

Article

On Solving Certain Functional Equations of Dynamic Programming in Relational-Metric Space Through a Non-Unique Fixed-Point Theorem

Doaa Filali ¹, Esmail Alshaban ², Adel Alatawi ², Bassam Z. Albalawi ², Fahad M. Alamrani ^{2,*}
and Faizan Ahmad Khan ^{2,*}

¹ Department of Mathematical Science, College of Sciences, Princess Nourah Bint Abdulrahman University, Riyadh 11671, Saudi Arabia; dkfilali@pnu.edu.sa

² Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia; ealshaban@ut.edu.sa (E.A.); amalatawi@ut.edu.sa (A.A.); b-albalawi@ut.edu.sa (B.Z.A.)

* Correspondence: fm.alomrani@ut.edu.sa (F.M.A.); fkhan@ut.edu.sa (F.A.K.)

Abstract

This article aims to describe the solvability of certain functional equations related to dynamic programming by fixed-point theorems in relational-metric spaces. To achieve our goal, we exhibit a non-unique fixed-point finding for a map of the Ćirić type in a relational-metric space. In this way, we consolidate and amend innumerable celebrated results in fixed-point theory. A couple of instances are supplied to emphasize the practical value of our findings.

Keywords: dynamic programming; symmetric relation; fixed points

MSC: 39B52; 47H10; 54H25

1. Introduction

One of most prominent advantages of fixed-point theory is addressing the issue of existence and uniqueness of solutions for differential equations, integral equations, or matrix equations. On the other hand, there are several nonlinear equations that possess more than one solution. Evidently, just the availability of a fixed point is within evaluation in the metric fixed-point theory, ignoring the uniqueness of a fixed point. In this regard, in 1974, Ćirić [1] proposed criteria for several operators with non-unique fixed points and delivered a reputable outcome. Influenced by this pioneering study, several authors have outlined non-unique fixed points for operators that meet various requirements, see, e.g., Achari [2,3], Pachpatte [4], Mishra [5], Achari [6,7], Dhage [8], Liu and Wang [9], Ćirić [10], Ćirić and Jotić [11], Liu et al. [12], Karapınar and Romaguera [13], Karapınar and Agarwal [14], Alqahtani et al. [15], Karapınar [16,17], Hussain [18] and references therein.

In the decade preceding, validation of fixed-point outcomes in relational MS have evolved into an attractive subject for research of metric fixed-point theory. The fact that comprehensive contraction conditions in these insights merely pertain to elements connected through a BR is a crucial aspect of relational-metric fixed-point theorems. Therefore, in comparison with typical functional contractions, relational functional contractions are relatively weaker. Additionally, these restrictions allow these results to be utilized in situations where standard fixed-point theorems are not feasible, such as typical integral equations, matrix equations, and boundary value problems fulfilling specific extra hypotheses. In 2015, Alam



Academic Editor: Włodzimierz Fechner

Received: 31 January 2026

Revised: 4 March 2026

Accepted: 8 March 2026

Published: 12 March 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

and Imdad [19] found a variation of *BCP* in the setting of relational *MS*. Consequently, a variety of conclusions are developed in this regard. For references to some of them, we turn to Alam and Imdad [20,21], Khan [22], Alam et al. [23], Hasanuzzaman et al. [24], Arif and Imdad, [25], Aljawi and Uddin [26], Wasey et al. [27], Alamrani et al. [28], and Alshaban et al. [29], alongside additional resources.

The paper aims to establish a non-unique fixed-point finding of Ćirić-type mapping in the setup of *MS* endowed with a *BR*. To exhibit utility of our outcomes, we provided two exemplary instances. We deal with our findings with a discussion of bounded solutions of certain types of functional equations that are encountered in dynamic programming.

2. Preliminaries

This section covers some pertinent concepts and fundamental findings that are incorporated in our subsequent text. We say that:

Definition 1 ([30]). Given any set $V \neq \emptyset$, a subset R of V^2 is a *BR* on V .

Trivial examples of *BR* on a set V are V^2 (called universal relation), $\emptyset$ (called empty relation) and $\Delta_V = \{(v, v) : v \in V\}$ (called identity relation or the diagonal relation). Naturally, $\leq, \geq, <$ and $>$ are standard *BR* on $\mathbb{R}$.

Remark 1. Let V be a set endowed with a *BRR*. Then, for each pair $v, u \in V$, one of the following criteria is true:

- (i) $(v, u) \in R$; that means “ v is *R*-related to u ”
- (ii) $(v, u) \notin R$; that means “ v is not *R*-related to u .”

Definition 2 ([30]). A *BR* R on V is

- amorphous if there is absolutely no particular feature of R ,
- reflexive if $(v, v) \in R \quad \forall v \in V$,
- transitive if whenever $(v, u) \in R$ and $(u, w) \in R$ then $(v, w) \in R$,
- symmetric if whenever $(v, u) \in R$ then $(u, v) \in R$,
- antisymmetric if whenever $(v, u) \in R$ and $(u, v) \in R$ then $v = u$,
- equivalence if R is reflexive, symmetric and transitive,
- partial order if R is reflexive, antisymmetric and transitive,
- complete if $(v, u) \in R$ or $(u, v) \in R \quad \forall v, u \in V$.

Definition 3 ([19]). Assuming that V is a set with a *BR* R and $\mathcal{P} : V \rightarrow V$ remains a map, then R is $\mathcal{P}$ -closed *BR* when

$$(v, u) \in R \Rightarrow (\mathcal{P}v, \mathcal{P}u) \in R.$$

Proposition 1 ([22]). If R serves $\mathcal{P}$ -closed, then for each $n \in \mathbb{N}_0$, R serves $\mathcal{P}^n$ -closed.

