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Generalized Laplace Transform for Higher-Order Hybrid Fractional Cauchy Problems: Theory and Applications to Memory-Dependent Dynamics

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Abstract

This paper develops a generalized Laplace transform framework on weighted function spaces $C_{\delta_\psi, \gamma}^n[a, b]$, establishing a symmetry between integer-order δ_ψ operators and fractional ψ -Hilfer derivatives at the level of transform representations. Explicit transformation formulas are derived for the n th-order δ_ψ -derivative and the ψ -Hilfer fractional derivative of order $\alpha \in (m-1, m)$, with $m \leq n$. These results form an analytical basis for the treatment of higher-order hybrid fractional Cauchy problems that systematically couple integer-order and fractional operators subject to mixed initial conditions. The general solution is expressed in closed form using a bivariate Mittag–Leffler function. To illustrate the utility of the approach, a representative second-order hybrid model is studied and compared numerically with its classical integer-order counterpart. The simulations reveal significant differences in the dynamical response, including variations in amplitude, damping behavior, and long-term evolution.

Keywords: generalized Laplace transform; weighted function spaces; ψ -Hilfer fractional derivative; δ_ψ -derivative; hybrid fractional Cauchy problems; bivariate Mittag–Leffler function

MSC: 34A08; 34A12; 44A10

1. Introduction

Systems exhibiting memory, nonlocality, and hereditary effects are effectively described using fractional differential equations (FDEs), which generalize traditional integer-order differential equations. Since the foundational contributions of Kilbas, Podlubny, and Samko [1–3], fractional calculus has been widely developed and applied across various scientific fields. By permitting derivatives of non-integer order, fractional operators broaden the notion of differentiation and provide a richer mathematical structure for modeling complex dynamical behavior.

Among the many fractional operators proposed in the literature, the ψ -Hilfer fractional derivative has gained increasing attention due to its broad unifying capability. Introduced by Sousa and Oliveira in 2018 as a generalization of the classical Hilfer derivative, the ψ -Hilfer operator incorporates an auxiliary function ψ , allowing it to encompass a wide range of fractional derivatives through appropriate choices of parameters and kernel functions [4]. In particular, it recovers the Riemann–Liouville, Caputo, and classical Hilfer derivatives as special cases, while also admitting more general formulations that extend beyond standard



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fractional models. By accommodating complex hereditary effects and interpolating between local and nonlocal dynamics, the ψ -Hilfer derivative offers a flexible and robust formulation for fractional-order systems.

The Laplace transform remains a central analytical tool for solving differential equations, owing to its ability to convert differential operators into algebraic expressions. However, classical Laplace theory is often inadequate for generalized fractional operators, particularly those defined on weighted spaces or involving auxiliary functions such as ψ . To address these challenges, several generalized transforms have been introduced in the literature, including the ρ -Laplace transform, the Katugampola transform, β -Laplace transform, and the q -Laplace transform, each designed for specific operator structures or functional settings [5–8].

For ψ -Hilfer fractional derivatives acting on weighted spaces of the form $C_{\delta_\psi, \gamma}^n[a, b]$, the construction of a compatible Laplace-type transform is nontrivial. Although generalized Laplace transforms have been developed for related fractional operators and function spaces [9–11], the explicit involvement of the auxiliary function ψ modifies the differential structure and the associated convolution kernels. Consequently, classical Laplace transforms and existing generalizations cannot be directly applied to the class of hybrid problems addressed in this work.

An added analytical challenge arises in fractional differential models that simultaneously involve integer-order and fractional operators. Such systems naturally arise in applications where instantaneous effects coexist with memory-dependent behavior [12–14]. In mechanical systems, for example, integer-order derivatives typically represent inertia and elastic restoring forces, while fractional derivatives are used to model viscoelastic damping with long-term memory. The interaction between these operators complicates the analytical treatment, as common approaches generally fail to simultaneously simplify both integer and fractional components within a common analytical setting.

Several approaches have been proposed to address fractional and hybrid oscillatory systems, including analytical and numerical techniques for solving fractional-order models arising in physical and engineering applications [15–18]. More recently, advanced rheological and hybrid dynamical models have been investigated using Laplace transforms and power series methods to obtain approximate analytical solutions for eigenvalue and forced vibration problems [19,20]. While these studies present valuable insights, they are often restricted to specific fractional operators or do not fully address the interaction between generalized fractional derivatives and higher-order integer derivatives.

In our previous work [21], we introduced a generalized Laplace transform designed for the weighted space $C_{\delta_\psi, \gamma}^1[a, b]$, enabling the analytical analysis of hybrid problems involving first-order δ_ψ -derivatives coupled with ψ -Hilfer fractional derivatives of order $\alpha \in (0, 1)$. Fundamental properties such as linearity, convolution theorems, and transformation rules for ψ -Hilfer operators were established, and the framework was applied to practical models including capacitor charging dynamics and hydraulic door-closer systems.

Building upon this foundation, the present study extends this theory in several significant directions:

1. **Higher-Order Operators and Generalized Transforms:** We develop the generalized Laplace transform for ψ -Hilfer fractional derivatives of order $\alpha \in (m-1, m)$ with $m \leq n$ and n -th-order δ_ψ -derivatives, generalizing the first-order case treated in [21]. Crucially, we prove new versions of the key operational formulas—the Laplace transforms of the n -th δ_ψ -derivative (Theorem 1), ψ -Caputo (Theorem 3), and ψ -Hilfer derivatives (Theorem 4)—for arbitrary $n \geq 1$, whereas only the case $n = 1$ was established previously. Additionally, we extend the regularity theorem for ψ -Riemann–

- Liouville fractional integration (Theorem 2) to the space $C_{\delta\psi,\gamma}^n[a, b]$. These generalized theorems form a sound foundation for solving higher-order hybrid problems.
2. **Extended Weighted Function Spaces:** The analysis is conducted in the space $C_{\delta\psi,\gamma}^n[a, b]$ with $m \leq n$, accommodating functions whose higher derivatives may exhibit singularities or memory behaviors not captured in the first-order space $C_{\delta\psi,\gamma}^1[a, b]$.
 3. **New Solution Forms:** The interaction of integer-order and fractional operators in higher-order hybrid Cauchy problems leads to solution representations involving bivariate Mittag–Leffler functions $E_{\alpha,\beta,\gamma}^\delta(x, y)$.
 4. **Broader Application Scope:** The framework is applied to second-order mechanical systems (e.g., mass–spring–damper with fractional damping), enabling the modeling of oscillatory systems with memory-dependent viscoelastic behavior, going beyond the first-order capacitor and hydraulic models studied previously.

Table 1 summarizes the key advances over [21].

Table 1. Comparison with previous work [21].

Aspect	Paper [21]	This Work
Function space	$C_{\delta\psi,\gamma}^1[a, b]$	$C_{\delta\psi,\gamma}^n[a, b], n \geq 1$
$\delta\psi$ -derivative order	$n = 1$	$n \geq 1$
Fractional order α	$0 < \Re(\alpha) < 1$	$m - 1 < \Re(\alpha) < m, 1 \leq m \leq n$
Laplace transform formulas	For $n = 1$ only	Generalized to arbitrary $n \geq 1$
Solution functions	Classical Mittag–Leffler $E_{\alpha,\beta}(z)$	Bivariate Mittag–Leffler $E_{\alpha,\beta,\gamma}^\delta(x, y)$
Applications	First-order systems (capacitor, hydraulic)	Higher-order oscillatory systems (mass–spring–damper)

These extensions are non-trivial. The transition from first-order to higher-order systems requires new regularity conditions, modified initial value formulations, and the introduction of multivariate special functions to capture the interplay between different orders of differentiation. The results presented here thus constitute a substantial generalization of the framework established in [21], enabling the analysis of a broader class of memory-dependent dynamical systems.

After introducing the required background in Section 2, the paper proceeds to develop the main analytical results in Section 3, where a higher-order generalized Laplace transform theory is established and applied to hybrid fractional Cauchy problems. Section 4 provides representative examples to demonstrate the proposed framework. The results are then placed in a broader context in Section 5, and the paper concludes with a summary of findings in Section 6.

2. Preliminaries

This section recalls the notation, function spaces, and auxiliary results required for the subsequent analysis. This section covers only the essential concepts required here. A detailed exposition appears in [21] and the cited references.

2.1. Function Spaces

This paper works with weighted function spaces defined on finite intervals $[a, b]$ where $0 \leq a < b < \infty$. The key spaces are introduced below, along with their fundamental properties. Throughout the work, the complex parameter γ satisfies $0 \leq \Re(\gamma) < 1$. We adopt the standard convention $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1 (ψ -Weighted n -times δ_ψ -Differentiable Space $C_{\delta_\psi, \gamma}^n[a, b]$, [4]). Let $n \in \mathbb{N}$, $\gamma \in \mathbb{C}$ and let $\psi \in C^n[a, b]$ such that $\psi'(t) > 0$ on $[a, b]$. We denote by $C_{\delta_\psi, \gamma}^n[a, b]$ the Banach space of functions $f(t)$ which have continuous δ_ψ -derivatives on $[a, b]$ up to order $n - 1$ and derivative $(\delta_\psi^n f)(t)$ on (a, b) of order n such that $(\delta_\psi^n f)(t) \in C_{\gamma, \psi}[a, b]$:

$$C_{\delta_\psi, \gamma}^n[a, b] = \left\{ f : [a, b] \rightarrow \mathbb{R} : \delta_\psi^{n-1} f(t) \in C[a, b]; \delta_\psi^n f(t) \in C_{\gamma, \psi}[a, b], \delta_\psi := \frac{1}{\psi'(t)} \frac{d}{dt} \right\}$$

with the norm

$$\|f\|_{C_{\delta_\psi, \gamma}^n[a, b]} = \sum_{k=0}^{n-1} \|\delta_\psi^k f\|_{C[a, b]} + \|\delta_\psi^n f\|_{C_{\gamma, \psi}[a, b]}.$$

We note that the spaces form a decreasing chain

$$C_{\delta_\psi, \gamma}^m[a, b] \subset C_{\delta_\psi, \gamma}^n[a, b] \subset C[a, b], \quad 1 \leq n \leq m.$$

Moreover, if $0 < \Re(\gamma) < 1$, then $C^n[a, b] \subsetneq C_{\delta_\psi, \gamma}^n[a, b]$, allowing controlled singular behavior at $t = a$.

A detailed characterization of the space $C_{\delta_\psi, \gamma}^n[a, b]$ via representation formulas is given by Lemma 3 in [21] (p. 9), and is not repeated here.

2.2. Fractional Differential Operators

This subsection defines the generalized fractional derivative operator that form the basis of our analysis. We consider operators of complex order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ and non-integer, denoting $[\Re(\alpha)]$ as its integer part. Definitions are given on the interval $[a, b]$ where $0 \leq a < b \leq \infty$, establishing the setting for our subsequent Laplace transform theory.

Definition 2 (ψ -Hilfer fractional derivative, [22]). Let $f, \psi \in C^n([a, b], \mathbb{R})$ be functions such that ψ is strictly increasing and $\psi'(t) \neq 0$ for all $t \in [a, b]$. The ψ -Hilfer fractional derivative of order $\alpha \in \mathbb{C}$ (with $\Re(\alpha) > 0$) and type β (with $0 \leq \beta \leq 1$) of a function f on $[a, b]$ is defined by

$$({}^H D_{a^+}^{\alpha, \beta; \psi} f)(t) = \left(I_{a^+}^{\beta(n-\alpha); \psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a^+}^{(1-\beta)(n-\alpha); \psi} f \right)(t), \quad (1)$$

where $n = [\Re(\alpha)] + 1$.

The ψ -Hilfer fractional derivative defined above can be written in the following form:

$$({}^H D_{a^+}^{\alpha, \beta; \psi} f)(t) = (I_{a^+}^{\gamma_H - \alpha; \psi} D_{a^+}^{\gamma_H; \psi} f)(t), \quad (2)$$

where $\gamma_H = \alpha + \beta(n - \alpha)$, and $I_{a^+}^{\gamma_H - \alpha; \psi}(\cdot)$ and $D_{a^+}^{\gamma_H; \psi}(\cdot)$ denote the ψ -Riemann–Liouville fractional integral (Definition 2.1 in [22]) and the ψ -Riemann–Liouville fractional derivative (Definition 2.2 in [22]), respectively.

The ψ -Hilfer fractional derivative provides a unified framework that generalizes numerous fractional operators through appropriate choices of α , β , and ψ , as detailed in Remark 3 of [21] (p. 7). Important special cases include the ψ -Caputo, ψ -Riemann–Liouville, and classical Caputo and Riemann–Liouville derivatives.

2.3. Generalized Laplace Transform on the Space $C_{\delta_\psi, \gamma}^1[a, b]$

This subsection recalls the generalized Laplace transform framework associated with the operator δ_ψ and the ψ -Hilfer fractional derivative, which forms the analytical foundation for the results developed in this work. Only the definitions and properties required for the subsequent analysis are summarized here; further details may be found in [21].

Definition 3 (Generalized Laplace transform, [9]). Let $f, \psi : [a, \infty) \rightarrow \mathbb{R}$ be functions such that $\psi'(t) > 0$ on $[a, \infty)$. The generalized Laplace transform of f is defined by

$$\mathcal{L}_\psi\{f(t)\}(s) = \int_a^\infty e^{-s(\psi(t)-\psi(a))} f(t) \psi'(t) dt,$$

for all values of s for which the integral converges.

The generalized Laplace transform formulas for the first-order δ_ψ -derivative, the ψ -Caputo fractional derivative, and the ψ -Hilfer fractional derivative of order $0 < \Re(\alpha) < 1$, together with regularity preservation results in the weighted space $C_{\delta_\psi, \gamma}^1[a, b]$, are taken from [21] and will be used throughout the paper without further proof.

For the reader's convenience, Tables 2 and 3 summarize the weighted function spaces and generalized fractional operators used throughout the paper. The spaces $C_{\delta_\psi, \gamma}^n[a, b]$ and operators such as the ψ -Hilfer derivative ${}^H D_{a^+}^{\alpha, \beta; \psi}$ generalize classical notions through the auxiliary function ψ and the parameters α and β , providing a unified framework for modeling memory effects and hereditary dynamics in nonlocal systems.

Table 2. Summary of main weighted function spaces with their norms.

Function Space	Mathematical Definition	Norm
$C^n[a, b]$	$\{f : [a, b] \rightarrow \mathbb{C} : f^{(n)} \in C[a, b]\}$	$\sum_{k=0}^n \max_{t \in [a, b]} f^{(k)}(t) $
$C_{\gamma, \psi}[a, b]$	$\{f : (a, b] \rightarrow \mathbb{R} : (\psi(t) - \psi(a))^\gamma f(t) \in C[a, b]\}$	$\max_{t \in [a, b]} (\psi(t) - \psi(a))^\gamma f(t) $
$C_{\delta_\psi, \gamma}^n[a, b]$	$\{f : [a, b] \rightarrow \mathbb{R} : \delta_\psi^{n-1} f \in C[a, b], \delta_\psi^n f \in C_{\gamma, \psi}[a, b]\}$	$\sum_{k=0}^{n-1} \ \delta_\psi^k f\ _{C[a, b]} + \ \delta_\psi^n f\ _{C_{\gamma, \psi}[a, b]}$

Table 3. Summary of main fractional operators.

Fractional Operator	Mathematical Definition
ψ -Riemann–Liouville fractional integral	$I_{a^+}^{\alpha; \psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau) (\psi(t) - \psi(\tau))^{\alpha-1} f(\tau) d\tau, t > a$
ψ -Riemann–Liouville fractional derivative	$D_{a^+}^{\alpha; \psi} f(t) = \left(\left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a^+}^{n-\alpha; \psi} f \right)(t), t > a$
ψ -Caputo fractional derivative	${}^C D_{a^+}^{\alpha; \psi} f(t) = \left(I_{a^+}^{n-\alpha; \psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n f \right)(t), t > a$
ψ -Hilfer fractional derivative	${}^H D_{a^+}^{\alpha, \beta; \psi} f(t) = \left(I_{a^+}^{\beta(n-\alpha); \psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a^+}^{(1-\beta)(n-\alpha); \psi} f \right)(t), t > a$

3. Results

This section develops a generalized Laplace transform theory for hybrid fractional systems of arbitrary order, extending the first-order framework established in [21]. The presentation follows the logical progression illustrated in Figure A1, moving from foundational regularity results in weighted function spaces to generalized transform formulas and, ultimately, to exact solution representations for hybrid fractional models.

A key advance in this section is the handling of interactions between higher-order integer derivatives and fractional operators. These interactions occur within weighted function spaces that permit controlled singular behavior at the initial point (Theorem 5 and 6).

3.1. Generalized Laplace Transform on the Space $C_{\delta_\psi, \gamma}^n[a, b]$

The generalized Laplace transform is extended here from the first-order space $C_{\delta_\psi, \gamma}^1[a, b]$ to the higher-order weighted spaces $C_{\delta_\psi, \gamma}^n[a, b]$ for any $n \geq 1$. Regularity preservation results for ψ -Riemann–Liouville fractional integrals are also extended from the first-order setting to arbitrary $n \geq 2$, establishing conditions under which solutions remain in $C_{\delta_\psi, \gamma}^n[a, b]$ (Theorem 2). Generalized Laplace transforms are established for three classes of higher-order operators: n th-order δ_ψ -derivatives (Theorem 1), ψ -Caputo derivatives

(Theorem 3), and ψ -Hilfer derivatives (Theorem 4). Each transform formula correctly incorporates multiple initial conditions. Hybrid differential equations that couple n -th-order δ_ψ -derivatives with ψ -Hilfer fractional operators consequently admit exact solutions written in terms of bivariate Mittag–Leffler functions.

Theorem 1 (Generalized Laplace transform of the n -th δ_ψ -derivative). *Let $f \in C_{\delta_\psi, \gamma}^n[a, b]$ be such that $\delta_\psi^i f$, $i = 0, 1, \dots, n - 1$, are of ψ -exponential order s_0 . Then the generalized Laplace transform of $\delta_\psi^n f(t)$ exists and is given by*

$$\mathcal{L}_\psi\{\delta_\psi^n f(t)\}(s) = s^n \mathcal{L}_\psi\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{n-k-1} (\delta_\psi^k f)(a),$$

for $\Re(s) > s_0$, where $\delta_\psi := \frac{1}{\psi'(t)} \frac{d}{dt}$.

