

Article

Different Bounds of Various Norms of Special Matrices with Generalized Pell–Lucas Sequence [†]

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Abstract

Norm bounds for circulant-type matrices associated with the bi-periodic Pell–Lucas sequence are examined from a symmetry-driven perspective. By incorporating alternating recurrence coefficients, the results clarify how periodicity and circulant structure affect spectral norm behavior through explicit bounds. This framework extends existing Pell–Lucas-type matrix inequalities and emphasizes the role of symmetry in spectral analysis.

Keywords: bi-periodic Pell–Lucas numbers; circulant-type matrices; spectral bounds; matrix norms

MSC: 15A60; 11B39; 15B05

1. Introduction

Circulant and symmetric circulant matrices often exhibit remarkable algebraic and spectral properties, which can be effectively exploited to derive closed-form expressions, and occupy a central position due to their close connection with discrete Fourier analysis and group theory. Circulant-type matrices naturally arise in a wide range of mathematical and applied contexts, including signal processing, numerical analysis, coding theory, and the solution of differential equations [1–4]. The r -circulant, symmetric r -circulant, and geometric circulant matrices extend the classical circulant framework by introducing additional parameters. Their symmetry-induced structure allows many matrix properties, such as eigenvalues, spectral norms to be expressed explicitly or estimated efficiently. These generalized structures provide a flexible setting for investigating how symmetry influences matrix norms and spectral behavior.

Matrix norms play an important role in perturbation analysis, state estimation and error estimation. In particular, they have important positions and applications in solving coding theory, discrete Fourier transform, various types of partial and ordinary differential equations, numerical analysis.

Special integer sequences occur in many disciplines, such as computer programming, architecture, and human anatomy. These sequences play an important role in constructing matrices with rich algebraic properties. Over the past decade, numerous researchers have examined the spectral norms of circulant matrices with special integer sequences. First, Solak studied the norms of circulant matrices with Fibonacci and Lucas sequences [5]. The authors extended this study to r -circulant matrices in [6]. Yazlık and Taskara generalized



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this study to generalized integer sequences in [7]. Shi also developed it for the norms of geometric circulant matrices in [8]. Kome and Yazlik studied spectral norms of r -circulant matrices with generalized sequences in [9]. Uygun and Aytar studied the inequalities for the norms of some specific matrices with another generalized sequence called the bi-periodic Jacobsthal sequence in [10]. Additionally, the upper bound estimation for the spectral norm of r -circulant matrices with the Padovan sequence was discovered by Sintunavaratin in [11]. Bozkurt and Tam found the determinants and inverses of r -circulant matrices associated with a number sequence in [12]. Bozkurt and Yilmaz investigated the determinants and inverses of circulant matrices with Pell and Pell–Lucas numbers in [13]. Existing studies on the spectral norms of circulant-type matrices have primarily focused on matrices generated by classical recurrence sequences with constant coefficients, such as Fibonacci, Lucas, Pell, Padovan, and generalized Horadam sequences. However, despite their increasing relevance, there is currently a lack of systematic analysis of circulant-type matrices constructed from bi-periodic sequences, particularly the bi-periodic Pell–Lucas sequence. Specifically, explicit expressions for Euclidean norms, sharp upper and lower bounds for spectral norms, and eigenvalue characterizations of circulant, r -circulant, symmetric r -circulant, and geometric circulant matrices associated with bi-periodic Pell–Lucas numbers have not been established. As a result, it is still unclear how alternating recurrence coefficients influence spectral behavior, norm growth, and stability properties in structured matrices. This gap motivates the present work, which provides a unified symmetry-based framework for analyzing matrix norms and spectral properties of circulant-type matrices generated by the bi-periodic Pell–Lucas sequence. By systematically exploiting the symmetry properties of these matrices, we derive new upper and lower bounds for their spectral norms using different analytical techniques. In addition, closed expressions for the eigenvalues of r -circulant matrices are obtained, highlighting the role of symmetry in their spectral distribution. The obtained results generalize and extend several known results in the literature on circulant matrices and special sequences, thereby enriching the theory of symmetry-based spectral analysis. Moreover, the impact of bi-periodicity on symmetry-preserving matrix operations, such as Kronecker and Hadamard products, are investigated.

One of the well-known special integer sequences is the Pell sequence defined recursively $p_n = 2p_{n-1} + p_{n-2}$, $p_0 = 0$, $p_1 = 1$. Similarly, the other sequence is the Pell–Lucas sequence defined by $q_n = 2q_{n-1} + q_{n-2}$ starting with the values of $q_0 = 2$, $q_1 = 2$ for $n \geq 2$ in [14]. The bi-periodic Pell–Lucas sequence represents a natural and nontrivial extension of the classical Pell–Lucas numbers by introducing alternating recurrence coefficients. The bi-periodic Pell–Lucas sequence preserves several fundamental properties of classical Pell–Lucas numbers while simultaneously enriching the sequence with additional structural complexity. Unlike constant-coefficient sequences, the bi-periodic structure captures two-phase dynamics, where the growth behavior alternates according to parity.

The three bounding methods considered in this paper are not intended as alternative formulations of the same estimate, but as complementary tools that emphasize different aspects of the spectral behavior. Method 1 exploits circulant symmetry and typically yields sharper bounds, Method 2 provides simple closed-form estimates independent of structural assumptions, and Method 3 highlights the influence of extremal entries. This multi-method approach allows a systematic comparison of tightness under different parameter choices of a, b, r , and n .

Definition 1 ([15]). Let a and b be any positive real numbers, the bi-periodic Pell–Lucas sequence denoted by $\{Q_n\}_{n=0}^{\infty}$ is defined by

$$Q_0 = 2, Q_1 = 2a, Q_n = \begin{cases} 2bQ_{n-1} + Q_{n-2}, & \text{if } n \text{ is even} \\ 2aQ_{n-1} + Q_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2. \quad (1)$$

When $a = b = 1$, the classic Pell–Lucas sequences are formed. The characteristic equation of the recurrence relation of the bi-periodic Pell–Lucas sequence is

$$x^2 - 2abx - ab = 0,$$

with the roots

$$\alpha = ab + \sqrt{a^2b^2 + ab}, \quad \beta = ab - \sqrt{a^2b^2 + ab}. \quad (2)$$

Roots α and β defined by (2) fulfill the following properties

$$\begin{aligned} (2\alpha + 1)(2\beta + 1) &= 1, & \alpha + \beta &= 2ab, & \alpha\beta &= -ab, \\ (2\alpha + 1) &= \frac{\alpha^2}{ab}, & (2\beta + 1) &= \frac{\beta^2}{ab}, \\ -(2\alpha + 1)\beta &= \alpha, & -(2\beta + 1)\alpha &= \beta. \end{aligned} \quad (3)$$

The Binet formula for the bi-periodic Pell–Lucas sequence is

$$Q_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} (\alpha^n + \beta^n), \quad (4)$$

where $\lfloor n \rfloor$ is the floor function of n , $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$ is the parity function. $\xi(n)$ is also defined as

$$\xi(n) = \begin{cases} 1, & \text{if } n \text{ is odd number} \\ 0, & \text{if } n \text{ is even number} \end{cases}.$$

Throughout the paper, all circulant-type matrices are assumed to be square and generated by non-degenerate bi-periodic Pell–Lucas sequences.