Definition 4 ([19]). Assuming that V is a set with a *BR* R , a sequence $\{v_n\} \subset V$ is *R*-preserving if it verifies

$$(v_n, v_{n+1}) \in R, \quad \forall n \in \mathbb{N}$$

Definition 5 ([20]). A *MS* (V, ϱ) endowed with a *BR* R is *R*-complete if any Cauchy sequence in V that remains *R*-preserving is convergent.

Definition 6 ([20]). A self-map $\mathcal{P}$ in a MS $(\mathbf{V}, \varrho)$ is $\mathbf{R}$ -continuous at $v \in \mathbf{V}$ if for all $\mathbf{R}$ -preserving sequence $\{v_n\} \subset \mathbf{V}$ with $v_n \xrightarrow{\varrho} v$,

$$\mathcal{P}(v_n) \xrightarrow{\varrho} \mathcal{P}(v).$$

Definition 7 ([20]). A self-map $\mathcal{P}$ in a MS $(\mathbf{V}, \varrho)$ is $\mathbf{R}$ -continuous if it is $\mathbf{R}$ -continuous at each point.

3. Main Results

Let $\mathbf{V}$ be a set equipped with a BR $\mathbf{R}$, and $\mathcal{P} : \mathbf{V} \rightarrow \mathbf{V}$ be a map. Then, we shall adopt the following notation.

$$\mathbf{V}(\mathcal{P}, \mathbf{R}) := \{v \in \mathbf{V} : (v, \mathcal{P}v) \in \mathbf{R}\}.$$

We right away propose a non-unique fixed-point finding to Ćirić-type contraction through an amorphous BR.

Theorem 1. Let $(\mathbf{V}, \varrho)$ be a MS equipped with a BR $\mathbf{R}$, and $\mathcal{P} : \mathbf{V} \rightarrow \mathbf{V}$ be a map. Furthermore,

- (a) $(\mathbf{V}, \varrho)$ is $\mathbf{R}$ -complete,
- (b) $\mathbf{R}$ is $\mathcal{P}$ -closed,
- (c) $\mathcal{P}$ is $\mathbf{R}$ -continuous,
- (d) $\mathbf{V}(\mathcal{P}, \mathbf{R}) \neq \emptyset$,
- (e) $\exists \kappa \in (0, 1)$ verifying

$$\min\{\varrho(\mathcal{P}v, \mathcal{P}u), \varrho(v, \mathcal{P}v), \varrho(u, \mathcal{P}u)\} - \min\{\varrho(v, \mathcal{P}u), \varrho(u, \mathcal{P}v)\} \leq \kappa \varrho(v, u), \quad \forall (v, u) \in \mathbf{R}.$$

Then $\mathcal{P}$ comprises at-least one fixed point. Moreover, for every $v_0 \in \mathbf{V}(\mathcal{P}, \mathbf{R})$, the sequence $\{\mathcal{P}^n v_0\}$ converges to a fixed point of $\mathcal{P}$.

Proof. In lieu of assumption (d), choose $v_0 \in \mathbf{V}(\mathcal{P}, \mathbf{R})$. Then, we have $(v_0, \mathcal{P}v_0) \in \mathbf{R}$. Construct the following sequence $\{v_n\}$:

$$v_n = \mathcal{P}^n(v_0) = \mathcal{P}(v_{n-1}), \quad \forall n \in \mathbb{N}_0. \quad (1)$$

By supposition (b) and Proposition 1, we conclude

$$(\mathcal{P}^n v_0, \mathcal{P}^{n+1} v_0) \in \mathbf{R},$$

i.e.,

$$(v_n, v_{n+1}) \in \mathbf{R}, \quad \forall n \in \mathbb{N}_0. \quad (2)$$

Thereby, $\{v_n\}$ is $\mathbf{R}$ -preserving sequence.

Define $\varrho_n := \varrho(v_{n-1}, v_n)$. Through assumption (e), (1) and (2), we find for every $n \in \mathbb{N}$ that

$$\begin{aligned} \min\{\varrho(\mathcal{P}v_{n-1}, \mathcal{P}v_n), \varrho(v_{n-1}, \mathcal{P}v_{n-1}), \varrho(v_n, \mathcal{P}v_n)\} &= \min\{\varrho(v_{n-1}, \mathcal{P}v_n), \varrho(v_n, \mathcal{P}v_{n-1})\} \\ &\leq \kappa \varrho(v_{n-1}, v_n) \end{aligned}$$

which by employing (1) becomes

$$\begin{aligned} \min\{\varrho(v_n, v_{n+1}), \varrho(v_{n-1}, v_n), \varrho(v_n, v_{n+1})\} &= \min\{\varrho(v_{n-1}, v_{n+1}), \varrho(v_n, v_n)\} \\ &\leq \kappa \varrho(v_{n-1}, v_n) \end{aligned}$$

or

$$\min\{\varrho_{n+1}, \varrho_n, \varrho_{n+1}\} - \min\{\varrho(v_{n-1}, v_{n+1}), 0\} \leq \kappa \varrho_n$$

which reduces to

$$\min\{\varrho_n, \varrho_{n+1}\} \leq \kappa \varrho_n. \quad (3)$$

If $\exists n_0 \in \mathbb{N}$ with $\varrho_{n_0} \leq \varrho_{n_0+1}$, then (3) yields that

$$\varrho_{n_0} \leq \kappa \varrho_{n_0}.$$

Since $\kappa < 1$, above inequality gives rise to $\varrho_{n_0} = 0$, which means that $v_{n_0-1} = v_{n_0} = \mathcal{P}(v_{n_0-1})$. Thus, v_{n_0-1} retains a fixed point of $\mathcal{P}$ and so as the proof is performed.

