetric matrices, $n\times n$ real anti-symmetric matrices, the commutative quaternions, and $m\times n$ commutative quaternion matrices. $\mathbb{HH}_c^{n\times n}$ and $\mathbb{AHH}_c^{n\times n}$ are the set of all $n\times n$ Hermitian and anti-Hermitian matrices over $\mathbb{H}_c$, respectively. The symbols I, $\overline{A}$, A^T, A^*, $A^\dagger$, $\mathrm{rank}(A)$, and $\|A\|$, respectively, represent an identity matrix with the appropriate size, the conjugate, the transpose, the conjugate transpose, the Moore–Penrose-generalized inverse, the rank, and the Frobenius norm of A. For $A\in\mathbb{C}^{m\times n}$, $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ denote the real part and the imaginary part of A.

2. Preliminaries

This section presents preliminary concepts, encompassing the complex representation and properties of commutative quaternion matrices, along with the structure of relevant operators.

For a matrix $A\in\mathbb{H}_c^{m\times n}$, it can be uniquely written as $A=A_0+A_1i+A_2j+A_3k$, where A_0, A_1, A_2, and $A_3\in\mathbb{R}^{m\times n}$. Moreover, it can also be uniquely formulated as $A=C_1+C_2j$, where $C_1=A_0+A_1i$, $C_2=A_2+A_3i\in\mathbb{C}^{m\times n}$.

Definition 1 ([11]). *For a matrix $A=A_1+A_2j\in\mathbb{H}_c^{m\times n}$ with $A_1,A_2\in\mathbb{C}^{m\times n}$, the complex representation of A is $\mathcal{F}(A)=\begin{pmatrix}A_1 & A_2\\ A_2 & A_1\end{pmatrix}\in\mathbb{C}^{2m\times 2n}$.*

Clearly, $\mathcal{F}(A)$ is uniquely determined by A. Furthermore, the complex representation of commutative quaternion matrices has the following properties.

Proposition 1 ([11]). *If $A,B\in\mathbb{H}_c^{n\times n}$, then*

(1) $A=B$ *if and only if* $\mathcal{F}(A)=\mathcal{F}(B)$;
(2) $\mathcal{F}(A+B)=\mathcal{F}(A)+\mathcal{F}(B)$;
(3) $\mathcal{F}(AB)=\mathcal{F}(A)\mathcal{F}(B)$;
(4) $\mathcal{F}(I_n)=I_{2n}$.

For matrices $A=(a_{ij})\in\mathbb{H}_c^{m\times n}$ and $B=(b_{ij})\in\mathbb{H}_c^{s\times t}$, the Kronecker product of A and B is defined as $A\otimes B=(a_{ij}B)\in\mathbb{H}_c^{ms\times nt}$. For commutative quaternion matrices A,B,C,D,E with compatible dimensions and a real scalar λ, the following properties hold:

$$(\lambda A)\otimes B=A\otimes(\lambda B)=\lambda(A\otimes B),$$

$$(A,B,C)\otimes D=(A\otimes D,B\otimes D,C\otimes D),$$

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}\otimes E=\begin{pmatrix}A\otimes E & B\otimes E\\ C\otimes E & D\otimes E\end{pmatrix}.$$

The vec-operator of a matrix $A=(a_{ij})_{m\times n}=(a_1,a_2,\ldots,a_n)$ is defined by $\mathrm{vec}(A)=(a_1,a_2,\ldots,a_n)^T$, where $a_j=(a_{1j},a_{2j},\ldots,a_{mj})$ for $j=1,2,\ldots,n$. For matrices A,B,C,D,X with compatible sizes, we have

$$\mathrm{vec}(A+C)=\mathrm{vec}(A)+\mathrm{vec}(C),$$

$$\operatorname{vec}(AXB) = (B^T \otimes A)\operatorname{vec}(X),$$

$$\operatorname{vec}(A,\ D) = \begin{pmatrix} \operatorname{vec}(A) \\ \operatorname{vec}(D) \end{pmatrix}.$$

To establish the Hermitian and anti-Hermitian solutions of matrix Equation (1) over the commutative quaternion algebra, we now recall essential definitions and properties. For a matrix $A = A_1 + A_2 j \in \mathbb{H}_c^{m\times n}$ with $A_1, A_2 \in \mathbb{C}^{m\times n}$, A can be equivalently represented as

$$A \cong \Phi_A = (A_1, A_2),$$

where the symbol $\cong$ denotes an equivalence relation.

For $B = (B_{ij}) \in \mathbb{C}^{m\times n}$, the Frobenius norm of B is

$$\|B\| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |b_{ij}|^2}, \quad |b_{ij}|^2 = (\operatorname{Re} b_{ij})^2 + (\operatorname{Im} b_{ij})^2.$$

The Frobenius norm for commutative quaternion matrix $A = A_1 + A_2 j \in \mathbb{H}^{m\times n}$ with $A_1, A_2 \in \mathbb{C}^{m\times n}$ is

$$\|A\| = \sqrt{\|\operatorname{Re} A_1\|^2 + \|\operatorname{Im} A_1\|^2 + \|\operatorname{Re} A_2\|^2 + \|\operatorname{Im} A_2\|^2},$$

where $\hat{A} = (\operatorname{Re}(A_1), \operatorname{Im}(A_1), \operatorname{Re}(A_2), \operatorname{Im}(A_2)$. Then, we get $\|\Phi_A\| = \|\hat{A}\| = \|\operatorname{vec}_{\Phi_A}\|$.

Proposition 2 ([28]). *Let $\lambda \in \mathbb{R}$, $A, B \in \mathbb{H}_c^{m\times n}$, and $C \in \mathbb{H}_c^{n\times s}$. Then,*

(1) $A = B$ *if and only if* $\Phi_A = \Phi_B$;
(2) $\Phi_{A+B} = \Phi_A + \Phi_B, \quad \Phi_{\lambda A} = \lambda \Phi_A$;
(3) $\Phi_{AC} = \Phi_A \mathcal{F}(C)$.