Definition 2. Let n be an arbitrary positive integer. A circulant matrix $Q = \text{circ}(c_0, c_1, \dots, c_{n-1})$ is defined as

$$Q = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{bmatrix}.$$

Definition 3. Let n be any positive integer, r be any complex number. Then an r -circulant matrix $Q_r = \text{circ}_r(c_0, c_1, \dots, c_{n-1})$ is defined as

$$Q_r = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & c_0 \end{bmatrix}.$$

The r -circulant matrices are determined by the parameter r and the first-row elements of the matrix. The matrix becomes a circulant matrix when the parameter satisfies $r = 1$.

Definition 4. An $n \times n$ geometric circulant matrix $Q_{r^*} = \text{circ}_{r^*}(c_0, c_1, \dots, c_{n-1})$ is defined as follows:

$$Q_{r^*} = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ r^2c_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1}c_1 & r^{n-2}c_2 & r^{n-3}c_3 & \cdots & c_0 \end{bmatrix}.$$

If the parameter satisfies the condition $r = 1$, the matrix is transformed into the classical circulant matrix.

Here, the parameters a and b govern the growth rate of the bi-periodic Pell–Lucas sequence, while the parameter r determines the relative contribution of cyclic shifts in the r -circulant matrices.

Lemma 1 ([5]). Let $Q_r = \text{circ}_r\left(\left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0, \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} Q_1, \dots, \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} Q_{n-1}\right)$ be an r -circulant matrix whose entries are generated by the bi-periodic Pell–Lucas numbers. If $\sum_{j=0}^{n-1} r^j \left(\frac{b}{a}\right)^{\frac{\xi(j)}{2}} Q_j \neq 0$, the r -circulant matrix is nonsingular.

Proof. The eigenvalues are demonstrated by

$$\lambda_k(Q_r) = \sum_{j=0}^{n-1} \left(\frac{b}{a}\right)^{\frac{\xi(j)}{2}} Q_j r^{\frac{j}{n}} w^{jk}.$$

where $w = e^{-\frac{2\pi i}{n}}$, $i = \sqrt{-1}$, $k = 0, 1, \dots, n-1$. The determinant of Q_r is expressed as

$$\det(Q_r) = \lambda_0 \lambda_1 \dots \lambda_{n-1}.$$

In particular, for $j = 0$ we have $\lambda_0 = \sum_{j=0}^{n-1} \left(\frac{b}{a}\right)^{\frac{\xi(j)}{2}} Q_j r^{\frac{j}{n}} \neq 0$ which implies that none of the eigenvalues responsible for the determinant vanishing is zero. So, $\det(Q_r) \neq 0$. For $a, b > 0$, the bi-periodic Pell–Lucas numbers are positive and increasing. Hence, except for degenerate parameter choices of r , the condition $\sum_{j=0}^{n-1} \left(\frac{b}{a}\right)^{\frac{\xi(j)}{2}} Q_j r^{\frac{j}{n}} \neq 0$ is naturally satisfied, implying that r -circulant matrices generated by the bi-periodic Pell–Lucas sequence are typically nonsingular. $\square$

Definition 5. An $n \times n$ matrix is called a symmetric r -circulant matrix if it is of the form

$$SQ_r = \begin{bmatrix} c_0 & c_1 & \cdots & c_2 & c_{n-1} \\ c_1 & c_2 & \cdots & c_{n-1} & rc_0 \\ c_2 & c_3 & \cdots & rc_0 & rc_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & rc_0 & \cdots & rc_{n-3} & rc_{n-2} \end{bmatrix}.$$

The circulant and symmetric r -circulant structures considered in this paper naturally exhibit rotational and reflection symmetries, which play a crucial role in determining their spectral behavior. Throughout the paper, we use the following notation:

$Q = \text{circ}(c_0, c_1, \dots, c_{n-1})$: circulant matrix,

$Q_r = \text{circ}(c_0, c_1, \dots, c_{n-1})$: r -circulant matrix,

$SQ_r = \text{circ}(c_0, c_1, \dots, c_{n-1})$: symmetric r -circulant matrix,

$Q_{r^*} = \text{circ}_r(c_0, c_1, \dots, c_{n-1})$: geometric circulant matrix.

The following lemma gives the sum of the squares of the bi-periodic Pell–Lucas numbers.

Lemma 2. *The sum of the squares of the first n terms of the bi-periodic Pell–Lucas sequence is found by*

$$\sum_{i=0}^{n-1} \left(\frac{b}{a}\right)^{\xi(i)} Q_i^2 = \frac{Q_n Q_{n-1}}{2a} + 2. \quad (5)$$

Proof. The Binet form of the bi-periodic Pell–Lucas sequence (4) gives us

$$\left(\frac{b}{a}\right)^{\xi(i)} Q_i^2 = \left(\frac{\alpha^2}{ab}\right)^i + \left(\frac{\beta^2}{ab}\right)^i + 2(-1)^i.$$

Using the sum of the geometric series, we find that

$$\sum_{i=0}^{n-1} \left(\frac{b}{a}\right)^{\xi(i)} Q_i^2 = \left[\frac{\left(\frac{\alpha^2}{ab}\right)^n - 1}{\left(\frac{\alpha^2}{ab}\right) - 1} + \frac{\left(\frac{\beta^2}{ab}\right)^n - 1}{\left(\frac{\beta^2}{ab}\right) - 1} - (-1)^n + 1 \right].$$

By the equalities in (3), we get

$$\begin{aligned} \sum_{i=0}^{n-1} \left(\frac{b}{a}\right)^{\xi(i)} Q_i^2 &= \left[\frac{\beta \left(\frac{\alpha^2}{ab}\right)^n + \beta + \alpha \left(\frac{\beta^2}{ab}\right)^n + \alpha}{2\alpha\beta} - (-1)^n + 1 \right] \\ &= \left[-ab \frac{\alpha^{2n-1} + \beta^{2n-1}}{-2ab(ab)^n} - (-1)^n + 2 \right]. \end{aligned}$$

Using the Binet form in (4), we have

$$\begin{aligned} \frac{Q_n Q_{n-1}}{2a} &= \frac{a^{\xi(n)}}{2a(ab)^{\lfloor \frac{n+1}{2} \rfloor}} (\alpha^n + \beta^n) \frac{a^{\xi(n-1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} (\alpha^{n-1} + \beta^{n-1}) \\ &= \frac{\alpha^{2n-1} + \beta^{2n-1} + (\alpha\beta)^{n-1}(\alpha + \beta)}{2(ab)^n} \\ &= \frac{\alpha^{2n-1} + \beta^{2n-1}}{2(ab)^n} + (-1)^{n-1} \end{aligned}$$