In contrast, we attain $\varrho_n \geq \varrho_{n+1}$, $\forall n \in \mathbb{N}$. From (3), we derive

$$\varrho_{n+1} \leq \kappa \varrho_n, \quad \forall n \in \mathbb{N}. \quad (4)$$

Utilizing induction, (4) provides that

$$\varrho_n \leq \kappa \varrho_{n-1} \leq \kappa^2 \varrho_{n-2} \leq \cdots \leq \kappa^n \varrho_0$$

so that

$$\varrho_n \leq \kappa^n \varrho_0, \quad \forall n \in \mathbb{N}. \quad (5)$$

For any $n < m$, through (5), we arrive at

$$\begin{aligned} \varrho(v_n, v_m) &\leq \varrho(v_n, v_{n+1}) + \varrho(v_{n+1}, v_{n+2}) + \cdots + \varrho(v_{m-1}, v_m) \\ &\leq (\kappa^n + \kappa^{n+1} + \cdots + \kappa^{m-1}) \varrho_0 \\ &= \kappa^n (1 + \kappa + \kappa^2 + \cdots + \kappa^{m-n-1}) \varrho_0 \\ &< \frac{\kappa^n}{1 - \kappa} \varrho_0 \\ &\rightarrow 0 \text{ as } m, n \rightarrow \infty. \end{aligned}$$

Because of this, the sequence $\{v_n\}$ forms Cauchy. Thus far, $\{v_n\}$ is $\mathbf{R}$ -preserving Cauchy sequence. By $\mathbf{R}$ -completeness of $(\mathbf{V}, \varrho)$, $\exists \bar{v} \in \mathbf{V}$ enjoying

$$v_n \xrightarrow{\varrho} \bar{v}.$$

Since $\mathcal{P}$ is $\mathbf{R}$ -continuous and $\{v_n\}$ is $\mathbf{R}$ -preserving along with $v_n \xrightarrow{\varrho} \bar{v}$, we therefore attain that $v_{n+1} = \mathcal{P}(v_n) \xrightarrow{\varrho} \mathcal{P}(\bar{v})$ yielding thereby $\mathcal{P}(\bar{v}) = \bar{v}$. So, $\bar{v}$ serves a fixed point of $\mathcal{P}$. $\square$

The following finding of Alam and Imdad [19] can be derived from Theorem 1.

Corollary 1 ([19]). *Let $(\mathbf{V}, \varrho)$ be a MS equipped with a BR $\mathbf{R}$, and $\mathcal{P} : \mathbf{V} \rightarrow \mathbf{V}$ be a map. Furthermore,*

- (a) $(\mathbf{V}, \varrho)$ is $\mathbf{R}$ -complete,
- (b) $\mathbf{R}$ is $\mathcal{P}$ -closed,
- (c) $\mathcal{P}$ is $\mathbf{R}$ -continuous,
- (d) $\exists v_0 \in \mathbf{V}$ with $(v_0, \mathcal{P}v_0) \in \mathbf{R}$,
- (e) $\exists \kappa \in (0, 1)$ verifying

$$\varrho(\mathcal{P}v, \mathcal{P}u) \leq \kappa \varrho(v, u), \quad \forall (v, u) \in \mathbf{R}.$$

Then $\mathcal{P}$ possesses at-least one fixed point.

Under universal BR, Theorem 1 deduces the following classical outcome of Ćirić [1].

Corollary 2 ([1]). Let $(\mathbf{V}, \varrho)$ be a complete MS and $\mathcal{P} : \mathbf{V} \rightarrow \mathbf{V}$ be a continuous map. If $\exists \kappa \in (0, 1)$ verifying

$$\min\{\varrho(\mathcal{P}v, \mathcal{P}u), \varrho(v, \mathcal{P}v), \varrho(u, \mathcal{P}u)\} - \min\{\varrho(v, \mathcal{P}u), \varrho(u, \mathcal{P}v)\} \leq \kappa \varrho(v, u), \quad \forall v, u \in \mathbf{V}.$$

Then $\mathcal{P}$ comprises at-least one fixed point.

4. Illustrative Examples

This portion consists of two scenarios that emphasize the significance of Theorem 1.

Example 1. Consider $\mathbf{V} = [0, 1]$ with standard metric $\varrho(v, u) = |v - u|$ and the BR $\mathbf{R} := \mathbb{R} \times \mathbb{Q}$. Then $(\mathbf{V}, \varrho)$ serves $\mathbf{R}$ -complete MS. Assuming that $\mathcal{P}$ retains the identity map on $\mathbf{V}$, naturally, $\mathbf{R}$ retains $\mathcal{P}$ -closed, while $\mathcal{P}$ retains $\mathbf{R}$ -continuous. Meanwhile, $\mathcal{P}$ conforms to presumption (e) of Theorem 1 for arbitrary $\kappa \in [0, 1)$. In this way, all of the prerequisites of Theorem 1 are achieved. It turns out that $\mathcal{P}$ owns a fixed point. Herein $F(\mathcal{P}) = [0, 1]$.