Proof. We proceed by induction on n . The case $n = 1$ follows from Theorem 4 in [21] (p. 13). Assume that the formula holds for some integer $m \geq 1$.

For $n = m + 1$, let $g = \delta_\psi^m f$. Then $g \in C_{\delta_\psi, \gamma}^1[a, b]$ and is of ψ -exponential order s_0 . Applying Theorem 4 in [21] (p. 13) to g yields

$$\mathcal{L}_\psi\{\delta_\psi^{m+1} f\}(s) = s \mathcal{L}_\psi\{\delta_\psi^m f\}(s) - (\delta_\psi^m f)(a), \quad \Re(s) > s_0.$$

Using the inductive hypothesis, for $\Re(s) > s_0$,

$$\begin{aligned} \mathcal{L}_\psi\{\delta_\psi^{m+1} f\}(s) &= s \left[s^m \mathcal{L}_\psi\{f\}(s) - \sum_{k=0}^{m-1} s^{m-k-1} (\delta_\psi^k f)(a) \right] - (\delta_\psi^m f)(a) \\ &= s^{m+1} \mathcal{L}_\psi\{f\}(s) - \sum_{k=0}^m s^{(m+1)-k-1} (\delta_\psi^k f)(a), \end{aligned}$$

which completes the induction. $\square$

Remark 1 (Interpretation of Theorem 1). *Theorem 1 extends the classical Laplace transform rule for integer-order derivatives to the δ_ψ -derivative framework. As in the classical setting, each application of δ_ψ corresponds to multiplication by s in the transform domain, while initial values of lower-order δ_ψ -derivatives generate correction terms. This result is fundamental for the analysis of higher-order hybrid systems, as it provides the operational backbone required to transform n -th-order δ_ψ -derivatives into algebraic expressions compatible with ψ -Hilfer fractional operators.*

We now present the general theorem on regularity preservation under ψ -Riemann–Liouville fractional integration. This result specifies conditions under which the fractional integral maintains the function’s membership in the space $C_{\delta_\psi, \gamma}^n[a, b]$.

Theorem 2 (Regularity preservation under ψ -Riemann–Liouville fractional integration). *Let $\alpha \in \mathbb{C}$ with $n - 1 < \Re(\alpha) < n$, where $n \in \mathbb{N}$, and let $\psi \in C^1[a, b]$ be strictly increasing with $\psi'(t) > 0$ for all $t \in [a, b]$. Suppose that $\gamma \in \mathbb{C}$ satisfies*

$$\Re(\alpha) - (n - 1) \leq \Re(\gamma) < 1.$$

Then, the following statements hold:

- I. *If $f \in C_{\delta_\psi, \gamma}^1[a, b]$, then $I_{a+}^{1-\alpha, \psi} f \in C_{\delta_\psi, \gamma}^1[a, b]$.*
- II. *If $n \geq 2$, $f \in C_{\delta_\psi, \gamma}^n[a, b]$, and $(\delta_\psi^k f)(a) = 0$ for all $k = 0, 1, \dots, n - 2$, then $I_{a+}^{n-\alpha, \psi} f \in C_{\delta_\psi, \gamma}^n[a, b]$.*

Proof. The proof of Case I follows directly from Theorem 9 in [21] (p. 19). We now turn to the proof of Case II.

Let $F(t) = (I_{a+}^{n-\alpha, \psi} f)(t)$. From Theorem 7 in [21] (p. 16), it can be represented as:

$$F(t) = \frac{1}{\Gamma(2n-\alpha)} \int_a^t (\psi(t) - \psi(s))^{2n-\alpha-1} (\delta_\psi^n f)(s) \psi'(s) ds + \sum_{k=0}^{n-1} \frac{(\delta_\psi^k f)(a)}{\Gamma(n-\alpha+k+1)} (\psi(t) - \psi(a))^{n-\alpha+k}.$$

The assumption $(\delta_\psi^k f)(a) = 0$ for $k = 0, 1, \dots, n-2$ plays a crucial role here: It eliminates all but the last term in the sum, yielding the simplified representation

$$F(t) = \frac{1}{\Gamma(2n-\alpha)} \int_a^t (\psi(t) - \psi(s))^{2n-\alpha-1} (\delta_\psi^n f)(s) \psi'(s) ds + \frac{(\delta_\psi^{n-1} f)(a)}{\Gamma(2n-\alpha)} (\psi(t) - \psi(a))^{2n-\alpha-1}.$$

This simplification is essential, as it ensures that after repeated application of δ_ψ , all remaining terms share a common asymptotic structure near $t = a$, which is necessary for verifying the weighted continuity conditions of $C_{\delta_\psi, \gamma}^n[a, b]$.

Applying the operator δ_ψ repeatedly, we obtain for $j = 1, \dots, n$

$$(\delta_\psi^j F)(t) = \frac{1}{\Gamma(2n-\alpha-j)} \int_a^t (\psi(t) - \psi(s))^{2n-\alpha-j-1} (\delta_\psi^n f)(s) \psi'(s) ds + \frac{(\delta_\psi^{n-1} f)(a)}{\Gamma(2n-\alpha-j)} (\psi(t) - \psi(a))^{2n-\alpha-j-1}.$$

In particular, for $j = n-1$ and $j = n$:

$$\delta_\psi^{n-1} F(t) = \frac{1}{\Gamma(n-\alpha+1)} \int_a^t (\psi(t) - \psi(s))^{n-\alpha} (\delta_\psi^n f)(s) \psi'(s) ds + \frac{(\delta_\psi^{n-1} f)(a)}{\Gamma(n-\alpha+1)} (\psi(t) - \psi(a))^{n-\alpha}, \quad (3)$$

$$\delta_\psi^n F(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (\psi(t) - \psi(s))^{n-\alpha-1} (\delta_\psi^n f)(s) \psi'(s) ds + \frac{(\delta_\psi^{n-1} f)(a)}{\Gamma(n-\alpha)} (\psi(t) - \psi(a))^{n-\alpha-1}. \quad (4)$$

To show that $F \in C_{\delta_\psi, \gamma}^n[a, b]$, it suffices to verify that $\delta_\psi^{n-1} F \in C[a, b]$ and $(\psi(t) - \psi(a))^\gamma \delta_\psi^n F \in C[a, b]$.

Part I: To show that $\delta_\psi^{n-1} F(t)$ is continuous on $[a, b]$. From (3), the second term

$$S(t) := \frac{(\delta_\psi^{n-1} f)(a)}{\Gamma(n-\alpha+1)} (\psi(t) - \psi(a))^{n-\alpha}$$

is continuous on $[a, b]$ since $n - \Re(\alpha) > 0$. It remains to show that the first term

$$I(t) := \frac{1}{\Gamma(n-\alpha+1)} \int_a^t (\psi(t) - \psi(s))^{n-\alpha} (\delta_\psi^n f)(s) \psi'(s) ds$$

is continuous on $[a, b]$.

1. *Continuity of $I(t)$ on $(a, b]$:* Let $t_0 \in (a, b]$. For $t > t_0$:

$$\begin{aligned} I(t) - I(t_0) &= \frac{1}{\Gamma(n-\alpha+1)} \left[\int_a^t (\psi(t) - \psi(s))^{n-\alpha} (\delta_\psi^n f)(s) \psi'(s) ds - \int_a^{t_0} (\psi(t_0) - \psi(s))^{n-\alpha} (\delta_\psi^n f)(s) \psi'(s) ds \right] \\ &= \frac{1}{\Gamma(n-\alpha+1)} \left[\int_{t_0}^t (\psi(t) - \psi(s))^{n-\alpha} (\delta_\psi^n f)(s) \psi'(s) ds + \int_a^{t_0} ((\psi(t) - \psi(s))^{n-\alpha} - (\psi(t_0) - \psi(s))^{n-\alpha}) (\delta_\psi^n f)(s) \psi'(s) ds \right]. \end{aligned}$$

Let

$$A(t) := \int_{t_0}^t (\psi(t) - \psi(s))^{n-\alpha} (\delta_\psi^n f)(s) \psi'(s) ds$$

and

$$B(t) := \int_a^{t_0} ((\psi(t) - \psi(s))^{n-\alpha} - (\psi(t_0) - \psi(s))^{n-\alpha}) (\delta_\psi^n f)(s) \psi'(s) ds$$

denote the two integrals.

For $A(t)$: Since $f \in C_{\delta_\psi, \gamma}^n[a, b]$, we have $(\delta_\psi^n f)(s) = g(s)(\psi(s) - \psi(a))^{-\gamma}$ with g continuous. For $s \in [t_0, t]$, $\psi(s) - \psi(a)$ is bounded away from zero (since $t_0 > a$), so $|(\delta_\psi^n f)(s)| \leq C$ for some constant C . Also, $(\psi(t) - \psi(s))^{n-\alpha}$ is continuous. Thus, the integrand is bounded, say by L . Then,

$$|A(t)| \leq \int_{t_0}^t L ds = L(t - t_0) \rightarrow 0 \quad \text{as } t \rightarrow t_0^+.$$

For $B(t)$: For each fixed $s \in [a, t_0]$, as $t \rightarrow t_0^+$, we have $(\psi(t) - \psi(s))^{n-\alpha} \rightarrow (\psi(t_0) - \psi(s))^{n-\alpha}$ uniformly (since ψ is continuous). Also, $(\delta_\psi^n f)(s)\psi'(s)$ is integrable on $[a, t_0]$ (as shown below). By the dominated convergence theorem, $B(t) \rightarrow 0$.

To justify dominated convergence, note

$$|(\psi(t) - \psi(s))^{n-\alpha} - (\psi(t_0) - \psi(s))^{n-\alpha}| \leq |(\psi(t_0) - \psi(s))^{n-\alpha}|$$

for t sufficiently close to t_0 . Now show the following:

$$\int_a^{t_0} |(\psi(t_0) - \psi(s))^{n-\alpha}| |(\delta_\psi^n f)(s)| \psi'(s) ds < \infty.$$

Substitute $(\delta_\psi^n f)(s) = g(s)(\psi(s) - \psi(a))^{-\gamma}$ with $|g(s)| \leq M$ for some positive M :

$$\int_a^{t_0} |(\psi(t_0) - \psi(s))^{n-\alpha}| |(\delta_\psi^n f)(s)| \psi'(s) ds \leq M \int_a^{t_0} (\psi(t_0) - \psi(s))^{n-\Re(\alpha)} (\psi(s) - \psi(a))^{-\Re(\gamma)} \psi'(s) ds.$$

Change variables: $u = \psi(s)$, $du = \psi'(s) ds$, $u_a = \psi(a)$, $u_{t_0} = \psi(t_0)$:

$$M \int_a^{t_0} (\psi(t_0) - \psi(s))^{n-\Re(\alpha)} (\psi(s) - \psi(a))^{-\Re(\gamma)} \psi'(s) ds = M \int_{u_a}^{u_{t_0}} (u_{t_0} - u)^{n-\Re(\alpha)} (u - u_a)^{-\Re(\gamma)} du.$$

Let $w = u_{t_0} - u_a = \psi(t_0) - \psi(a)$, and substitute $v = u - u_a$:

$$\begin{aligned} M \int_{u_a}^{u_{t_0}} (u_{t_0} - u)^{n-\Re(\alpha)} (u - u_a)^{-\Re(\gamma)} du &= M \int_0^w (w - v)^{n-\Re(\alpha)} v^{-\Re(\gamma)} dv \\ &= Mw^{n-\Re(\alpha)+1-\Re(\gamma)} \int_0^1 (1-z)^{n-\Re(\alpha)} z^{-\Re(\gamma)} dz \\ &= M(\psi(t_0) - \psi(a))^{n-\Re(\alpha)+1-\Re(\gamma)} B(n - \Re(\alpha) + 1, 1 - \Re(\gamma)), \end{aligned}$$

where $v = wz$. The Beta function $B(n - \Re(\alpha) + 1, 1 - \Re(\gamma))$ converges since $n - \Re(\alpha) + 1 > 0$ and $1 - \Re(\gamma) > 0$. Thus, the integral is finite, and by the dominated convergence theorem

$$\lim_{t \rightarrow t_0^+} B(t) = \int_a^{t_0} \lim_{t \rightarrow t_0^+} ((\psi(t) - \psi(s))^{n-\alpha} - (\psi(t_0) - \psi(s))^{n-\alpha}) (\delta_\psi^n f)(s) \psi'(s) ds = 0.$$

Hence, $I(t) - I(t_0) \rightarrow 0$ as $t \rightarrow t_0^+$, with a similar argument holding for $t < t_0$, so $I(t)$ is continuous on $(a, b]$.

2. *Continuity of $I(t)$ at $t = a$* : We had $(\delta_\psi^n f)(s) = g(s)(\psi(s) - \psi(a))^{-\gamma}$ with $|g(s)| \leq M$ for $s \in [a, b]$. Similar to the computation in the previous part, we have

$$\begin{aligned} |I(t)| &\leq \frac{1}{|\Gamma(n - \alpha + 1)|} \int_a^t (\psi(t) - \psi(s))^{n-\Re(\alpha)} |(\delta_\psi^n f)(s)| \psi'(s) ds \\ &\leq \frac{M}{|\Gamma(n - \alpha + 1)|} \int_a^t (\psi(t) - \psi(s))^{n-\Re(\alpha)} (\psi(s) - \psi(a))^{-\Re(\gamma)} \psi'(s) ds \\ &= \frac{M}{|\Gamma(n - \alpha + 1)|} (\psi(t) - \psi(a))^{n-\Re(\alpha)+1-\Re(\gamma)} B(n - \Re(\alpha) + 1, 1 - \Re(\gamma)). \end{aligned}$$

Since $n - \Re(\alpha) + 1 - \Re(\gamma) > 0$ (because $\Re(\alpha) < n$, $\Re(\gamma) < 1$), we have

$$|I(t)| \leq K(\psi(t) - \psi(a))^{n-\Re(\alpha)+1-\Re(\gamma)} \rightarrow 0, \quad \text{as } t \rightarrow a^+,$$

where $K = \frac{M}{|\Gamma(n-\alpha+1)|} B(n - \Re(\alpha) + 1, 1 - \Re(\gamma))$. Thus, $I(t)$ is continuous at $t = a$.

Therefore, $\delta_\psi^{n-1}F(t) = I(t) + S(t)$ is continuous on $[a, b]$.

Part II: To show that $(\psi(t) - \psi(a))^\gamma \delta_\psi^n F(t)$ is continuous on $[a, b]$. From (4), multiplied by $(\psi(t) - \psi(a))^\gamma$ to the second term gives

$$\tilde{S}(t) = \frac{(\delta_\psi^{n-1}f)(a)}{\Gamma(n-\alpha)} (\psi(t) - \psi(a))^{n-\alpha-1+\gamma}.$$

Since $n - \Re(\alpha) - 1 + \Re(\gamma) \geq 0$ (because $\Re(\gamma) \geq \Re(\alpha) - (n - 1)$), this is continuous on $[a, b]$. The first term

$$\tilde{I}(t) := \frac{1}{\Gamma(n-\alpha)} (\psi(t) - \psi(a))^\gamma \int_a^t (\psi(t) - \psi(s))^{n-\alpha-1} (\delta_\psi^n f)(s) \psi'(s) ds$$

is shown to be continuous on $[a, b]$ as follows.

1. *Continuity of $\tilde{I}(t)$ on $(a, b]$.* Let $t_0 \in (a, b]$. For $t > t_0$,

$$\begin{aligned} \tilde{I}(t) - \tilde{I}(t_0) &= \frac{1}{\Gamma(n-\alpha)} \left[(\psi(t) - \psi(a))^\gamma \int_a^t (\psi(t) - \psi(s))^{n-\alpha-1} (\delta_\psi^n f)(s) \psi'(s) ds \right. \\ &\quad \left. - (\psi(t_0) - \psi(a))^\gamma \int_a^{t_0} (\psi(t_0) - \psi(s))^{n-\alpha-1} (\delta_\psi^n f)(s) \psi'(s) ds \right] \\ &= \frac{1}{\Gamma(n-\alpha)} \left[\int_{t_0}^t K(t, s) (\delta_\psi^n f)(s) \psi'(s) ds + \int_a^{t_0} (K(t, s) - K(t_0, s)) (\delta_\psi^n f)(s) \psi'(s) ds \right], \end{aligned}$$

where we define the kernel function

$$K(t, s) := (\psi(t) - \psi(a))^\gamma (\psi(t) - \psi(s))^{n-\alpha-1}.$$

Now define the two integrals $\tilde{A}(t)$ and $\tilde{B}(t)$ as follows:

$$\tilde{A}(t) := \int_{t_0}^t K(t, s) (\delta_\psi^n f)(s) \psi'(s) ds, \quad \text{and} \quad \tilde{B}(t) := \int_a^{t_0} (K(t, s) - K(t_0, s)) (\delta_\psi^n f)(s) \psi'(s) ds.$$

For $\tilde{A}(t)$: Since $f \in C_{\delta_\psi, \gamma}^n[a, b]$ we have $\delta_\psi^n f$ continuous on $[t_0, b]$, and therefore bounded there; ψ' is continuous and positive on $[a, b]$, so it is bounded above and bounded away from 0 on $[t_0, b]$. Thus, there exist constants $N > 0$ and $m > 0$ such that

$$|(\delta_\psi^n f)(s) \psi'(s)| \leq N, \quad \psi'(s) \geq m > 0, \quad s \in [t_0, b].$$

Therefore,

$$|\tilde{A}(t)| \leq \int_{t_0}^t |K(t, s)| |(\delta_\psi^n f)(s) \psi'(s)| ds \leq N (\psi(t) - \psi(a))^{\Re(\gamma)} \int_{t_0}^t (\psi(t) - \psi(s))^{n-\Re(\alpha)-1} ds.$$

Make the change of variable $u = \psi(t) - \psi(s)$. Then $du = -\psi'(s) ds$ and, using $\psi'(s) \geq m$, we get

$$\int_{t_0}^t (\psi(t) - \psi(s))^{n-\Re(\alpha)-1} ds \leq \frac{1}{m} \int_0^{\psi(t)-\psi(t_0)} u^{n-\Re(\alpha)-1} du = \frac{1}{m(n-\Re(\alpha))} (\psi(t) - \psi(t_0))^{n-\Re(\alpha)}.$$

Here, $n - \Re(\alpha) > 0$. Combining the estimates yields

$$|\tilde{A}(t)| \leq P(\psi(t) - \psi(a))^{\Re(\gamma)}(\psi(t) - \psi(t_0))^{n-\Re(\alpha)},$$

for a constant $P := \frac{N}{m(n-\Re(\alpha))}$. As $t \rightarrow t_0^+$ we have $\psi(t) - \psi(t_0) \rightarrow 0$, while $(\psi(t) - \psi(a))^{\Re(\gamma)}$ remains bounded near t_0 ; hence $|\tilde{A}(t)| \rightarrow 0$. This proves $\tilde{A}(t) \rightarrow 0$ as $t \rightarrow t_0^+$.