The equality of the results completes the proof. $\square$

Lemma 3. *The following property holds for the bi-periodic Pell–Lucas sequence*

$$\sum_{i=0}^{n-1} \left(\frac{b}{a}\right)^{\xi(i)} \left(\frac{Q_i}{|r|^i}\right)^2 = \frac{Q_{2n-2} - |r|^2 Q_{2n} - |r|^{2n} (4ab+2) + 2|r|^{2n+2}}{|r|^{2n-2} (1+|r|^4 - (4ab+2)|r|^2)} + 2 \frac{|r|^{2n} - (-1)^n}{|r|^{2n-2} (1+|r|^2)}. \quad (6)$$

Proof. The proof follows a similar procedure to the proof of the previous lemma. Using the Binet forms of the bi-periodic Pell–Lucas sequence, we have

$$\left(\frac{b}{a}\right)^{\xi(i)} \left(\frac{Q_i}{|r|^i}\right)^2 = \left[\left(\frac{\alpha^2}{ab|r|^2}\right)^i + \left(\frac{\beta^2}{ab|r|^2}\right)^i + 2\left(\frac{-1}{|r|^2}\right)^i \right].$$

By (5) and the sum of the geometric series, we have that

$$\begin{aligned} & \sum_{i=0}^{n-1} \left(\frac{b}{a}\right)^{\xi(i)} \left(\frac{Q_i}{|r|^i}\right)^2 \\ &= \frac{(ab)^2(\alpha^{2n-2} + \beta^{2n-2}) - ab|r|^2(\alpha^{2n} + \beta^{2n}) - (ab|r|^2)^n ab(4ab + 2) + 2(ab|r|^2)^{n+1}}{(ab|r|^2)^{n-1} [1 + |r|^4 - (4ab + 2)|r|^2] (ab)^2} + 2 \frac{|r|^{2n} - (-1)^n}{|r|^{2n-2}(1 + |r|^2)} \end{aligned}$$

We get the result by (4). $\square$

These summation identities provide an explicit measure of how the bi-periodic parameters control the Frobenius norm of the matrix and, indirectly, its spectral norm behavior.

For any $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$, the Frobenius (or Euclidean) norm of matrix A is displayed by the following equality:

$$\|A\|_E = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}},$$

and the spectral norm of matrix A is shown as

$$\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(A^H A)}$$

where A^H is the conjugate transpose of matrix A and $\lambda_i(A^H A)$ is any eigenvalue of $A^H A$.

Let A be $n \times n$ matrix, then the following inequalities hold between the Euclid and spectral norms [16–18].

$$\frac{1}{\sqrt{n}} \|A\|_E \leq \|A\|_2 \leq \|A\|_E. \quad (7)$$

The spectral norm of matrix A also satisfies

$$\max_{1 \leq i, j \leq n} |a_{ij}| \leq \|A\|_2 \leq \max_{1 \leq i, j \leq n} n |a_{ij}|, \quad (8)$$

Different norms for matrices such as 1-norm and ∞ -norm are defined as

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|, \quad \|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|,$$

with the following property

$$\|A\|_2 \leq \sqrt{\|A\|_1 \|A\|_\infty}. \quad (9)$$

Suppose that $A, B \in M_{m,n}(\mathbb{C})$, then the Hadamard product of A and B is the $m \times n$ matrix of element wise products

$$A \circ B = (a_{ij} b_{ij}).$$

The following property is satisfied:

$$\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2. \quad (10)$$

$r_1(A)$ the maximum row length norm, $c_1(B)$ the maximum column length norm are defined as $r_1(A) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^n |a_{ij}|^2}$ and $c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^n |b_{ij}|^2}$ with the following property

$$\|A \circ B\|_2 \leq r_1(A) c_1(B). \quad (11)$$

Given $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$, the Kronecker product of A and B is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

and has the following property:

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2. \quad (12)$$

The Kronecker and Hadamard products are fundamental matrix operations due to their ability to preserve and transfer structural properties between matrices. The Kronecker product provides a natural mechanism for constructing large-scale matrices from smaller building blocks while maintaining spectral and algebraic relationships. In particular, it is essential in tensor representations, block matrix formulations, and the study of linear operators on product spaces. Many spectral properties, including eigenvalue distributions and norm inequalities can be explicitly characterized through the Kronecker product. Unlike the Kronecker product, the Hadamard product preserves the dimension of the original matrices and is closely linked to positivity, sparsity, and correlation structures. Classical results such as the Schur product theorem highlight its importance in maintaining positive semidefiniteness. Moreover, the Hadamard product is indispensable in studying perturbations, weighted matrices, and elementwise transformations, where global matrix operations are unsuitable. Consequently, analyzing matrix norms and eigenvalues under these products offers deep insight into how structural symmetries propagate or are altered through matrix operations, which is particularly relevant for circulant-type matrices generated by special integer sequences.

2. Bounds of r -Circulant Matrices Involving Bi-Periodic Pell–Lucas Numbers

In this section, we consider the r -circulant matrix Q_r generated by the bi-periodic Pell–Lucas sequence. The following bounds are derived by explicitly exploiting the r -circulant structure and the cyclic repetition of the generating sequence. For clarity, we briefly summarize the role of each method. Method 1 relies on Frobenius norm identities and symmetry-based decompositions and is generally sharper for moderate values of $|r|$. Method 2 yields easily computable bounds that are effective when closed-form simplicity is preferred. Method 3 produces coarse but transparent bounds that emphasize growth with respect to maximal entries.

Theorem 1. (Method 1: Spectral norm bound for r -circulant matrices): Let $r \in \mathbb{C}$ and $Q_r = \text{circ}_r\left(\left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0, \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} Q_1, \dots, \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} Q_{n-1}\right)$ be an r -circulant matrix with the bi-periodic Pell–Lucas numbers. The upper and lower bounds for the spectral norm of Q_r are computed as follows:

(i) If $|r| \geq 1$,

$$\sqrt{\frac{Q_n Q_{n-1}}{2a}} + 2 \leq \|Q_r\|_2 \leq \sqrt{1 + (n-1)|r|^2} \sqrt{\frac{Q_n Q_{n-1}}{2a}} + 2.$$

(ii) If $|r| < 1$,

$$|r| \sqrt{\frac{Q_n Q_{n-1}}{2a}} + 2 \leq \|Q_r\|_2 \leq \sqrt{n \left(\frac{Q_n Q_{n-1}}{2a} + 2 \right)}.$$

Proof. The r -circulant matrix Q_r is of the form

$$Q_r = \begin{bmatrix} \left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0 & \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} Q_1 & \left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} Q_2 & \cdots & \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} Q_{n-1} \\ r \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} Q_{n-1} & \left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0 & \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} Q_1 & \cdots & \left(\frac{b}{a}\right)^{\frac{\xi(n-2)}{2}} Q_{n-2} \\ r \left(\frac{b}{a}\right)^{\frac{\xi(n-2)}{2}} Q_{n-2} & r \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} Q_{n-1} & \left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0 & \cdots & \left(\frac{b}{a}\right)^{\frac{\xi(n-3)}{2}} Q_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} Q_1 & r \left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} Q_2 & r \left(\frac{b}{a}\right)^{\frac{\xi(3)}{2}} Q_3 & \cdots & \left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0 \end{bmatrix}.$$