Example 2. Consider $\mathbf{V} = [0, 1]$ with standard metric $\varrho(v, u) = |v - u|$ and the usual BR $\mathbf{R} := \geq$. Then, $(\mathbf{V}, \varrho)$ forms $\mathbf{R}$ -complete MS. Define the map $\mathcal{P} : \mathbf{V} \rightarrow \mathbf{V}$ by

$$\mathcal{P}(v) = \begin{cases} 0, & \text{if } v = 1 \\ 2/3, & \text{otherwise.} \end{cases}$$

Then, $\mathbf{R}$ retains $\mathcal{P}$ -closed, and $\mathcal{P}$ retains $\mathbf{R}$ -continuous. Meanwhile, $\mathcal{P}$ conforms to presumption (e) of Theorem 1 for any $\kappa \geq 2/3$. In this way, all of the prerequisites of Theorem 1 are achieved. It turns out that $\mathcal{P}$ owns a fixed point.

Example 3. Take $\mathbf{V} = [0, 2]$ with standard metric ϱ . Define a map $\mathcal{P} : \mathbf{V} \rightarrow \mathbf{V}$ by

$$\mathcal{P}(v) = \begin{cases} v/2, & \text{if } 0 \leq v < 1 \\ 2, & \text{if } 1 \leq v \leq 2. \end{cases}$$

Consider a BR $\mathbf{R} = \{(0, 1), (0, 2)\}$ on $\mathbf{V}$. Clearly, $(\mathbf{V}, \varrho)$ retains $\mathbf{R}$ -complete. Also, $\mathbf{R}$ forms $\mathcal{P}$ -closed while $\mathcal{P}$ retains $\mathbf{R}$ -continuous. Also, $\mathcal{P}$ satisfies the contractivity condition (e). Indeed, $\mathcal{P}$ admits two fixed points: $\check{v} = 0$ and $\check{v} = 2$.

Remark 2. Take $v_0 = 0$ and $u_0 = 1$. Then, we have

$$\varrho(\mathcal{P}v_0, \mathcal{P}u_0) = 2.$$

Also, for any arbitrary $\kappa \in (0, 1)$, we have

$$\kappa \varrho(v_0, u_0) = \kappa.$$

It follows that

$$\varrho(\mathcal{P}v_0, \mathcal{P}u_0) > \kappa \varrho(v_0, u_0).$$

Thus, for this pair, the contraction conditions of Alam and Imdad [19] are never satisfied. Since here $\mathbf{R}$ is not a universal relation, this example is therefore also not covered by the results of Ćirić [1] and Karapınar [17]. This emphasises the efficiency and inventiveness of Theorem 1 over the corresponding discoveries of Alam and Imdad [19], Ćirić [1] and Karapınar [17].

5. An Application in Dynamic Programming

Various methods of the fixed-point theory are widely used in the field of mathematical optimization. It is commonly recognized that dynamic programming offers practical resources for computer programming and mathematical optimization. Under this scenario, the dynamic programming problem associated with a multistage process reduces to the solution of a specific functional equation described in the forthcoming lines.

Throughout the section, assume that A and B are BS, $\mathcal{H} \subset A$ is state space and $\mathcal{D} \subset B$ retains decision space. Let $\psi : \mathcal{H} \times \mathcal{D} \rightarrow \mathcal{H}$, $\hbar : \mathcal{H} \times \mathcal{D} \rightarrow \mathbb{R}$ and $\Phi : \mathcal{H} \times \mathcal{D} \times \mathbb{R} \rightarrow \mathbb{R}$ be known functions. Then return function $\varphi : \mathcal{H} \rightarrow \mathbb{R}$ of continuous decision process is determined by following functional equation:

$$\varphi(t) = \sup_{s \in \mathcal{D}} \{ \hbar(t, s) + \Phi(t, s, \varphi(\psi(t, s))) \}, \quad \text{for each } t \in \mathcal{H}. \quad (6)$$

Now, using Theorem 1, we examine existence of bounded solution of functional Equation (6). In doing so, we are essentially motivated by Bhakta and Mitra [31].

Theorem 2. Assuming that the functions $\hbar$ and Φ retain continuous and bounded while ψ is continuous. Also,

(i) $\exists \kappa \in (0, 1)$ such that for all $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$, we have

$$0 \leq \Phi(t, s, \beta) - \Phi(t, s, \alpha) \leq \kappa(\beta - \alpha), \quad \forall t \in \mathcal{H} \text{ and } \forall s \in \mathcal{D}.$$

(ii) $\exists v_0 : \mathcal{H} \rightarrow \mathbb{R}$ and $s_0 \in \mathcal{D}$ verifying

$$v_0(t) \leq \hbar(t, s_0) + \Phi(t, s_0, v_0(\psi(t, s_0))), \quad \forall t \in \mathcal{H}.$$

Then, the functional Equation (6) admits a bounded solution.

Proof. Let $\mathbb{B}(\mathcal{H})$ form the collection of all bounded real functions on $\mathcal{H}$. On $\mathbb{B}(\mathcal{H})$, define a metric ϱ :

$$\varrho(v, u) = \sup_{t \in \mathcal{H}} |v(t) - u(t)|, \quad \forall v, u \in \mathbb{B}(\mathcal{H}).$$

As the convergence in $\mathbb{B}(\mathcal{H})$ is uniform, for any Cauchy sequence $\{v_n\} \subset \mathbb{B}(\mathcal{H})$, $\{v_n\}$ remains uniform convergent to a bounded function v^* , and we conclude that $v^* \in \mathbb{B}(\mathcal{H})$. On $\mathbb{B}(\mathcal{H})$, define the following BR

$$\mathbf{R} := \{(v, u) : v(t) \leq u(t), \quad \forall t \in \mathcal{H}\}.$$

Define a map $\mathcal{P} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ by

$$(\mathcal{P}v)(t) = \sup_{s \in \mathcal{D}} \{ \hbar(t, s) + \Phi(t, s, v(\psi(t, s))) \}, \quad \forall v \in \mathcal{B}(\mathcal{H}) \text{ and } t \in \mathcal{H}.$$

(a) $(\mathcal{B}(\mathcal{H}), \varrho)$ being a complete MS is also $\mathbf{R}$ -complete.