For $\tilde{B}(t)$: For each fixed $s \in [a, t_0]$, the integrand converges pointwise to 0 as $t \rightarrow t_0^+$. To dominate, note

$$\begin{aligned} |K(t, s) - K(t_0, s)| &= |(\psi(t) - \psi(a))^{\gamma}(\psi(t) - \psi(s))^{n-\alpha-1} - (\psi(t_0) - \psi(a))^{\gamma}(\psi(t_0) - \psi(s))^{n-\alpha-1}| \\ &\leq |(\psi(t_0) - \psi(a))^{\gamma}(\psi(t_0) - \psi(s))^{n-\alpha-1}| = |K(t_0, s)|, \end{aligned}$$

for t sufficiently close to t_0 . Then, by a computation similar to one in the proof of Theorem 9 in [21] (p. 19):

$$\int_a^{t_0} |K(t_0, s)| |(\delta_{\psi}^n f)(s)| \psi'(s) ds \leq MB(n - \Re(\alpha), 1 - \Re(\gamma))(\psi(t_0) - \psi(a))^{n-\Re(\alpha)} < \infty.$$

Thus, by the dominated convergence theorem, $\tilde{B}(t) \rightarrow 0$ as $t \rightarrow t_0^+$. Thus, $\tilde{I}(t) - \tilde{I}(t_0) \rightarrow 0$ as $t \rightarrow t_0^+$. A similar argument holds for $t < t_0$. Hence, $\tilde{I}(t)$ is continuous on $(a, b]$.

2. Continuity of $\tilde{I}(t)$ at $t = a$. Substitute $|(\delta_{\psi}^n f)(s)| \leq M(\psi(s) - \psi(a))^{-\Re(\gamma)}$ for some $M > 0$ on $[a, b]$, we have

$$\begin{aligned} |\tilde{I}(t)| &\leq \frac{1}{|\Gamma(n - \alpha)|} (\psi(t) - \psi(a))^{\Re(\gamma)} \int_a^t (\psi(t) - \psi(s))^{n-\Re(\alpha)-1} |(\delta_{\psi}^n f)(s)| \psi'(s) ds. \\ &\leq \frac{M}{|\Gamma(n - \alpha)|} (\psi(t) - \psi(a))^{\Re(\gamma)} \int_a^t (\psi(t) - \psi(s))^{n-\Re(\alpha)-1} (\psi(s) - \psi(a))^{-\Re(\gamma)} \psi'(s) ds. \end{aligned}$$

By computing the above integral using arguments similar to those in the proof of Theorem 9 in [21] (p. 19),

$$|\tilde{I}(t)| \leq R(\psi(t) - \psi(a))^{\Re(\gamma)} \rightarrow 0, \quad \text{as } t \rightarrow a^+,$$

where $R = \frac{M}{|\Gamma(n - \alpha)|} B(n - \Re(\alpha), 1 - \Re(\gamma))$. Thus, $\tilde{I}(t)$ is continuous at $t = a$.

Therefore, $(\psi(t) - \psi(a))^{\gamma} \delta_{\psi}^n F(t) = \tilde{I}(t) + \tilde{S}(t)$ is continuous on $[a, b]$.

From Part I and Part II, we conclude that $F(t) \in C_{\delta_{\psi}, \gamma}^n[a, b]$. This completes the proof for case II. $\square$

While mathematically necessary for our proof, these conditions correspond physically to systems starting from rest (zero initial states up to order $n - 2$). In practice, they are often satisfied in boundary value problems or systems beginning from equilibrium.

Remark 2 (Interpretation of Theorem 2). This theorem shows that ψ -Riemann–Liouville fractional integration preserves the regularity structure of functions in the weighted spaces $C_{\delta_{\psi}, \gamma}^n[a, b]$, provided suitable compatibility conditions between the fractional order α , the weight parameter γ , and the initial data are satisfied. Part I treats the first-order case, while Part II extends the result to higher orders under additional zero initial conditions. The constraint $\Re(\alpha) - (n - 1) \leq \Re(\gamma)$ ensures that the singular behavior introduced by fractional integration remains admissible within the weighted space, preventing the loss of regularity near the initial point. This preservation property is essential for the well-posedness of higher-order fractional models and guarantees that subsequent operations, such as further differentiation or Laplace transformation, remain well defined.

The generalized Laplace transform of the ψ -Caputo fractional derivative is now established for arbitrary $n \geq 1$, extending the $n = 1$ case from [21]. This formula directly enables the ψ -Hilfer transform in the following theorem.

Theorem 3 (Generalized Laplace transform of ψ -Caputo fractional derivative). *Let $\alpha \in \mathbb{C}$ with $n - 1 < \Re(\alpha) < n$, $n \in \mathbb{N}$ and $f \in C_{\delta_\psi, \gamma}^n[a, b]$ for any $b > a$. Assume $\psi \in C^1[a, \infty)$ is a strictly increasing function with $\psi'(t) > 0$ for all $t \geq a$ and $\delta_\psi^k f, k = 0, 1, \dots, n$ be of $\psi(t)$ -exponential order s_0 . Then*

$$\mathcal{L}_\psi \{({}^C D_{a^+}^{\alpha; \psi} f)(t)\}(s) = s^\alpha \mathcal{L}_\psi \{f(t)\}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} (\delta_\psi^k f)(a^+),$$

for $\Re(s) > s_0$, where $\delta_\psi := \frac{1}{\psi'(t)} \frac{d}{dt}$.

Proof. We begin by taking the definition of ψ -Caputo fractional derivative from Table 3,

$$({}^C D_{a^+}^{\alpha; \psi} f)(t) = (I_{a^+}^{n-\alpha; \psi} \delta_\psi^n f)(t),$$

where $I_{a^+}^{n-\alpha; \psi}(\cdot)$ denotes the ψ -Riemann–Liouville fractional integral of order $n - \alpha$. Now taking the generalized Laplace transform on both sides, we have

$$\mathcal{L}_\psi \{({}^C D_{a^+}^{\alpha; \psi} f)(t)\}(s) = \mathcal{L}_\psi \{(I_{a^+}^{n-\alpha; \psi} \delta_\psi^n f)(t)\}(s), \quad (5)$$

for $\Re(s) > s_0$. Since $f \in C_{\delta_\psi, \gamma}^n[a, b]$, it follows that $\delta_\psi^n f \in C_{\gamma, \psi}[a, b]$. Moreover, by hypothesis, $\delta_\psi^n f$ is of ψ -exponential order s_0 . Therefore, applying Corollary 2.II of [21] (p. 22), to the right-hand side of (5) yields

$$\mathcal{L}_\psi \{(I_{a^+}^{n-\alpha; \psi} \delta_\psi^n f)(t)\}(s) = s^{-(n-\alpha)} \mathcal{L}_\psi \{(\delta_\psi^n f)(t)\}(s), \quad (6)$$

for $\Re(s) > s_0$. As $f \in C_{\delta_\psi, \gamma}^n[a, b]$, and by hypothesis, $\delta_\psi^k f, k = 0, 1, \dots, n - 1$ are of ψ -exponential order s_0 , we invoke Theorem 1 to obtain

$$\mathcal{L}_\psi \{({}^C D_{a^+}^{\alpha; \psi} f)(t)\}(s) = s^{-(n-\alpha)} \left[s^n \mathcal{L}_\psi \{f(t)\}(s) - \sum_{k=0}^{n-1} s^{n-k-1} (\delta_\psi^k f)(a^+) \right] = s^\alpha \mathcal{L}_\psi \{f(t)\}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} (\delta_\psi^k f)(a^+).$$

for $\Re(s) > s_0$. This completes the proof. $\square$

The most general result of this section is the generalized Laplace transform of the ψ -Hilfer fractional derivative for order $\alpha \in (n - 1, n)$ with $n \geq 2$. This theorem unifies both the ψ -Riemann–Liouville ($\beta = 0$) and ψ -Caputo ($\beta = 1$) cases presented earlier, extending the first-order framework from [21] to higher-order hybrid problems. The result provides a comprehensive transformation formula that accommodates the intermediate parameter β and associated initial conditions.

Theorem 4 (Generalized Laplace transform of the ψ -Hilfer fractional derivative). *Let $n \in \mathbb{N} \setminus \{1\}$ and let*

$$n - 1 < \Re(\alpha) < n, \quad \beta \in [0, 1].$$

Define $\gamma := n - \gamma_H$, where $\gamma_H = \alpha + \beta(n - \alpha)$. Assume that

$$\frac{1}{2} \leq \Re(\gamma) < 1.$$

Let $f \in C_{\delta_\psi, \gamma}^n[a, b]$ for any $b > a$, and assume $\psi \in C^1[a, \infty)$ is strictly increasing with $\psi'(t) > 0$ for all $t \geq a$. Assume that f and $\delta_\psi^k I_{a+}^{\gamma; \psi} f$, $k = 0, 1, \dots, n$, are of ψ -exponential order s_0 , and that $(\delta_\psi^k f)(a) = 0$ for $k = 0, 1, \dots, n-2$. Then, for $\Re(s) > s_0$,

$$\mathcal{L}_\psi \left\{ ({}^H D_{a+}^{\alpha, \beta; \psi} f)(t) \right\}(s) = s^\alpha \mathcal{L}_\psi \{f(t)\}(s) - s^{\alpha - \gamma_H} \sum_{k=0}^{n-1} s^{n-k-1} (\delta_\psi^k I_{a+}^{n-\gamma_H; \psi} f)(a^+),$$

where $\delta_\psi = \frac{1}{\psi'(t)} \frac{d}{dt}$.

Proof. Let $n \geq 2$ and $n-1 < \Re(\alpha) < n$. By Remark 2 (Relationship between the ψ -Hilfer and ψ -Caputo fractional derivatives) in [21] (p. 7) define

$$g(t) := (I_{a+}^{(1-\beta)(n-\alpha); \psi} f)(t) = (I_{a+}^{\gamma; \psi} f)(t), \quad \gamma = n - \gamma_H = (1 - \beta)(n - \alpha),$$

we have the identity

$$({}^H D_{a+}^{\alpha, \beta; \psi} f)(t) = ({}^C D_{a+}^{\mu; \psi} g)(t), \quad \mu = n(1 - \beta) + \beta\alpha.$$

To apply Theorem 2.II to $I_{a+}^{\gamma; \psi} f$, we set $\alpha_* := n - \gamma$, so that $I_{a+}^{\gamma; \psi} f = I_{a+}^{n-\alpha_*; \psi} f$ and $n-1 < \Re(\alpha_*) < n$ whenever $0 \leq \Re(\gamma) < 1$. Moreover, the condition in Theorem 2 that

$$\Re(\alpha_*) - (n-1) \leq \Re(\gamma) < 1$$

becomes

$$1 - \Re(\gamma) \leq \Re(\gamma) < 1,$$

which holds provided that $\frac{1}{2} \leq \Re(\gamma) < 1$. (Equivalently, since $\Re(\gamma) = (1 - \beta)(n - \Re(\alpha))$, this requires $\Re(\alpha) \leq n - \frac{1}{2}$ and $\beta \in \left[0, 1 - \frac{1}{2(n - \Re(\alpha))}\right]$.) Hence, since $f \in C_{\delta_\psi, \gamma}^n[a, b]$ and $(\delta_\psi^k f)(a) = 0$ for $k = 0, 1, \dots, n-2$, Theorem 2.II implies that

$$g = I_{a+}^{\gamma; \psi} f \in C_{\delta_\psi, \gamma}^n[a, b],$$

so g satisfies the regularity assumptions required for the ψ -Caputo derivative.

Taking the generalized Laplace transform gives, for $\Re(s) > s_0$,

$$\mathcal{L}_\psi \left\{ ({}^H D_{a+}^{\alpha, \beta; \psi} f)(t) \right\}(s) = \mathcal{L}_\psi \left\{ ({}^C D_{a+}^{\mu; \psi} g)(t) \right\}(s). \quad (7)$$

Moreover, by hypothesis, $\delta_\psi^k I_{a+}^{\gamma; \psi} f$, $k = 0, 1, \dots, n$ are of ψ -exponential order s_0 . Therefore, applying Theorem 3 to the right-hand side of (7) yields

$$\mathcal{L}_\psi \left\{ ({}^C D_{a+}^{\mu; \psi} g)(t) \right\}(s) = s^\mu \mathcal{L}_\psi \{g(t)\}(s) - \sum_{k=0}^{n-1} s^{\mu-k-1} (\delta_\psi^k g)(a^+), \quad \Re(s) > s_0. \quad (8)$$

Next, using the generalized Laplace transform of the ψ -Riemann–Liouville fractional integral (Theorem 10 in [21] (p. 22)) and the fact that $0 \leq \Re(\gamma) < 1$, we obtain

$$\mathcal{L}_\psi \{g(t)\}(s) = \mathcal{L}_\psi \left\{ (I_{a+}^{\gamma; \psi} f)(t) \right\}(s) = s^{-\gamma} \mathcal{L}_\psi \{f(t)\}(s), \quad \Re(s) > s_0. \quad (9)$$

Substituting (9) into (8) and then into (7) yields

$$\mathcal{L}_\psi \left\{ ({}^H D_{a+}^{\alpha, \beta; \psi} f)(t) \right\}(s) = s^\mu s^{-\gamma} \mathcal{L}_\psi \{f(t)\}(s) - \sum_{k=0}^{n-1} s^{\mu-k-1} (\delta_\psi^k I_{a+}^{\gamma; \psi} f)(a^+).$$

Since $\mu - \gamma = \alpha$, the first term becomes $s^\alpha \mathcal{L}_\psi\{f(t)\}(s)$. Finally, using $\gamma_H = n - \gamma$ (equivalently $\mu = \alpha + \gamma = \alpha + n - \gamma_H$), we rewrite

$$s^{\mu-k-1} = s^{\alpha+\gamma-k-1} = s^{\alpha+n-\gamma_H-k-1} = s^{\alpha-\gamma_H} s^{n-k-1}.$$

Therefore,

$$\mathcal{L}_\psi\left\{{}^H D_{a^+}^{\alpha,\beta;\psi} f\right\}(t)(s) = s^\alpha \mathcal{L}_\psi\{f(t)\}(s) - s^{\alpha-\gamma_H} \sum_{k=0}^{n-1} s^{n-k-1} (\delta_\psi^k I_{a^+}^{n-\gamma_H;\psi} f)(a^+), \quad \Re(s) > s_0,$$

which completes the proof. $\square$

Remark 3. Since $\Re(\gamma) = (1 - \beta)(n - \Re(\alpha))$, the condition $\Re(\gamma) \geq \frac{1}{2}$ is equivalent to

$$n - 1 < \Re(\alpha) \leq n - \frac{1}{2}, \quad \beta \in \left[0, 1 - \frac{1}{2(n - \Re(\alpha))}\right].$$

Thus, for fixed α , the regularity requirement implicitly restricts the admissible values of β .

The following corollary provides the generalized Laplace transform of the ψ -Riemann–Liouville fractional derivative under appropriate regularity assumptions.

Corollary 1 (Generalized Laplace transform of the ψ -Riemann–Liouville fractional derivative). Let $\alpha \in \mathbb{C}$ with

$$n - 1 < \Re(\alpha) \leq n - \frac{1}{2},$$

where $n \in \mathbb{N} \setminus \{1\}$. Let $f \in C_{\delta_\psi, \gamma}^n[a, b]$ for any $b > a$, where $\gamma := n - \alpha$, and assume that $\psi \in C^1[a, \infty)$ is strictly increasing with $\psi'(t) > 0$ for all $t \geq a$. Assume that f and $\delta_\psi^k I_{a^+}^{n-\alpha;\psi} f$, $k = 0, 1, \dots, n$, are of ψ -exponential order s_0 , and that $(\delta_\psi^k f)(a) = 0$ for $k = 0, 1, \dots, n - 2$. Then, for $\Re(s) > s_0$,

$$\mathcal{L}_\psi\left\{{}^H D_{a^+}^{\alpha;\psi} f\right\}(t)(s) = s^\alpha \mathcal{L}_\psi\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{n-k-1} (\delta_\psi^k I_{a^+}^{n-\alpha;\psi} f)(a^+),$$

where $\delta_\psi := \frac{1}{\psi'(t)} \frac{d}{dt}$.

Proof. Set $\beta = 0$ in Theorem 4. Then

$$\gamma_H = \alpha + \beta(n - \alpha) = \alpha, \quad \gamma = n - \gamma_H = n - \alpha.$$

The additional hypothesis of Theorem 4, namely $\frac{1}{2} \leq \Re(\gamma) < 1$, becomes

$$\frac{1}{2} \leq \Re(n - \alpha) < 1,$$

which is equivalent to $n - 1 < \Re(\alpha) \leq n - \frac{1}{2}$. Substituting $\beta = 0$ (hence, $\gamma_H = \alpha$) into the formula of Theorem 4 gives

$$\mathcal{L}_\psi\left\{{}^H D_{a^+}^{\alpha,0;\psi} f\right\}(t)(s) = s^\alpha \mathcal{L}_\psi\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{n-k-1} (\delta_\psi^k I_{a^+}^{n-\alpha;\psi} f)(a^+),$$

since $s^{\alpha-\gamma_H} = s^{\alpha-\alpha} = 1$. Finally, for $\beta = 0$ the ψ -Hilfer derivative coincides with the ψ -Riemann–Liouville fractional derivative, so the result follows. $\square$

A distinctive feature of the higher-order hybrid problems considered here is the appearance of the bivariate Mittag–Leffler function $E_{\alpha,\beta,\gamma}^\delta(x,y)$ in the explicit solutions. This function, defined in Definition 5, naturally arises when inverting Laplace transforms of the form $(s^n + \lambda_1 s^\alpha + \lambda_2)^{-1}$ that characterize hybrid systems combining integer-order δ_ψ -derivatives with ψ -Hilfer fractional operators.

