

Article

A Pre-Conditioning CQ Algorithm with Double Inertia and Self-Updated Stepsizes for Split Feasibility Problems in Hilbert Spaces

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Abstract

In this work, we propose a double inertial pre-conditioning CQ Algorithm for split feasibility problem in real Hilbert spaces, in which, we use the double inertial steps and new stepsizes to speed up the convergent rate. We also compute the only one projection onto the nonempty closed convex set C in our proposed method. These help improve the numerical results. Next, we establish the weak convergence of the sequence generated by our method. Finally, we use a numerical experiment to demonstrate our theoretical results.

Keywords: split feasibility problem ; double inertial steps; new stepsize strategy; CQ algorithm; weak convergence

MSC: 47J20; 49J40; 35A15; 47J25

1. Introduction

In this paper, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. The split feasibility problem (SFP) is as follows.

Find $g^* \in C$ such that $Ag^* \in Q$.

Its solution set is suggested as

$$\Omega = \{g^* \in C : Ag^* \in Q\}.$$

Here, C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 , respectively, and $A : H_1 \rightarrow H_2$ is a bounded linear operator.

The split feasibility problem (SFP) proposed by Censor and Elfving [1] was used to study inverse problems, and then it was applied to signal recovery [2] and image deblurring problems [3]. From then, the SFP has attracted attention, and many related algorithms have been created to investigate it deeply. For instance, Byrne [2,3] stated a classical CQ algorithm as follows:

$$j_{n+1} = P_C(j_n - \Gamma A^*(I - P_Q)A j_n), \quad \forall n \geq 1, \quad (1)$$

where $\Gamma \in \left(0, \frac{2}{\|A\|^2}\right)$, I represents an identity operator, P_C and P_Q are the projections onto C and Q , respectively, and A^* means the adjoint of A . For the CQ algorithms, more details can



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be found in [4–10]. We know that the algorithm (1) begins the iteration; it first computes the operator norm $\|A\|$, which creates a small stepsize Γ , and thus, the convergent rate of the algorithm is slow. Meanwhile, it is not easy to calculate the operator norm $\|A\|$ in practice. From this consideration, López [11] suggested the following self-adaptive stepsize by modifying Yang [12].

$$\Gamma_n = \frac{tU(j_n)}{\|\nabla U(j_n)\|^2}. \quad (2)$$

Here, $\inf t(4-t) > 0$, and $U(j) = \frac{1}{2}\|(I - P_Q)Aj\|^2$. Under the stepsize (2), the weak convergence of the algorithm (1) is proved. Meanwhile, several CQ algorithms with self-updated stepsizes have been proposed to solve the SFP; (see Refs. [13–17] and the references therein).

To introduce an efficient stepsize, Altıparmak et al. [18] presented a pre-conditioning CQ algorithm for solving the SFP, as follows:

$$\begin{cases} y_n = P_C^B(j_n - \Gamma_n \nabla U_{\tilde{E}}(j_n)); j_{n+1} = P_C^B(y_n - \gamma_n \nabla U_{\tilde{E}}(y_n)); \\ \text{where the stepsizes } \Gamma_n \text{ and } \gamma_n \text{ is updated respectively via} \\ \Gamma_n \|\nabla U(y_n) - \nabla U(j_n)\| \leq \delta \|y_n - j_n\|, 0 < \delta < 1, \\ \gamma_n = \frac{tU(y_n)}{\|\nabla U_{\tilde{E}}(y_n)\|^2}, 0 < t < 4, \end{cases} \quad (3)$$

where $\nabla U_{\tilde{E}}(j) = EA^*(I - P_Q)Aj$, $\nabla U(j) = A^*(I - P_Q)Aj$, E is defined in Section 3, $\Gamma_n = \sigma \rho^{m_n}$, $\sigma > 0$, $0 < \rho < 1$, and m_n is the smallest nonnegative integer. The authors analyzed the weak convergence of the algorithm (3) under these stepsizes. Pre-conditioning methods are usually applied to speed up the convergence of the algorithms, for more details, see [19–21].

On the other hand, the *inertial extrapolation technique* can be used to accelerate the convergence rate of the algorithms; for this, Nesterov [22] proposed the following classical iterative scheme

$$\begin{aligned} v_n &= j_n + u_n(j_n - j_{n-1}) \\ j_{n+1} &= F_n - \Gamma_n F(v_n), n \geq 1, \end{aligned} \quad (4)$$

where $u_n \in [0, 1)$, and $\Gamma_n > 0$. The notation v_n is the inertial step. Inspired by the inertial idea [23–27], some inertial splitting algorithms have been proposed to solve the SFP [28–31]. Recently, Wang et al. [32] proposed the following double inertial splitting methods to solve monotone inclusion problems.

$$\begin{cases} v_n = j_n + u_n(j_n - j_{n-1}), \\ s_n = j_n + q_n(j_n - j_{n-1}), \\ y_n = (I + \Gamma_n Y)^{-1}(I - \Gamma_n L)v_n; \\ j_{n+1} = (1 - \alpha_n)s_n + \alpha_n(y_n - \lambda_n(Ly_n - Lv_n)), \end{cases} \quad (5)$$

where the stepsize Γ_n is updated by

$$\Gamma_{n+1} = \begin{cases} \min\left\{\frac{(\mu_n + \delta)\|v_n - y_n\|}{\|Lv_n - Ly_n\|}, \Gamma_n + \hat{P}_n\right\}, & \text{if } Lv_n - Ly_n \neq 0, \\ \Gamma_n + \hat{P}_n, & \text{otherwise,} \end{cases}$$

Here, I is the identity operator on H . $Y : H \rightarrow 2^H$ is maximally monotone mapping, $L : H \rightarrow H$ is monotone and Lipschitz continuous mapping, $\delta \in (0, 1)$, $u_n \in [0, 1]$,

$0 \leq q_n \leq q_{n+1} < \frac{3+2d-\sqrt{8d+17}}{2d}, d \in (1, +\infty); 0 < \alpha_n \leq \alpha_{n+1} \leq \frac{1}{1+d}, d \in (1, +\infty)$, and $\sum_{n=1}^{\infty} \hat{P}_n < \infty$. Its convergence is established under suitable conditions in real Hilbert spaces.

It is observed that, in algorithm (3), the proper choice of the stepsize Γ_n is required to satisfy the Armijo-like linesearch techniques; that is, the projections of the iterate point Ay_n onto Q and the projections onto C may be calculated many times at each iteration. On the other hand, the algorithm (3) needs to compute the double projections onto C at least for finishing an iteration. These considerations affect its effectiveness. Furthermore, their weak convergence is guaranteed by employing some strong conditions such as the firm nonexpansiveness of $I - P_Q$. However, the investigation of the algorithm (3) with double inertial effects (5) has yet to be conducted. The following question is asked.

Question: Can we develop new modifications of the algorithm (3) such that numerical improvement and new convergence results can be obtained under weaker conditions than the firm nonexpansiveness of $I - P_Q$?

To answer this question, our contributions in this article are as follows:

- To adopt double inertial steps to speed up the convergent rate of the algorithm (3) and create a new convergence result;
- To design a new stepsize not only to reduce the computational projections of y_n onto Q but also to avoid the Armijo-like linesearch techniques [13,14,18]. Moreover, the projections P_C^B for the computation of the next iteration j_n can be removed, and thus, an accelerated convergent algorithm is generated;
- To provide some practical examples such as signal recovery problems for demonstration and illustration.

The paper is organized as follows. In Section 2, we list some useful results for the convergence analysis of the algorithm. In Section 3, we propose a double inertial preconditioning CQ algorithm and prove its weak convergence. Numerical experiments are described in Section 4.