For $|r| \geq 1$, the application of (5) and (7) yields

$$\begin{aligned} \|Q_r\|_E^2 &= \sum_{k=0}^{n-1} (n-k) \left(\frac{b}{a}\right)^{\xi(k)} Q_k^2 + \sum_{k=0}^{n-1} k |r|^2 \left(\frac{b}{a}\right)^{\xi(k)} Q_k^2 \\ &\geq \sum_{k=0}^{n-1} (n-k) \left(\frac{b}{a}\right)^{\xi(k)} Q_k^2 + \sum_{k=0}^{n-1} k \left(\frac{b}{a}\right)^{\xi(k)} Q_k^2. \end{aligned}$$

$$\|Q_r\|_2 \geq \frac{\|Q_r\|_E}{\sqrt{n}} \geq \sqrt{\frac{Q_n Q_{n-1}}{2a}} + 2.$$

On the other hand, let $Q_r = B \circ C$ where $B = [b_{ij}]$ and $C = [c_{ij}]$ are defined as

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ r & 1 & 1 & \cdots & 1 \\ r & r & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & r & r & \cdots & 1 \end{bmatrix},$$

and

$$C = \begin{bmatrix} \left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0 & \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} Q_1 & \left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} Q_2 & \cdots & \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} Q_{n-1} \\ \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} Q_{n-1} & \left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0 & \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} Q_1 & \cdots & \left(\frac{b}{a}\right)^{\frac{\xi(n-2)}{2}} Q_{n-2} \\ \left(\frac{b}{a}\right)^{\frac{\xi(n-2)}{2}} Q_{n-2} & \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} Q_{n-1} & \left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0 & \cdots & \left(\frac{b}{a}\right)^{\frac{\xi(n-3)}{2}} Q_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} Q_1 & \left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} Q_2 & \left(\frac{b}{a}\right)^{\frac{\xi(3)}{2}} Q_3 & \cdots & \left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0 \end{bmatrix}.$$

It is satisfied by the norm of the maximum row and column lengths of these matrices that

$$r_1(B) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^n |b_{ij}|^2} = \sqrt{1 + (n-1)|r|^2},$$

$$c_1(C) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^n |c_{ij}|^2} = \sqrt{\frac{Q_n Q_{n-1}}{2a} + 2}.$$

Using Equation (11), we get

$$\|Q_r\|_2 \leq r_1(B)c_1(C) = \sqrt{1 + (n-1)|r|^2} \sqrt{\frac{Q_n Q_{n-1}}{2a} + 2}.$$

(iii) For $|r| \leq 1$, by using (4) and (6), it is obtained that

$$\begin{aligned} \|Q_r\|_E^2 &= \sum_{k=0}^{n-1} (n-k) \left(\frac{b}{a}\right)^{\xi(k)} Q_k^2 + \sum_{k=1}^{n-1} k|r|^2 \left(\frac{b}{a}\right)^{\xi(k)} Q_k^2 \\ &\geq \sum_{k=0}^{n-1} n|r|^2 \left(\frac{b}{a}\right)^{\xi(k)} Q_k^2 = n|r|^2 \left(\frac{Q_n Q_{n-1}}{2a} + 2\right). \end{aligned}$$

Hence, we get

$$\|Q_r\|_2 \geq \frac{\|Q_r\|_E}{\sqrt{n}} \geq |r| \sqrt{\frac{Q_n Q_{n-1}}{2a} + 2}.$$

On the other hand, let $Q_r = B \circ C$ where B, C are given as above. By the maximum row and column length norm of these matrices, it is satisfied that

$$r_1(B) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^n |b_{ij}|^2} = \sqrt{n},$$

$$c_1(C) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^n |c_{ij}|^2} = \sqrt{\frac{Q_n Q_{n-1}}{2a} + 2}.$$

Hence we obtain the second part of the proof. $\square$

This method is particularly effective when $|r|$ is moderate and the circulant symmetry is dominant. The proposed bounds are consistent with the classical results for standard r -circulant matrices, and the presence of symmetry may lead to sharper estimates in certain parameter ranges, although this effect is not observed in all cases.

Lemma 4. *The bi-periodic Pell–Lucas numbers satisfies the following property*

$$\sum_{k=0}^{n-1} \left(\frac{b}{a}\right)^{\frac{\xi(k)}{2}} Q_k = \begin{cases} \frac{Q_n}{2a} + \frac{Q_{n-1}}{2\sqrt{ab}} + 1 - \frac{1}{\sqrt{ab}}, & n \text{ is odd} \\ \frac{Q_n}{2\sqrt{ab}} + \frac{Q_{n-1}}{2a} + 1 - \frac{1}{\sqrt{ab}}, & n \text{ is even} \end{cases} \quad (13)$$

The following method provides a structure-independent bound, included for comparison with symmetry-based estimates.

Theorem 2. (Method 2: Structure-independent norm estimate) Let Q_r be an r -circulant matrix whose entries are the bi-periodic Pell–Lucas numbers. We can find upper bounds for the spectral norm of Q_r in a different method. For $|r| \leq 1$, we get

$$\|Q_r\|_2 \leq \begin{cases} \frac{Q_n}{2a} + \frac{Q_{n-1}}{2\sqrt{ab}} + 1 - \frac{1}{\sqrt{ab}}, & n \text{ is odd} \\ \frac{Q_n}{2\sqrt{ab}} + \frac{Q_{n-1}}{2a} + 1 - \frac{1}{\sqrt{ab}}, & n \text{ is even} \end{cases}$$

and for $|r| \geq 1$, we have

$$\|Q_r\|_2 \leq 2 + |r| \left[\begin{cases} \frac{Q_n}{2a} + \frac{Q_{n-1}}{2\sqrt{ab}} - 1 - \frac{1}{\sqrt{ab}}, & n \text{ is odd} \\ \frac{Q_n}{2\sqrt{ab}} + \frac{Q_{n-1}}{2a} - 1 - \frac{1}{\sqrt{ab}}, & n \text{ is even} \end{cases} \right].$$

Proof. For $|r| \leq 1$, we have from (9) and (13)

$$\begin{aligned} \|Q_r\|_2 &\leq \sqrt{\|Q_r\|_1 \|Q_r\|_\infty} = \sqrt{\left(\max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}| \right) \left(\max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}| \right)} \\ &= \sum_{k=0}^{n-1} \left(\frac{b}{a} \right)^{\frac{\xi(k)}{2}} Q_k = \begin{cases} \frac{Q_n}{2a} + \frac{Q_{n-1}}{2\sqrt{ab}} + 1 - \frac{1}{\sqrt{ab}}, & n \text{ is odd} \\ \frac{Q_n}{2\sqrt{ab}} + \frac{Q_{n-1}}{2a} + 1 - \frac{1}{\sqrt{ab}}, & n \text{ is even} \end{cases} \end{aligned}$$

and for $|r| \geq 1$, by means of (9) and (13) we obtain

$$\|Q_r\|_2 \leq 2 + |r| \left[\begin{cases} \frac{Q_n}{2a} + \frac{Q_{n-1}}{2\sqrt{ab}} - 1 - \frac{1}{\sqrt{ab}}, & n \text{ is odd} \\ \frac{Q_n}{2\sqrt{ab}} + \frac{Q_{n-1}}{2a} - 1 - \frac{1}{\sqrt{ab}}, & n \text{ is even} \end{cases} \right].$$

□

This bound is useful in large-scale settings where structural exploitation is less critical and simplicity is desired.