Definition 2 ([26]). *For the matrix $A = (a_{ij}) \in \mathbb{H}_c^{n\times n}$, define $a_1 = (a_{11}, \sqrt{2}a_{21}, \ldots, \sqrt{2}a_{n1})$, $a_2 = (a_{22}, \sqrt{2}a_{32}, \ldots, \sqrt{2}a_{n2}), \ldots, a_{n-1} = (a_{(n-1)(n-1)}, \sqrt{2}a_{n(n-1)})$, and $a_n = a_{nn}$, and denote the following vector by $vec_S(A)$,*

$$\operatorname{vec}_S(A) = (a_1, a_2, \ldots, a_{n-1}, a_n)^T \in \mathbb{H}_c^{\frac{n(n+1)}{2}}.$$

Definition 3 ([26]). *For the matrix $B = (b_{ij}) \in \mathbb{H}_c^{n\times n}$, define $b_1 = (b_{21}, b_{31}, \ldots, b_{n1})$, $b_2 = (b_{32}, b_{42}, \ldots, b_{n2}), \ldots, b_{n-2} = (b_{(n-1)(n-2)}, b_{n(n-2)})$, $b_{n-1} = b_{n(n-1)}$, and denote the following vector by $\operatorname{vec}_A(B)$,*

$$\operatorname{vec}_A(B) = \sqrt{2}(b_1, b_2, \ldots, b_{n-2}, b_{n-1})^T \in \mathbb{H}_c^{\frac{n(n-1)}{2}}.$$

Proposition 3 ([29]). *Suppose that $X \in \mathbb{R}^{n\times n}$ and e_i is the ith column of the identity matrix I_n; then,*

(1)

$$X \in \mathbb{SR}^{n\times n} \Leftrightarrow \operatorname{vec}(X) = K_S \operatorname{vec}_S(X),$$

where the matrix $K_S \in \mathbb{R}^{n^2\times(n(n+1)/2)}$ takes the following form:

$$K_S = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}e_1 & e_2 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & e_1 & \cdots & 0 & 0 & \sqrt{2}e_2 & e_3 & \cdots & e_n & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & e_2 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e_1 & 0 & 0 & 0 & \cdots & 0 & \cdots & \sqrt{2}e_{n-1} & e_n & 0 \\ 0 & 0 & \cdots & 0 & e_1 & 0 & 0 & \cdots & e_2 & \cdots & 0 & e_{n-1} & \sqrt{2}e_n \end{pmatrix};$$

(2)

$$X \in \mathbb{ASR}^{n\times n} \Leftrightarrow \mathrm{vec}(X) = K_A \,\mathrm{vec}_A(X),$$

where $K_A \in \mathbb{R}^{n^2\times(n(n-1)/2)}$ *takes the following form:*

$$K_A = \frac{1}{\sqrt{2}} \begin{pmatrix} e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ -e_1 & 0 & \cdots & 0 & 0 & e_3 & \cdots & e_{n-1} & e_n & \cdots & 0 \\ 0 & -e_1 & \cdots & 0 & 0 & -e_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -e_1 & 0 & 0 & \cdots & -e_2 & 0 & \cdots & e_n \\ 0 & 0 & \cdots & 0 & -e_1 & 0 & \cdots & 0 & -e2 & \cdots & -e_{n-1} \end{pmatrix}.$$

(3) The matrices K_S *and* K_A *satisfy* $K_S^T K_S = I_{\frac{n(n+1)}{2}}$, $K_A^T K_A = I_{\frac{n(n-1)}{2}}$.

Let $X = X_1 + X_2 j \in \mathbb{HH}_c^{n\times n}$, where $X_1, X_2 \in \mathbb{C}^{n\times n}$. Then, it follows that

$$X \in \mathbb{HH}_c^{n\times n} \Leftrightarrow \begin{cases} \mathrm{Re}(X_1)^T = \mathrm{Re}(X_1), \mathrm{Im}(X_1)^T = -\mathrm{Im}(X_1) \\ \mathrm{Re}(X_2)^T = -\mathrm{Re}(X_2), \mathrm{Im}(X_2)^T = -\mathrm{Im}(X_2) \end{cases}. \tag{3}$$

That is, for $X = X_1 + X_2 j \in \mathbb{HH}_c^{n\times n}$, $\mathrm{Re}(X_1)$ is real symmetric, and $\mathrm{Im}(X_1)$, $\mathrm{Re}(X_2)$, and $\mathrm{Im}(X_2)$ are real anti-symmetric matrices. If $Y = Y_1 + Y_2 j \in \mathbb{AHH}_c^{n\times n}$ with $Y_1, Y_2 \in \mathbb{C}^{n\times n}$, then

$$Y \in \mathbb{AHH}_c^{n\times n} \Leftrightarrow \begin{cases} \mathrm{Re}(Y_1)^T = -\mathrm{Re}(Y_1), \mathrm{Im}(Y_1)^T = \mathrm{Im}(X_1) \\ \mathrm{Re}(Y_2)^T = \mathrm{Re}(Y_2), \mathrm{Im}(Y_2)^T = \mathrm{Im}(X_2) \end{cases}, \tag{4}$$

i.e., $\mathrm{Re}(Y_1)$ is real anti-symmetric, and $\mathrm{Im}(Y_1)$, $\mathrm{Re}(Y_2)$, and $\mathrm{Im}(Y_2)$ are real symmetric matrices. By Proposition 3, we have the following:

Theorem 1. (1) *Assume that* $X = X_1 + X_2 j \in \mathbb{HH}_c^{n\times n}$, *where* $X_1, X_2 \in \mathbb{C}^{n\times n}$. *Then,*

$$\begin{pmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{pmatrix} = M \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix}, \tag{5}$$

$$\begin{pmatrix} \mathrm{vec}(\overline{X_1}) \\ \mathrm{vec}(-X_2) \end{pmatrix} = N \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix}, \tag{6}$$

where

$$M = \begin{pmatrix} K_S & iK_A & 0 & 0 \\ 0 & 0 & K_A & iK_A \end{pmatrix}, N = \begin{pmatrix} K_S & -iK_A & 0 & 0 \\ 0 & 0 & -K_A & -iK_A \end{pmatrix}. \tag{7}$$

(2) *Assume that* $Y = Y_1 + Y_2 j \in \mathbb{AHH}_c^{n\times n}$, *where* $Y_1, Y_2 \in \mathbb{C}^{n\times n}$. *Then,*

$$\begin{pmatrix} \mathrm{vec}(Y_1) \\ \mathrm{vec}(Y_2) \end{pmatrix} = P \begin{pmatrix} \mathrm{vec}_A(\mathrm{Re}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Re}(Y_2)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_2)) \end{pmatrix}, \tag{8}$$

where

$$P = \begin{pmatrix} K_A & iK_S & 0 & 0 \\ 0 & 0 & K_S & iK_S \end{pmatrix}. \tag{9}$$