2. Preliminaries

For the reader's convenience, we first introduce two notations; that is, $\rightharpoonup$ stands for weak convergence, while $\rightarrow$ stands for strong convergence. Second, we let H be a real Hilbert space, and we recall some well-known concepts related to the SFP.

Definition 1. Let C be a subset of H and $U : C \rightarrow H$ be an operator; then,

(i) U is called nonexpansive on C if

$$\|Uj - Uy\| \leq \|j - y\|, \forall j, y \in C.$$

(ii) F is said to be firmly nonexpansive on C if

$$\langle Uj - Uy, j - y \rangle \geq \|Uj - Uy\|^2, \forall j, y \in C.$$

Definition 2 ([18]). Let H be a real Hilbert space; a differentiable function $F : H \rightarrow \mathbb{R}$ is convex if and only if

$$U(y) - U(j) \geq \langle \nabla U(j), y - j \rangle, \forall j, y \in H.$$

Lemma 1. For all $j, y, z \in H$, then

- $\|\tilde{\alpha}j + (1 - \tilde{\alpha})y\|^2 = \tilde{\alpha}\|j\|^2 + (1 - \tilde{\alpha})\|y\|^2 - \tilde{\alpha}(1 - \tilde{\alpha})\|j - y\|^2, \tilde{\alpha} \in (0, 1)$.
- $\langle j - y, j - z \rangle = \frac{1}{2}\|j - y\|^2 + \frac{1}{2}\|j - z\|^2 - \frac{1}{2}\|y - z\|^2$.

Next, we provide some concepts about a positive-definite operator and a self-adjoint operator.

Let $B : H \rightarrow H$ be a bounded linear operator on a real Hilbert space. B is named self adjoint if $B^* = B$, where B^* is the adjoint operator. Moreover, a self adjoint operator is called positive definite if

$$\langle j, Bj \rangle > 0, \forall j \in H, j \neq 0.$$

For more details, see [33,34].

Let B be a self-adjoint positive-definite operator, and $E = B^{-1}$. Thereby, we define the norm $\|\cdot\|_B$ as

$$\|j\|_B^2 = \langle j, Bj \rangle = \langle j, j \rangle_B, \forall j \in H,$$

as in [18]. The B —projection operator on C related to the norm $\|\cdot\|_B$ denotes P_C^B ; that is,

$$P_C^B(j) = \operatorname{argmin}_{z \in C} \{\|j - z\|_B\}.$$

Now, we give the following related properties.

Lemma 2 ([18]). *Let C be a nonempty closed convex subset of a real Hilbert space H ; the B —projection operator on C fulfills the following relationships.*

- (i) $\langle B(j - P_C^B j), y - P_C^B j \rangle \leq 0, \forall j \in H$ and $y \in C$.
- (ii) $\|j \pm z\|_B^2 = \|j\|_B^2 \pm 2\langle j, Bz \rangle + \|z\|_B^2, \forall j, z \in H$.
- (iii) $\|P_C^B j - P_C^B y\|_B^2 \leq \|j - y\|_B^2 - \|(P_C^B j - j) - (P_C^B y - y)\|_B^2, \forall j, y \in H$.
- (iv) $\|P_C^B j - y\|_B^2 \leq \|j - y\|_B^2 - \|j - P_C^B j\|_B^2, \forall j, y \in H$.

Lemma 3 ([29,35]). *Let $\{\varphi_n\}$ be a sequence of nonnegative numbers fulfilling*

$$\varphi_{n+1} \leq \varphi_n \varphi_n + U_n, \forall n \in \mathbb{N},$$

where $\{\varphi_n\}$ and $\{U_n\}$ are sequences of nonnegative numbers such that $\{\varphi_n\} \subset [1, +\infty)$, $\sum_{n=1}^{\infty} (\varphi_n - 1) < \infty$, and $\sum_{n=1}^{\infty} U_n < \infty$. Then, $\lim_{n \rightarrow \infty} \varphi_n$ exists.

Lemma 4 ([31]). *Let $C \subset H$ be a non-empty set, and let $\{j_n\}$ be a sequence in H that fulfills the following conditions:*

- (i) $\lim_{n \rightarrow \infty} \|j_n - j\|$ exists for all $j \in C$;
 - (ii) Every weak sequential cluster point of $\{j_n\}$ belongs to C .
- Then, $\{j_n\}$ converges weakly to a point in C .

Lemma 5 (Lemma 2.2 [31]). *Let $\{Q_n\} \subset [0, \infty)$ and $\{\tilde{b}_n\} \subset [0, \infty)$ be the sequences such that*

- (i) $Q_{n+1} - Q_n \leq \varrho_n(Q_n - Q_{n-1}) + \tilde{b}_n$;
- (ii) $\sum_{n=1}^{\infty} \tilde{b}_n < \infty$;
- (iii) $\{\varrho_n\} \subset [0, \varrho]$, where $\varrho \in [0, 1)$.

Then, $\{Q_n\}$ is a converging sequence, and $\sum_{n=1}^{\infty} [Q_n - Q_{n-1}]_+ < \infty$, where $[\zeta]_+ = \max\{\zeta, 0\}$ (for any $\zeta \in \mathbb{R}$).

3. Weak Convergence

In this section, we propose a double inertial pre-conditioning CQ algorithm for solving SFP and analyze a weak convergence result under mild conditions. Next, we present some assumptions.

(P₁) The solution set of SFP is nonempty; that is, $\Omega \neq \emptyset$.

(P₂) Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 , respectively, and $A : H_1 \rightarrow H_2$ be a bounded linear operator with its adjoint A^* . Let B and $\tilde{E}$ be self adjoint positive-definite operators such that $A^*\tilde{E} = EA^*$, with $E = B^{-1}$.

(P₃) Let $\kappa_{\min}(B)$ and $\kappa_{\max}(B)$ be the minimum and the largest eigenvalues of B , respectively. For the $\|\cdot\|_B^2$, the result is as follows:

$$\kappa_{\min}(B)\|z\|^2 \leq \|z\|_B^2 \leq \kappa_{\max}(B)\|z\|^2, \quad \forall z \in H,$$

which is suggested in Ref. [36].

(P₄) The sequences $\{u_n\}$, $\{q_n\}$, $\{\alpha_n\}$, and $\{h_n\}$ have the following inequalities.

(i) $0 \leq u_n \leq 1$;

(ii) $0 \leq q_n \leq q_{n+1} \leq q < \frac{3+2d-\sqrt{8d+17}}{2d}$, $d \in (1, \infty)$;

(iii) $0 < \alpha < \alpha_n \leq \alpha_{n+1} \leq \frac{1}{1+d}$, $d \in (1, \infty)$;

(iv) $h_n = (1 - \alpha_n)q_n + \alpha_n u_n$ is a non-decreasing sequence.

(P₅) Let P_Q be $\tilde{E}$ -projection on Q related to the norm $\|\cdot\|_{\tilde{E}}$. For the continuously differentiable convex functions $U, U_{\tilde{E}} : H_1 \rightarrow R$, we define

$$U(j) = \frac{1}{2}\|(I - P_Q)Ax\|^2,$$

and

$$U_{\tilde{E}}(j) = \frac{1}{2}\|(I - P_Q)Aj\|_{\tilde{E}}^2 = \frac{1}{2}\langle \tilde{E}((I - P_Q)Aj), (I - P_Q)Aj \rangle;$$

their corresponding gradients are the following, respectively.

$$\nabla U_{\tilde{E}}(j) = EA^*(I - P_Q)Aj, \quad \nabla U(j) = A^*(I - P_Q)Aj.$$

Proposition 1 ([18]). Assume that the condition P_1 holds. The results are equivalent below.

(i) $g^* \in \Omega$,

(ii) $g^* \in C$ and $F(g^*) = 0$;

(iii) $g^* \in C$ and $\nabla F(g^*) = 0$.