Theorem 3. (Method 3: Entrywise extremal bound) The bounds for the spectral norm of Q_r is established by (12) as

$$\max_{1 \leq i, j \leq n} |a_{ij}| \leq \|Q_r\|_2 \leq n \max_{1 \leq i, j \leq n} |a_{ij}|$$

Although generally looser, this method provides explicit control over extremal growth and is informative when entrywise dominance is relevant.

Remark 1. The three methods differ in both precision and complexity: the Frobenius-norm-based approach yields the sharpest bounds, the $\|Q_r\|_1 \|Q_r\|_\infty$

Method provides a simple and broadly applicable estimate, while the entrywise bound offers a coarser but easily computable alternative.

Corollary 1. Let $A = B = Q_r = \text{circ}_r\left(\left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0, \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} Q_1, \dots, \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} Q_{n-1}\right)$. Then the lower and upper bounds for the spectral norm of the Kronecker product of A and B are demonstrated by

(i) If $|r| \geq 1$, then

$$\frac{Q_n Q_{n-1}}{2a} + 2 \leq \|A \otimes B\|_2 \leq \left[1 + |r|^2(n-1)\right] \left(\frac{Q_n Q_{n-1}}{2a} + 2\right).$$

(ii) If $|r| \leq 1$, then

$$|r|^2 \left(\frac{Q_n Q_{n-1}}{2a} + 2 \right) \leq \|A \otimes B\|_2 \leq n \left(\frac{Q_n Q_{n-1}}{2a} + 2 \right).$$

Proof. The proof is obtained by $\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$. $\square$

Corollary 2. Let $A = B = Q_r$ be r -circulant matrices whose entries are the bi-periodic Pell–Lucas numbers. Then the upper bounds for the spectral norm of the Hadamard product of A and B are demonstrated by

(i) If $|r| \geq 1$, then

$$\|A \circ B\|_2 \leq \left[1 + |r|^2(n-1) \right] \left(\frac{Q_n Q_{n-1}}{2a} + 2 \right).$$

(ii) If $|r| \leq 1$, then

$$\|A \circ B\|_2 \leq n \left(\frac{Q_n Q_{n-1}}{2a} + 2 \right).$$

Proof. The proof is easily seen by $\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2$. $\square$

Theorem 4. Let Q_r be an r -circulant matrix with the bi-periodic Pell–Lucas numbers. The eigenvalues are calculated as

$$\begin{aligned} \lambda_j(Q_r) = & \frac{\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} \left(r^{\frac{n+3}{n}} w^{jn+3j} Q_{n-1} - r^{\frac{n+1}{n}} w^{jn+j} Q_{n+1} \right) \\ & + \left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} \left(r^{\frac{n+2}{n}} w^{jn+2j} Q_{n-2} - r w^{jn} Q_n \right) \\ & + 2 - (4ab + 2)r^{\frac{2}{n}} w^{2j} + 2a \left(r^{\frac{3}{n}} w^{3j} + r^{\frac{1}{n}} w^j \right)}{r^{\frac{4}{n}} w^{4j} - 2(2ab + 1)r^{\frac{2}{n}} w^{2j} + 1} \end{aligned}$$

where $w = e^{-\frac{2\pi i}{n}}$, $i = \sqrt{-1}$, $j = 0, 1, \dots, n-1$.

Proof. The proof follows from the Binet representation of the bi-periodic Pell–Lucas sequence and the separation of even and odd indices. After rewriting the weighted sums as geometric series in terms of the characteristic roots, the resulting expressions are simplified using the algebraic identities satisfied by these roots. Routine algebraic manipulations then lead to the stated closed-form expression. If n is even, we obtain that

$$\begin{aligned} \lambda_j(Q_r) &= \sum_{k=0}^{n-1} \left(\frac{b}{a}\right)^{\frac{\xi(k)}{2}} Q_k r^{\frac{k}{n}} w^{jk} \\ &= \frac{\alpha^n r w^{jn} - (ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1} (\alpha^2 r^{\frac{2}{n}} w^{2j} - ab)} + \frac{\beta^n r w^{jn} - (ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1} (\beta^2 r^{\frac{2}{n}} w^{2j} - ab)} \\ &\quad + \frac{1}{\sqrt{ab}} \left[\frac{\alpha^{n+1} r^{1+\frac{1}{n}} w^{jn+j} - \alpha r^{\frac{1}{n}} w^j (ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1} (\alpha^2 r^{\frac{2}{n}} w^{2j} - ab)} + \frac{\beta^{n+1} r^{1+\frac{1}{n}} w^{jn+j} - \beta r^{\frac{1}{n}} w^j (ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1} (\beta^2 r^{\frac{2}{n}} w^{2j} - ab)} \right]. \end{aligned}$$

At last, we obtain

$$\lambda_j(Q_r) = \frac{r^{1+\frac{2}{n}}w^{jn+2j}Q_{n-2} - rw^{jn}Q_n - (4ab+2)r^{\frac{2}{n}}w^{2j} + 2}{r^{\frac{4}{n}}w^{4j} - 2(2ab+1)r^{\frac{2}{n}}w^{2j} + 1} + \sqrt{\frac{b}{a}} \frac{r^{1+\frac{3}{n}}w^{jn+3j}Q_{n-1} - r^{1+\frac{1}{n}}w^{jn+j}Q_{n+1} + 2a(r^{\frac{3}{n}}w^{3j} + r^{\frac{1}{n}}w^j)}{r^{\frac{4}{n}}w^{4j} - 2(2ab+1)r^{\frac{2}{n}}w^{2j} + 1}.$$

Similarly, if n is an odd number, we get

$$\sum_{k=0}^{n-1} \left(\frac{b}{a}\right)^{\frac{\xi(k)}{2}} Q_k r^{\frac{k}{n}} w^{jk} = \sum_{k=0}^{\frac{n-1}{2}} \frac{\alpha^{2k} + \beta^{2k}}{(ab)^k} r^{\frac{2k}{n}} w^{2jk} + \frac{1}{\sqrt{ab}} \sum_{k=0}^{\frac{n-3}{2}} \frac{\alpha^{2k+1} + \beta^{2k+1}}{(ab)^k} r^{\frac{2k+1}{n}} w^{j(2k+1)}$$