(b) Take $(v, u) \in \mathbf{R}$. Then for each $t \in \mathcal{H}$ and for each $s \in \mathcal{D}$, we attain

$$v(\psi(t, s)) \leq u(\psi(t, s)).$$

From (i), we conclude

$$\Phi(t, s, v(\psi(t, s))) \leq \Phi(t, s, u(\psi(t, s))),$$

implying thereby

$$\hbar(t, s) + \Phi(t, s, v(\psi(t, s))) \leq \hbar(t, s) + \Phi(t, s, u(\psi(t, s))),$$

i.e.,

$$(\mathcal{P}v, \mathcal{P}u) \in \mathbf{R}.$$

It follows that $\mathbf{R}$ is $\mathcal{P}$ -closed.

(c) Since the functions $\hbar$, ψ and Φ are continuous, thereby $\mathcal{P}$ is continuous. Consequently, $\mathcal{P}$ must be $\mathbf{R}$ -continuous.

(d) By (ii), we have $v_0 : \mathcal{H} \rightarrow \mathbb{R}$ and $s_0 \in \mathcal{D}$ verifying

$$v_0(t) \leq \sup_{s \in \mathcal{D}} \{ \hbar(t, s) + \Phi(t, s, v_0(\psi(t, s))) \} = (\mathcal{P}v_0)(t), \quad \forall t \in \mathcal{H}$$

so that

$$(v_0, \mathcal{P}v_0) \in \mathbf{R}.$$

(e) Take $(v, u) \in \mathbf{R}$. Then $v(t) \leq u(t)$, $\forall t \in \mathcal{H}$. Since $v, u \in \mathcal{B}(\mathcal{H})$, $\forall t \in \mathcal{H}$, we therefore conclude

$$(\mathcal{P}v)(t) = \sup_{s \in \mathcal{D}} \{ \hbar(t, s) + \Phi(t, s, v(\psi(t, s))) \}$$

and

$$(\mathcal{P}u)(t) = \sup_{s \in \mathcal{D}} \{ \hbar(t, s) + \Phi(t, s, u(\psi(t, s))) \}.$$

Choose an arbitrary $\epsilon > 0$. Then, $\exists s_1, s_2 \in \mathcal{D}$ such that

$$(\mathcal{P}v)(t) < \hbar(t, s_1) + \Phi(t, s_1, v(t_1)) + \epsilon, \quad (7)$$

$$(\mathcal{P}u)(t) < \hbar(t, s_2) + \Phi(t, s_2, u(t_2)) + \epsilon, \quad (8)$$

where $t_1 := \psi(t, s_1)$ and $t_2 := \psi(t, s_2)$ are parameters. Also, we have

$$(\mathcal{P}v)(t) \geq \hbar(t, s_2) + \Phi(t, s_2, v(t_2)), \quad (9)$$

$$(\mathcal{P}u)(t) \geq \hbar(t, s_1) + \Phi(t, s_1, u(t_1)). \quad (10)$$

Utilizing (7), (10) and (ii), we attain

$$\begin{aligned}(\mathcal{P}v)(t) - (\mathcal{P}u)(t) &< \Phi(t, s_1, v(t_1)) - \Phi(t, s_1, u(t_1)) + \epsilon \\&\leq |\Phi(t, s_1, v(t_1)) - \Phi(t, s_1, u(t_1))| + \epsilon \\&\leq \kappa |v(t_1) - u(t_1)| + \epsilon \\&\leq \kappa \varrho(v, u) + \epsilon.\end{aligned}$$

yielding thereby

$$(\mathcal{P}v)(t) - (\mathcal{P}u)(t) < \kappa d(v, u) + \epsilon. \quad (11)$$

Now, from (8), (9) and (ii), we get

$$\begin{aligned}(\mathcal{P}v)(t) - (\mathcal{P}u)(t) &> \Phi(t, s_2, v(t_2)) - \Phi(t, s_2, u(t_2)) - \epsilon \\&\geq -|\Phi(t, s_2, v(t_2)) - \Phi(t, s_2, u(t_2))| - \epsilon \\&\geq -\kappa |v(t_2) - u(t_2)| - \epsilon \\&\geq -\kappa \varrho(v, u) - \epsilon\end{aligned}$$

so that

$$(\mathcal{P}v)(t) - (\mathcal{P}u)(t) > -\kappa \varrho(v, u) - \epsilon. \quad (12)$$

Combining the inequalities (11) and (12), we conclude that

$$|(\mathcal{P}v)(t) - (\mathcal{P}u)(t)| < \kappa \varrho(v, u) + \epsilon. \quad (13)$$

Define

$$M := \min\{\varrho(\mathcal{P}v, \mathcal{P}u), \varrho(v, \mathcal{P}v), \varrho(u, \mathcal{P}u)\}$$

and

$$N := \min\{\varrho(v, \mathcal{P}u), \varrho(u, \mathcal{P}v)\}.$$

Now, the following three main cases arise.