In contrast, the first-order solutions in [21] involved only the classical two-parameter Mittag–Leffler function $E_{\alpha,\beta}(z)$. The bivariate extension captures the coupling between the integer-order dynamics (through δ_ψ^n) and the fractional memory effects (through ${}^H D_{a^+}^{\alpha,\beta;\psi}$), reflecting the richer algebraic structure of higher-order hybrid systems.

We recall both the classical two-parameter Mittag–Leffler function and its bivariate extension, as they will be used extensively to express the exact solutions obtained in the following subsections.

Definition 4 (Two-parameter Mittag–Leffler function, [2]). *The Mittag–Leffler function involving two parameters is given by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad z \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0.$$

Definition 5 (Bivariate Mittag–Leffler function, [23]). *Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ be complex parameters with $\Re(\alpha) > 0$ and $\Re(\beta) > 0$. We define the following bivariate Mittag–Leffler function for general complex numbers x and y :*

$$E_{\alpha,\beta,\gamma}^\delta(x,y) := \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\delta)_{k+l}}{\Gamma(\alpha k + \beta l + \gamma)} \cdot \frac{x^k y^l}{k!l!},$$

where the numerator $(\delta)_{k+l}$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\dots(a+n-1).$$

Remark 4. For the bivariate Mittag–Leffler function $E_{\alpha,\beta,\gamma}^1(x,y)$, when $y = 0$, it reduces to the classical two-parameter Mittag–Leffler function:

$$E_{\alpha,\beta,\gamma}^1(x,0) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(1)_{k+l}}{\Gamma(\alpha k + \beta l + \gamma)} \cdot \frac{x^k 0^l}{k!l!} = \sum_{k=0}^{\infty} \frac{(1)_k}{\Gamma(\alpha k + \gamma)} \cdot \frac{x^k}{k!},$$

where we use the facts that $0^l = 0$ for $l > 0$ and $0^0 = 1$ (by convention), and only the $l = 0$ terms survive. Finally, using $(1)_k = k!$, we obtain

$$E_{\alpha,\beta,\gamma}^1(x,0) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \gamma)} = E_{\alpha,\gamma}(x).$$

3.2. Generalized Laplace Transform Applied to Hybrid Fractional Cauchy Problems

Building on the extended transform theory of Section 3.1, we derive exact solutions to hybrid fractional Cauchy problems of arbitrary order. Theorems 5 and 6 provide closed-form solutions expressed through bivariate Mittag–Leffler functions for systems combining n -th-order δ_ψ -derivatives with ψ -Hilfer fractional operators. These solutions capture the richer algebraic structure of higher-order hybrid dynamics, moving beyond the first-order case (Theorem 14 in [21] (p. 27)), through the appearance of genuinely bivariate functions that reflect the coupling between integer and fractional derivative terms.

For notational convenience, we introduce the shorthand notations

$$\Delta_\tau^\psi t := \psi(t) - \psi(\tau) \quad \text{and} \quad \Delta_a^\psi t := \psi(t) - \psi(a),$$

which will be used throughout the remainder of this work.

Based on Theorem 2 I and II, we establish our main results in two theorems. The first provides the explicit solution to a hybrid fractional Cauchy problem involving the composition of an n -th-order δ_ψ -derivative and a ψ -Hilfer fractional derivative of order $0 < \Re(\alpha) < 1$, while the second addresses the case $m - 1 < \Re(\alpha) < m$ for $m \geq 2$. The solutions are expressed in terms of bivariate Mittag-Leffler functions and systematically incorporate all specified mixed initial conditions.

Theorem 5. Let $\alpha \in \mathbb{C}$ with $0 < \Re(\alpha) < 1$, $\beta \in [0, 1]$, and $\gamma = \alpha + \beta(1 - \alpha)$. Suppose $\psi \in C^1[a, \infty)$ is a strictly increasing function with $\psi'(t) > 0$ for all $t \geq a$, and let $u \in C_{\delta_\psi, \gamma}^n[a, b]$ for any $b > a$, $n \in \mathbb{N}$, and $n \geq 1$. Assume further that u and $\delta_\psi^l I_{a+}^{1-\gamma; \psi} u$ for $l = 0, 1$ are of ψ -exponential order s_0 . Then, for any $g \in C_{\delta_\psi, \gamma}^n[a, b]$ of ψ -exponential order s_1 , the general solution of the hybrid Cauchy problem

$$\begin{cases} u^{[n]}(t) + \lambda_1 ({}^H D_{a+}^{\alpha, \beta; \psi} u)(t) + \lambda_2 u(t) = g(t), & \lambda_1, \lambda_2 \in \mathbb{R}, t > a, \\ u^{[l]}(a) = c_l, & c_l \in \mathbb{R}, l = 0, 1, \dots, n-1, \\ (I_{a+}^{1-\gamma; \psi} u)(a^+) = d_0, & d_0 \in \mathbb{R}, \end{cases} \quad (10)$$

where $u^{[n]} := \delta_\psi^n u$, is given by

$$\begin{aligned} u(t) = & \int_a^t (\Delta_\tau^\psi t)^{n-1} E_{n-\alpha, n, n}^1(-\lambda_1 (\Delta_\tau^\psi t)^{n-\alpha}, -\lambda_2 (\Delta_\tau^\psi t)^n) g(\tau) \psi'(\tau) d\tau \\ & + \sum_{l=0}^{n-1} c_l (\Delta_a^\psi t)^l E_{n-\alpha, n, l+1}^1(-\lambda_1 (\Delta_a^\psi t)^{n-\alpha}, -\lambda_2 (\Delta_a^\psi t)^n) \\ & + \lambda_1 d_0 (\Delta_a^\psi t)^{n-\alpha+\gamma-1} E_{n-\alpha, n, n-\alpha+\gamma}^1(-\lambda_1 (\Delta_a^\psi t)^{n-\alpha}, -\lambda_2 (\Delta_a^\psi t)^n), \quad t \geq a. \end{aligned} \quad (11)$$

Note: For $0 < \Re(\alpha) < 1$, we will take $\gamma = \gamma_H$.

Proof. Under the stated assumptions, all terms in (10) admit generalized Laplace transforms. Since $u \in C_{\delta_\psi, \gamma}^n[a, b]$ is of ψ -exponential order s_0 , Theorem 1 ensures that $\mathcal{L}_\psi\{u^{[n]}\}(s)$ exists for $\Re(s) > s_0$. Moreover, the ψ -exponential order of $\delta_\psi^l I_{a+}^{1-\gamma; \psi} u$, $l = 0, 1$, implies, by Theorem 13 in [21] (p. 25), the existence of $\mathcal{L}_\psi\{{}^H D_{a+}^{\alpha, \beta; \psi} u\}(s)$. Finally, the continuity and ψ -exponential order s_1 of g guarantee that $\mathcal{L}_\psi\{g\}(s)$ exists.

Taking the generalized Laplace transform of both sides of the first equation in (10), and using its linearity as stated in Theorem 3.5 of [9] together with the initial conditions in (10), we obtain

$$\begin{aligned} \mathcal{L}_\psi\{g(t)\}(s) &= \mathcal{L}_\psi\{u^{[n]}(t)\}(s) + \lambda_1 \mathcal{L}_\psi\{{}^H D_{a+}^{\alpha, \beta; \psi} u(t)\}(s) + \lambda_2 \mathcal{L}_\psi\{u(t)\}(s) \\ &= s^n \mathcal{L}_\psi\{u(t)\}(s) - \sum_{l=0}^{n-1} s^{n-l-1} (\delta_\psi^l u)(a) + \lambda_1 [s^\alpha \mathcal{L}_\psi\{u(t)\}(s) - s^{\alpha-\gamma} (I_{a+}^{1-\gamma; \psi} u)(a^+)] + \lambda_2 \mathcal{L}_\psi\{u(t)\}(s) \\ &= s^n \mathcal{L}_\psi\{u(t)\}(s) - \sum_{l=0}^{n-1} c_l s^{n-l-1} + \lambda_1 [s^\alpha \mathcal{L}_\psi\{u(t)\}(s) - d_0 s^{\alpha-\gamma}] + \lambda_2 \mathcal{L}_\psi\{u(t)\}(s), \end{aligned}$$

for $\Re(s) > \max\{s_0, s_1\}$. Letting $\mathcal{L}_\psi\{u(t)\}(s) = U(s)$ and $\mathcal{L}_\psi\{g(t)\}(s) = G(s)$, we have

$$(s^n + \lambda_1 s^\alpha + \lambda_2)U(s) = G(s) + \sum_{l=0}^{n-1} c_l s^{n-l-1} + \lambda_1 d_0 s^{\alpha-\gamma}.$$

Solving for $U(s)$ yields

$$U(s) = \frac{1}{s^n + \lambda_1 s^\alpha + \lambda_2} G(s) + \sum_{l=0}^{n-1} c_l \frac{s^{n-l-1}}{s^n + \lambda_1 s^\alpha + \lambda_2} + d_0 \frac{\lambda_1 s^{\alpha-\gamma}}{s^n + \lambda_1 s^\alpha + \lambda_2}.$$

The desired expression follows by inverting the generalized Laplace transform term by term.

$$u(t) = \mathcal{L}_\psi^{-1} \left\{ \frac{1}{s^n + \lambda_1 s^\alpha + \lambda_2} G(s) \right\} (t) + \sum_{l=0}^{n-1} c_l \mathcal{L}_\psi^{-1} \left\{ \frac{s^{n-l-1}}{s^n + \lambda_1 s^\alpha + \lambda_2} \right\} (t) + \lambda_1 d_0 \mathcal{L}_\psi^{-1} \left\{ \frac{s^{\alpha-\gamma}}{s^n + \lambda_1 s^\alpha + \lambda_2} \right\} (t). \quad (12)$$

Beginning with the first term of (12), using Theorem 6 of [21] (p. 15), we have

$$\mathcal{L}_\psi^{-1} \left\{ \frac{1}{s^n + \lambda_1 s^\alpha + \lambda_2} G(s) \right\} (t) = \mathcal{L}_\psi^{-1} \left\{ \frac{1}{s^n + \lambda_1 s^\alpha + \lambda_2} \right\} (t) *_\psi \mathcal{L}_\psi^{-1} \{ G(s) \} (t). \quad (13)$$

To compute the first inverse generalized Laplace transform on the right-hand side of (13), we use the series expansion $\frac{1}{A+B} = \frac{1}{A} \sum_{k=0}^{\infty} (-1)^k \frac{B^k}{A^k}$, valid for $\left| \frac{B}{A} \right| < 1$. Taking $A = s^n$ and $B = \lambda_1 s^\alpha + \lambda_2$, we obtain

$$\mathcal{L}_\psi^{-1} \left\{ \frac{1}{s^n + \lambda_1 s^\alpha + \lambda_2} \right\} (t) = \mathcal{L}_\psi^{-1} \left\{ \frac{1}{s^n} \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{s^n} \right)^k (\lambda_1 s^\alpha + \lambda_2)^k \right\} (t), \quad (14)$$

for $\left| \frac{\lambda_1 s^\alpha + \lambda_2}{s^n} \right| < 1$. Applying the binomial expansion $(a+b)^k = \sum_{j=0}^k \binom{k}{j} a^{k-j} b^j$ with $a = \lambda_2$ and $b = \lambda_1 s^\alpha$, and then using Lemma 3.6, item 2, in [9], we obtain

$$\begin{aligned} \mathcal{L}_\psi^{-1} \left\{ \frac{1}{s^n} \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{s^n} \right)^k (\lambda_1 s^\alpha + \lambda_2)^k \right\} (t) &= \mathcal{L}_\psi^{-1} \left\{ \frac{1}{s^n} \sum_{k=0}^{\infty} \sum_{j=0}^k (-1)^k \binom{k}{j} \left(\frac{1}{s^n} \right)^k \lambda_2^{k-j} (\lambda_1 s^\alpha)^j \right\} (t) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^k (-1)^k \binom{k}{j} \lambda_1^j \lambda_2^{k-j} \mathcal{L}_\psi^{-1} \left\{ \frac{1}{s^{n(k+1)-j\alpha}} \right\} (t) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^k (-1)^k \binom{k}{j} \lambda_1^j \lambda_2^{k-j} \frac{(\Delta_a^\psi t)^{n(k+1)-1-j\alpha}}{\Gamma(n(k+1)-j\alpha)}. \end{aligned} \quad (15)$$

Substituting (14) and (15) into (13), and using Theorem 5, Definition 15 of [21] (p. 14), and Definition 5, we obtain

$$\begin{aligned} \mathcal{L}_\psi^{-1} \left\{ \frac{1}{s^n + \lambda_1 s^\alpha + \lambda_2} G(s) \right\} (t) &= \int_a^t \sum_{k=0}^{\infty} \sum_{j=0}^k (-1)^k \binom{k}{j} \lambda_1^j \lambda_2^{k-j} \frac{(\Delta_\tau^\psi t)^{n(k+1)-1-j\alpha}}{\Gamma(n(k+1)-j\alpha)} g(\tau) \psi'(\tau) d\tau \\ &= \int_a^t \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} (-1)^k \binom{k}{j} \lambda_1^j \lambda_2^{k-j} \frac{(\Delta_\tau^\psi t)^{n(k+1)-1-j\alpha}}{\Gamma(n(k+1)-j\alpha)} g(\tau) \psi'(\tau) d\tau \\ &= \int_a^t \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (-1)^{i+j} \binom{i+j}{j} \lambda_1^j \lambda_2^i \frac{(\Delta_\tau^\psi t)^{n(i+j+1)-1-j\alpha}}{\Gamma(n(i+j+1)-j\alpha)} g(\tau) \psi'(\tau) d\tau \\ &= \int_a^t (\Delta_\tau^\psi t)^{n-1} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (-1)^{i+j} \frac{(i+j)!}{j!i!} \lambda_1^j \lambda_2^i \frac{(\Delta_\tau^\psi t)^{(n-\alpha)j+ni}}{\Gamma((n-\alpha)j+ni+n)} g(\tau) \psi'(\tau) d\tau \\ &= \int_a^t (\Delta_\tau^\psi t)^{n-1} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (-1)^{i+j} \frac{(1)_{i+j}}{j!i!} \lambda_1^j \lambda_2^i \frac{(\Delta_\tau^\psi t)^{(n-\alpha)j+ni}}{\Gamma((n-\alpha)j+ni+n)} g(\tau) \psi'(\tau) d\tau \\ &= \int_a^t (\Delta_\tau^\psi t)^{n-1} E_{n-\alpha, n, n}^1(-\lambda_1 (\Delta_\tau^\psi t)^{n-\alpha}, -\lambda_2 (\Delta_\tau^\psi t)^n) g(\tau) \psi'(\tau) d\tau. \end{aligned} \quad (16)$$

Similarly, for the second and third terms of (12), taking $\left| \frac{\lambda_1 s^\alpha + \lambda_2}{s^n} \right| < 1$, using Lemma 3.6 in [9] and Definition 5, we obtain

$$\begin{aligned} \mathcal{L}_\psi^{-1} \left\{ \frac{s^{n-l-1}}{s^n + \lambda_1 s^\alpha + \lambda_2} \right\} (t) &= \sum_{k=0}^{\infty} \sum_{j=0}^k (-1)^k \binom{k}{j} \lambda_1^j \lambda_2^{k-j} \frac{(\Delta_a^\psi t)^{nk+l-j\alpha}}{\Gamma(nk+l+1-j\alpha)} \\ &= (\Delta_a^\psi t)^l E_{n-\alpha, n, l+1}^1 (-\lambda_1 (\Delta_a^\psi t)^{n-\alpha}, -\lambda_2 (\Delta_a^\psi t)^n), \end{aligned} \quad (17)$$

$$\begin{aligned} \mathcal{L}_\psi^{-1} \left\{ \frac{s^{\alpha-\gamma}}{s^n + \lambda_1 s^\alpha + \lambda_2} \right\} (t) &= \sum_{k=0}^{\infty} \sum_{j=0}^k (-1)^k \binom{k}{j} \lambda_1^j \lambda_2^{k-j} \frac{(\Delta_a^\psi t)^{n(k+1)-1-(j+1)\alpha+\gamma}}{\Gamma(n(k+1)-(j+1)\alpha+\gamma)} \\ &= (\Delta_a^\psi t)^{n-\alpha+\gamma-1} E_{n-\alpha, n, n-\alpha+\gamma}^1 (-\lambda_1 (\Delta_a^\psi t)^{n-\alpha}, -\lambda_2 (\Delta_a^\psi t)^n). \end{aligned} \quad (18)$$

Substituting (16)–(18) into (12) yields the general solution (11). $\square$

Theorem 6. Let $\alpha \in \mathbb{C}$ with $m-1 < \Re(\alpha) < m$, $m \in \mathbb{N} \setminus \{1\}$, and let $\beta \in [0, 1]$. Define $\gamma = m - \gamma_H$, where $\gamma_H = \alpha + \beta(m - \alpha)$. Suppose $\psi \in C^1[a, \infty)$ is a strictly increasing function with $\psi'(t) > 0$ for all $t \geq a$. Let $n \in \mathbb{N}$ with $n \geq m$, and let $u \in C_{\delta_\psi, \gamma}^n[a, b]$ for any $b > a$. Assume that

$$\frac{1}{2} \leq \Re(\gamma) < 1.$$

Assume further that u and $\delta_\psi^l I_{a^+}^{m-\gamma_H; \psi} u$, $l = 0, 1, \dots, m$, are of ψ -exponential order s_0 . Then, for any $g \in C_{\delta_\psi, \gamma}^n[a, b]$ of ψ -exponential order s_1 , the general solution of the hybrid Cauchy problem

$$\begin{cases} u^{[n]}(t) + \lambda_1 ({}^H D_{a^+}^{\alpha, \beta; \psi} u)(t) + \lambda_2 u(t) = g(t), & \lambda_1, \lambda_2 \in \mathbb{R}, t > a, \\ u(a) = u^{[1]}(a) = \dots = u^{[m-2]}(a) = 0, u^{[l]}(a) = c_l, & c_l \in \mathbb{R}, l = m-1, m, \dots, n-1, \\ (I_{a^+}^{m-\gamma_H; \psi} u)^{[l]}(a^+) = d_l, & d_l \in \mathbb{R}, \forall l = 0, 1, \dots, m-1, \end{cases} \quad (19)$$

where $u^{[n]} := \delta_\psi^n u$, is given by

$$\begin{aligned} u(t) &= \int_a^t (\Delta_\tau^\psi t)^{n-1} E_{n-\alpha, n, n}^1 (-\lambda_1 (\Delta_\tau^\psi t)^{n-\alpha}, -\lambda_2 (\Delta_\tau^\psi t)^n) g(\tau) \psi'(\tau) d\tau \\ &+ \sum_{l=m-1}^{n-1} c_l (\Delta_a^\psi t)^l E_{n-\alpha, n, l+1}^1 (-\lambda_1 (\Delta_a^\psi t)^{n-\alpha}, -\lambda_2 (\Delta_a^\psi t)^n) \\ &+ \lambda_1 \sum_{l=0}^{m-1} d_l (\Delta_a^\psi t)^{n-\alpha+\gamma_H-m+l-2} E_{n-\alpha, n, n-\alpha+\gamma_H-m+l-1}^1 (-\lambda_1 (\Delta_a^\psi t)^{n-\alpha}, -\lambda_2 (\Delta_a^\psi t)^n), \quad t \geq a. \end{aligned} \quad (20)$$

Proof. Under the stated hypotheses, each component appearing in (19) admits a well-defined generalized Laplace transform. In particular, membership of u in $C_{\delta_\psi, \gamma}^n[a, b]$ together with its ψ -exponential growth bound s_0 implies, by Theorem 1, that $\mathcal{L}_\psi \{u^{[n]}\}(s)$ exists for $\Re(s) > s_0$. Furthermore, the assumed ψ -exponential order of $\delta_\psi^l I_{a^+}^{m-\gamma_H; \psi} u$ for $l = 0, 1, \dots, m$, combined with the vanishing initial conditions $u^{[l]}(a) = 0$ for $l = 0, 1, \dots, m-2$, ensures, by Theorem 4, the existence of the generalized Laplace transform of the ψ -Hilfer derivative $\mathcal{L}_\psi \{{}^H D_{a^+}^{\alpha, \beta; \psi} u\}(s)$. Finally, since g is continuous and of ψ -exponential order s_1 , its generalized Laplace transform $\mathcal{L}_\psi \{g\}(s)$ is also well defined.