Proof. (1) For the proof of equality (5), Theorem 2 can be seen in [26]. Now we provide the proof of equality (6). According to Proposition 3 and the equality (3),

$$\begin{aligned} \begin{pmatrix} \mathrm{vec}(\overline{X_1}) \\ \mathrm{vec}(-X_2) \end{pmatrix} &= \begin{pmatrix} \mathrm{vec}(\mathrm{Re}(X_1) - \mathrm{Im}(X_1)i) \\ \mathrm{vec}(-\mathrm{Re}(X_2) - \mathrm{Im}(X_2)i) \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{vec}(\mathrm{Re}(X_1)) - i\,\mathrm{vec}(\mathrm{Im}(X_1)) \\ -\mathrm{vec}(\mathrm{Re}(X_2)) - i\,\mathrm{vec}(\mathrm{Im}(X_2)) \end{pmatrix} \\ &= \begin{pmatrix} K_S\,\mathrm{vec}_S(\mathrm{Re}(X_1)) - iK_A\,\mathrm{vec}_A(\mathrm{Im}(X_1)) \\ -K_A\,\mathrm{vec}_A(\mathrm{Re}(X_2)) - iK_A\,\mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix} \\ &= N \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix}, \end{aligned}$$

i.e., equality (6) holds.

(2) For $Y = Y_1 + Y_2 j \in \mathbb{AHH}_c^{n\times n}$, by Proposition 3 and the equality (4),

$$\begin{aligned} \begin{pmatrix} \mathrm{vec}(Y_1) \\ \mathrm{vec}(Y_2) \end{pmatrix} &= \begin{pmatrix} \mathrm{vec}(\mathrm{Re}(Y_1)) + \mathrm{vec}(\mathrm{Im}(Y_1))i \\ \mathrm{vec}(\mathrm{Re}(Y_2)) + \mathrm{vec}(\mathrm{Im}(Y_2))i \end{pmatrix} \\ &= \begin{pmatrix} K_A\,\mathrm{vec}_A(\mathrm{Re}(Y_1)) + iK_S\,\mathrm{vec}_S(\mathrm{Im}(Y_1)) \\ K_S\,\mathrm{vec}_S(\mathrm{Re}(Y_2)) + iK_S\,\mathrm{vec}_S(\mathrm{Im}(Y_2)) \end{pmatrix} \\ &= P \begin{pmatrix} \mathrm{vec}_A(\mathrm{Re}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Re}(Y_2)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_2)) \end{pmatrix}, \end{aligned}$$

i.e., the equality (8) holds. □

Theorem 2. *Suppose that* $A = A_1 + A_2 j \in \mathbb{H}_c^{m\times n}$, $B = B_1 + B_2 j \in \mathbb{H}_c^{n\times s}$, $C = C_1 + C_2 j \in \mathbb{H}_c^{m\times n}$, $D = D_1 + D_2 j \in \mathbb{H}_c^{n\times s}$, $E = E_1 + E_2 j \in \mathbb{H}_c^{m\times n}$, $F = F_1 + F_2 j \in \mathbb{H}_c^{n\times s}$, $X = X_1 + X_2 j \in \mathbb{HH}_c^{n\times n}$, $Y = Y_1 + Y_2 j \in \mathbb{AHH}_c^{n\times n}$, *where* $A_i, C_i, E_i \in \mathbb{C}^{m\times n}$, $B_i, D_i, F_i \in \mathbb{C}^{n\times s}$, $X_i, Y_i \in \mathbb{C}^{n\times s}$, $i = 1, 2$. *Then,*

$$\mathrm{vec}(\Phi_{AXB}) = \mathcal{F}\Big[(B_1^T \otimes A_1 + B_2^T \otimes A_2) + (B_2^T \otimes A_1 + B_1^T \otimes A_2)j\Big] M \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix}, \tag{10}$$

$$\mathrm{vec}(\Phi_{C\overline{X}D}) = \mathcal{F}\Big[(D_1^T \otimes C_1 + D_2^T \otimes C_2) + (D_2^T \otimes C_1 + D_1^T \otimes C_2)j\Big] N \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix}, \tag{11}$$

$$\mathrm{vec}(\Phi_{EYF}) = \mathcal{F}\Big[(F_1^T \otimes E_1 + F_2^T \otimes E_2) + (F_2^T \otimes E_1 + F_1^T \otimes E_2)j\Big] P \begin{pmatrix} \mathrm{vec}_A(\mathrm{Re}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Re}(Y_2)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_2)) \end{pmatrix}. \tag{12}$$

Proof. Compared with Theorem 2.2 in [30], by Propositions 1 and 2, and Theorem 3, we have

$$\begin{aligned}
\mathrm{vec}(\Phi_{AXB}) &= \mathrm{vec}(\Phi_A \mathcal{F}(X)\mathcal{F}(B)) \\
&= \mathrm{vec}\left(\begin{pmatrix} A_1 & A_2 \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_2 & X_1 \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix}\right) \\
&= \mathrm{vec}(A_1X_1B_1 + A_2X_2B_1 + A_1X_2B_2 + A_2X_1B_2, A_1X_1B_2 + A_2X_2B_2 + A_1X_2B_1 + A_2X_1B_1) \\
&= \begin{pmatrix} (B_1^T \otimes A_1)\,\mathrm{vec}(X_1) + (B_1^T \otimes A_2)\,\mathrm{vec}(X_2) + (B_2^T \otimes A_1)\,\mathrm{vec}(X_2) + (B_2^T \otimes A_2)\,\mathrm{vec}(X_1) \\ (B_2^T \otimes A_1)\,\mathrm{vec}(X_1) + (B_2^T \otimes A_2)\,\mathrm{vec}(X_2) + (B_1^T \otimes A_1)\,\mathrm{vec}(X_2) + (B_1^T \otimes A_2)\,\mathrm{vec}(X_1) \end{pmatrix} \\
&= \begin{pmatrix} B_1^T \otimes A_1 & B_2^T \otimes A_1 \\ B_2^T \otimes A_1 & B_1^T \otimes A_1 \end{pmatrix}\begin{pmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{pmatrix} + \begin{pmatrix} B_2^T \otimes A_2 & B_1^T \otimes A_2) \\ B_1^T \otimes A_2 & B_2^T \otimes A_2 \end{pmatrix}\begin{pmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{pmatrix} \\
&= \Big(\mathcal{F}(B)^T \otimes A_1, \mathcal{F}(Bj)^T \otimes A_2\Big)\begin{pmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \\ \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{pmatrix} \\
&= \Big(\mathcal{F}(B)^T \otimes A_1, \mathcal{F}(Bj)^T \otimes A_2\Big)\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{pmatrix} \\
&= \mathcal{F}\Big[(B_1^T \otimes A_1 + B_2^T \otimes A_2) + (B_2^T \otimes A_1 + B_1^T \otimes A_2)j\Big]\begin{pmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{pmatrix} \\
&= \mathcal{F}\Big[(B_1^T \otimes A_1 + B_2^T \otimes A_2) + (B_2^T \otimes A_1 + B_1^T \otimes A_2)j\Big] M \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix},
\end{aligned}$$