In what follows, we illustrate that our designed stepsize scheme is well-defined.

Lemma 6. Let $\{\Gamma_n\}$ be the sequence produced by Algorithm 1. Then, we need to prove that $\lim_{n \rightarrow \infty} \Gamma_n$ exists and denote $\lim_{n \rightarrow \infty} \Gamma_n = \tilde{\Gamma}$, where $\tilde{\Gamma} \geq \min\left\{\frac{\delta}{\|A\|^2}, \Gamma_1\right\}$ and $\Gamma_1 > 0$, is an initial stepsize.

Proof. In the case of $M_n > 0$, and since P_Q is of firm nonexpansiveness, we derive

$$\begin{aligned} \langle \nabla U(y_n) - \nabla U(v_n), y_n - v_n \rangle &= \langle A^*(I - P_Q)Ay_n - A^*(I - P_Q)Av_n, y_n - v_n \rangle \\ &= \langle (I - P_Q)Ay_n - (I - P_Q)Av_n, Ay_n - Av_n \rangle \\ &= \|Ay_n - Av_n\|^2 - \langle P_Q(Ay_n) - P_Q(Av_n), Ay_n - Av_n \rangle \\ &\leq \|Ay_n - Av_n\|^2 - \|P_Q(Ay_n) - P_Q(Av_n)\|^2 \\ &\leq \|Ay_n - Av_n\|^2 \\ &\leq \|A\|^2\|y_n - v_n\|^2, \end{aligned}$$

which shows that

$$\frac{\delta\|y_n - v_n\|^2}{\langle \nabla F(y_n) - \nabla F(v_n), y_n - v_n \rangle} \geq \frac{\delta}{\|A\|^2}.$$

This further discloses that

$$\Gamma_{n+1} = \min\left\{\frac{\delta\|y_n - v_n\|^2}{\langle \nabla U(y_n) - \nabla U(v_n), y_n - v_n \rangle}, \phi_n \Gamma_n\right\} \geq \min\left\{\frac{\delta}{\|A\|^2}, \Gamma_n\right\},$$

where $\phi_n \geq 1$. By induction, the sequence $\{\Gamma_n\}$ has a lower bound $\min\left\{\frac{\delta}{\|A\|^2}, \Gamma_1\right\}$. From (8), it verifies that

$$\Gamma_{n+1} \leq \phi_n \Gamma_n.$$

Thanks to Lemma 3, we see that $\lim_{n \rightarrow \infty} \Gamma_n$ exists and denote $\lim_{n \rightarrow \infty} \Gamma_n = \tilde{\Gamma}$. Since the sequence $\{\Gamma_n\}$ has the lower bound $\min\left\{\frac{\delta}{\|A\|^2}, \Gamma_1\right\}$, we disclose that $\tilde{\Gamma} > 0$. $\square$

Now, our algorithm is described as follows.

Algorithm 1 A Pre-conditioning CQ Algorithm With Double Inertia and Self-updated Stepsizes For Solving SFP

Step 0. Given $j_0, j_1 \in H_1, \Gamma_1 > 0, \delta \in \left(0, \frac{\kappa_{\min}(B)}{2}\right)$. Take the sequence $\{\phi_n\}$ fulfilling Lemma 2.3.

Step 1. Given j_{n-1}, j_n ($n \geq 1$) and compute

$$v_n = j_n + u_n(j_n - j_{n-1}), s_n = j_n + q_n(j_n - j_{n-1}), \quad (6)$$

$$y_n = P_C^B(v_n - \Gamma_n \nabla U_{\tilde{E}}(v_n)), j_{n+1} = (1 - \alpha_n)s_n + \alpha_n(y_n - \gamma_n \nabla U_{\tilde{E}}(y_n)), \quad (7)$$

where the stepsize Γ_n is computed by

$$\Gamma_{n+1} = \begin{cases} \min\left\{\frac{\delta \|y_n - v_n\|^2}{\langle \nabla U(y_n) - \nabla U(v_n), y_n - v_n \rangle}, \phi_n \Gamma_n\right\}, & \text{if } M_n > 0, \\ \phi_n \Gamma_n, & \text{otherwise,} \end{cases} \quad (8)$$

where $M_n = \langle \nabla U(y_n) - \nabla U(v_n), y_n - v_n \rangle$.

And where the stepsize γ_n is computed by

$$\gamma_n = \frac{tU(y_n)}{\|\nabla U_{\tilde{E}}(y_n)\|_B^2}, \quad 0 < t < 2. \quad (9)$$

If $y_n = v_n$, then stop, y_n is a solution of SFP. Otherwise, go to Step 1.

Remark 1. From Algorithm 1, we observe that we can directly compute the iterative point j_n by using the step (7), which is not involved in the projection P_C^B , compared to that in the original algorithm [18]. Thus, the numerical result is improved.

Theorem 1. Assume that conditions (P_1) – (P_5) hold. The sequence $\{j_n\}$ produced by Algorithm 1 converges weakly to a point $z \in \Omega$.

Proof. Let $z \in \Omega$; it yields that $\nabla F(z) = 0, z = P_C^B z$, and $P_Q A z = A z$. Now, we have the following result by using the Pythagorean theorem.

$$\begin{aligned} & \|y_n - \gamma_n \nabla U_{\tilde{E}}(y_n) - z\|_B^2 \\ &= \|y_n - z\|_B^2 + \gamma_n^2 \|\nabla U_{\tilde{E}}(y_n)\|_B^2 - 2\gamma_n \langle y_n - z, \nabla U_{\tilde{E}}(y_n) \rangle_B. \end{aligned} \quad (10)$$

By Lemma 1(ii), we show that

$$\begin{aligned} \langle y_n - z, -\nabla U_{\tilde{E}}(y_n) \rangle_B &= \langle y_n - z, -\nabla U(y_n) \rangle = \langle Ay_n - Az, (P_Q - I)Ay_n \rangle \\ &= \langle Ay_n - P_Q(Ay_n) + P_Q(Ay_n) - Az, (P_Q - I)Ay_n \rangle \\ &= \langle P_Q(Ay_n) - Az, P_Q(Ay_n) - Ay_n \rangle - \|P_Q(Ay_n) - Ay_n\|^2 \\ &= \frac{1}{2} \left[\|P_Q(Ay_n) - Az\|^2 + \|P_Q(Ay_n) - Ay_n\|^2 - \|Ay_n - Az\|^2 \right] \\ &\quad - \|P_Q(Ay_n) - Ay_n\|^2 \\ &\leq -\frac{1}{2} \|P_Q(Ay_n) - Ay_n\|^2; \end{aligned} \quad (11)$$

we also prove

$$\langle v_n - z, -\nabla U_{\tilde{E}}(v_n) \rangle_B \leq -\frac{1}{2} \|P_Q(Av_n) - Av_n\|^2. \quad (12)$$

By the iterate y_n , we have

$$\begin{aligned} \|y_n - z\|_B^2 &= \|P_C^B(v_n - \Gamma_n \nabla U_{\tilde{E}}(v_n)) - z\|_B^2 \\ &\leq \|v_n - \Gamma_n \nabla U_{\tilde{E}}(v_n) - z\|_B^2 - \|y_n - v_n + \Gamma_n \nabla U_{\tilde{E}}(v_n)\|_B^2 \\ &= \|v_n - z\|_B^2 - 2\Gamma_n \langle v_n - z, \nabla U_{\tilde{E}}(v_n) \rangle_B - \|y_n - v_n\|_B^2 - 2\Gamma_n \langle y_n - v_n, \nabla U_{\tilde{E}}(v_n) \rangle_B. \end{aligned} \quad (13)$$