The generalized Laplace transform is applied to the first equation in (19), and linearity (Theorem 3.5 in [9]) yields

$$\begin{aligned}
\mathcal{L}_\psi\{g(t)\}(s) &= \mathcal{L}_\psi\{u^{[n]}(t)\}(s) + \lambda_1 \mathcal{L}_\psi\{{}^H D_{a^+}^{\alpha,\beta;\psi} u(t)\}(s) + \lambda_2 \mathcal{L}_\psi\{u(t)\}(s) \\
&= s^n \mathcal{L}_\psi\{u(t)\}(s) - \sum_{l=0}^{n-1} s^{n-l-1} (\delta_\psi^l u)(a) + \lambda_1 \left[s^\alpha \mathcal{L}_\psi\{u(t)\}(s) - s^{\alpha-\gamma_H} \sum_{l=0}^{m-1} s^{m-l-1} (\delta_\psi^l I_{a^+}^{m-\gamma_H;\psi} u)(a^+) \right] + \lambda_2 \mathcal{L}_\psi\{u(t)\}(s) \\
&= s^n \mathcal{L}_\psi\{u(t)\}(s) - \sum_{l=m-1}^{n-1} c_l s^{n-l-1} + \lambda_1 \left[s^\alpha \mathcal{L}_\psi\{u(t)\}(s) - s^{\alpha-\gamma_H} \sum_{l=0}^{m-1} d_l s^{m-l-1} \right] + \lambda_2 \mathcal{L}_\psi\{u(t)\}(s),
\end{aligned}$$

for $\Re(s) > \max\{s_0, s_1\}$. The last line is obtained by applying the initial conditions $u(a) = u^{[1]}(a) = \dots = u^{[m-2]}(a) = 0$ and $u^{[l]}(a) = c_l$ for $l = m-1, m, \dots, n-1$, where $c_l \in \mathbb{R}$.

Letting $\mathcal{L}_\psi\{u(t)\}(s) = U(s)$ and $\mathcal{L}_\psi\{g(t)\}(s) = G(s)$, we have

$$(s^n + \lambda_1 s^\alpha + \lambda_2)U(s) = G(s) + \sum_{l=m-1}^{n-1} c_l s^{n-l-1} + \lambda_1 s^{\alpha-\gamma_H} \sum_{l=0}^{m-1} d_l s^{m-l-1}.$$

Solving for $U(s)$ yields

$$U(s) = \frac{1}{s^n + \lambda_1 s^\alpha + \lambda_2} G(s) + \sum_{l=m-1}^{n-1} c_l \frac{s^{n-l-1}}{s^n + \lambda_1 s^\alpha + \lambda_2} + \lambda_1 s^{\alpha-\gamma_H} \sum_{l=0}^{m-1} \frac{d_l s^{m-l-1}}{s^n + \lambda_1 s^\alpha + \lambda_2}.$$

Termwise inversion yields

$$u(t) = \mathcal{L}_\psi^{-1} \left\{ \frac{1}{s^n + \lambda_1 s^\alpha + \lambda_2} G(s) \right\}(t) + \sum_{l=m-1}^{n-1} c_l \mathcal{L}_\psi^{-1} \left\{ \frac{s^{n-l-1}}{s^n + \lambda_1 s^\alpha + \lambda_2} \right\}(t) + \lambda_1 \sum_{l=0}^{m-1} d_l \mathcal{L}_\psi^{-1} \left\{ \frac{s^{\alpha-\gamma_H+m-l-1}}{s^n + \lambda_1 s^\alpha + \lambda_2} \right\}(t). \quad (21)$$

The inverse Laplace transforms of the first and second terms are identical to the corresponding terms in (12) from Theorem 5, yielding (16) and (17), respectively. Similarly, for the third term of (21), taking $\left| \frac{\lambda_1 s^{\alpha+\lambda_2}}{s^n} \right| < 1$, using Lemma 3.6 in [9], and Definition 5, we obtain

$$\begin{aligned}
\mathcal{L}_\psi^{-1} \left\{ \frac{s^{\alpha-\gamma_H+m-l-1}}{s^n + \lambda_1 s^\alpha + \lambda_2} \right\}(t) &= \sum_{k=0}^{\infty} \sum_{j=0}^k (-1)^k \binom{k}{j} \lambda_1^j \lambda_2^{k-j} \frac{(\Delta_a^\psi t)^{n(k+1)-2-(j+1)\alpha+\gamma_H-m+l}}{\Gamma(n(k+1)-(j+1)\alpha+\gamma_H-m+l-1)} \\
&= (\Delta_a^\psi t)^{n-\alpha+\gamma_H-m+l-2} E_{n-\alpha, n, n-\alpha+\gamma_H-m+l-1}^1(-\lambda_1 (\Delta_a^\psi t)^{n-\alpha}, -\lambda_2 (\Delta_a^\psi t)^n). \quad (22)
\end{aligned}$$

Substituting (16), (17) and (22) into (21) yields the general solution (20). $\square$

As an important special case of our main results with $n = 1$ in Theorem 5, we obtain the following explicit solution expressed in terms of the bivariate Mittag-Leffler function.

Corollary 2. Let $\alpha \in \mathbb{C}$ with $0 < \Re(\alpha) < 1$, $\beta \in [0, 1]$, and $\gamma = \alpha + \beta(1 - \alpha)$. Suppose $\psi \in C^1[a, \infty)$ is a strictly increasing function with $\psi'(t) > 0$ for all $t \geq a$, and let $u \in C_{\delta_\psi, \gamma}^1[a, b]$ for any $b > a$. Assume further that u and $\delta_\psi^k I_{a^+}^{1-\gamma;\psi} u$ for $k = 0, 1$ are of ψ -exponential order s_0 . Then, for any $g \in C_{\delta_\psi, \gamma}^1[a, b]$ of ψ -exponential order s_1 , the general solution of the hybrid fractional Cauchy problem

$$\begin{cases} u^{[1]}(t) + \lambda_1 ({}^H D_{a^+}^{\alpha,\beta;\psi} u)(t) + \lambda_2 u(t) = g(t), & \lambda_1, \lambda_2 \in \mathbb{R}, t > a, \\ u(a) = c_0, & (I_{a^+}^{1-\gamma;\psi} u)(a^+) = d_0, & c_0, d_0 \in \mathbb{R}, \end{cases} \quad (23)$$

where $u^{[1]} := \delta_\psi u$, is given by

$$u(t) = \int_a^t E_{1-\alpha,1,1}^1(-\lambda_1(\Delta_\tau^\psi t)^{1-\alpha}, -\lambda_2(\Delta_\tau^\psi t))g(\tau)\psi'(\tau)d\tau + c_0 E_{1-\alpha,1,1}^1(-\lambda_1(\Delta_a^\psi t)^{1-\alpha}, -\lambda_2(\Delta_a^\psi t)) \\ + \lambda_1 d_0 (\Delta_a^\psi t)^{\gamma-\alpha} E_{1-\alpha,1,1-\alpha+\gamma}^1(-\lambda_1(\Delta_a^\psi t)^{1-\alpha}, -\lambda_2(\Delta_a^\psi t)), \quad t \geq a. \quad (24)$$

Remark 5. Corollary 2 generalizes Theorem 14 of [21] (p. 27) to the hybrid Cauchy problem (23) with $\lambda_2 = 0$. When $\lambda_2 = 0$, the solution (24), expressed via the bivariate Mittag–Leffler function, simplifies, by Remark 4, to a form involving the two-parameter Mittag–Leffler function:

$$u(t) = \int_a^t E_{1-\alpha,1,1}^1(-\lambda_1(\Delta_\tau^\psi t)^{1-\alpha}, 0)g(\tau)\psi'(\tau)d\tau + c_0 E_{1-\alpha,1,1}^1(-\lambda_1(\Delta_a^\psi t)^{1-\alpha}, 0) \\ + \lambda_1 d_0 (\Delta_a^\psi t)^{\gamma-\alpha} E_{1-\alpha,1,1-\alpha+\gamma}^1(-\lambda_1(\Delta_a^\psi t)^{1-\alpha}, 0) \\ = \int_a^t E_{1-\alpha,1}(-\lambda_1(\Delta_\tau^\psi t)^{1-\alpha})g(\tau)\psi'(\tau)d\tau + c_0 E_{1-\alpha,1}(-\lambda_1(\Delta_a^\psi t)^{1-\alpha}) \\ + \lambda_1 d_0 (\Delta_a^\psi t)^{\gamma-\alpha} E_{1-\alpha,\gamma-\alpha+1}(-\lambda_1(\Delta_a^\psi t)^{1-\alpha}), \quad t \geq a.$$

We now present important special cases of our main results. For $n = 2$ (Theorem 5), the solutions simplify to the following explicit forms involving the bivariate Mittag–Leffler function.

Corollary 3. Let $\alpha \in \mathbb{C}$ with $0 < \Re(\alpha) < 1$, $\beta \in [0, 1]$, and $\gamma = \alpha + \beta(1 - \alpha)$. Suppose $\psi \in C^1[a, \infty)$ is a strictly increasing function with $\psi'(t) > 0$ for all $t \geq a$, and let $u \in C_{\delta_\psi, \gamma}^2[a, b]$ for any $b > a$. Assume further that u and $\delta_\psi^k I_{a+}^{1-\gamma;\psi} u$ for $k = 0, 1$ are of ψ -exponential order s_0 . Then, for any $g \in C_{\delta_\psi, \gamma}^2[a, b]$ of ψ -exponential order s_1 , the general solution of the hybrid Cauchy problem

$$\begin{cases} u^{[2]}(t) + \lambda_1 ({}^H D_{a+}^{\alpha, \beta; \psi} u)(t) + \lambda_2 u(t) = g(t), & \lambda_1, \lambda_2 \in \mathbb{R}, \quad t > a, \\ u^{[l]}(a) = c_l, & c_l \in \mathbb{R}, \quad l = 0, 1, \\ (I_{a+}^{1-\gamma; \psi} u)(a^+) = d_0, & d_0 \in \mathbb{R}, \end{cases} \quad (25)$$

where $u^{[2]} := \delta_\psi^2 u$, is given by

$$u(t) = \int_a^t (\Delta_\tau^\psi t) E_{2-\alpha, 2, 2}^1(-\lambda_1(\Delta_\tau^\psi t)^{2-\alpha}, -\lambda_2(\Delta_\tau^\psi t)^2)g(\tau)\psi'(\tau)d\tau + \sum_{l=0}^1 c_l (\Delta_a^\psi t)^l E_{2-\alpha, 2, l+1}^1(-\lambda_1(\Delta_a^\psi t)^{2-\alpha}, -\lambda_2(\Delta_a^\psi t)^2) \\ + \lambda_1 d_0 (\Delta_a^\psi t)^{1-\alpha+\gamma} E_{2-\alpha, 2, 2-\alpha+\gamma}^1(-\lambda_1(\Delta_a^\psi t)^{2-\alpha}, -\lambda_2(\Delta_a^\psi t)^2), \quad t \geq a. \quad (26)$$

4. Illustrative Example

The theoretical framework established in the preceding subsections, particularly the generalized Laplace transform and the solution theorems for hybrid fractional Cauchy problems, provides a powerful analytical toolset for modeling real-world dynamical systems with memory and hereditary effects. In this section, we apply our generalized Laplace transform framework to a fundamental mechanical system: the mass–spring–damper oscillator. By replacing the classical damping term with a ψ -Hilfer fractional derivative, we extend the model to capture memory-dependent viscoelastic behavior. Using Corollary 3, we derive explicit solutions in terms of bivariate Mittag–Leffler functions under various forcing conditions, constant, linear, harmonic, and exponentially decaying harmonic, demonstrating the impact of fractional damping on system dynamics in real-world applications such as material testing and vibration control.

4.1. Mechanical Model and Fractional Generalization

Figure 1 shows a mass–spring–damper system, a basic model in mechanical vibrations. According to Newton’s second law, the balance of inertial, damping, and elastic forces yields the linear second-order ODE:

$$mx''(t) + cx'(t) + kx(t) = F(t). \quad (27)$$

The model parameters have the following physical meanings and dimensions:

- m : system mass (inertia), $[m] = M$
- t : time, $[t] = T$
- k : spring stiffness; restoring force is proportional to displacement, $[k] = MT^{-2}$
- c : damping coefficient; energy dissipation is proportional to velocity, $[c] = MT^{-1}$
- $x(t)$: displacement from equilibrium, $[x] = L$
- $F(t)$: external forcing function acting on the mass, $[F] = MLT^{-2}$

The analytical solution of (27) with initial conditions

$$x(0) = x_0 \quad \text{and} \quad x'(0) = v_0, \quad (28)$$

can be expressed using the impulse response function. The corresponding homogeneous solution is

$$x_h(t) = e^{-\zeta\omega_n t} (A \cos(\omega_d t) + B \sin(\omega_d t))$$

where

- $\omega_n = \sqrt{\frac{k}{m}}$ (Natural frequency) has dimension $[\omega_n] = T^{-1}$.
- $\zeta = \frac{c}{2m\omega_n}$ (Damping ratio) is dimensionless: $[\zeta] = 1$.
- $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ (Damped natural frequency) has dimension $[\omega_d] = T^{-1}$.
- $A = x_0$ has dimension $[A] = L$.
- $B = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}$ with $[B] = L$.

The specific form of $F(t)$ determines the solution to (27) and (28). Standard techniques like variation of parameters, undetermined coefficients, or convolution integrals yield

$$x(t) = x_h(t) + \frac{1}{m\omega_d} \int_0^t e^{-\zeta\omega_n(t-\tau)} \sin(\omega_d(t-\tau)) F(\tau) d\tau, \quad t \geq 0. \quad (29)$$

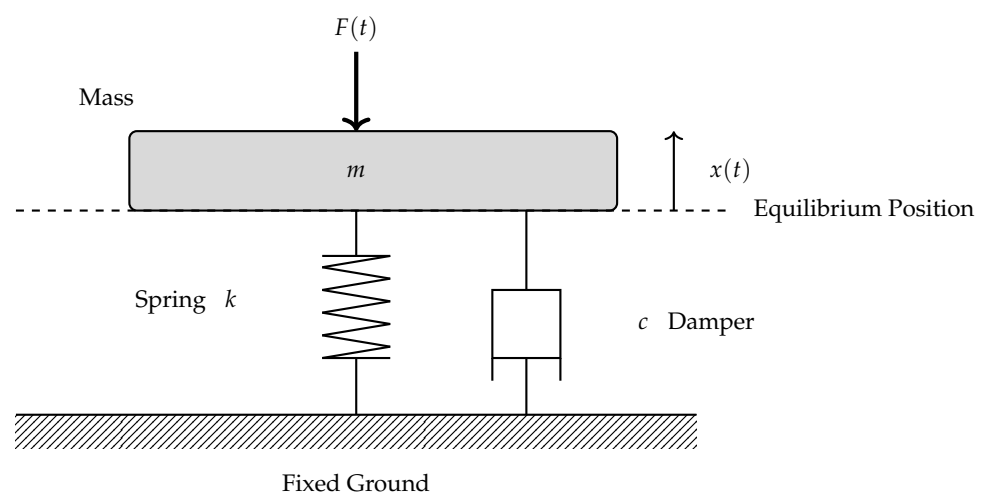


Figure 1. The mass–spring–damper system.

Classical viscous damping $cx'(t)$ assumes an instantaneous, memory-less response: an idealization. Real materials often display viscoelastic behavior with hereditary effects

and power-law memory, which integer-order derivatives cannot capture. Fractional calculus addresses this by replacing the classical damping term with a fractional derivative. Dimensional consistency must be maintained, however; $\frac{d}{dt}$ has dimension T^{-1} , whereas ${}^H D_{0+}^{\alpha,\beta;\psi}$ has $T^{-\alpha}$. Following [15], a characteristic time σ with $[\sigma] = T$ is introduced, giving the replacement

$$\frac{d}{dt} \longrightarrow \sigma^{\alpha-1} {}^H D_{0+}^{\alpha,\beta;\psi}.$$

This ensures that both sides have the same physical dimension, with the classical case recovered as $\alpha \rightarrow 1$ since $\sigma^{\alpha-1} \rightarrow 1$.