i.e., equality (10) holds. By using a similar derivation process as (10) and Theorem 3, we get

$$\begin{aligned}
\mathrm{vec}(\Phi_{C\overline{X}D}) &= \mathcal{F}\Big[(D_1^T \otimes C_1 + D_2^T \otimes C_2) + (D_2^T \otimes C_1 + D_1^T \otimes C_2)j\Big]\begin{pmatrix} \mathrm{vec}(\overline{X_1}) \\ \mathrm{vec}(-X_2) \end{pmatrix} \\
&= \mathcal{F}\Big[(D_1^T \otimes C_1 + D_2^T \otimes C_2) + (D_2^T \otimes C_1 + D_1^T \otimes C_2)j\Big] N \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\mathrm{vec}(\Phi_{EYF}) &= \mathcal{F}\Big[(F_1^T \otimes E_1 + F_2^T \otimes E_2) + (F_2^T \otimes E_1 + F_1^T \otimes E_2)j\Big]\begin{pmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{pmatrix} \\
&= \mathcal{F}\Big[(F_1^T \otimes E_1 + F_2^T \otimes E_2) + (F_2^T \otimes E_1 + F_1^T \otimes E_2)j\Big] P \begin{pmatrix} \mathrm{vec}_A(\mathrm{Re}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Re}(Y_2)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_2)) \end{pmatrix}.
\end{aligned}$$

□

Note that the results of $\mathrm{vec}(\Phi_{AXB})$, $\mathrm{vec}(\Phi_{C\overline{X}D})$, and $\mathrm{vec}(\Phi_{EYF})$ are very important for calculating the solutions of commutative quaternion matrix Equation (2).

Lemma 1 ([31]). *The matrix equation $Ax = b$ with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ has a solution $x \in \mathbb{R}^n$ if and only if*

$$AA^{\dagger}b = b.$$

In this case, it has a general solution that can be written as

$$x = A^{\dagger}b + (I_n - A^{\dagger}A)y,$$

where $y \in \mathbb{R}^n$ is an arbitrary vector. Moreover, $Ax = b$ has a unique solution

$$x = A^{\dagger}b$$

if $\mathrm{rank}(A) = n$.

3. A Pair of Hermitian and Anti-Hermitian Solutions to the Matrix Equation (2)

This section focuses on obtaining a pair of Hermitian and anti-Hermitian solutions to commutative quaternion matrix Equation (2). We outline the main results of the paper as follows.

Theorem 3. *Let $A = A_1 + A_2 j \in \mathbb{H}_c^{m\times n}$, $B = B_1 + B_2 j \in \mathbb{H}_c^{n\times s}$, $C = C_1 + C_2 j \in \mathbb{H}_c^{m\times n}$, $D = D_1 + D_2 j \in \mathbb{H}_c^{n\times s}$, $E = E_1 + E_2 j \in \mathbb{H}_c^{m\times n}$, $F = F_1 + F_2 j \in \mathbb{H}_c^{n\times s}$ be given, where $A_i, C_i, E_i \in \mathbb{C}^{m\times n}$, and $B_i, D_i, F_i \in \mathbb{C}^{n\times s}$. $X = X_1 + X_2 j \in \mathbb{HH}_c^{n\times n}$ and $Y = Y_1 + Y_2 j \in \mathbb{AHH}_c^{n\times n}$ are unknown, where $X_i, Y_i \in \mathbb{C}^{n\times n}$, $i = 1, 2$. Matrices M, N, P are defined as (7) and (9). Assume $W = R + S$; thus,*

$$R = \mathcal{F}\Big[(B_1^T \otimes A_1 + B_2^T \otimes A_2) + (B_2^T \otimes A_1 + B_1^T \otimes A_2)j\Big]M,$$

$$S = \mathcal{F}\Big[(D_1^T \otimes C_1 + D_2^T \otimes C_2) + (D_2^T \otimes C_1 + D_1^T \otimes C_2)j\Big]N,$$

$$T = \mathcal{F}\Big[F_1^T \otimes E_1 + F_2^T \otimes E_2) + (F_2^T \otimes E_1 + F_1^T \otimes E_2)j\Big]P,$$

$$\mathcal{T}_1 = \begin{pmatrix}\mathrm{Re}(W) & \mathrm{Re}(T)\end{pmatrix},\ \mathcal{T}_2 = \begin{pmatrix}\mathrm{Im}(W) & \mathrm{Im}(T)\end{pmatrix},\ \omega = \begin{pmatrix}\mathrm{Re}(\mathrm{vec}(\Phi_G))\\ \mathrm{Im}(\mathrm{vec}(\Phi_G))\end{pmatrix},$$

$$\mathcal{K} = \begin{pmatrix}K_S & 0 & 0 & 0\\ 0 & K_A & 0 & 0\\ 0 & 0 & K_A & 0\\ 0 & 0 & 0 & K_A\end{pmatrix},\ \mathcal{L} = \begin{pmatrix}K_A & 0 & 0 & 0\\ 0 & K_S & 0 & 0\\ 0 & 0 & K_S & 0\\ 0 & 0 & 0 & K_S\end{pmatrix},\ \mathcal{M} = \begin{pmatrix}\mathcal{K} & 0\\ 0 & \mathcal{L}\end{pmatrix},$$