Now, we need to deduce the above part $\langle y_n - v_n, \nabla U_{\tilde{E}}(v_n) \rangle_B$. By the differentiable convexity of U , we obtain

$$\begin{aligned} \langle y_n - v_n, \nabla U_{\tilde{E}}(v_n) \rangle_B &= \langle y_n - v_n, \nabla U(v_n) \rangle \\ &= \langle y_n - v_n, \nabla U(v_n) - \nabla U(y_n) \rangle + \langle y_n - v_n, \nabla U(y_n) \rangle \\ &= -\langle y_n - v_n, \nabla U(y_n) - \nabla U(v_n) \rangle + \langle y_n - v_n, \nabla U(y_n) \rangle \\ &\geq -\langle y_n - v_n, \nabla U(y_n) - \nabla U(v_n) \rangle + U(y_n) - U(v_n) \\ &\geq -\langle y_n - v_n, \nabla U(y_n) - \nabla U(v_n) \rangle - U(v_n). \end{aligned} \quad (14)$$

After arrangement by substituting (12) and (14) into (13), we have

$$\begin{aligned} \|y_n - z\|_B^2 &\leq \|v_n - z\|_B^2 + 2\Gamma_n \langle y_n - v_n, \nabla U(y_n) - \nabla U(v_n) \rangle - \|y_n - v_n\|_B^2 \\ &\leq \|v_n - z\|_B^2 + 2\frac{\Gamma_n}{\Gamma_{n+1}} \Gamma_{n+1} \langle y_n - v_n, \nabla U(y_n) - \nabla U(v_n) \rangle - \|y_n - v_n\|_B^2 \\ &\leq \|v_n - z\|_B^2 + 2\frac{\Gamma_n}{\Gamma_{n+1}} \delta \|y_n - v_n\| - \|y_n - v_n\|_B^2 \\ &= \|v_n - z\|_B^2 - \left(\kappa_{\min}(B) - 2\frac{\Gamma_n}{\Gamma_{n+1}} \delta \right) \|y_n - v_n\|^2. \end{aligned} \quad (15)$$

Since

$$\lim_{n \rightarrow \infty} \left(\kappa_{\min}(B) - 2\frac{\Gamma_n}{\Gamma_{n+1}} \delta \right) = \kappa_{\min}(B) - 2\delta > 0,$$

where $\delta \in \left(0, \frac{\kappa_{\min}(B)}{2}\right)$, then $\exists N \geq 0, \forall n \geq N, \kappa_{\min}(B) - 2\frac{\Gamma_n}{\Gamma_{n+1}} \delta > 0$. So, for any $n \geq N$,

$$\|y_n - z\|_B^2 \leq \|v_n - z\|_B^2 - (\kappa_{\min}(B) - 2\delta) \|y_n - v_n\|^2. \quad (16)$$

By Lemma 1(i), we deduce from the iterative step j_{n+1} that

$$\begin{aligned} \|j_{n+1} - z\|_B^2 &= \|(1 - \alpha_n)s_n + \alpha_n(y_n - \gamma_n \nabla U_{\tilde{E}}(y_n)) - z\|_B^2 \\ &= (1 - \alpha_n) \|s_n - z\|_B^2 + \alpha_n \|y_n - \gamma_n \nabla U_{\tilde{E}}(y_n) - z\|_B^2 \\ &\quad - (1 - \alpha_n) \alpha_n \|s_n - y_n + \gamma_n \nabla U_{\tilde{E}}(y_n)\|_B^2 \\ &= (1 - \alpha_n) \|s_n - z\|_B^2 + \alpha_n (\|y_n - z\|_B^2 + \gamma_n^2 \|\nabla U_{\tilde{E}}(y_n)\|_B^2 - 2\gamma_n \langle y_n - z, \nabla U_{\tilde{E}}(y_n) \rangle_B) \\ &\quad - (1 - \alpha_n) \alpha_n \|s_n - y_n + \gamma_n \nabla U_{\tilde{E}}(y_n)\|_B^2 \end{aligned} \quad (17)$$

Substituting (11) and (16) into (17), and for any $n \geq N$, one has

$$\begin{aligned} \|j_{n+1} - z\|_B^2 &\leq (1 - \alpha_n) \|s_n - z\|_B^2 + \alpha_n (\|v_n - z\|_B^2 - (\kappa_{\min}(B) - 2\delta) \|y_n - v_n\|^2) \\ &\quad + \alpha_n (\gamma_n^2 \|\nabla U_{\tilde{E}}(y_n)\|_B^2 - 2\gamma_n \langle y_n - z, \nabla U_{\tilde{E}}(y_n) \rangle_B) - (1 - \alpha_n) \alpha_n \|s_n - y_n + \gamma_n \nabla U_{\tilde{E}}(y_n)\|_B^2 \\ &= (1 - \alpha_n) \|s_n - z\|_B^2 + \alpha_n \|v_n - z\|_B^2 - \alpha_n (\kappa_{\min}(B) - 2\delta) \|y_n - v_n\|^2 \\ &\quad + \alpha_n \left(\frac{t^2 U^2(y_n)}{\|\nabla U_{\tilde{E}}(y_n)\|_B^4} \|\nabla U_{\tilde{E}}(y_n)\|_B^2 - 2 \frac{tU(y_n)}{\|\nabla U_{\tilde{E}}(y_n)\|_B^2} U(y_n) \right) - (1 - \alpha_n) \alpha_n \|s_n - y_n + \gamma_n \nabla U_{\tilde{E}}(y_n)\|_B^2 \\ &= (1 - \alpha_n) \|s_n - z\|_B^2 + \alpha_n \|v_n - z\|_B^2 - \alpha_n (\kappa_{\min}(B) - 2\delta) \|y_n - v_n\|^2 \\ &\quad - \alpha_n t(2 - t) \frac{U^2(y_n)}{\|\nabla U_{\tilde{E}}(y_n)\|_B^2} - (1 - \alpha_n) \alpha_n \|s_n - y_n + \gamma_n \nabla U_{\tilde{E}}(y_n)\|_B^2. \end{aligned} \quad (18)$$

Next, the inertial steps s_n and v_n respectively yield that

$$\begin{aligned}\|s_n - z\|_B^2 &= \|j_n + q_n(j_n - j_{n-1}) - z\|_B^2 \\ &= \|(1 + q_n)(j_n - z) - q_n(j_{n-1} - z)\|_B^2 \\ &= (1 + q_n)\|j_n - z\|_B^2 - q_n\|j_{n-1} - z\|_B^2 + q_n(1 + q_n)\|j_n - j_{n-1}\|_B^2,\end{aligned}\quad (19)$$

and

$$\|v_n - z\|_B^2 = (1 + u_n)\|j_n - z\|_B^2 - u_n\|j_{n-1} - z\|_B^2 + u_n(1 + u_n)\|j_n - j_{n-1}\|_B^2. \quad (20)$$

Substituting (19) and (20) into (18), and for any $n \geq N$, we directly deduce that

$$\begin{aligned}\|j_{n+1} - z\|_B^2 &\leq (1 - \alpha_n)((1 + q_n)\|j_n - z\|_B^2 - q_n\|j_{n-1} - z\|_B^2 + q_n(1 + q_n)\|j_n - j_{n-1}\|_B^2) \\ &\quad + \alpha_n((1 + u_n)\|j_n - z\|_B^2 - u_n\|j_{n-1} - z\|_B^2 + u_n(1 + u_n)\|j_n - j_{n-1}\|_B^2) \\ &\quad - \alpha_n(\kappa_{\min}(B) - 2\delta)\|y_n - v_n\|^2 \\ &\quad - \alpha_n t(2 - t) \frac{U^2(y_n)}{\|\nabla U_E(y_n)\|_B^2} - (1 - \alpha_n)\alpha_n\|s_n - y_n + \gamma_n \nabla U_E(y_n)\|_B^2.\end{aligned}\quad (21)$$