At last, we obtain

$$\lambda_j(Q_r) = \frac{r^{1+\frac{3}{n}}w^{jn+3j}Q_{n-1} - r^{1+\frac{1}{n}}w^{jn+j}Q_{n+1} - (4ab+2)r^{\frac{2}{n}}w^{2j} + 2}{(r^{\frac{4}{n}}w^{4j} - 2(2ab+1)r^{\frac{2}{n}}w^{2j} + 1)} + \sqrt{\frac{b}{a}} \frac{r^{1+\frac{2}{n}}w^{jn+2j}Q_{n-2} - rw^{jn}Q_n + 2ar^{\frac{3}{n}}w^{3j} + 2ar^{\frac{1}{n}}w^j}{r^{\frac{4}{n}}w^{4j} - 2(2ab+1)r^{\frac{2}{n}}w^{2j} + 1}.$$

The combination of the results gives us the proof. $\square$

Since $\lambda_j(Q_r)$ is expressed as a function of w^j the spectrum inherits the usual rotational and conjugate symmetry of r -circulant matrices, with eigenvalues distributed regularly along parametrized curves in the complex plane. When the matrix entries are real, and they lie on closed curves in the complex plane determined by the bi-periodic parameters a, b and the modulus of r . The extremal spectral norm is attained at low-frequency modes (typically $j = 0$ or $j = n/2$).

3. Estimation of the Bounds of Symmetric r -Circulant Matrices with Bi-Periodic Pell–Lucas Numbers

In this section, we consider the symmetric r -circulant matrix SQ_r generated by the bi-periodic Pell–Lucas sequence.

Theorem 5. (Spectral norm bound for symmetric r -circulant matrices) Let $SQ_r = \text{scirc}_r\left(\left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}}Q_0, \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}}Q_1, \dots, \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}Q_{n-1}\right)$ be a symmetric r -circulant matrix with the bi-periodic Pell–Lucas numbers, then the upper bounds for the spectral norm of SQ_r are given by

$$\|SQ_r\|_2 \leq \begin{cases} \sqrt{n\left(\frac{Q_n Q_{n-1}}{2a} + 2\right)}, & |r| < 1 \\ \sqrt{\left[\frac{Q_n Q_{n-1}}{2a} + 2\right] \left[(n-1)|r|^2 + 1\right]}, & |r| \geq 1 \end{cases}$$

Proof. Let $SQ_r = BoC$ where $C = [c_{ij}]$ be given above in the proof of Theorem 8 and $B = [b_{ij}]$ defined as

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & r & r \\ 1 & r & \cdots & r & r \end{bmatrix}.$$

The maximum row and column length norm of these matrices satisfies that

$$\begin{aligned} r_1(B) &= \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^n |b_{ij}|^2} = \begin{cases} \sqrt{n}, & |r| < 1 \\ \sqrt{(n-1)|r|^2 + 1}, & |r| \geq 1 \end{cases}, \\ c_1(C) &= \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^n |c_{ij}|^2} = \sqrt{\frac{Q_n Q_{n-1}}{2a} + 2}. \end{aligned}$$

By (11), we get the result. $\square$

Corollary 3. Let $A = B = SQ_r = \text{scirc}_r\left(\left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0, \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} Q_1, \dots, \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} Q_{n-1}\right)$ be symmetric r -circulant matrices with the bi-periodic Pell–Lucas numbers. Then the upper bounds for spectral norm of the Kronecker product of A and B are demonstrated by

$$\|A \otimes B\|_2 \leq \begin{cases} n\left(\frac{Q_n Q_{n-1}}{2a} + 2\right), & |r| < 1 \\ \left(\frac{Q_n Q_{n-1}}{2a} + 2\right)((n-1)|r|^2 + 1), & |r| \geq 1 \end{cases}$$

The Hadamard product of the matrices gives a similar result.

Remark 2. When compared with the bounds for r -circulant matrices generated by classical constant-coefficient sequences, the bounds obtained in this study for r -circulant matrices with the bi-periodic Pell–Lucas sequences explicitly incorporate the alternating structure of the generating sequence. In particular, the dependence on parity-separated terms reflects the influence of bi-periodicity and leads, in many cases, to sharper upper and lower bounds for the Euclidean and spectral norms.

While bi-periodicity and structural symmetry often reduce overestimation in the norm bounds especially for symmetric r -circulant and geometric circulant matrices. There exist cases in which the bounds coincide with these circulant matrices. This typically occurs when the alternating coefficients effectively reduce to a constant-coefficient behavior. Hence, the improvement induced by symmetry is conditional and depends on the interaction between the bi-periodic parameters and the underlying circulant structure.

4. Bounds for the Spectral Norm of the Geometric Circulant Matrices by Bi-Periodic Pell–Lucas Numbers

In this section, we consider the geometric circulant matrix Q_{r^*} generated by the bi-periodic Pell–Lucas sequence. In geometric circulant matrices, the inclusion of the geometric weighting factor r^k provides additional flexibility by allowing the influence of matrix entries to vary with their position. This factor amplifies higher-index terms when $|r| \geq 1$ and suppresses them when $|r| < 1$ thereby capturing non-uniform growth or decay patterns that cannot be modeled by standard or r -circulant matrices. When combined with the bi-periodic Pell–Lucas structure, the geometric weighting interacts naturally with circulant symmetry, offering a transparent framework for analyzing how growth behavior propagates to spectral properties and norm bounds.

Theorem 6. (Method 1: Spectral norm bound for geometric circulant matrices) Let $Q_{r^*} = \text{circ}_{r^*} \left(\left(\frac{b}{a} \right)^{\frac{\xi(0)}{2}} Q_0, \left(\frac{b}{a} \right)^{\frac{\xi(1)}{2}} Q_1, \dots, \left(\frac{b}{a} \right)^{\frac{\xi(n-1)}{2}} Q_{n-1} \right)$ be a geometric circulant matrix involving the bi-periodic Pell–Lucas numbers. Then the upper and lower bounds for the spectral norm of Q_{r^*} are obtained as .