$$\mathrm{vec}(\vec{X}) = \begin{pmatrix}\mathrm{vec}_S(\mathrm{Re}(X_1))\\ \mathrm{vec}_A(\mathrm{Im}(X_1))\\ \mathrm{vec}_A(\mathrm{Re}(X_2))\\ \mathrm{vec}_A(\mathrm{Im}(X_2))\end{pmatrix},\quad \mathrm{vec}(\vec{Y}) = \begin{pmatrix}\mathrm{vec}_A(\mathrm{Re}(Y_1))\\ \mathrm{vec}_S(\mathrm{Im}(Y_1))\\ \mathrm{vec}_S(\mathrm{Re}(Y_2))\\ \mathrm{vec}_S(\mathrm{Im}(Y_2))\end{pmatrix},$$

$$\mathrm{vec}(\hat{X}) = \begin{pmatrix}\mathrm{vec}(\mathrm{Re}(X_1))\\ \mathrm{vec}(\mathrm{Im}(X_1))\\ \mathrm{vec}(\mathrm{Re}(X_2))\\ \mathrm{vec}(\mathrm{Im}(X_2))\end{pmatrix},\quad \mathrm{vec}(\hat{Y}) = \begin{pmatrix}\mathrm{vec}(\mathrm{Re}(Y_1))\\ \mathrm{vec}(\mathrm{Im}(Y_1))\\ \mathrm{vec}(\mathrm{Re}(Y_2))\\ \mathrm{vec}(\mathrm{Im}(Y_2))\end{pmatrix}.$$

Then, the matrix Equation (2) has a pair of Hermitian and anti-Hermitan solutions $\begin{pmatrix} X, & Y \end{pmatrix}$ *if and only if*

$$\begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix} \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix}^{\dagger} \omega = \omega. \tag{13}$$

In this case, the set of solutions is expressed as

$$\Gamma = \left\{ \begin{pmatrix} X, & Y \end{pmatrix} \middle| \begin{pmatrix} \mathrm{vec}(\hat{X}) \\ \mathrm{vec}(\hat{Y}) \end{pmatrix} = \mathcal{M} \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix}^{\dagger} \omega + \mathcal{M} \left(I_{4n^2} - \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix}^{\dagger} \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix} \right) \mathrm{y} \right\}, \tag{14}$$

where y *is an arbitrary vector with an appropriate size. Furthermore, commutative quaternion matrix Equation (2) has a unique pair of Hermitian and anti-Hermitian solutions* $\begin{pmatrix} X & Y \end{pmatrix} \in \Gamma$ *if and only if*

$$\mathrm{rank} \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix} = 4n^2, \tag{15}$$

and the corresponding solution set can be written as

$$\Gamma = \left\{ \begin{pmatrix} X, & Y \end{pmatrix} \middle| \begin{pmatrix} \mathrm{vec}(\hat{X}) \\ \mathrm{vec}(\hat{Y}) \end{pmatrix} = \mathcal{M} \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix}^{\dagger} \omega \right\}. \tag{16}$$

Proof. According to Proposition 2 and Theorem 2, it follows that

$$\begin{aligned}
& AXB + C\overline{X}D + EYF = G \\
\Leftrightarrow\ & \Phi_{AXB+C\overline{X}D+EYF} = \Phi_G \\
\Leftrightarrow\ & \Phi_{AXB} + \Phi_{C\overline{X}D} + \Phi_{EYF} = \Phi_G \\
\Leftrightarrow\ & \mathrm{vec}(\Phi_{AXB}) + \mathrm{vec}(\Phi_{C\overline{X}D}) + \mathrm{vec}(\Phi_{EYF}) = \mathrm{vec}(\Phi_G) \\
\Leftrightarrow\ & R \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix} + S \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix} + T \begin{pmatrix} \mathrm{vec}_A(\mathrm{Re}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Re}(Y_2)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_2)) \end{pmatrix} = \mathrm{vec}(\Phi_G) \\
\Leftrightarrow\ & W \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix} + T \begin{pmatrix} \mathrm{vec}_A(\mathrm{Re}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Re}(Y_2)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_2)) \end{pmatrix} = \mathrm{vec}(\Phi_G) \\
\Leftrightarrow\ & (\mathrm{Re}(W) + i\,\mathrm{Im}(W)) \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix} + (\mathrm{Re}(T) + i\,\mathrm{Im}(T)) \begin{pmatrix} \mathrm{vec}_A(\mathrm{Re}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Re}(Y_2)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_2)) \end{pmatrix} \\
& \quad = \mathrm{Re}(\mathrm{vec}(\Phi_G)) + i\,\mathrm{Im}(\mathrm{vec}(\Phi_G)) \\
\Leftrightarrow\ & \begin{pmatrix} \mathrm{Re}(W) & \mathrm{Re}(T) \\ \mathrm{Im}(W) & \mathrm{Im}(T) \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\vec{X}) \\ \mathrm{vec}(\vec{Y}) \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(\mathrm{vec}(\Phi_G)) \\ \mathrm{Im}(\mathrm{vec}(\Phi_G)) \end{pmatrix} \\
\Leftrightarrow\ & \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\vec{X}) \\ \mathrm{vec}(\vec{Y}) \end{pmatrix} = \omega.
\end{aligned}$$

Based on the results derived above, matrix Equation (2) has a pair of Hermitian and anti-Hermitian solutions if and only if the matrix equation

$$\begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \end{pmatrix} = \omega \tag{17}$$

is solvable. By Lemma 1, (17) has a solution if and only if (13) holds, and the solution can be represented as

$$\begin{pmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \end{pmatrix} = \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix}^{\dagger} \omega + \left(I_{4n^2} - \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix}^{\dagger} \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix} \right) \mathrm{y},$$

where y is an arbitrary vector with an appropriate size. Additionally,

$$\mathrm{vec}(\hat{X}) = \begin{pmatrix} \mathrm{vec}(\mathrm{Re}(X_1)) \\ \mathrm{vec}(\mathrm{Im}(X_1)) \\ \mathrm{vec}(\mathrm{Re}(X_2)) \\ \mathrm{vec}(\mathrm{Im}(X_2)) \end{pmatrix} = \begin{pmatrix} K_S & 0 & 0 & 0 \\ 0 & K_A & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_A \end{pmatrix} \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix} = \mathcal{K}\,\mathrm{vec}(\overrightarrow{X}),$$

$$\mathrm{vec}(\hat{Y}) = \begin{pmatrix} \mathrm{vec}(\mathrm{Re}(Y_1)) \\ \mathrm{vec}(\mathrm{Im}(Y_1)) \\ \mathrm{vec}(\mathrm{Re}(Y_2)) \\ \mathrm{vec}(\mathrm{Im}(Y_2)) \end{pmatrix} = \begin{pmatrix} K_A & 0 & 0 & 0 \\ 0 & K_S & 0 & 0 \\ 0 & 0 & K_S & 0 \\ 0 & 0 & 0 & K_S \end{pmatrix} \begin{pmatrix} \mathrm{vec}_A(\mathrm{Re}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_1)) \\ \mathrm{vec}_S(\mathrm{Re}(Y_2)) \\ \mathrm{vec}_S(\mathrm{Im}(Y_2)) \end{pmatrix} = \mathcal{L}\,\mathrm{vec}(\overrightarrow{Y}).$$