Case-1. If

$$M = \varrho(\mathcal{P}v, \mathcal{P}u)$$

and

$$N = \min\{\varrho(v, \mathcal{P}u), \varrho(u, \mathcal{P}v)\},$$

then in both possibilities for N , we conclude

$$\begin{aligned}M - N &= \varrho(\mathcal{P}v, \mathcal{P}u) - \min\{\varrho(v, \mathcal{P}u), \varrho(u, \mathcal{P}v)\} \\&\leq \varrho(\mathcal{P}v, \mathcal{P}u)\end{aligned}$$

Case-2. If

$$M = \varrho(v, \mathcal{P}v)$$

and

$$N = \varrho(v, \mathcal{P}u),$$

then by triangle inequality, we attain

$$\varrho(v, \mathcal{P}v) \leq \varrho(v, \mathcal{P}u) + \varrho(\mathcal{P}u, \mathcal{P}v)$$

implying thereby

$$M - N = \varrho(v, \mathcal{P}v) - \varrho(v, \mathcal{P}u) \leq \varrho(\mathcal{P}u, \mathcal{P}v).$$

Observe that the case

$$M = \varrho(u, \mathcal{P}u)$$

and

$$N = \varrho(u, \mathcal{P}v),$$

reduces to Case-2 if we interchange the roles of v and u .

Case-3. Assume that

$$M = \varrho(v, \mathcal{P}v)$$

and

$$N = \varrho(u, \mathcal{P}v).$$

Thus, we have $\varrho(v, \mathcal{P}v) \leq \varrho(u, \mathcal{P}u)$. This yields

$$M - N \leq \varrho(u, \mathcal{P}u) - \varrho(u, \mathcal{P}v) \leq \varrho(\mathcal{P}v, \mathcal{P}u).$$

Observe that the case

$$M = \varrho(u, \mathcal{P}u)$$

and

$$N = \varrho(v, \mathcal{P}u)$$

reduces to Case-3 if we interchange the roles of v and u .

Thus, in all the cases, we have

$$M - N \leq \varrho(\mathcal{P}v, \mathcal{P}u). \quad (14)$$

Taking supremum over $t \in \mathcal{H}$ in (13) and using (14), we arrive at

$$M - N < \kappa \varrho(v, u) + \epsilon. \quad (15)$$

Since $\epsilon > 0$ is considered arbitrary, we may instantly infer that

$$M - N \leq \kappa \varrho(v, u).$$

It follows that the operator $\mathcal{P}$ satisfies (e). Consequently, by Theorem 1, $\mathcal{P}$ comprises a fixed point, $\bar{v} \in \mathcal{B}(\mathcal{H})$ and hence $\bar{v}$ forms a bounded solution of the functional Equation (6). $\square$

We pursue the following scenario to explain Theorem 2.

Example 4. Consider

$$\mathcal{H} = \mathcal{D} = [0, 1] \subset \mathbb{R}.$$

Define the functions $\psi : \mathcal{H} \times \mathcal{D} \rightarrow \mathcal{H}$, $\hbar : \mathcal{H} \times \mathcal{D} \rightarrow \mathbb{R}$ and $\Phi : \mathcal{H} \times \mathcal{D} \times \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$\psi(t, s) = ts, \quad \hbar(t, s) = t + s, \quad \text{and} \quad \Phi(t, s, r) = \frac{1}{2}r.$$

Then the functional Equation (6) takes the form:

$$\varphi(t) = \sup_{s \in [0, 1]} \left\{ t + s + \frac{1}{2} \varphi(ts) \right\}. \quad (16)$$

It can be verified that $\hbar(t, s) \in [0, 2]$ so that $\hbar$ is bounded. Also, $\Phi(t, s, r) \in [0, \frac{1}{2}]$ so that Φ is bounded. Furthermore, $\hbar$, Φ and ψ are continuous.

Take $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$. Then, we conclude

$$\Phi(t, s, \beta) - \Phi(t, s, \alpha) = \frac{1}{2}\beta - \frac{1}{2}\alpha = \frac{1}{2}(\beta - \alpha).$$

Thus, supposition (i) of Theorem 2 holds for $\kappa = \frac{1}{2}$.

Choose $v_0(t) = 0$ and $s_0 = 0$. Then, $\forall t \in [0, 1]$, we conclude

$$h(t, 0) + \Phi(t, 0, v_0(\psi(t, 0))) = t + 0 + \frac{1}{2} \cdot 0 = t \geq 0 = v_0(t).$$

Thus, assumption (ii) of Theorem 2 is verified. Therefore, all suppositions of Theorem 2 hold. Consequently, functional Equation (16) admits a bounded solution. By routine calculation, it is easy to verify that the function $\bar{v}(t) = 2(t + 1)$, $\forall t \in [0, 1]$ forms a solution of the problem.

6. Conclusions

The present article described the metrical non-unique fixed-point outcomes by means of a BR over Ćirić-type contraction. Non-unique fixed-point findings may be advantageous in multiple fields of qualitative sciences alongside nonlinear analysis. It should be emphasized that our outcome extended, enriched, improved, unified and modified a number of prior findings. We laid out a few situations demonstrating our findings. To draw out the practical utility of proposal and wide scope of our insights, we employed our findings to specialized functional equations.