Since

$$\begin{aligned}\alpha_n^2\|s_n - y_n + \gamma_n \nabla U_E(y_n)\|_B^2 &= \|j_{n+1} - s_n\|_B^2 \\ &= \|j_{n+1} - j_n - q_n(j_n - j_{n-1})\|_B^2 \\ &\geq \|j_{n+1} - j_n\|_B^2 + q_n^2\|j_n - j_{n-1}\|_B^2 - 2q_n\|j_{n+1} - j_n\|_B\|j_n - j_{n-1}\|_B \\ &\geq (1 - q_n)\|j_{n+1} - j_n\|_B^2 + (q_n^2 - q_n)\|j_n - j_{n-1}\|_B^2,\end{aligned}\quad (22)$$

after arrangement by substituting (22) into (21), and for any $n \geq N$, the obtained result is the following:

$$\begin{aligned}\|j_{n+1} - z\|_B^2 &\leq (1 - \alpha_n)((1 + q_n)\|j_n - z\|_B^2 - q_n\|j_{n-1} - z\|_B^2 + q_n(1 + q_n)\|j_n - j_{n-1}\|_B^2) \\ &\quad + \alpha_n((1 + u_n)\|j_n - z\|_B^2 - u_n\|j_{n-1} - z\|_B^2 + u_n(1 + u_n)\|j_n - j_{n-1}\|_B^2) \\ &\quad - \alpha_n(\kappa_{\min}(B) - 2\delta)\|y_n - v_n\|^2 \\ &\quad - \alpha_n t(2 - t) \frac{U^2(y_n)}{\|\nabla U_E(y_n)\|_B^2} - \frac{1 - \alpha_n}{\alpha_n}((1 - q_n)\|j_{n+1} - j_n\|_B^2 + (q_n^2 - q_n)\|j_n - j_{n-1}\|_B^2) \\ &\leq (1 + (1 - \alpha_n)q_n + \alpha_n u_n)\|j_n - z\|_B^2 - ((1 - \alpha_n)q_n + \alpha_n u_n)\|j_{n-1} - z\|_B^2 \\ &\quad + \left((1 - \alpha_n)(1 + q_n)q_n + \alpha_n(1 + u_n)u_n - \frac{1 - \alpha_n}{\alpha_n}(q_n^2 - q_n)\right)\|j_n - j_{n-1}\|_B^2 \\ &\quad - \frac{1 - \alpha_n}{\alpha_n}(1 - q_n)\|j_{n+1} - j_n\|_B^2 \\ &\leq (1 + h_n)\|j_n - z\|_B^2 - h_n\|j_{n-1} - z\|_B^2 + g_n\|j_n - j_{n-1}\|_B^2 - c_n\|j_{n+1} - j_n\|_B^2,\end{aligned}\quad (23)$$

where

$$0 < t < 2, h_n = (1 - \alpha_n)q_n + \alpha_n u_n, g_n = (1 - \alpha_n)(1 + q_n)q_n + \alpha_n(1 + u_n)u_n - \frac{1 - \alpha_n}{\alpha_n}(q_n^2 - q_n), c_n = \frac{1 - \alpha_n}{\alpha_n}(1 - q_n).$$

Define

$$\Lambda_n = \|j_n - z\|_B^2 - h_n\|j_{n-1} - z\|_B^2 + g_n\|j_n - j_{n-1}\|_B^2.$$

Since the sequence h_n is non-decreasing, by (23), one obtains

$$\begin{aligned}\Lambda_n - \Lambda_{n-1} &= \|j_{n+1} - z\|_B^2 - (1 + h_{n+1})\|j_n - z\|_B^2 + h_n\|j_{n-1} - z\|_B^2 \\ &\quad - g_n\|j_n - j_{n-1}\|_B^2 + g_{n+1}\|j_{n+1} - j_n\|_B^2 \\ &\leq \|j_{n+1} - z\|_B^2 - (1 + h_n)\|j_n - z\|_B^2 + h_n\|j_{n-1} - z\|_B^2 \\ &\quad - g_n\|j_n - j_{n-1}\|_B^2 + g_{n+1}\|j_{n+1} - j_n\|_B^2 \\ &\leq -(c_n - g_{n+1})\|j_{n+1} - j_n\|_B^2.\end{aligned}\quad (24)$$

According to A_4 , we discuss that

$$\begin{aligned} c_n - g_{n+1} &= \frac{1-\alpha_n}{\alpha_n}(1-q_n) - (1-q_{n+1})(1+q_{n+1})q_{n+1} - q_{n+1}(1+u_{n+1})u_{n+1} \\ &\quad + \frac{1-q_{n+1}}{q_{n+1}}(q_{n+1}^2 - q_{n+1}) \\ &\geq \frac{1-q_{n+1}}{q_{n+1}}(1-2q_{n+1}+q_{n+1}^2) - (1-q_{n+1})(q_{n+1}+q_{n+1}^2) - 2q_{n+1} \\ &\geq d(1-2q+q^2) - \frac{d}{1+d}(q+q^2) - \frac{2}{1+d} \\ &= \frac{1}{1+d}(d^2q^2 - (3d+2d^2)q + (d^2+d-2)). \end{aligned} \quad (25)$$

The work (24) with (25) implies that

$$\Lambda_n - \Lambda_{n-1} \leq -\tilde{\kappa} \|j_{n+1} - j_n\|_B^2, \quad (26)$$

where $\tilde{\kappa} = \frac{d^2q^2 - (3d+2d^2)q + (d^2+d-2)}{1+d}$. We find that $q < \frac{3+2d-\sqrt{8d+17}}{2d}$, $d \in (1, \infty)$, and one has $\tilde{\kappa} > 0$. With the help of (26), we see that

$$\Lambda_n - \Lambda_{n-1} \leq 0. \quad (27)$$

Thereby, the sequence Λ_n is non-increasing. We know from A_4 that $g_n > 0$. Therefore,

$$\begin{aligned} \Lambda_n &= \|j_n - z\|_B^2 - h_n \|j_{n-1} - z\|_B^2 + g_n \|j_n - j_{n-1}\|_B^2 \\ &\geq \|j_n - z\|_B^2 - h_n \|j_{n-1} - z\|_B^2. \end{aligned} \quad (28)$$

This is the equivalent to the following analysis:

$$\begin{aligned} \|j_n - z\|_B^2 &\leq h_n \|j_{n-1} - z\|_B^2 + \Lambda_n \\ &\leq h \|j_{n-1} - z\|_B^2 + \Lambda_n \\ &\leq \cdots \leq h^n \|j_0 - z\|_B^2 + \Lambda_1(1+h+\cdots+h^{n-1}) \\ &\leq h^n \|j_0 - z\|_B^2 + \frac{\Lambda_1}{1-h}, \end{aligned} \quad (29)$$

where $h = \frac{5+2d-\sqrt{8d+17}}{2+2d} < 1$.