Then,

$$\begin{pmatrix} \mathrm{vec}(\hat{X}) \\ \mathrm{vec}(\hat{Y}) \end{pmatrix} = \begin{pmatrix} \mathcal{K} & 0 \\ 0 & \mathcal{L} \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \end{pmatrix} = \mathcal{M} \begin{pmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \end{pmatrix} = \mathcal{M} \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix}^{\dagger} \omega + \mathcal{M} \left(I_{4n^2} - \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix}^{\dagger} \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix} \right) \mathrm{y}.$$

This implies that (14) holds. Furthermore, if (13) holds, then commutative matrix Equation (2) has a unique pair of Hermitian and anti-Hermitian solutions if and only if

$$\begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix}^{\dagger} \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix} = I_{4n^2},$$

i.e., (15) holds, and the expression of the solution can be formulated as (16). □

In order to obtain the structure of a pair of Hermitian and anti-Hermitian solutions of matrix Equation (2), we consider the Moore–Penrose-generalized inverse of the column block matrix $\begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix}^{\dagger}$. According to the result of the Moore–Penrose-generalized inverse of the block matrix in [32], we suppose

$$\begin{cases} q = 2sm, \\ H = (I_{4n^2} - \mathcal{T}_1^{\dagger}\mathcal{T}_1)\mathcal{T}_2^T, \\ U = \left[I_q + (I_q - H^{\dagger}H)\mathcal{T}_2\mathcal{T}_1^{\dagger}\mathcal{T}_1^{\dagger T}\mathcal{T}_2^T(I_q - H^{\dagger}H) \right]^{-1}, \\ J = H^{\dagger} + (I_q - H^{\dagger}H)U\mathcal{T}_2\mathcal{T}_1^{\dagger}\mathcal{T}_1^{\dagger T}(I_{4n^2} - \mathcal{T}_2^T H^{\dagger}), \\ R_{11} = I_q - \mathcal{T}_1\mathcal{T}_1^{\dagger} + \mathcal{T}_1^{\dagger T}\mathcal{T}_2^T U(I_q - H^{\dagger}H)\mathcal{T}_2\mathcal{T}_1^{\dagger}, \\ R_{12} = -\mathcal{T}_1^{\dagger T}\mathcal{T}_2^T(I_q - H^{\dagger}H)U, \\ R_{22} = (I_q - H^{\dagger}H)U. \end{cases} \tag{18}$$

We get

$$\begin{pmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{pmatrix}^{\dagger} = \begin{pmatrix}\mathcal{T}_1^{\dagger} - J^T\mathcal{T}_2\mathcal{T}_1^{\dagger}, & J^T\end{pmatrix},$$
$$\begin{pmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{pmatrix}^{\dagger}\begin{pmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{pmatrix} = \mathcal{T}_1^{\dagger}\mathcal{T}_1 + HH^{\dagger},$$
$$I_{2q} - \begin{pmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{pmatrix}\begin{pmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{pmatrix}^{\dagger} = \begin{pmatrix}R_{11} & R_{12}\\ R_{12}^T & R_{22}\end{pmatrix}. \tag{19}$$

Corollary 1. *Commutative quaternion matrix Equation (2) with $X \in \mathbb{HH}_c^{n\times n}$, $Y \in \mathbb{AHH}_c^{n\times n}$ exists if and only if*

$$\begin{pmatrix}R_{11} & R_{12}\\ R_{12}^T & R_{22}\end{pmatrix}\omega = 0, \tag{20}$$

where R_{11}, R_{12}, R_{22} are from (19) and (18). In this case, the set of pairs of Hermitian and anti-Hermitian solutions is as follows:

$$\Gamma = \left\{\begin{pmatrix}X, & Y\end{pmatrix}\middle|\begin{pmatrix}\mathrm{vec}(\hat{X})\\ \mathrm{vec}(\hat{Y})\end{pmatrix} = \mathcal{M}\begin{pmatrix}\mathcal{T}_1^{\dagger} - J^T\mathcal{T}_2\mathcal{T}_1^{\dagger}, & J^T\end{pmatrix}\omega + \mathcal{M}\left(I_{4n^2} - \mathcal{T}_1^{\dagger}\mathcal{T}_1 - HH^{\dagger}\right)\mathrm{y}\right\}, \tag{21}$$

in which y *is an arbitrary vector with an appropriate size. Moreover, commutative quaternion matrix Equation (2) has a unique pair of Hermitian and anti-Hermitian solutions $\begin{pmatrix}X, & Y\end{pmatrix} \in \Gamma$ if and only if (15) holds, and the corresponding solution set can be written as*

$$\Gamma = \left\{\begin{pmatrix}X, & Y\end{pmatrix}\middle|\begin{pmatrix}\mathrm{vec}(\hat{X})\\ \mathrm{vec}(\hat{Y})\end{pmatrix} = \mathcal{M}\begin{pmatrix}\mathcal{T}_1^{\dagger} - J^T\mathcal{T}_2\mathcal{T}_1^{\dagger}, & J^T\end{pmatrix}\omega\right\} \tag{22}$$

Remark 1. *We can use the results of Theorem 3 and Corollary 1 to derive the necessary and sufficient conditions and the expressions of solutions to the following classical four generalized Sylvester matrix equations (i)–(iv) and two matrix equations (v), (vi) over commutative quaternion algebra.*

(i) $AXB + C\overline{X}D = G$, *with X being Hermitian;*
(ii) $AX + \overline{X}D = G$, *with X being Hermitian;*
(iii) $AXB + EYF = G$, *with (X, Y), in which X is Hermitian and Y is anti-Hermitian;*
(iv) $AX + YF = G$, *with X being Hermitian and Y being anti-Hermitian;*
(v) $AXB = G$, *with X being Hermitian;*
(vi) $EYF = G$, *with Y being anti-Hermitian.*

4. Numerical Exemplification

In this section, we provide two numerical algorithms and three numerical examples for finding a pair of Hermitian and anti-Hermitian solutions of commutative quaternion matrix Equation (2).