The resulting fractional model is:

$$m x^{[2]}(t) + c \sigma^{\alpha-1} ({}^H D_{0+}^{\alpha,\beta;\psi} x)(t) + k x(t) = F(t), \quad \alpha \in (0, 1), \quad t > 0 \quad (30)$$

where $x^{[2]} := \delta_\psi^2 x$ and ${}^H D_{0+}^{\alpha,\beta;\psi} x$ denotes the ψ -Hilfer fractional derivative of order α and type β of a function x . For the fractional ψ -Hilfer mass–spring–damper model, choosing $[\psi(t)] = T$ yields a dimensionless $\psi'(t)$ and maintains the dimensional structure of Newton's second law. Both the inertia term $m x^{[2]}(t)$ and the fractional damping term $c \sigma^{\alpha-1} ({}^H D_{0+}^{\alpha,\beta;\psi} x)(t)$ then carry the physical dimension of force, $[F(t)] = MLT^{-2}$. Dimensional consistency across all terms of (30) is thereby achieved, preserving the physical meaning of the fractional extension.

Mixed initial conditions for the fractional model are constructed to align with the classical conditions (28):

$$x(0) = x_0, \quad x^{[1]}(0) = \frac{v_0}{\psi'(0)}, \quad \text{and} \quad (I_{0+}^{1-\gamma;\psi} x)(0^+) = d_0, \quad (31)$$

where d_0 may be chosen arbitrarily, and $\gamma = \alpha + \beta(1 - \alpha) \in [\alpha, 1)$. The solution to the fractional model (30), subject to conditions (31), follows from Corollary 3 with $\lambda_1 = 2\zeta\omega_n\sigma^{\alpha-1}$, $\lambda_2 = \omega_n^2$, $a = 0$, and $g(t) = F(t)/m$:

$$\begin{aligned} x(t) = & \frac{1}{m} \int_0^t (\Delta_\tau^\psi t) E_{2-\alpha,2,2}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_\tau^\psi t)^{2-\alpha}, -\omega_n^2(\Delta_\tau^\psi t)^2) F(\tau) \psi'(\tau) d\tau \\ & + x_0 E_{2-\alpha,2,1}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_0^\psi t)^{2-\alpha}, -\omega_n^2(\Delta_0^\psi t)^2) + \frac{v_0}{\psi'(0)} (\Delta_0^\psi t) E_{2-\alpha,2,2}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_0^\psi t)^{2-\alpha}, -\omega_n^2(\Delta_0^\psi t)^2) \\ & + 2\zeta\omega_n\sigma^{\alpha-1} d_0 (\Delta_0^\psi t)^{1-\alpha+\gamma} E_{2-\alpha,2,2-\alpha+\gamma}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_0^\psi t)^{2-\alpha}, -\omega_n^2(\Delta_0^\psi t)^2), \quad t \geq 0. \end{aligned} \quad (32)$$

The parameter σ represents the characteristic memory time scale of the fractional damping. It measures how far back in time past velocities affect the present damping force, producing hereditary effects in the dynamics. A dimensionally consistent relation between the fractional order α and σ arises naturally for the mass–spring–damper system. Following the approach of Gómez-Aguilar et al. [15], we define this relation as

$$\alpha = \sigma\omega_n, \quad 0 < \sigma \leq \frac{1}{\omega_n}, \quad (33)$$

where $\omega_n = \sqrt{k/m}$ is the undamped natural frequency of the system. This expression ensures dimensional consistency, as both sides are dimensionless: $[\alpha] = 1$, $[\sigma] = T$, and $[\omega_n] = T^{-1}$, yielding $[\sigma\omega_n] = 1$. Moreover, the bound $0 < \sigma \leq 1/\omega_n$ ensures that α remains within the physically meaningful interval $(0, 1]$. This relation bounds σ by the intrinsic oscillation period $T_n = 2\pi/\omega_n$, keeping $\alpha \in (0, 1]$. As $\alpha \rightarrow 1$ (so $\sigma \rightarrow 1/\omega_n$), the fractional damping term collapses to classical viscous damping $c x'(t)$, recovering the integer-order model. Smaller α values produce stronger memory effects and more pronounced viscoelastic behavior, mimicking the power-law relaxation seen in polymers, biological tissues, and complex fluids. The fractional framework therefore generalizes the

classical oscillator, continuously interpolating between purely elastic ($\alpha \rightarrow 0$) and purely viscous ($\alpha = 1$) damping in a single constitutive model.

Comparing solutions (29) and (32) reveals how memory effects influence various physical systems, illustrating the broader modeling power of fractional calculus.

4.2. Physical Interpretation of the ψ -Hilfer Parameters

The fractional order $\alpha \in (0, 1)$ directly governs the amount of temporal memory in the ψ -Hilfer derivative's evolution law. In contrast to classical derivatives, which are local operators depending only on the instantaneous state, fractional derivatives are nonlocal and incorporate a weighted history of past states.

This hereditary structure follows directly from the definition of the ψ -Hilfer derivative

$${}^H D_{a^+}^{\alpha, \beta; \psi} f(t) = I_{a^+}^{\beta(1-\alpha); \psi} \left(\delta_{\psi} I_{a^+}^{(1-\beta)(1-\alpha); \psi} f \right)(t), \quad \delta_{\psi} = \frac{1}{\psi'(t)} \frac{d}{dt},$$

which is expressed in terms of ψ -Riemann–Liouville fractional integrals. Recall that for $\rho > 0$,

$$(I_{a^+}^{\rho; \psi} f)(t) = \frac{1}{\Gamma(\rho)} \int_a^t (\psi(t) - \psi(\tau))^{\rho-1} f(\tau) \psi'(\tau) d\tau.$$

Hence, the operator inherently involves an integration over the entire past interval $[a, t]$, so that all previous states $f(\tau)$ contribute to the present value.

The memory effect is governed by the kernel

$$(\psi(t) - \psi(\tau))^{\rho-1}, \quad 0 < \rho < 1,$$

whose exponent $\rho - 1 \in (-1, 0)$ produces a power-law decay. Power-law kernels decay much more slowly than exponential kernels; therefore, even remote past states retain a non-negligible influence on the current dynamics. This slow decay is the mathematical origin of long-range temporal memory in fractional models.

Since the effective fractional orders in the ψ -Hilfer derivative are proportional to $(1 - \alpha)$, decreasing α increases the strength of the history dependence. Physically, this corresponds to systems whose present response depends strongly on accumulated past deformation or motion. As a consequence, one typically observes

- slower relaxation toward equilibrium,
- reduced oscillation amplitudes,
- non-exponential (often algebraic) decay of transients.

The limiting cases clarify the role of α as a memory index.

As $\alpha \rightarrow 1^-$, the orders $\beta(1 - \alpha)$ and $(1 - \beta)(1 - \alpha)$ tend to zero. Using the property

$$\lim_{\rho \rightarrow 0^+} I_{a^+}^{\rho; \psi} f(t) = f(t)$$

for sufficiently smooth f , both fractional integrals reduce to the identity operator. Consequently,

$$\lim_{\alpha \rightarrow 1^-} {}^H D_{a^+}^{\alpha, \beta; \psi} f(t) = \delta_{\psi} f(t).$$

The operator becomes local in time, and the hereditary contribution disappears. In the special case $\psi(t) = t$, one recovers the classical first derivative and thus standard viscous damping with exponential relaxation. This shows that the fractional model continuously connects to the classical memoryless case.

When $\alpha \rightarrow 0^+$, the fractional integral orders approach their maximal values, and the weighting of past states becomes nearly uniform over long intervals. The system effectively

accumulates its entire history, resembling viscoelastic materials with strong hereditary effects. Relaxation becomes very slow, and the response may approach a quasi-static regime.

Therefore, α provides a continuous interpolation between memoryless viscous dynamics and strongly hereditary viscoelastic behavior. From a modeling perspective, α can be interpreted as a memory index that tunes how strongly past states influence present motion.

The parameter $\beta \in [0, 1]$ plays a different role. It does not change the overall strength of memory but redistributes it between the inner and outer fractional integrals. In this sense, β interpolates between ψ -Riemann–Liouville-type and ψ -Caputo-type behavior. Its primary influence appears in how initial history is weighted and how initial conditions enter the model, while the long-term decay and stability properties are governed mainly by α .

In the present mechanical model with mixed initial conditions, the contribution of β is therefore confined to the γ -dependent term in the general solution. When $d_0 = 0$, this term vanishes, and the solution becomes independent of β , explaining why variations in β do not affect the numerical responses reported here. In contrast, for $d_0 \neq 0$, the parameter β modulates the influence of fractional initial data on the early-time dynamics, allowing finer control of transient behavior associated with pre-existing memory or stress history.

The characteristic time σ : As introduced in (33), the parameter $\sigma > 0$ represents the intrinsic memory time scale of the material. It determines the temporal range over which past velocities contribute meaningfully to the present damping force. In this formulation, the relation $\alpha = \sigma\omega_n$ couples the fractional order to the system's natural time scale $1/\omega_n$, ensuring dimensional consistency and a physically interpretable parameterization. When σ is comparable to $1/\omega_n$, the system exhibits short memory, and α approaches 1, leading to behavior close to classical viscous damping. In contrast, smaller values of σ indicate long-lived memory effects, resulting in smaller α and enhanced hereditary behavior, as observed in materials with complex internal microstructure such as polymers and soft gels.

4.3. Analytical Solutions for Representative Forcing Functions

To keep the first term as a classical derivative, we set $\psi(t) = t$. With this choice,

$$x^{[2]} = (\delta_\psi)^2 x = \delta_\psi(\delta_\psi x) = \frac{1}{\psi'(t)} \frac{d}{dt} \left[\frac{1}{\psi'(t)} \frac{dx}{dt} \right] = \frac{1}{[\psi'(t)]^2} \frac{d^2 x}{dt^2} - \frac{\psi''(t)}{[\psi'(t)]^3} \frac{dx}{dt}$$

reduces to $d^2 x / dt^2$. Consequently, the derivative appearing in the second term $({}^H D_{0+}^{\alpha, \beta; t} x)(t)$ of Equation (30) reduces to the standard Hilfer fractional derivative.

For numerical illustration, we adopt the parameter set

$$m = 1 \text{ kg}, \quad k = 4 \text{ N/m}, \quad c = 0.2 \text{ kg/s}, \quad x_0 = 0 \text{ m}, \quad v_0 = 0 \text{ m/s}, \quad d_0 = 0, \quad \beta = 0.5$$

and vary $\alpha \in \{0.1, 0.5, 0.7, 0.9\}$. The following cases illustrate the qualitative impact of varying α .

Case I: Consider a constant forcing function $F(t) = F_0$, where F_0 (in N) represents a time-invariant external load. This scenario models static loading, such as a suspended weight or constant pressure, where F_0/k corresponds to the steady-state deflection of the spring.

Using (29) with $F(t) = F_0$, the solution of the classical problem (27) and (28) is

$$x(t) = \frac{F_0}{k} + e^{-\zeta\omega_n t} \left[\left(x_0 - \frac{F_0}{k} \right) \cos(\omega_d t) + \frac{1}{\omega_d} \left(v_0 + \zeta\omega_n \left(x_0 - \frac{F_0}{k} \right) \right) \sin(\omega_d t) \right], \quad t \geq 0. \quad (34)$$

The solution to the fractional problem (30) and (31) follows from (32) with $F(t) = F_0$:

$$\begin{aligned}
 x(t) = & \frac{F_0}{m} \int_0^t (\Delta_\tau^t) E_{2-\alpha,2,2}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_\tau^t)^{2-\alpha}, -\omega_n^2(\Delta_\tau^t)^2) d\tau \\
 & + x_0 E_{2-\alpha,2,1}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_0^t)^{2-\alpha}, -\omega_n^2(\Delta_0^t)^2) + v_0(\Delta_0^t) E_{2-\alpha,2,2}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_0^t)^{2-\alpha}, -\omega_n^2(\Delta_0^t)^2) \\
 & + 2\zeta\omega_n\sigma^{\alpha-1}d_0(\Delta_0^t)^{1-\alpha+\gamma} E_{2-\alpha,2,2-\alpha+\gamma}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_0^t)^{2-\alpha}, -\omega_n^2(\Delta_0^t)^2), \quad t \geq 0.
 \end{aligned} \quad (35)$$

For numerical illustration and graphical presentation, we adopt the following parameter set:

$$m = 1 \text{ kg}, \quad k = 4 \text{ N/m}, \quad c = 0.2 \text{ kg/s}, \quad x_0 = 0 \text{ m}, \quad v_0 = 0 \text{ m/s}, \quad d_0 = 0, \quad \beta = 0.5,$$

and vary the fractional order $\alpha \in \{0.1, 0.5, 0.7, 0.9\}$. The corresponding γ values, computed from $\gamma = \alpha + \beta(1 - \alpha)$, are 0.55, 0.75, 0.85, and 0.95, respectively.

Figure 2 presents a comparative analysis of the dynamic response of a mass–damper–spring system to a constant external force $F(t) = F_0 = 1 \text{ N}$, representing moderate static loading. The classical integer-order model (dashed line) with viscous damping ($\alpha = 1$) follows the analytical solution (34), exhibiting underdamped oscillations that converge exponentially to the steady-state displacement $F_0/k = 0.25 \text{ m}$. The fractional models (solid lines) incorporate memory effects through Hilfer fractional derivatives, with solutions given by (35). Lower values of α correspond to stronger memory effects in the fractional derivative, which for the chosen parameters result in reduced oscillation amplitudes compared to the classical model.

A constant external force $F(t) = F_0$ arises in many practical scenarios involving sustained loading. A typical example is the viscoelastic creep experiment, where a mass attached to a polymeric or biological spring suddenly bears a constant load—an extra weight or uniform pressure. After an initial transient governed by inertia and damping, the system slowly approaches equilibrium displacement F_0/k . In these materials, damping is inherently nonlocal and exhibits long-range memory, so relaxation deviates from the exponential decay predicted by classical integer-order models. The fractional formulation with Hilfer derivatives reproduces the characteristic power-law attenuation and memory-dependent settling seen in real viscoelastic media: polymers, elastomers, soft tissues, and engineered metamaterials. Hence, the constant-force case offers a physically meaningful benchmark for comparing classical viscous damping with the richer hereditary dynamics captured by the fractional model.

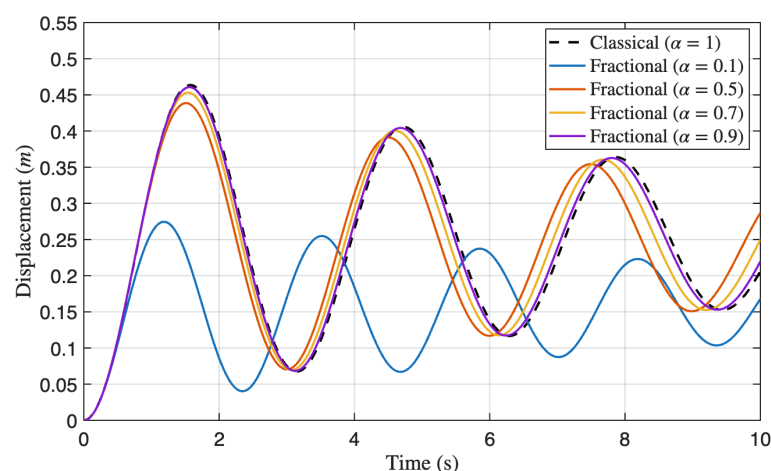


Figure 2. Dynamic response of a mass–damper–spring system to constant force $F(t) = F_0$: comparison between classical viscous damping ($\alpha = 1$) and fractional damping ($\alpha \in (0, 1)$).

Case II: Linearly increasing force $F(t) = F_0 + Rt$, where F_0 (in N) is a constant base load, and R (in N/s) is the ramp rate. This models gradual loading in structural tests or pressure buildup in fluid systems.

From (29) with $F(t) = F_0 + Rt$, the classical problem (27) and (28) gives

$$x(t) = \frac{F_0}{k} + \frac{R}{k}t - \frac{2R\zeta\omega_n}{k^2} + e^{-\zeta\omega_n t} \left[A \cos(\omega_d t) + \frac{B - A\zeta\omega_n}{\omega_d} \sin(\omega_d t) \right], \quad t \geq 0, \quad (36)$$

where

$$A = x_0 - \frac{F_0}{k} + \frac{2R\zeta\omega_n}{k^2}, \quad B = v_0 + 2\zeta\omega_n \left(x_0 - \frac{F_0}{k} \right) + \frac{R}{k}(4\zeta^2 - 1).$$

Using (32) with $F(t) = F_0 + Rt$, we obtain the solution to the fractional problem (30) and (31):

$$\begin{aligned} x(t) = & \frac{1}{m} \int_0^t (\Delta_\tau^t) E_{2-\alpha,2,2}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_\tau^t)^{2-\alpha}, -\omega_n^2(\Delta_\tau^t)^2)(F_0 + R\tau) d\tau \\ & + x_0 E_{2-\alpha,2,1}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_0^t)^{2-\alpha}, -\omega_n^2(\Delta_0^t)^2) + v_0(\Delta_0^t) E_{2-\alpha,2,2}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_0^t)^{2-\alpha}, -\omega_n^2(\Delta_0^t)^2) \\ & + 2\zeta\omega_n\sigma^{\alpha-1}d_0(\Delta_0^t)^{1-\alpha+\gamma} E_{2-\alpha,2,2-\alpha+\gamma}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_0^t)^{2-\alpha}, -\omega_n^2(\Delta_0^t)^2), \quad t \geq 0. \end{aligned} \quad (37)$$

The same parameter set as in Case I is used for numerical illustration, with identical α values. Figure 3 compares dynamic responses to a linearly increasing force $F(t) = F_0 + Rt$, where $F_0 = 1$ N, and the ramp rate is $R = 0.5$ N/s. This loading scenario models gradual force applications such as structural testing or pressure buildup. The classical integer-order model (dashed line, $\alpha = 1$) follows the analytical solution (36), exhibiting exponentially decaying transient oscillations around the quasi-static response $(F_0 + Rt)/k$. The fractional-order models (solid lines), with α values as specified above, incorporate memory effects through Hilfer fractional derivatives, yielding solutions given by (37). As α approaches 1, the fractional response converges to the classical behavior; lower values of α correspond to stronger memory effects, resulting in reduced oscillation amplitudes relative to the classical model.