Using the definition of the sequence Λ_n , it shows that

$$\begin{aligned} \Lambda_{n+1} &= \|j_{n+1} - z\|_B^2 - h_{n+1} \|j_{n-1} - z\|_B^2 + b_{n+1} \|j_n - j_{n-1}\|_B^2 \\ &\geq -h_{n+1} \|j_n - z\|_B^2. \end{aligned} \quad (30)$$

Combining (29) together with (30), we deduce that

$$-\Lambda_{n+1} \leq h_{n+1} \|j_n - z\|_B^2 \leq h \|j_n - z\|_B^2 \leq h^{n+1} \|j_0 - z\|_B^2 + \frac{h\Lambda_1}{1-h}. \quad (31)$$

Using (26) and (31) again, we see that

$$\begin{aligned} \tilde{\kappa} \sum_{k=1}^n \|j_{k+1} - j_k\|_B^2 &\leq \Lambda_1 - \Lambda_{n+1} \leq h^{n+1} \|j_0 - z\|_B^2 + \frac{\Lambda_1}{1-h} \\ &\leq \|j_0 - z\|_B^2 + \frac{\Lambda_1}{1-h}. \end{aligned} \quad (32)$$

So, we conclude that

$$\sum_{k=1}^n \|j_{k+1} - j_k\|_B^2 < \infty. \quad (33)$$

Thereby,

$$\lim_{n \rightarrow \infty} \|j_{n+1} - j_n\|_B = 0, \quad (34)$$

which combined with the boundedness of u_n , directly shows that

$$\begin{aligned} \|j_{n+1} - v_n\|_B^2 &= \|j_{n+1} - j_n\|_B^2 + u_n^2 \|j_n - j_{n-1}\|_B^2 \\ &\quad - 2u_n \langle j_{n+1} - j_n, j_n - j_{n-1} \rangle \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (35)$$

We also have

$$\|j_n - v_n\|_B \leq \|j_n - j_{n+1}\|_B + \|j_{n+1} - v_n\|_B \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (36)$$

From (23), we know that

$$\begin{aligned} \|j_{n+1} - z\|_B &\leq (1 + h_n) \|j_n - z\|_B^2 - h_n \|j_{n-1} - z\|_B^2 + g_n \|j_n - j_{n-1}\|_B^2 - c_n \|j_{n+1} - j_n\|_B^2 \\ &\leq (1 + h_n) \|j_n - z\|_B^2 - h_n \|j_{n-1} - z\|_B^2 + g_n \|j_n - j_{n-1}\|_B^2 \\ &= \|j_n - z\|_B^2 + h_n (\|j_n - z\|_B^2 - \|j_{n-1} - z\|_B^2) + g_n \|j_n - j_{n-1}\|_B^2 \end{aligned} \quad (37)$$

Owing to $0 \leq h_n \leq h < 1$ and the boundedness of g_n , and by Lemma 2.5 and (33), we seek a point $g^* \in [0, \infty)$ such that

$$\lim_{n \rightarrow \infty} \|j_n - z\|_B = g^*. \quad (38)$$

In view of (19) and the boundedness of q_n , we arrive at

$$\begin{aligned} \|s_n - z\|_B^2 &= (1 + q_n) \|j_n - z\|_B^2 - q_n \|j_{n-1} - z\|_B^2 + q_n (1 + q_n) \|j_n - j_{n-1}\|_B^2 \\ &= \|j_n - z\|_B^2 + q_n (\|j_n - z\|_B^2 - \|j_{n-1} - z\|_B^2) + q_n (1 + q_n) \|j_n - j_{n-1}\|_B^2 \rightarrow g^* \text{ as } n \rightarrow \infty. \end{aligned} \quad (39)$$

Similarly, and by the boundedness of u_n , one gets

$$\lim_{n \rightarrow \infty} \|v_n - z\|_B = g^*. \quad (40)$$

The inequality (18) can be rewritten as

$$\begin{aligned} \alpha_n \left((\kappa_{\min}(B) - 2\delta) \|y_n - v_n\|^2 + t(2 - t) \frac{U^2(y_n)}{\|\nabla U_E(y_n)\|_B^2} + (1 - \alpha_n) \|s_n - y_n + \Lambda_n \nabla U_E(y_n)\|_B^2 \right) \\ \leq (1 - \alpha_n) \|s_n - z\|_B^2 + \alpha_n \|v_n - z\|_B^2 - \|j_{n+1} - z\|_B. \end{aligned} \quad (41)$$

This work, together with (38)–(40), and $0 < \alpha_n < \frac{1}{1+d}$, $d \in (1, \infty)$, shows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|y_n - v_n\|^2 &= 0, \\ \lim_{n \rightarrow \infty} \frac{U^2(y_n)}{\|\nabla U_E(y_n)\|_B^2} &= 0, \\ \lim_{n \rightarrow \infty} \|s_n - y_n + \Lambda_n \nabla U_E(y_n)\|_B^2 &= 0. \end{aligned} \quad (42)$$

Since the gradient mapping ∇U is Lipschitzian, ∇U_E is Lipschitzian too. Thus, $\|\nabla U_E(y_n)\|$ is bounded. Therefore,

$$\lim_{n \rightarrow \infty} U(y_n) = 0. \quad (43)$$

By $\lim_{n \rightarrow \infty} \|y_n - v_n\|^2 = 0$, we get

$$\lim_{n \rightarrow \infty} \|y_n - v_n\|_B^2 = 0. \quad (44)$$

Let $\bar{z} \in C$ and $\bar{z} \in w_w(j_n) \subset \Omega$, and thus, by the boundedness of the sequence j_n , there exists a subsequence j_{n_i} of j_n such that $j_{n_i} \rightarrow \bar{z}$ as $i \rightarrow \infty$. From (36) and (44), we know that $y_{n_i} \rightarrow \bar{z}$ as $i \rightarrow \infty$. Using (43), one sees that

$$\lim_{i \rightarrow \infty} U(y_{n_i}) = U(\bar{z}) = \|A\bar{z} - P_Q A\bar{z}\| = 0, \quad (45)$$

which indicates $A\bar{z} \in Q$, and from Proposition 1(ii) and Lemma 4, it proves that $\bar{z} \in \Omega$. $\square$

4. Numerical Experiments

In this section, some numerical experiments are providing for comparing our proposed Algorithm 1 (abbreviated, Alg.1) with algorithm 3.2 [18] (PCQ), algorithm 3.1 [31] (Sahu 2021), and algorithm 1 [37] (Suantai 2022). All codes were written by MATLAB R2018a and run on a PC Desktop Intel(R) Core(TM) i7-6700 CPU @ 3.40 GHZ computer with RAM 8.00 GB.

Example 1 (LASSO problem [38]). *The concrete form of the LASSO problem is as follows:*

$$\min \left\{ \frac{1}{2} \|Aj - V\|^2 : j \in \mathbb{R}^N, \|j\|_1 \leq \kappa \right\}, \quad (46)$$

where $A \in \mathbb{R}^{M \times N}$, $M < N$, $V \in \mathbb{R}^M$, and $\kappa > 0$. This problem is devoted to finding a sparse solution to the SFP. The sampling matrix A is produced by a standard normal distribution with mean zero and unit variance. The truly sparse signal j^* is built by distributing W nonzero elements uniformly at random in the interval $[-2, 2]$, while the rest is kept zero. The sample data $V = Aj^*$.

In consideration of SFP, we define $C = \{\tilde{j} : \|\tilde{j}\|_1 \leq \kappa\}$, $\kappa = W$, and $Q = \{V\}$. We suggest the subgradient projection, because of the fact that there is no closed-form expression for the projection of a closed convex set C . Let $c(\tilde{j}) = \|\tilde{j}\|_1 - \kappa$ and

$$C_n = \{\tilde{j} : c(v_n) + \langle \omega_n, \tilde{j} - v_n \rangle \leq 0\},$$

where $\omega_n \in \partial c(v_n)$. The orthogonal projection of a point $\tilde{p} \in \mathbb{R}^N$ onto C_n can be achieved by

$$P_{C_n}(\tilde{j}) = \begin{cases} j, & \text{if } c(v_n) + \langle \omega_n, \tilde{j} - v_n \rangle \leq 0, \\ \tilde{j} - \frac{c(v_n) + \langle \omega_n, \tilde{j} - v_n \rangle}{\|\omega_n\|^2} \omega_n, & \text{otherwise,} \end{cases}$$

The subdifferential ∂c at v_n is

$$\partial c(v_n) = \begin{cases} 1, & \text{if } v_n > 0, \\ [-1, 1], & \text{if } v_n = 0, \\ -1, & \text{if } v_n < 0. \end{cases}$$

We initialize the algorithms at the original and define

$$E_n = \frac{\|j_n - j^*\|}{\max\{1, \|j_n\|\}}.$$

To test all methods under the same iteration error E_n for different M , N , and W , we limit the number of iterations to 7000 and a smaller value of E_n ; the algorithms behave better. Meanwhile, the corresponding results are reported in Table 1.