6.1. Limitations

Alam and Imdad [19] employed the $\mathbf{R}$ -connected condition to establish a unique fixed-point theorem corresponding to the existence theorem. However, we are not able to prove such a uniqueness theorem due to the involvement of left-hand terms in our contraction condition. On account of the same reason, we cannot utilize the ϱ -self-closedness condition (cf. [19]) that was usually adopted as an alternate hypothesis to $\mathbf{R}$ -continuity (assumption (c)) in fixed-point theorems under certain relational contractions.

6.2. Further Application

Consider the subsequent Fredholm integral equation of the second kind

$$\mu(t) = \phi(t) + \lambda \int_0^1 \kappa(s, t) F(s, \mu(s)) ds \quad (17)$$

where $\phi : [0, 1] \rightarrow \mathbb{R}$ retains a continuous function, $F : [0, 1] \times \mathcal{C}[0, 1] \rightarrow \mathbb{R}$ is a given nonlinear function, $\kappa : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is the kernel, $\lambda \in \mathbb{R}$ is constant and $\mu \in \mathcal{C}[0, 1]$ is the unknown function.

Recall that $\underline{\mu} \in \mathcal{C}[0, 1]$ is termed as a lower solution of (17) if

$$\underline{\mu}(t) \leq \phi(t) + \lambda \int_0^1 \kappa(s, t) F(s, \underline{\mu}(s)) ds.$$

Also, $\bar{\mu} \in \mathcal{C}[0, 1]$ is termed as a lower solution of (17) if

$$\bar{\mu}(t) \geq \phi(t) + \lambda \int_0^1 \kappa(s, t) F(s, \bar{\mu}(s)) ds.$$

On $\mathcal{C}[0, 1]$, define the subsequent BR

$$\mathbf{R} := \{(\mu, \vartheta) : \mu(t) \leq \vartheta(t), \forall t \in [a, b]\}.$$

Let ϱ be the metric on $\mathcal{C}[0, 1]$ induced by $\|\cdot\|_\infty$, defined by

$$\varrho(\mu, \vartheta) = \|\mu(t) - \vartheta(t)\|_\infty = \sup_{t \in [0, 1]} |\mu(t) - \vartheta(t)|.$$

Clearly, $(\mathcal{C}[0, 1], \varrho)$ forms an **R**-complete **MS**.

Define a map $\mathcal{P} : \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 1]$ by

$$\mathcal{P}\mu(t) = \phi(t) + \lambda \int_0^1 \kappa(s, t) F(s, \mu(s)) ds. \quad (18)$$

Then, $\mathcal{P}$ being continuous is an **R**-continuous map.

Along with integral Equation (17), assume that $\exists$ a function $f : [0, 1] \rightarrow \mathbb{R}$ enjoying $x, y \in \mathbb{R}$ with $x \geq y$, the function F verifies the condition

$$0 \leq F(s, x) - F(s, y) \leq |f(s)| (x - y), \quad \forall s \in [0, 1].$$

Also, suppose that

$$\int_0^1 |\kappa(s, t) f(s)|^2 ds \leq L^2,$$

where $0 < |\lambda|L < 1$. Then assumptions (b) and (e) of Theorem 1 are met. If Equation (17) admits a lower solution, then we can demonstrate that assumption (d) of Theorem 1 holds. Thereby, in view of Theorem 1, the map $\mathcal{P}$ represented by (18) comprises a fixed point, which forms a solution of (17).

6.3. Possible Future Directions

With regard to the impact of the relation-theoretic fixed-point strategy, we examine the subsequent potential avenues for further research:

- To expand our findings over several abstract spaces, e.g., quasimetric space, semimetric space, cone MS, b-metric space, etc., by means of a BR;
- To enhance the contraction-condition employing auxiliary functions;
- To flesh out our findings for two maps by proving coincidence point and common fixed-point theorems;
- To implement our findings in fractional differential equations following the work of [32,33].

Author Contributions: Conceptualization, E.A., F.M.A. and F.A.K.; methodology, E.A.; investigation, B.Z.A.; formal analysis, D.F.; writing—original draft preparation, A.A. and F.M.A.; writing—review and editing, B.Z.A. and F.M.A.; project administration, D.F.; supervision, F.A.K.; funding acquisition, D.F. and A.A. All authors have read and agreed to the published version of the manuscript.

Funding: The first author expresses gratitude to the Princess Nourah bint Abdulrahman University Researchers Supporting Project (Number: PNURSP2026R174), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Data Availability Statement: The data handled over present investigation are included in this article. With a legitimate inquiry, more details can be retrieved solely from the corresponding authors.

Conflicts of Interest: The authors affirm they possess no competing interests.

Abbreviations

In the current work, the aforementioned acronyms are prevalent:

$\mathbb{N}$ Set of positive integers;
 $\mathbb{N}_0 :=$ $\mathbb{N} \cup \{0\};$

$\mathbb{R}$	Set of real numbers;
$\mathbb{R}^+ :=$	$[0, \infty)$;
BR	binary relation;
BCP	Banach contraction principle;
RHS	right hand side;
MS	metric space;
BS	Banach space;
$F(\mathcal{P})$	the set of all fixed points of map $\mathcal{P}$.