A linearly increasing external force of the form $F(t) = F_0 + Rt$ arises naturally in controlled ramp-loading experiments widely used in materials science and viscoelasticity. In such tests, a mechanical loading device (e.g., an Instron or MTS machine) is programmed to increase the applied force at a constant rate in order to probe the rate-dependent response of polymers, biological tissues, and other materials exhibiting memory effects. The system initially experiences a transient inertial motion, after which the displacement evolves under the combined influence of the ramped loading and the material's hereditary dissipation. Fractional-order damping provides a realistic description of this behavior, as the resulting deformation typically follows power-law patterns that cannot be captured by classical integer-order models.

Case III: Consider a periodic force: $F(t) = F_0 \sin(\omega_f t)$, where F_0 (in N) is the force amplitude and ω_f (in rad/s) is the forcing frequency. Harmonic excitation like rotating machinery unbalance, seismic oscillations, or alternating electromagnetic forces is modeled here. Resonance occurs when ω_f nears the natural frequency ω_n .

From (29) with $F(t) = F_0 \sin(\omega_f t)$, the solution to the classical problem (27) and (28) is given by

$$x(t) = e^{-\zeta\omega_n t} [A \cos(\omega_d t) + B \sin(\omega_d t)] + X \sin(\omega_f t - \phi), \quad t \geq 0, \quad (38)$$

where

$$X = \frac{F_0/m}{\sqrt{(\omega_n^2 - \omega_f^2)^2 + (2\zeta\omega_n\omega_f)^2}}, \quad \phi = \arctan\left(\frac{2\zeta\omega_n\omega_f}{\omega_n^2 - \omega_f^2}\right),$$

$$A = x_0 + X \sin \phi, \quad B = \frac{1}{\omega_d} \left(v_0 + \zeta\omega_n(x_0 + X \sin \phi) - \omega_f X \cos \phi \right).$$

Using (32) with $F(t) = F_0 \sin(\omega_f t)$, we obtain the solution to the fractional problem (30) and (31):

$$x(t) = \frac{F_0}{m} \int_0^t (\Delta_\tau^t) E_{2-\alpha,2,2}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_\tau^t)^{2-\alpha}, -\omega_n^2(\Delta_\tau^t)^2) \sin(\omega_f \tau) d\tau$$

$$+ x_0 E_{2-\alpha,2,1}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_0^t)^{2-\alpha}, -\omega_n^2(\Delta_0^t)^2) + v_0(\Delta_0^t) E_{2-\alpha,2,2}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_0^t)^{2-\alpha}, -\omega_n^2(\Delta_0^t)^2) \quad (39)$$

$$+ 2\zeta\omega_n\sigma^{\alpha-1}d_0(\Delta_0^t)^{1-\alpha+\gamma} E_{2-\alpha,2,2-\alpha+\gamma}^1(-2\zeta\omega_n\sigma^{\alpha-1}(\Delta_0^t)^{2-\alpha}, -\omega_n^2(\Delta_0^t)^2), \quad t \geq 0.$$

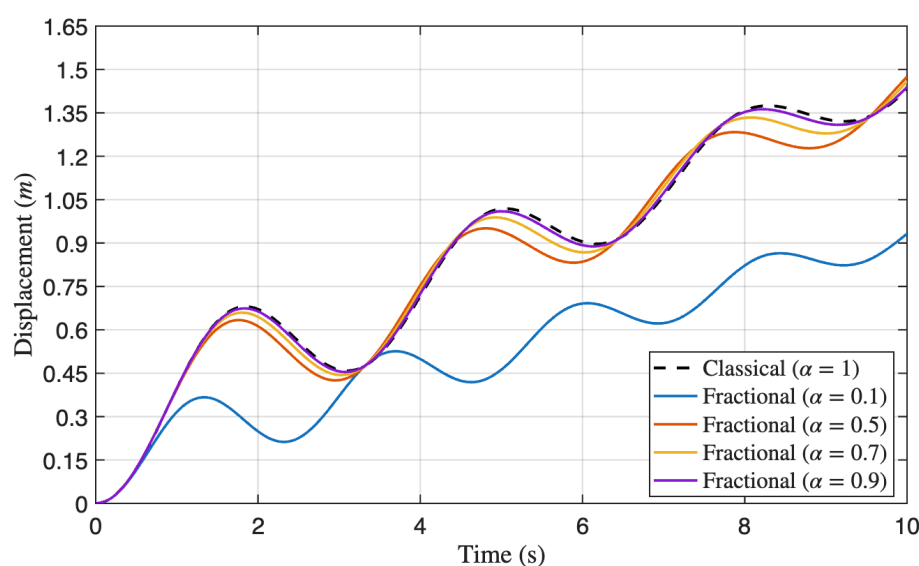


Figure 3. Dynamic response of a mass–damper–spring system to linearly increasing force $F(t) = F_0 + Rt$: comparison between classical viscous damping ($\alpha = 1$) and fractional damping ($\alpha \in (0, 1)$).

Using the same parameters as Case I, Figure 4 compares dynamic responses to harmonic excitation $F(t) = F_0 \sin(\omega_f t)$. The forcing frequency $\omega_f = 0.95\omega_n$ rad/s represents near-resonance conditions, producing significant amplification while avoiding destructive vibrations. The classical integer-order model (dashed line, $\alpha = 1$) follows (38), exhibiting a steady-state amplitude $X \approx 1.84$ m and phase lag $\phi \approx 0.77$ rad, with exponentially decaying transients superimposed on the steady-state motion. In contrast, the fractional-order models (solid lines) incorporate memory effects through Hilfer fractional derivatives, as given by (39). Lower α values correspond to stronger memory effects, which, for the parameters used here, result in reduced steady-state amplitudes and broader resonance peaks compared to the classical model.

A harmonic external force of the form $F(t) = F_0 \sin(\omega_f t)$ arises naturally in mechanical systems subjected to rotating imbalance. In electric motors, ventilation fans, pumps, and turbine rotors, even a small eccentricity in the mass distribution produces a sinusoidal inertial force acting on the supporting structure at the rotation frequency. This imbalance-induced harmonic loading is one of the most common sources of forced vibration in engineering practice and is responsible for resonance phenomena when the forcing frequency approaches the natural frequency of the system. In such applications,

fractional-order damping provides a more realistic representation of energy dissipation in viscoelastic mounts, elastomeric supports, and composite structures than classical integer-order damping models.

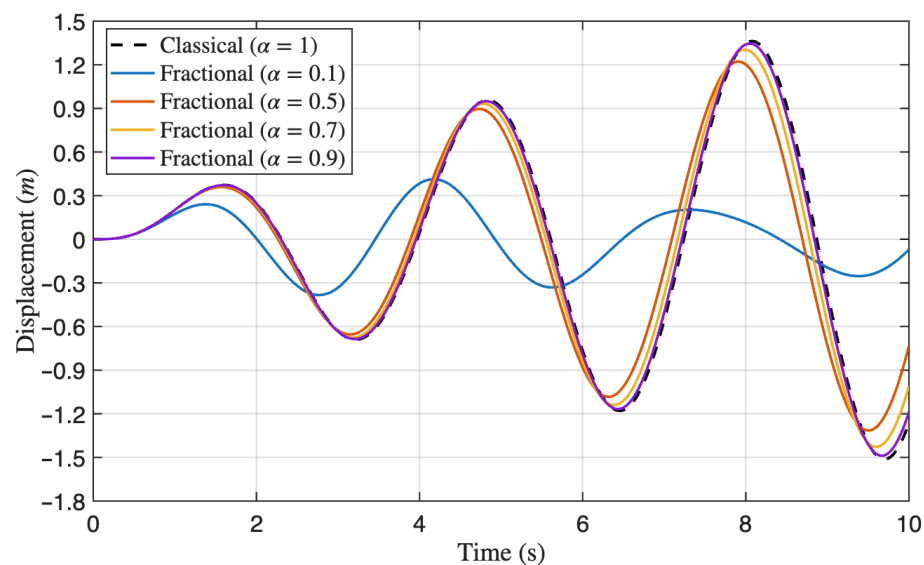


Figure 4. Dynamic response of a mass–damper–spring system to harmonic force $F(t) = F_0 \sin(\omega_f t)$: comparison between classical viscous damping ($\alpha = 1$) and fractional damping ($\alpha \in (0, 1)$).

Case IV: Consider an exponentially decaying harmonic force $F(t) = Ae^{-\theta t} \sin(\omega_f t)$, where A (in N) is the initial amplitude, θ (in s^{-1}) is the decay rate, and ω_f (in rad/s) is the forcing frequency. This models transient oscillatory loads such as impact-induced vibrations, damped wave forces, or pulsed energy inputs, where the excitation amplitude decays exponentially while maintaining oscillatory character.

From (29) with $F(t) = Ae^{-\theta t} \sin(\omega_f t)$, we obtain the solution to the classical problem (27) and (28):

$$x(t) = e^{-\zeta\omega_n t} [C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)] + \frac{A}{m} \cdot \frac{e^{-\theta t} [(\omega_n^2 - \theta^2 - \omega_f^2) \sin(\omega_f t) + 2\omega_f(\zeta\omega_n - \theta) \cos(\omega_f t)]}{(\omega_n^2 - \theta^2 - \omega_f^2)^2 + 4\omega_f^2(\zeta\omega_n - \theta)^2}, \quad t \geq 0, \quad (40)$$

where

$$C_1 = x_0 - \frac{A}{m} \cdot \frac{2\omega_f(\zeta\omega_n - \theta)}{(\omega_n^2 - \theta^2 - \omega_f^2)^2 + 4\omega_f^2(\zeta\omega_n - \theta)^2}, \quad C_2 = \frac{1}{\omega_d} \left(v_0 + \zeta\omega_n C_1 + \frac{A}{m} \cdot \frac{\omega_f(\omega_n^2 - \theta^2 - \omega_f^2) + 2\theta\omega_f(\zeta\omega_n - \theta)}{(\omega_n^2 - \theta^2 - \omega_f^2)^2 + 4\omega_f^2(\zeta\omega_n - \theta)^2} \right).$$

The solution to the fractional problem (30) and (31) follows from (32) with $F(t) = Ae^{-\theta t} \sin(\omega_f t)$:

$$\begin{aligned} x(t) = & \frac{A}{m} \int_0^t (\Delta_\tau^t) E_{2-\alpha, 2, 2}^1(-2\zeta\omega_n \sigma^{\alpha-1}(\Delta_\tau^t)^{2-\alpha}, -\omega_n^2(\Delta_\tau^t)^2) e^{-\theta\tau} \sin(\omega_f \tau) d\tau \\ & + x_0 E_{2-\alpha, 2, 1}^1(-2\zeta\omega_n \sigma^{\alpha-1}(\Delta_0^t)^{2-\alpha}, -\omega_n^2(\Delta_0^t)^2) + v_0(\Delta_0^t) E_{2-\alpha, 2, 2}^1(-2\zeta\omega_n \sigma^{\alpha-1}(\Delta_0^t)^{2-\alpha}, -\omega_n^2(\Delta_0^t)^2) \\ & + 2\zeta\omega_n \sigma^{\alpha-1} d_0(\Delta_0^t)^{1-\alpha+\gamma} E_{2-\alpha, 2, 2-\alpha+\gamma}^1(-2\zeta\omega_n \sigma^{\alpha-1}(\Delta_0^t)^{2-\alpha}, -\omega_n^2(\Delta_0^t)^2), \quad t \geq 0. \end{aligned} \quad (41)$$

Based on the parameter specification of Case I, Figure 5 presents a comparative analysis of the dynamic response of a mass-damper-spring system to an exponentially decaying harmonic force $F(t) = Ae^{-\theta t} \sin(\omega_f t)$. The initial force amplitude $A = 1$ N, decay rate $\theta = 0.5$ s^{-1} , and forcing frequency $\omega_f = 0.95\omega_n$ rad/s model transient oscillatory loads such as impact-induced vibrations or damped wave forces. The classical integer-order model (dashed line, $\alpha = 1$) follows (40), exhibiting oscillations that initially track the

decaying excitation before transitioning to free vibration decay governed by the system's natural damping. In contrast, the fractional-order models (solid lines) incorporate memory effects through Hilfer fractional derivatives, as given by (41). Lower α values correspond to stronger memory effects, which, for the parameters used here, result in reduced oscillation amplitudes and modified decay patterns compared to the classical model.

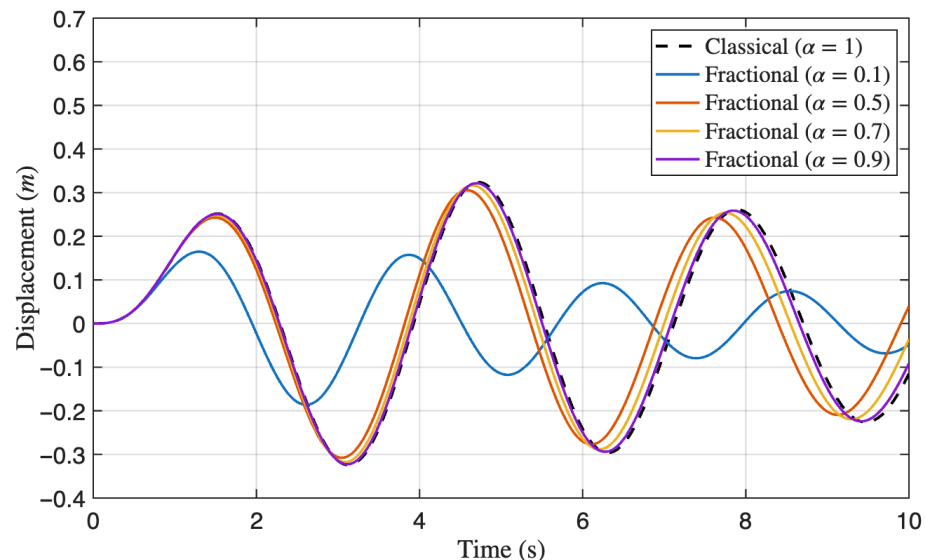


Figure 5. Dynamic response of a mass–damper–spring system to exponentially decaying harmonic force $F(t) = Ae^{-\theta t} \sin(\omega_f t)$: comparison between classical viscous damping ($\alpha = 1$) and fractional damping ($\alpha \in (0, 1)$).

An exponentially decaying harmonic force of the form $F(t) = Ae^{-\theta t} \sin(\omega_f t)$ naturally models transient seismic excitation. This force $F(t)$ originates from ground acceleration transmitted through the foundation to the structure. During an earthquake, ground acceleration exhibits an initial burst of oscillatory motion that decays rapidly due to soil damping, wave scattering, and loss of coherence in the seismic wave packet. A widely used idealization in structural dynamics represents this ground motion as a damped sinusoid, which induces an equivalent inertial force of the same form on a single-degree-of-freedom (SDOF) structure, the simplest lumped-mass model where the seismic response is characterized entirely by one horizontal displacement coordinate. This exponentially decaying harmonic loading captures the essential transient character of seismic events and provides a realistic, yet analytically tractable, framework for studying resonance, peak structural response, and the role of memory-dependent fractional damping mechanisms. The corresponding numerical solution Algorithm A1 for the SDOF system under this forcing function is detailed in Appendix B.

4.4. Quantitative Analysis and Stability Considerations

Key dynamic metrics from the constant-force response (Case I, Figure 2) quantify how the fractional order α affects behavior. The metrics are:

- Settling Time (T_s): time until the response stays within $\pm 5\%$ of $x_{ss} = F_0/k$
- Maximum Overshoot:

$$100 \times \frac{x_{\max} - x_{ss}}{x_{ss}}.$$

- Oscillation Period: time between successive peaks

Within the 0–10 s simulation window, none of the responses in Figure 2 settle completely inside the $\pm 5\%$ band, so $T_s > 10$ s for every case.

Table 4 reveals several clear trends. Overshoot diminishes with decreasing α , and the oscillation period also becomes somewhat shorter. Oscillations persist across all tested α values throughout the simulation. The fractional order therefore acts primarily as a damping-strength adjuster, not as a switch that completely suppresses oscillations.

Table 4. Metrics for constant-force response (Figure 2).

α	T_s (s)	Max. Overshoot (%)	Period (s)
1.0 (Classical)	>10	≈ 84	≈ 3.2
0.9	>10	≈ 82	≈ 3.2
0.7	>10	≈ 80	≈ 3.1
0.5	>10	≈ 76	≈ 3.0
0.1	>10	≈ 10	≈ 2.5

Stability behavior differs between fractional and classical oscillators. Classical oscillations ($\alpha = 1$) appear only when $\zeta < 1$. With $\zeta = 0.05$ held constant, reducing α raises effective damping yet maintains oscillations, even at $\alpha = 0.1$, where amplitude becomes quite small. Essentially, α introduces an additional damping-like parameter. Instead of a sharp overdamped transition, the system undergoes smooth, continuous changes in decay rate and frequency.

4.5. Discussion and Application Outlook

Parameter choices for α , β , and ψ follow the guidelines in Section 3.4 of [21] to ensure physical consistency. Lower α values produce solutions that depart noticeably from the classical integer-order response, indicating stronger memory and more prominent hereditary effects. As α nears 1, the hybrid fractional Cauchy solutions closely approach the classical ordinary-derivative model, showing only weak memory. This continuous interpolation between purely elastic ($\alpha \rightarrow 0$) and purely viscous ($\alpha = 1$) damping illustrates how the fractional framework can represent materials with intermediate viscoelastic properties.

We set $d_0 = 0$ in the mixed initial conditions (30) for all four cases. Given that $d_0 = 0$, the γ -dependent term in solution (32) vanishes, making the solution independent of the type parameter β . Hence, when no extra fractional-order initial information is supplied ($d_0 = 0$), β does not affect the dynamics. If $d_0 \neq 0$, however, β would directly enter through the γ -dependent terms, offering additional memory tuning and a broader parameter space for experimental data fitting. Experimental measurements ultimately determine suitable parameter values.