(i) If $|r| > 1$, then

$$\sqrt{\frac{Q_n Q_{n-1}}{2a} + 2} \leq \|Q_{r^*}\|_2 \leq \sqrt{\left(\frac{Q_n Q_{n-1}}{2a} + 2 \right) \frac{1 - |r|^{2n}}{1 - |r|^2}},$$

(ii) If $|r| \leq 1$, then

$$\begin{aligned} & |r|^n \sqrt{\frac{Q_{2n-2} - |r|^2 Q_{2n} - |r|^{2n} (4ab + 2) + 2|r|^{2n+2}}{(1 + |r|^4 - (4ab + 2)|r|^2)}} + 2 \frac{|r|^{2n} - (-1)^n}{1 + |r|^2} \\ & \leq \|Q_{r^*}\|_2 \leq \sqrt{n \left(\frac{Q_n Q_{n-1}}{2a} + 2 \right)}. \end{aligned}$$

Proof. The geometric circulant matrix Q_{r^*} is of the form

$$Q_{r^*} = \begin{bmatrix} \left(\frac{b}{a} \right)^{\frac{\xi(0)}{2}} Q_0 & \left(\frac{b}{a} \right)^{\frac{\xi(1)}{2}} Q_1 & \left(\frac{b}{a} \right)^{\frac{\xi(2)}{2}} Q_2 & \cdots & \left(\frac{b}{a} \right)^{\frac{\xi(n)}{2}} Q_{n-1} \\ r \left(\frac{b}{a} \right)^{\frac{\xi(n-1)}{2}} Q_{n-1} & \left(\frac{b}{a} \right)^{\frac{\xi(0)}{2}} Q_0 & \left(\frac{b}{a} \right)^{\frac{\xi(1)}{2}} Q_1 & \cdots & \left(\frac{b}{a} \right)^{\frac{\xi(n-2)}{2}} Q_{n-2} \\ r^2 \left(\frac{b}{a} \right)^{\frac{\xi(n-2)}{2}} Q_{n-2} & r \left(\frac{b}{a} \right)^{\frac{\xi(n-1)}{2}} Q_{n-1} & \left(\frac{b}{a} \right)^{\frac{\xi(0)}{2}} Q_0 & \cdots & \left(\frac{b}{a} \right)^{\frac{\xi(n-3)}{2}} Q_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} \left(\frac{b}{a} \right)^{\frac{\xi(1)}{2}} Q_1 & r^{n-2} \left(\frac{b}{a} \right)^{\frac{\xi(2)}{2}} Q_2 & r^{n-3} \left(\frac{b}{a} \right)^{\frac{\xi(3)}{2}} Q_3 & \cdots & \left(\frac{b}{a} \right)^{\frac{\xi(0)}{2}} Q_0 \end{bmatrix}.$$

For $|r| > 1$, Using the definition of the Frobenius norm and (5), we have the following equation

$$\begin{aligned} \|Q_{r^*}\|_E^2 &= \sum_{k=0}^{n-1} (n-k) \left(\frac{b}{a} \right)^{\xi(k)} Q_k^2 + \sum_{k=1}^{n-1} k |r^{n-k}|^2 \left(\frac{b}{a} \right)^{\xi(k)} Q_k^2 \\ &\geq \sum_{k=0}^{n-1} (n-k) \left(\frac{b}{a} \right)^{\xi(k)} Q_k^2 + \sum_{k=1}^{n-1} k \left(\frac{b}{a} \right)^{\xi(k)} Q_k^2 \\ &= n \left(\frac{Q_n Q_{n-1}}{2a} + 2 \right). \end{aligned}$$

From the equality (7), we get

$$\sqrt{\frac{Q_n Q_{n-1}}{2a} + 2} \leq \|Q_{r^*}\|_2.$$

These lower bounds are included mainly to provide explicit parameter-dependent estimates and to facilitate comparison with the corresponding upper bounds. Then, let the matrices B be represented by

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ r & 1 & 1 & \cdots & 1 \\ r^2 & r & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \cdots & 1 \end{bmatrix}$$

and C is given the proof of Theorem 8 that is, $Q_{r^*} = BoC$. The maximum row and column length norm of these matrices are represented as

$$r_1(B) = \sqrt{\frac{1 - |r|^{2n}}{1 - |r|^2}}, \quad c_1(C) = \sqrt{\frac{Q_n Q_{n-1}}{2a} + 2}.$$

By (11) we have

$$\|Q_{r^*}\|_2 \leq \sqrt{\left(\frac{Q_n Q_{n-1}}{2a} + 2\right) \frac{1 - |r|^{2n}}{1 - |r|^2}}.$$

For $|r| \leq 1$, we get the following inequality by (6) and Lemma 3

$$\begin{aligned} \|Q_{r^*}\|_E^2 &= \sum_{k=0}^{n-1} (n-k) \left(\frac{b}{a}\right)^{\xi(k)} Q_k^2 + \sum_{k=1}^{n-1} k |r^{n-k}|^2 \left(\frac{b}{a}\right)^{\xi(k)} Q_k^2 \\ &\geq n \sum_{k=0}^{n-1} |r^{n-k}|^2 \left(\frac{b}{a}\right)^{\xi(k)} Q_k^2 \\ &= n |r|^{2n} \left[\frac{Q_{2n-2} - |r|^2 Q_{2n} - |r|^{2n} (4ab + 2 - 2|r|^2)}{1 + |r|^4 - (4ab + 2)|r|^2} + 2 \frac{|r|^{2n} - (-1)^n}{|r|^2 + 1} \right]. \end{aligned}$$

For the above matrices B and C , we have $Q_{r^*} = BoC$ and it is obtained that

$$r_1(B) = \sqrt{\sum_{j=1}^n |b_{1j}|^2} = \sqrt{n}, \quad c_1(C) = \sqrt{\frac{Q_n Q_{n-1}}{2a} + 2}.$$

Using Equation (11), we get the result. $\square$

Lemma 5. The following sum property is satisfied for the bi-periodic Pell–Lucas numbers

$$\begin{aligned} \sum_{k=0}^{n-1} \left(\frac{b}{a}\right)^{\frac{\xi(k)}{2}} \frac{Q_k}{|r|^k} &= \sqrt{\frac{b}{a}} \frac{\xi(n)}{2} \left[\frac{Q_{n-2} - |r|^2 Q_n}{|r|^{n-2} M} \right] \\ &\quad + \sqrt{\frac{b}{a}} \frac{\xi(n+1)}{2} \left[\frac{Q_{n-1} - |r|^2 Q_{n+1}}{|r|^{n-1} M} \right] \\ &\quad + \frac{2|r|^4 - |r|^2(4ab+2) + 2a|r| + 2a|r|^3}{M} \end{aligned} \quad (14)$$

where $M = 1 - |r|^2(4ab + 2) + |r|^4$.

Proof. The proof is obtained from the Binet formula of the bi-periodic Pell–Lucas sequence and the separation of even and odd indices. After rewriting the weighted sums as geometric series in terms of the characteristic roots, the resulting expressions are simplified using the algebraic identities. Algebraic manipulations lead to the closed-form expression. Let n be odd, then we have

$$\begin{aligned} \sum_{k=0}^{n-1} \left(\frac{b}{a}\right)^{\frac{\xi(k)}{2}} \frac{Q_k}{|r|^k} &= \sum_{k=0}^{\frac{n-1}{2}} \frac{\alpha^{2k} + \beta^{2k}}{(ab)^k |r|^{2k}} + \sqrt{\frac{b}{a}} \frac{a}{abr} \sum_{k=0}^{\frac{n-3}{2}} \frac{\alpha^{2k+1} + \beta^{2k+1}}{(ab)^k |r|^{2k}} \\ &= \left[\frac{Q_{n-1} - r^2 Q_{n+1} - r^{n+1} (4ab + 2) + 2r^{n+3}}{r^{n-1} (1 - r^2 (4ab + 2) + r^4)} \right] \\ &\quad + \sqrt{\frac{b}{a}} \left[\frac{Q_{n-2} - r^2 Q_n + 2ar^{n-1} + 2ar^{n+1}}{r^{n-2} (1 - r^2 (4ab + 2) + r^4)} \right] \end{aligned}$$