References

- Ćirić, L.B. On some maps with a non-unique fixed point. *Publ. Inst. Math.* **1974**, *17*, 52–58.
- Achari, J. On Ćirić's non-unique fixed points. *Mat. Vesnik* **1976**, *13*, 255–257.
- Achari, J. Non-unique fixed points in L -spaces. *Publ. Inst. Math.* **1977**, *21*, 5–7.
- Pachpatte, B.G. On Ćirić type maps with a nonunique fixed point. *Indian J. Pure Appl. Math.* **1979**, *10*, 1039–1043.
- Mishra, S.N. On fixed points of orbitally continuous maps. *Nanta Math.* **1979**, *12*, 83–90.
- Achari, J. Common nonunique fixed points of two mappings. *Mathematica* **1980**, *22*, 5–8.
- Achari, J. On the generalization of Pachpatte's nonunique fixed point theorem. *Indian J. Pure Appl. Math.* **1982**, *13*, 299–302.
- Dhage, B.C. Some results for the maps with a nonunique fixed point. *Indian J. Pure Appl. Math.* **1985**, *16*, 245–256.
- Liu, Z.; Wang, H. Some results on a nonunique fixed point. *J. Liaoning Norm. Univ.* **1986**, *9*, 12–15.
- Ćirić, L.B. Remarks on some theorems of Mishra, Dhage and Pathak. *Pure Appl. Math. Sci.* **1990**, *32*, 27–29.
- Ćirić, L.B.; Jotić, N. A further extension of maps with non-unique fixed points. *Mat. Vesnik* **1998**, *50*, 1–4.
- Liu, Z.; Guo, Z.; Kang, S.M.; Lee, S.K. On Ćirić type mappings with nonunique fixed and periodic points. *Int. J. Pure Appl. Math.* **2006**, *26*, 399–408.
- Karapinar, E.; Romaguera, S. Nonunique fixed point theorems in partial metric spaces. *Filomat* **2013**, *27*, 1305–1314. [\[CrossRef\]](#)
- Karapinar, E.; Agarwal, R.P. A note on Ćirić type nonunique fixed point theorems. *Fixed Point Theory Appl.* **2017**, *2017*, 20. [\[CrossRef\]](#)
- Alqahtani, B.; Fulga, A.; Karapinar, E. Non-unique fixed point results in extended b -metric space. *Mathematics* **2018**, *6*, 68. [\[CrossRef\]](#)
- Karapinar, E. Ćirić type nonunique fixed points results: A review. *Appl. Comput. Math.* **2019**, *18*, 3–21.
- Karapinar, E. Recent advances on the results for nonunique fixed in various spaces. *Axioms* **2019**, *8*, 72. [\[CrossRef\]](#)
- Hussain, S. Non-unique fixed point theorems in modular metric spaces. *Symmetry* **2019**, *11*, 549. [\[CrossRef\]](#)
- Alam, A.; Imdad, M. Relation-theoretic contraction principle. *J. Fixed Point Theory Appl.* **2015**, *17*, 693–702. [\[CrossRef\]](#)
- Alam, A.; Imdad, M. Relation-theoretic metrical coincidence theorems. *Filomat* **2017**, *31*, 4421–4439. [\[CrossRef\]](#)
- Alam, A.; Imdad, M. Nonlinear contractions in metric spaces under locally T -transitive binary relations. *Fixed Point Theory* **2018**, *19*, 13–24. [\[CrossRef\]](#)
- Khan, F.A. Almost contractions under binary relations. *Axioms* **2022**, *11*, 441. [\[CrossRef\]](#)
- Alam, A.; George, R.; Imdad, M. Refinements to relation-theoretic contraction principle. *Axioms* **2022**, *11*, 316. [\[CrossRef\]](#)
- Hasanuzzaman, M.; Imdad, M.; Saleh, H.N. On modified $\mathcal{L}$ -contraction via binary relation with an application. *Fixed Point Theory* **2022**, *23*, 267–278. [\[CrossRef\]](#)
- Arif, M.; Imdad, M. Fixed point results under nonlinear Suzuki $(F, \mathcal{R}^{\neq})$ -contractions with an application. *Filomat* **2022**, *36*, 3155–3165. [\[CrossRef\]](#)
- Aljawi, S.; Uddin, I. Relation-theoretic nonlinear almost contractions with an application to boundary value problems. *Mathematics* **2024**, *12*, 1275. [\[CrossRef\]](#)
- Wasey, A.; Othman, W.A.M.; Alshaban, E.; Wong, K.B.; Alatawi, A. Proinov-type relational contractions and applications to boundary value problems. *AIMS Math.* **2025**, *10*, 13393–13408. [\[CrossRef\]](#)
- Alamrani, F.M.; Algehyne, E.A.; Alshaban, E.; Alatawi, A.; Mohammed, H.I.A.; Khan, F.A. Relational almost (ϕ, ψ) -contractions and applications to nonlinear Fredholm integral equations. *Axioms* **2025**, *14*, 1.
- Alshaban, E.; Alatawi, A.; Alamrani, F.M.; Alamer, A.; Alrshidi, N.N.; Khan, F.A. Nonlinear almost contractions of Pant type under binary relations with an application to boundary value problems. *Mathematics* **2025**, *13*, 906. [\[CrossRef\]](#)
- Lipschutz, S. *Schaum's Outlines of Theory and Problems of Set Theory and Related Topics*; McGraw-Hill: New York, NY, USA, 1964.
- Bhakta, P.C.; Mitra, S. Some existence theorems for functional equations arising in dynamic programming. *J. Math. Anal. Appl.* **1984**, *98*, 348–362. [\[CrossRef\]](#)

32. Shiri, B. Well-posedness of the mild solutions for incommensurate systems of delay fractional differential equations. *Fractal Fract.* **2025**, *9*, 60. [[CrossRef](#)]
33. Shiri, B.; Shi, Y.-G.; Baleanu, D. The well-posedness of incommensurate FDEs in the space of continuous functions. *Symmetry* **2024**, *16*, 1058. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.