Next, we outline some numerical examples. To make sure that commutative quaternion matrix Equation (2) is consistent, we suppose $G = A\widetilde{X}B + C\overline{\widetilde{X}}D + E\widetilde{Y}F$ with given matrices $A \in \mathbb{H}_c^{m\times n}$, $B \in \mathbb{H}_c^{n\times s}$, $C \in \mathbb{H}_c^{m\times n}$, $D \in \mathbb{H}_c^{n\times s}$, $E \in \mathbb{H}_c^{m\times n}$, $F \in \mathbb{H}_c^{n\times s}$, $\widetilde{X} \in \mathbb{HH}_c^{n\times n}$, and $\widetilde{Y} \in \mathbb{AHH}_c^{n\times n}$. The norms

$$n_1 = \left\|\begin{pmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{pmatrix}\begin{pmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{pmatrix}^{+}\omega - \omega\right\|, \; n_2 = \left\|\begin{pmatrix}R_{11} & R_{12}\\ R_{12}^T & R_{22}\end{pmatrix}\omega\right\|, \; n_3 = \left\|I_{2q} - \begin{pmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{pmatrix}\begin{pmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{pmatrix}^{\dagger} - \begin{pmatrix}R_{11} & R_{12}\\ R_{12}^T & R_{22}\end{pmatrix}\right\|$$

are small. Example 1 shows that when $\operatorname{rank}\begin{pmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{pmatrix} = 4n^2$, commutative quaternion matrix Equation (2) has a unique pair of Hermitian and anti-Hermitian solutions with $X \in \mathbb{HH}_c^{n\times n}$ and $Y \in \mathbb{AHH}_c^{n\times n}$. Example 2 shows the case when $\operatorname{rank}\begin{pmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{pmatrix} < 4n^2$ and shows the unique minimizer of the optimization problem. The application of the obtained results to Hopfield networks is presented in Example 3.

Example 1. *Let* $m = 3, n = 2, s = 2$, *and*

$$A = \begin{pmatrix} i+j & 1-2k \\ 2-i & 0.5+i+k \\ j & 1-i-j \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ i & j \\ 1-2k & -i \end{pmatrix}, E = \begin{pmatrix} 1+i & j \\ k & 0 \\ 0 & 2 \end{pmatrix},$$

$$B = \begin{pmatrix} 1+i & 1-j & 0 \\ 0.5k & 0.75j & i \end{pmatrix}, D = \begin{pmatrix} 1 & 0 & -1 \\ i & j & k \end{pmatrix}, F = \begin{pmatrix} 1 & j & k \\ i & -1 & 0 \end{pmatrix}.$$

Take

$$\widetilde{X} = \begin{pmatrix} 1 & 1+j+k \\ 1-j-k & 2 \end{pmatrix}, \ \widetilde{Y} = \begin{pmatrix} 0 & 1+2i-j \\ -1+2i-j & 0 \end{pmatrix}.$$

Obviously, $\widetilde{X} \in \mathbb{HH}_c^{n\times n}$ *and* $\widetilde{Y} \in \mathbb{AHH}_c^{n\times n}$. *Let*

$$\Phi_A \mathcal{F}(\widetilde{X})\mathcal{F}(B) + \Phi_C \mathcal{F}(\overline{\widetilde{X}})\mathcal{F}(D) + \Phi_E \mathcal{F}(\widetilde{Y})\mathcal{F}(F) = \Phi_G.$$

Using Matlab R2017a and Algorithms 1 and 2, we get

$$\operatorname{rank}\begin{pmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{pmatrix} = 16 = 4n^2, \ \ n_2 = 3.9658 \times 10^{-14}.$$

According to Algorithm 2, commutative quaternion matrix Equation (2) has a unique pair of Hermitian and anti-Hermitian solutions $(X, Y) \in \Gamma$, $n_1 = 2.5230 \times 10^{-14}$, $n_3 = 2.4907 \times 10^{-14}$, *and* $\left\|\Phi_{(X,Y)} - \Phi_{(\widetilde{X},\widetilde{Y})}\right\| = 8.7285 \times 10^{-15}$.

Algorithm 1: For the matrix equation (2)

1. Input the matrices $A \in \mathbb{H}_c^{m\times n}$, $B \in \mathbb{H}_c^{n\times s}$, $C \in \mathbb{H}_c^{m\times n}$, $D \in \mathbb{H}_c^{n\times s}$, $E \in \mathbb{H}_c^{m\times n}$, $F \in \mathbb{H}_c^{n\times s}$;
2. Compute $\mathcal{T}_1$, $\mathcal{T}_2$, and ω;
3. If both (13) and (15) hold, then calculate $(X, Y) \in \Gamma$ by (16);
4. If (13) holds, then calculate $(X, Y) \in \Gamma$ with (14); otherwise, stop.
5. Output the solution (X, Y), $X \in \mathbb{HH}_c^{n\times n}$, $Y \in \mathbb{AHH}_c^{n\times n}$.

Algorithm 2: For the matrix equation (2)

1. Input the matrices $A \in \mathbb{H}_c^{m\times n}$, $B \in \mathbb{H}_c^{n\times s}$, $C \in \mathbb{H}_c^{m\times n}$, $D \in \mathbb{H}_c^{n\times s}$, $E \in \mathbb{H}_c^{m\times n}$, $F \in \mathbb{H}_c^{n\times s}$;
2. Compute $\mathcal{T}_1$, $\mathcal{T}_2$, H, U, J, R_{11}, R_{12}, R_{22} and ω;
3. If (20) and (15) hold, then calculate $(X, Y) \in \Gamma$ by (22);
4. If (20) holds, then calculate $(X, Y) \in \Gamma$ by (21); otherwise stop.
5. Output the solution (X, Y), $X \in \mathbb{HH}_c^{n\times n}$, $Y \in \mathbb{AHH}_c^{n\times n}$.