The fractional mass–spring–damper model presented here generalizes the classical second-order ODE (27) by replacing local viscous damping with a ψ -Hilfer fractional derivative that incorporates hereditary effects. Adding the characteristic time σ maintains dimensional consistency; the classical model is recovered exactly as $\alpha \rightarrow 1$, ensuring compatibility with standard vibration theory. Unlike classical exponential decay, fractional solutions involve bivariate Mittag–Leffler functions whose algebraic asymptotics capture long-range memory. This difference appears in every forcing scenario studied: Lower α yields slower relaxation, smaller overshoot, and modified oscillatory behavior that cannot be reproduced by adjusting the classical damping coefficient alone. Therefore, the fractional model better describes systems with viscoelastic or history-dependent dissipation, while still containing the classical formulation as a special limit.

5. Discussion

Moving to the space $C_{\delta_\psi, \gamma}^n[a, b]$ greatly expands what the generalized Laplace transform framework from [21] can analyze. Earlier approaches, including our own, were limited to first-order hybrid systems. The present framework now handles interactions between arbitrary integer-order δ_ψ derivatives and ψ -Hilfer fractional operators of order $\alpha \in (m-1, m)$. This step from the special case $n = m = 1$ to the general condition $m \leq n$ significantly widens the scope, allowing memory effects distributed across multiple differentiation orders to be modeled.

Functions whose n th derivative shows singular or memory-dependent behavior fit naturally into $C_{\delta_\psi, \gamma}^n[a, b]$. Classical spaces like $C^n[a, b]$ demand continuity of all standard derivatives. Here, continuity is required only for the weighted derivatives $\delta_\psi^k f$ ($k = 1, \dots, n-1$), while $\delta_\psi^n f$ must belong to the weighted space $C_{\gamma, \psi}[a, b]$ rather than be classically continuous. Consequently, solutions that are singular in their highest derivative remain mathematically well-defined.

In Theorem 2 II, the assumptions $(\delta_\psi^k f)(a) = 0$ ($k = 0, \dots, n-2$) serve two roles. Mathematically, they ensure the fractional integral $I_{a+}^{n-\alpha, \psi} f$ inherits the δ_ψ -differentiability of f . Physically, they correspond to vanishing initial generalized fluxes. These conditions are not just technical conveniences; they prevent incompatible singularities when integer-order and fractional operations are composed. In applications, such conditions arise naturally for systems starting from equilibrium (e.g., zero initial displacement/velocity in mechanics) or in boundary-value problems where initial histories are prescribed. Although restrictive for arbitrary initial data, they reflect the intrinsic coupling between integer-order initial conditions and fractional memory in hybrid systems.

By accommodating strong memory effects in physical fractional models, the proposed framework goes well beyond classical methods. The generalized Laplace transform also acts as a unifying analytical tool for a wide range of fractional problems, including those with single fractional operators, cascaded fractional derivatives, and hybrid systems mixing integer- and fractional-order dynamics.

An important observation from studying higher-order systems is the difficulty purely numerical methods face with hybrid fractional equations. Coupling integer-order dynamics to fractional memory can produce sensitive dependence on initial and intermediate states, which numerical schemes may struggle to capture accurately without analytical guidance. The exact solutions obtained here therefore serve a dual purpose: They provide benchmarks for assessing numerical methods, and they reveal structural properties of the solutions themselves.

There are several key symmetry properties in the developed framework. First, operator dualities emerge through the generalized Laplace transform, where integer-order δ_ψ -derivatives and fractional ψ -Hilfer derivatives map to algebraic terms ($\delta_\psi^n \leftrightarrow s^n$, ${}^H D_{a+}^{\alpha, \beta; \psi} \leftrightarrow s^\alpha$). This symmetric structure simplifies complex hybrid models. Second, the ψ -Hilfer derivative shows parameter symmetry via β , which continuously interpolates between Riemann–Liouville ($\beta = 0$) and Caputo ($\beta = 1$) types, revealing how different fractional formulations connect within a single unified description. Third, the resulting bivariate Mittag–Leffler solutions $E_{\alpha, \beta, \gamma}^\delta(x, y)$ exhibit series symmetry that couples integer and fractional orders. These symmetries offer both mathematical elegance and practical benefits: they yield exact analytical solutions that benchmark numerical methods, expose conservation-like structures through operational calculus, and provide insight into system behavior through parameter studies. Our work thus contributes to exploring the interplay between symmetry and fractional calculus.

6. Conclusions

This paper introduced a generalized Laplace transform framework for higher-order hybrid fractional Cauchy problems. Our contributions are threefold:

- (1) We derived generalized Laplace transforms for the n th-order δ_ψ -derivative and for higher-order ψ -Hilfer fractional derivatives within the weighted space $C_{\delta_\psi, \gamma}^n[a, b]$;
- (2) Using these transforms, we solved hybrid fractional Cauchy problems, which combine the n th-order δ_ψ -derivatives with ψ -Hilfer fractional operators of order $\alpha \in (m - 1, m)$ for $m \leq n$ under mixed initial conditions, obtaining exact solutions expressed in terms of bivariate Mittag–Leffler functions; and
- (3) We illustrated the mathematical framework by analyzing a mass–spring–damper system with fractional damping under various forcing conditions.

The proposed approach provides

- A systematic methodology for analyzing higher-order hybrid fractional systems
- Exact analytical solutions that can serve as benchmarks for numerical methods
- A mathematical structure that is suitable for describing memory-dependent phenomena.

While the solution forms (particularly the bivariate Mittag–Leffler functions) are mathematically well-suited to capturing hereditary effects, making them potentially relevant for viscoelastic systems, vibration analysis, and other applications involving memory, direct physical modeling would require additional validation through case studies and parameter estimation from experimental data. The example presented serves to demonstrate the mathematical methodology rather than to establish validated physical models.

The proposed transform offers a foundation for further analysis by reducing higher-order ψ -Hilfer fractional equations to Volterra integral equations in $C_{\delta_\psi, \gamma}^n[a, b]$, enabling fixed-point studies of well-posedness for broad classes of fractional systems.

The approach also offers a useful template for building or refining generalized Laplace transforms that work with newer fractional operators likely to appear in future studies. Weighted spaces with controlled singularities and carefully chosen differential structures provide a clear path for extending transform methods beyond the ψ -Hilfer family, potentially handling coming innovations in fractional calculus.

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Appendix A. Theorem Dependency Chart

Figure A1 shows how the results build logically from basic regularity findings to final solution theorems for hybrid fractional systems. The left column holds first-order foundations: Theorem 12 in [21] (p. 25) gives the $n = 1$ ψ -Caputo transform, and Theorem 2.I describes first-order regularity preservation. These are extended to arbitrary order $n \geq 1$ via Theorem 2.II (higher-order regularity) and Theorem 3 ($n \geq 1$ ψ -Caputo transform).

The middle column traces the development of ψ -Hilfer transform formulas. Theorem 13 in [21] (p. 25) for $n = 1$ draws on the first-order transform results, while Theorem 4 for $n > 1$ depends on higher-order regularity and the corresponding transform formulas. Theorem 1, centrally positioned, provides the essential transform for n -th-order δ_ψ -derivatives.

The right column presents the main applications: Theorem 5 solves hybrid problems with $0 < \Re(\alpha) < 1$, while Theorem 6 handles the case $m - 1 < \Re(\alpha) < m$ for $m \geq 2$. Both final theorems directly depend on the transform for n -th-order derivatives (Theorem 1), demonstrating how higher-order extensions enable solving more complex hybrid systems.

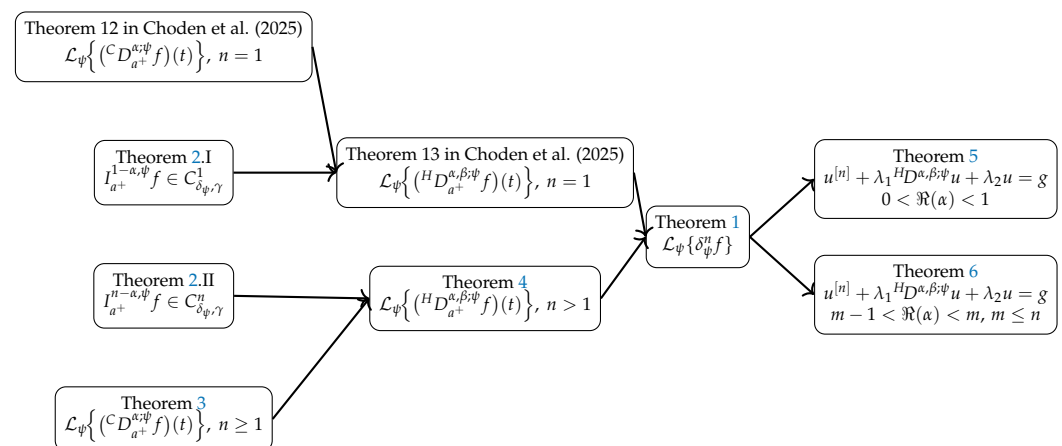


Figure A1. Dependency structure of the main results. Horizontal arrows between the first two columns indicate that the theorem at the tail (left) is used to prove the theorem at the head (right). Arrows connecting the second column to the central third column (Theorem 1) enable the proofs of the final theorems in the right column. Regularity and first-order transform results (left) extend to arbitrary order and ψ -Hilfer operators (middle). The generalized δ_ψ^n transform forms the central link leading to hybrid fractional Cauchy problem solutions (right). References: Theorem 12 and 13 from [21].

Appendix B. Numerical Implementation and Computational Details

This appendix describes the numerical strategy used to evaluate the analytical solutions derived in Section 4 for both the classical and fractional forced oscillator models. Although the fractional responses admit exact representations in terms of generalized (bivariate) Mittag–Leffler functions, their evaluation involves non-elementary kernels and convolution integrals that must be computed numerically at discrete time instants.

To ensure efficiency and accuracy, a precomputation–interpolation approach is adopted for the Mittag–Leffler kernels appearing in the fractional solution. For each fractional order, the required kernel functions are first evaluated on a uniform grid of time differences using truncated double-series representations. These precomputed values are then interpolated during the numerical evaluation of the convolution integrals, substantially reducing the computational cost compared to direct kernel evaluation at every time step.

The forcing term is treated in a unified manner, allowing constant, harmonic, ramp-type, and exponentially modulated excitations to be handled within the same computational framework. All convolution integrals are approximated using the trapezoidal rule, which provides a robust balance between accuracy and efficiency for the smooth integrands considered here. The complete MATLAB R2025b implementations used to generate the numerical results and figures reported in this work are made available to ensure full reproducibility.

Algorithm A1 Unified Numerical Framework for Classical and Fractional Forced Oscillators.

- 1: **Input:** (m, k, ζ) (or (m, k, c)), initial data (x_0, v_0, d_0) , orders $\{\alpha_i\}$, type parameter β , forcing $F(t)$, start time a , time grid $\{t_j\}$
- 2: Compute $\omega_n = \sqrt{k/m}$ and $\omega_d = \omega_n \sqrt{1 - \zeta^2}$
- 3: Choose quadrature nodes $\{\tau_q\} \subset [a, t_{\max}]$ and a time-difference grid $\{\Delta t_p\}$ with $\Delta t_1 > 0$
- 4: **Classical response:**
- 5: **if** a closed-form classical solution is available for the chosen forcing $F(t)$ **then**
- 6: **for** each t_j **do**
- 7: Evaluate $x_{\text{cl}}(t_j)$ using the known analytic formula
- 8: **end for**
- 9: **else**
- 10: **for** each t_j **do**
- 11: Approximate $x_{\text{cl}}(t_j)$ numerically on $\tau_q \leq t_j$ (e.g., by direct quadrature of the standard classical integral representation)
- 12: **end for**
- 13: **end if**
- 14: **Fractional response:**
- 15: **for** each α_i **do**
- 16: Set $\sigma(\alpha_i)$ and $\gamma(\alpha_i) = \alpha_i + \beta(1 - \alpha_i)$
- 17: Precompute kernel tables on $\{\Delta t_p\}$:

$$K_{\alpha_i}^{(2)}(\Delta t_p), \quad K_{\alpha_i}^{(1)}(\Delta t_p), \quad K_{\alpha_i}^{(\text{ic})}(\Delta t_p)$$
- 18: (using truncated bivariate Mittag-Leffler double sums)
- 19: **for** each t_j **do**
- 20: Restrict to $\tau_q \leq t_j$ and form $\Delta t_{jq} = t_j - \tau_q$
- 21: Interpolate $K_{\alpha_i}^{(2)}(\Delta t_{jq})$ from the table
- 22: Compute fractional convolution term:

$$I(t_j) \leftarrow \frac{1}{m} \int_a^{t_j} (t_j - \tau) K_{\alpha_i}^{(2)}(t_j - \tau) F(\tau) d\tau$$
- 23: Add initial-condition terms from the closed-form fractional solution:

$$x_{\alpha_i}(t_j) \leftarrow I(t_j) + x_0 K_{\alpha_i}^{(1)}(t_j) + v_0 t_j K_{\alpha_i}^{(2)}(t_j) + (\text{optional } d_0 \text{ term})$$
- 24: **end for**
- 25: **end for**
- 26: **Output:** $\{x_{\text{cl}}(t_j)\}$ and $\{x_{\alpha_i}(t_j)\}_i$

References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies; Elsevier: Amsterdam, The Netherlands, 2006.
2. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
3. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives*; Gordon and Breach Science Publishers: Yverdon, Switzerland, 1993.
4. Sousa, J.V.C.; Oliveira, E.C. On the Ψ -Hilfer Fractional Derivative. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *60*, 72–91. [\[CrossRef\]](#)
5. Jarad, F.; Abdeljawad, T. A Modified Laplace Transform for Certain Generalized Fractional Operators. *Res. Nonlinear Anal.* **2018**, *1*, 88–98.
6. El-Khatib, M.S.; Salim, T.O.; Hany, A.A. On Katugampola Laplace Transform. *Gen. Lett. Math.* **2020**, *9*, 93–100. [\[CrossRef\]](#)
7. Agarwal, G.; Gaur, A.; Nisar, K.S.; Abusufian, A.H. On Some Properties of β -Laplace Integral Transform. *J. Math. Comput. Sci.* **2020**, *23*, 315–320. [\[CrossRef\]](#)
8. Sadek, L.; Algefary, A. On Quantum Trigonometric Fractional Calculus. *Alex. Eng. J.* **2025**, *120*, 371–377. [\[CrossRef\]](#)
9. Jarad, F.; Abdeljawad, T. Generalized Fractional Derivatives and Laplace Transform. *Discret. Contin. Dyn. Syst. Ser. S* **2020**, *13*, 709–722. [\[CrossRef\]](#)

10. Medved', M.; Pospíšil, M. Generalized Laplace Transform and Tempered ψ -Caputo Fractional Derivative. *Math. Model. Anal.* **2023**, *28*, 146–162. [[CrossRef](#)]
11. Alsulami, M.; Al-Mazmumy, M.; Alyami, M.A.; Alsulami, A.S. Generalized Laplace Transform with Adomian Decomposition Method for Solving Fractional Differential Equations Involving ψ -Caputo Derivative. *Mathematics* **2024**, *12*, 3499. [[CrossRef](#)]
12. Torvik, P.J.; Bagley, R.L. On the Appearance of the Fractional Derivative in the Behavior of Real Materials. *J. Appl. Mech.* **1984**, *51*, 294–298. [[CrossRef](#)]
13. Mainardi, F.; Pironi, P.; Tampieri, F. On a Generalization of the Basset Problem via Fractional Calculus. In Proceedings of the Canadian Congress of Applied Mechanics, Edmonton, AB, Canada, 29 May–2 June 1995; pp. 836–837.
14. Díaz, M.V. The Basset Force and the Persistence of Hydrodynamic Memory. *CQD Rev. Eletrônica Paul. Mat.* **2022**, *22*, 211–219. [[CrossRef](#)]
15. Gómez-Aguilar, J.F.; Rosales-García, J.J.; Bernal-Alvarado, J.J.; Córdova-Fraga, T.; Guzmán-Cabrera, R. Fractional Mechanical Oscillators. *Rev. Mex. Fis.* **2012**, *58*, 348–352.
16. Sadek, L. Two Numerical Approaches for Solving Fractional Model of Chemical Kinetics Problem via Chebyshev Polynomials. *Prog. Fract. Differ. Appl.* **2024**, *10*, 601–613. [[CrossRef](#)]
17. Sadek, L.; Akgül, A. New Properties for Conformable Fractional Derivative and Applications. *Prog. Fract. Differ. Appl.* **2024**, *10*, 335–344. [[CrossRef](#)]
18. Gómez-Aguilar, J.; Yépez-Martínez, H.; Calderón-Ramón, C.; Cruz-Orduña, I.; Escobar-Jiménez, R.; Olivares-Peregrino, V. Modeling of a Mass-Spring-Damper System by Fractional Derivatives with and without a Singular Kernel. *Entropy* **2015**, *17*, 6289–6303. [[CrossRef](#)]
19. Hedrih, K.R.; Milovanović, G.V. Elements of Mathematical Phenomenology and Analogies of Electrical and Mechanical Oscillators of the Fractional Type with Finite Number of Degrees of Freedom of Oscillations: Linear and Nonlinear Modes. *Commun. Anal. Mech.* **2024**, *16*, 738–785. [[CrossRef](#)]
20. Hedrih, K.R. Rheologic Fractional Oscillators or Creepers. *Fractal Fract.* **2025**, *9*, 552. [[CrossRef](#)]
21. Choden, S.; Sompong, J.; Thailert, E.; Ntouyas, S.K. A New Laplace-Type Transform on Weighted Spaces with Applications to Hybrid Fractional Cauchy Problems. *Fractal Fract.* **2025**, *9*, 751. [[CrossRef](#)]
22. Sugumaran, H.; Ibrahim, R.; Kanagarajan, K. On ψ -Hilfer Fractional Differential Equation with Complex Order. *Univers. J. Math. Appl.* **2018**, *1*, 33–38. [[CrossRef](#)]
23. Fernandez, A.; Kürt, C.; Özarslan, M.A. A Naturally Emerging Bivariate Mittag–Leffler Function and Associated Fractional-Calculus Operators. *Comput. Appl. Math.* **2020**, *39*, 3. [[CrossRef](#)]

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