The second problem is the recovery of the signal j^* when $M = 960$, $N = 4096$, $W = 120$, and $M \times N$ matrix A is randomly obtained with independent samples of standard Gaussian distribution. The original signal j^* contains 120 randomly placed ± 1 spikes. The initial point $j_0 = 0$, and we adopt the following mean square error for measuring the restoration accuracy:

$$\text{MSE} = \frac{1}{N} \|j_n - j^*\|^2.$$

The smaller the MSE, the better the signal recovery.

The same parameters for all algorithms were chosen in the following manner:

For Algorithm 1 $q_n = 0.1 - \frac{1}{1000+n}$, $\alpha_n = 0.46 - \frac{1}{1000+n}$, and $\phi_n = \frac{1}{(n+1)^{1.1}} + 1$;

For PCQ, $\sigma = 1$, $\rho = 0.5$, $t = 1.2$, and $0 < \delta < \frac{\kappa_{\min}(B)}{2}$, which is suggested in [18];

For the Sahu 2021, $\theta = 0.3$, and $\delta = 0.2$, which is suggested in [31];

For Suantai 2022, we set $\sigma = 1$, $\rho = 0.5$, $\delta = 0.3$, $t = 2 \times 10^{-4}$, and $\theta = 0.3$ in [37].

We present the original signal and restored signal, as shown in Figure 1. Then, we compare the CPU time and E_n for all the algorithms as shown in Table 1.

Remark 2. From Figure 1, all algorithms finish the signal recovery successfully; we observe that Algorithm 1 uses less CPU time to obtain a smaller MSE than Sahu 2021 [31], Suantai 2022 [37], and PCQ. Thereby, our proposed algorithm runs faster than the others. According to Table 1, the E_n we obtain is larger than the other compared algorithms under the same parameters. However, it is difficult for our proposed algorithm to recover the signal, if we choose a larger or smaller parameter than the fixed parameter that properly makes our algorithm achieve the signal recovery.

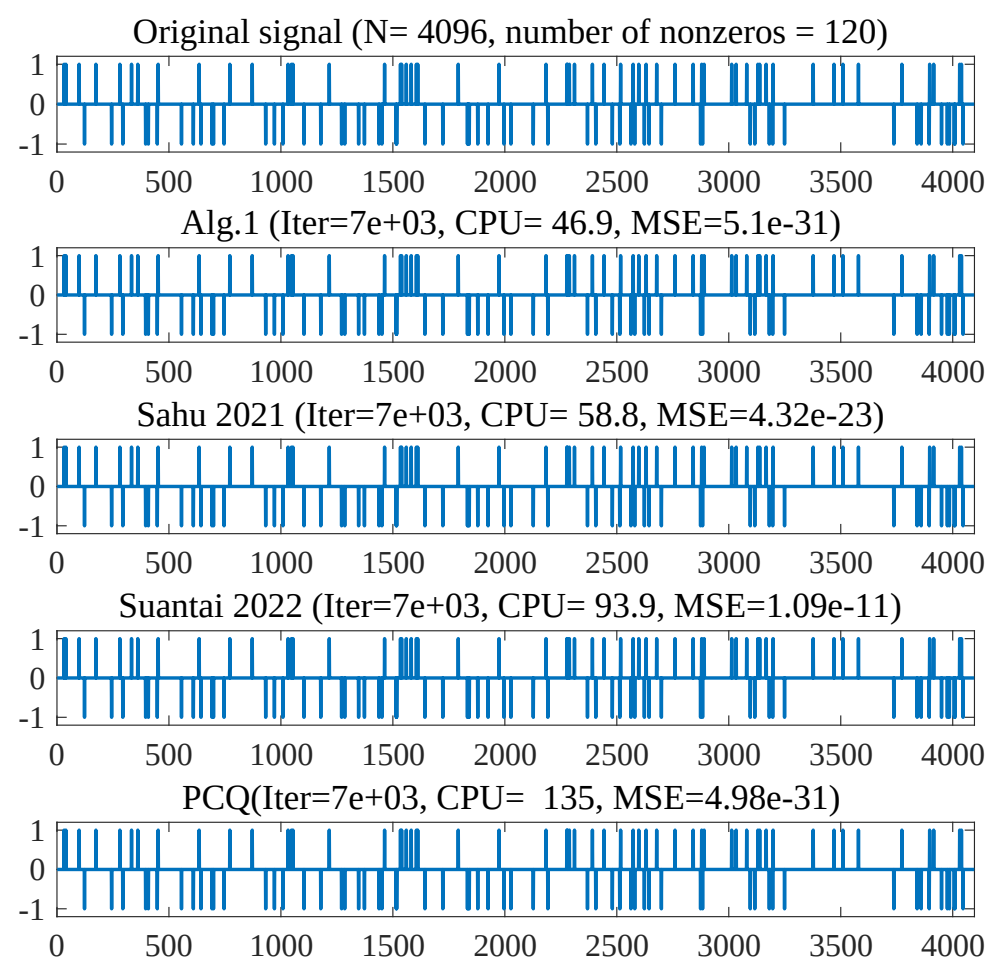


Figure 1. Comparison of signal processing ([18,31,37]).

Table 1. Results for Example 1.

(M, N, W)	Algorithm 1		Sahu 2021 [31]		Suantai 2022 [37]		PCQ	
	E_n	Time	E_n	Time	E_n	Time	E_n	Time
(240, 1024, 30)	5.4772	1.7094	1.3328×10^{-9}	2.0715	4.2953×10^{-5}	2.9206	6.8805×10^{-16}	3.9255
(480, 2048, 60)	7.7460	5.9345	1.7791×10^{-11}	7.8245	2.5638×10^{-5}	14.2064	7.3943×10^{-16}	32.5748
(720, 3072, 90)	9.4868	51.0043	6.8132×10^{-12}	70.7382	2.1654×10^{-5}	117.7369	6.3745×10^{-16}	134.6671
(960, 4096, 120)	10.9545	46.2216	3.8179×10^{-11}	58.2518	1.9982×10^{-5}	92.2356	6.6815×10^{-16}	132.1087

5. Conclusions

In this paper, we use the double inertia and new stepsizes to modify the pre-conditioning CQ algorithm for solving SFP. The weak convergence of the algorithm is obtained under mild conditions. Then, we use the LASSO problem to show that our algorithm behaves well.