Example 2. *Let* $m = 3, n = 3, s = 2$, *and*

$$A = \begin{pmatrix} i & 1 & j \\ k & j & 1 \\ 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & k \\ j & 0 & i \\ 0 & 1 & 0 \end{pmatrix}, E = \begin{pmatrix} 1 & 0 & i \\ i & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} j & k \\ 1 & i \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} k & -j \\ 0 & 0 \\ 1 & i \end{pmatrix}, F = \begin{pmatrix} i & -1 \\ k & -j \\ 1 & i \end{pmatrix}.$$

Taking

$$\widetilde{X} = \begin{pmatrix} 1 & 1-j & i \\ 1+j & 0 & k \\ -i & -k & 2 \end{pmatrix}, \ \widetilde{Y} = \begin{pmatrix} 0 & j & 1+k \\ j & 0 & i \\ -1+k & i & 0 \end{pmatrix}.$$

The matrices $\widetilde{X} \in \mathbb{HH}_c^{n\times n}$ *and* $\widetilde{Y} \in \mathbb{AHH}_c^{n\times n}$. *Using Matlab and Algorithm 2, we get*

$$\operatorname{rank}\begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix} = 12 < 4n^2 = 36, \ \ n_2 = 6.2430 \times 10^{-15}.$$

Commutative quaternion matrix Equation (2) has infinite pairs of Hermitian and anti-Hermitian solutions $(X, Y) \in \Gamma$, $n_1 = 8.8008 \times 10^{-15}$, $n_3 = 5.3643 \times 10^{-15}$. *Furthermore, the optimization problem*

$$\min_{(X,Y)\in\Gamma} (\left\|\Phi_{(X,Y)}\right\|)$$

has a unique minimizer $(\check{X}\,\check{Y})$ *with the expression*

$$\begin{pmatrix} \operatorname{vec}(\mathring{X}) \\ \operatorname{vec}(\mathring{Y}) \end{pmatrix} = \mathcal{M}\left(\mathcal{T}_1^{\dagger} - J^T\mathcal{T}_2\mathcal{T}_1^{\dagger}, \quad J^T\right)\omega;$$

in detail,

$$\check{X} = \begin{pmatrix} 0.8652 & 0.1429 - 0.3241i - 0.2119j + 0.1429k & 0.1078 + 0.6357i - 0.7071j - 0.3613k \\ 0.1429 + 0.3241i + 0.2119j - 0.1429k & 0.2812 & -0.1214 + 0.3590i - 0.0750j + 0.0929k \\ 0.1078 - 0.6357i + 0.7071j + 0.3613k & -0.1214 - 0.3590i + 0.0750j - 0.0929k & 0.4326 \end{pmatrix},$$

$$\check{Y} = \begin{pmatrix} 0.0000 - 0.1857i - 0.2619j + 0.3661k & -0.1310 + 0.1831i + 0.8562j - 0.0929k & 0.4000 - 0.2876i - 0.0328j + 0.0714k \\ 0.1310 + 0.1831i + 0.8562j - 0.0929k & 0.0000 - 0.0000i + 0.0000j + 0.0000k & -0.1503 + 0.2024i - 0.3071j - 1.1438k \\ -0.4000 - 0.2876i - 0.0328j + 0.0714k & 0.1503 + 0.2024i - 0.3071j - 1.1438k & 0.0000 - 0.6143i - 0.4048j + 0.3006k \end{pmatrix}.$$

In addition, $\left\|\Phi_{(\check{X},\check{Y})} - \Phi_{(\widetilde{X},\widetilde{Y})}\right\| = 4.0956.$

Example 3. *A Hopfield network is a recurrent neural network whose stability is characterized by an energy function. The state update equation is given by*

$$\mu(t+1) = \operatorname{sgn}(W\mu(t) + b),$$

where W denotes the weight matrix, b the bias vector, and sgn *the sign function. Convergence to stable states requires the weight matrix to be Hermitian, i.e.,* $W = W^*$ *(see [33]). When the network is extended to commutative quaternion algebra* $\mathbb{H}_c$, *the state vector* $\mu(t)$ *and the weight matrix W take values in* $\mathbb{H}_c$. *This extension allows us to representmulti-dimensional data, such as color images, where each pixel with RGB can be encoded as a commutative quaternion* $p = R + Gi + Bj$. *The Hermitian condition* $W = W^*$ *remains the stability criterion. In practical scenarios, dissipative effects or external perturbations may be present, which can be modeled by an anti-Hermitian component* $Y = -Y^*$. *When dissipative effects are present, the weight matrix admits a decomposition into Hermitian and anti-Hermitian parts, i.e.,* $W = X + Y$, *where X is*

Hermitian and Y is anti-Hermitian. The Hermitian part X ensures the stability of the network, while the anti-Hermitian part Y accounts for non-conservative effects. When the network is subject to additional linear constraints arising from design specifications, the determination of X and Y corresponds to solving a generalized Sylvester matrix equation of the form $AXB + CYD = E$, with known coefficient matrices A, B, C, D, E determined by the design requirements.

Consider the coefficient matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, E = \begin{pmatrix} 2 & 1+i+j \\ 1-i+j & 3 \end{pmatrix}.$$

Applying Algorithm 1, the matrix equation $AXB + CYD = E$ has a unique pair of solutions with

$$X = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}.$$

It is easy to verify that X is Hermitian and Y is anti-Hermitian.

5. Conclusions

In this paper, we established the necessary and sufficient conditions for the existence of a pair of Hermitian and anti-Hermitian solutions to commutative quaternion matrix Equation (2). Additionally, we provided an explicit expression of the solution set when it is consistent. Numerical algorithms and examples were given to illustrate the results of the paper. In future work, we will study (anti-)$\eta-$Hermitian solutions to matrix equations and develop iterative methods for large-scale matrix equations over relevant quaternion algebras.

Author Contributions: Conceptualization, methodology, validation, writing—original draft, H.C. and L.X.; software, L.L.; funding acquisition, H.C. and L.L.; discussion, writing—finaldraft, H.C., L.X. and L.L. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the key scientific research projects of universities in Anhui province (Grant No. 2023AH050476).

Data Availability Statement: All data is contained within the article.

Acknowledgments: The authors are very much indebted to the anonymous referees for their constructive and valuable comments and suggestions, which greatly improved this paper.

Conflicts of Interest: The authors declare no conflicts of interest.

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