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References

1. Censor, Y.; Elfving, T. A multi-projection algorithm using Bregman projection in a product space. *Numer. Algorithms* **1994**, *8*, 221–239. [\[CrossRef\]](#)
2. Byrne, C. A unified treatment of some iterative algorithms in signal processing and image reconstruction. *Inverse Probl.* **2004**, *20*, 103–120. [\[CrossRef\]](#)
3. Byrne, C. Iterative oblique projection onto convex sets and the split feasibility problem. *Inverse Probl.* **2002**, *18*, 441–453. [\[CrossRef\]](#)
4. Ha, N.S. Weak and strong convergence theorems for the mixed split feasibility problem in Hilbert spaces. *J. Glob. Optim.* **2025**, *93*, 1053–1077. [\[CrossRef\]](#)
5. Zhang, D.; Ye, M. Ball selected relaxation inertial projection algorithm for multiple-sets split feasibility problem. *Optim. Eng.* **2025**. [\[CrossRef\]](#)
6. Cao, Y.; Peng, Y.; Chen, Y.; Shi, L. Several relaxed CQ-algorithms for the split feasibility problem with multiple output sets. *J. Appl. Math. Comput.* **2025**, *71*, 5231–5258. [\[CrossRef\]](#)
7. Zhan, W.; Yu, H. A Novel Relaxed Method for the Split Feasibility Problem in Hilbert Spaces. *Bull. Iran. Math. Soc.* **2025**, *51*, 34. [\[CrossRef\]](#)
8. Huong, V.T.; Xu, H.K.; Yen, N.D. Stability analysis of split equality and split feasibility problems. *J. Glob. Optim.* **2025**, *92*, 411–429. [\[CrossRef\]](#)
9. Tong, X.; Ling, T.; Shi, L. Self-adaptive relaxed CQ algorithms for solving split feasibility problem with multiple output sets. *J. Appl. Math. Comput.* **2024**, *70*, 1441–1469. [\[CrossRef\]](#)
10. Tuyen, T.M.; Ha, N.S. An explicit iterative algorithm for solving the split common solution problem with multiple output sets. *Math. Meth. Oper. Res.* **2025**, *102*, 105–129. [\[CrossRef\]](#)
11. López, G.; Martín, V.; Wang, F.; Xu, H.K. Solving the split feasibility problem without prior knowledge of matrix norms. *Inverse Probl.* **2012**, *28*, 85004. [\[CrossRef\]](#)
12. Yang, Q. On variable-step relaxed projection algorithm for variational inequalities. *J. Math. Anal. Appl.* **2005**, *302*, 166–179. [\[CrossRef\]](#)
13. Bnouhachem, A.; Noor, M.A.; Khalfaoui, M.; Zhaohan, S. On descent-projection method for solving the split feasibility problems. *J. Glob. Optim.* **2012**, *54*, 627–639. [\[CrossRef\]](#)
14. Dong, Q.L.; Yao, Y.; He, S. Weak convergence theorems of the modified relaxed projection algorithms for the split feasibility problem in Hilbert spaces. *Optim. Lett.* **2014**, *8*, 1031–1046. [\[CrossRef\]](#)
15. Gibali, A.; Liu, L.W.; Tang, Y.C. Note on the modified relaxation CQ algorithm for the split feasibility problem. *Optim. Lett.* **2018**, *12*, 817–830. [\[CrossRef\]](#)
16. Kesornprom, S.; Pholasa, N.; Cholamjiak, P. On the convergence analysis of the gradient-CQ algorithms for the split feasibility problem. *Numer. Algorithms* **2020**, *84*, 997–1017. [\[CrossRef\]](#)

17. Alvarez, F.; Attouch, H. An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. *Set-Valued Anal.* **2001**, *9*, 3–11. [\[CrossRef\]](#)
18. Altıparmak, E.; Jolaoso, L.O.; ur Rehman, H. Pre-conditioning CQ algorithm for solving the split feasibility problem and its application to image restoration problem. *Optimization* **2025**, *13*, 3123–3141. [\[CrossRef\]](#)
19. Bauschke, H.H.; Borwein, J.M. On projection algorithms for solving convex feasibility problems. *SIAM Rev.* **1996**, *38*, 367–426. [\[CrossRef\]](#)
20. Byrne, C.L. Iterative projection onto convex sets using multiple Bregman distances. *Inverse Probl.* **1999**, *15*, 1295–1313. [\[CrossRef\]](#)
21. Wang, P.; Zhou, H. A preconditioning method of the CQ algorithm for solving an extended split feasibility problem. *J. Inequal. Appl.* **2014**, *1*, 1–11. [\[CrossRef\]](#)
22. Nesterov, Y. A method for solving the convex programming problem with convergence rate $O(1/k^2)$. *Dokl. Akad. Nauk SSSR* **1983**, *269*, 543–547.
23. Attouch, H.; Peypouquet, J. The Rate of Convergence of Nesterov's Accelerated Forward-Backward Method is Actually Faster Than $1/k^2$. *SIAM J. Optim.* **2016**, *26*, 1824–1834. [\[CrossRef\]](#)
24. Attouch, H.; Peypouquet, J.; Redonta, P. Fast convex optimization via inertial dynamics with Hessian driven damping. *J. Differ. Equ.* **2016**, *261*, 5734–5783. [\[CrossRef\]](#)
25. Attouch, H.; Chbani, Z.; Peypouquet, J.; Redont, P. Fast convergence of inertial dynamics and algorithms with asymptotic vanishing viscosity. *Math. Program.* **2018**, *168*, 123–175. [\[CrossRef\]](#)
26. Attouch, H.; Peypouquet, J. Convergence of inertial dynamics and proximal algorithms governed by maximally monotone operators. *Math. Program.* **2019**, *174*, 391–432. [\[CrossRef\]](#)
27. Suebcharoen, T.; Suparatulatorn, R.; Chaobankoh, T. Kunwai, K.; Mouktonglang, T. An Inertial Relaxed CQ Algorithm with Two Adaptive Step Sizes and Its Application for Signal Recovery. *Mathematics* **2024**, *12*, 2406. [\[CrossRef\]](#)
28. Alvarez, F. Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert spaces. *SIAM J. Optim.* **2004**, *14*, 773–782. [\[CrossRef\]](#)
29. Ma, X.; Jia, Z.; Li, Q. On inertial non-lipschitz stepsize algorithms for split feasibility problems. *Comp. Appl. Math.* **2024**, *43*, 431. [\[CrossRef\]](#)
30. Ma, X.; Liu, H.; Li, X. The iterative method for solving the proximal split feasibility problem with an application to LASSO problem. *Comp. Appl. Math.* **2022**, *41*, 5. [\[CrossRef\]](#)
31. Sahu, D.R.; Cho, Y.J.; Dong, Q.L.; Kashyap, M.R.; Li, X.H. Inertial relaxed CQ algorithms for solving a split feasibility problem in Hilbert spaces. *Numer. Algorithms* **2021**, *87*, 1075–1095. [\[CrossRef\]](#)
32. Wang, Z.B.; Zhen, Y.L.; Long, X.; Chen, Z.-y. A Modified Tseng Splitting Method with Double Inertial Steps for Solving Monotone Inclusion Problems. *J. Sci. Comput.* **2023**, *96*, 1–29. [\[CrossRef\]](#)
33. Limaye, B.V. *Functional Analysis*; New Age: New Delhi, India, 1996.
34. Mondal, S.; Sivakumar, K.C. A Survey of Z-Matrices and New Results on the Subclass of F_0 -Matrices. In *Inverse Problems, Regularization Methods and Related Topics. Industrial and Applied Mathematics*; Pereverzyev, S.V., Radha, R., Sivananthan, S., Eds.; Springer: Singapore, 2025.
35. Osilike, M.O.; Aniagbosor, S.C. Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings. *Math. Comput. Model.* **2000**, *32*, 1181–1191. [\[CrossRef\]](#)
36. Facchinei, F.; Pang, J.S. *Finite-Dimensional Variational Inequality and Complementarity Problems*; Springer: New York, NY, USA, 2003; Volume I.
37. Suantai, S.; Panyanak, B.; Kesornprom, S.; Chalamjiak, P. Inertial projection and contraction methods for split feasibility problem applied to compressed sensing and image restoration. *Optim. Lett.* **2022**, *16*, 1725–1744. [\[CrossRef\]](#)
38. Tibshirani, R. Regression shrinkage and selection via the Lasso. *J. R. Stat. Soc. Ser. B* **1996**, *58*, 267–288. [\[CrossRef\]](#)

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