Let n be even. Then we get

$$\begin{aligned} \sum_{k=0}^{n-1} \left(\frac{b}{a}\right)^{\frac{\xi(k)}{2}} \frac{Q_k}{|r|^k} &= \sum_{k=0}^{\frac{n-2}{2}} \frac{a^{2k} + \beta^{2k}}{(ab)^k |r|^{2k}} + \sqrt{ab} \sum_{k=0}^{\frac{n-2}{2}} \frac{a^{2k+1} + \beta^{2k+1}}{(ab)^{k+1} |r|^{2k+1}} \\ &= \frac{Q_{n-2} - |r|^2 Q_n - |r|^n (4ab + 2) + 2|r|^{n+2}}{|r|^{n-2} (1 - |r|^2 (4ab + 2) + |r|^4)} \\ &\quad + \sqrt{\frac{b}{a}} \left[\frac{Q_{n-1} - |r|^2 Q_{n+1} + 2a|r|^n + 2a|r|^{n+2}}{|r|^{n-1} (1 - |r|^2 (4ab + 2) + |r|^4)} \right]. \end{aligned}$$

□

Theorem 7. (Method 2) Let $r \in \mathbb{C}$ and Q_{r^*} be a geometric circulant matrix whose entries are the bi-periodic Pell–Lucas numbers. We can find upper bounds for the spectral norm of Q_{r^*} in a different method. For $|r| \leq 1$, we get

$$\|Q_{r^*}\|_2 \leq \begin{cases} \frac{Q_n}{2a} + \frac{Q_{n-1}}{2\sqrt{ab}} + 1 - \frac{1}{\sqrt{ab}}, & n \text{ is odd} \\ \frac{Q_n}{2\sqrt{ab}} + \frac{Q_{n-1}}{2a} + 1 - \frac{1}{\sqrt{ab}}, & n \text{ is even} \end{cases}$$

and for $|r| \geq 1$, we have

$$\begin{aligned} \|Q_{r^*}\|_2 &\leq |r|^n \sqrt{\frac{b}{a}}^{\xi(n)} \left[\frac{Q_{n-2} - |r|^2 Q_n}{|r|^{n-2} M} \right] + |r|^n \sqrt{\frac{b}{a}}^{\xi(n+1)} \left[\frac{Q_{n-1} - |r|^2 Q_{n+1}}{|r|^{n-1} M} \right] \\ &\quad + |r|^n \frac{2|r|^4 - |r|^2 (4ab + 2) + 2a|r| + 2a|r|^3}{M} \end{aligned}$$

Proof. For $|r| \leq 1$, it follows from (9) and (13) that

$$\begin{aligned} \|Q_{r^*}\|_2 &\leq \sqrt{\|Q_{r^*}\|_1 \|Q_{r^*}\|_\infty} = \sqrt{\left(\max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}| \right) \left(\max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}| \right)} \\ &= \sum_{k=0}^{n-1} \left(\frac{b}{a}\right)^{\frac{\xi(k)}{2}} Q_k = \begin{cases} \frac{Q_n}{2a} + \frac{Q_{n-1}}{2\sqrt{ab}} + 1 - \frac{1}{\sqrt{ab}}, & n \text{ is odd} \\ \frac{Q_n}{2\sqrt{ab}} + \frac{Q_{n-1}}{2a} + 1 - \frac{1}{\sqrt{ab}}, & n \text{ is even} \end{cases} \end{aligned}$$

For $|r| \geq 1$, by means of (9), (14) we obtain the result. □

Theorem 8. (Method 3) The bounds for the spectral norm of Q_{r^*} are established by (8) as

$$\max_{1 \leq i, j \leq n} |a_{ij}| \leq \|Q_{r^*}\|_2 \leq n \max_{1 \leq i, j \leq n} |a_{ij}|$$

The best approximation for the bounds of the spectral norm of Q_r in the above three methods varies depending on the value a , b and $|r|$.

The bounds for the spectral norm of Q_{r^*} is established by (12) as

$$\max_{1 \leq i, j \leq n} |a_{ij}| \leq \|Q_r\|_2 \leq n \max_{1 \leq i, j \leq n} |a_{ij}|$$

Corollary 4. Let $A = B = Q_{r^*}$ be geometric circulant matrices with the bi-periodic Pell–Lucas numbers. Then the lower and upper bounds for spectral norm of the Kronecker product of A and B are demonstrated by

(i) If $|r| > 1$, then

$$\frac{Q_n Q_{n-1}}{2a} + 2 \leq \|A \otimes B\|_2 \leq \left(\frac{Q_n Q_{n-1}}{2a} + 2 \right) \frac{1 - |r|^{2n}}{1 - |r|^2}.$$

(ii) If $|r| \leq 1$, then

$$|r|^2 \left[\frac{Q_{2n-2} - |r|^2 Q_{2n} - |r|^{2n} (4ab + 2) + 2|r|^{2n+2}}{(1 + |r|^4 - (4ab + 2)|r|^2)} + 2 \frac{|r|^{2n} - (-1)^n}{(1 + |r|^2)} \right] \leq \|A \otimes B\|_2$$

$$\|A \otimes B\|_2 \leq n \left(\frac{Q_n Q_{n-1}}{2a} + 2 \right).$$

Proof. The proof is easy to see by $\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$. $\square$

Corollary 5. Let $A = B = Q_{r^*}$ be geometric circulant matrices with the bi-periodic Pell–Lucas numbers. Then the upper bounds for spectral norm of Hadamard product of A and B are demonstrated by

(i) If $|r| > 1$, then

$$\|A \circ B\|_2 \leq \left(\frac{Q_n Q_{n-1}}{2a} + 2 \right) \frac{1 - |r|^{2n}}{1 - |r|^2}.$$

(ii) If $|r| \leq 1$, then

$$\|A \circ B\|_2 \leq n \left(\frac{Q_n Q_{n-1}}{2a} + 2 \right).$$

Proof. The proof is obtained by $\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2$. $\square$

By assigning weights of the form r^k , the influence of terms associated with higher indices is either amplified or damped, depending on the magnitude of r . This weighting mechanism allows the matrix to capture non-uniform growth and decay effects that cannot be modeled by standard or r -circulant matrices.

From a spectral perspective, the geometric factor alters the generating polynomial by introducing an exponential scaling, which shifts the balance between dominant and subordinate terms in the eigenvalue expressions. In the context of bi-periodic Pell–Lucas sequences, this interaction is particularly meaningful, as the geometric weighting complements the alternating recurrence coefficients and highlights how local growth patterns propagate through the circulant matrices. Consequently, the geometric factor provides a transparent and mathematically intuitive way to control spectral behavior and norm estimates.

Example 1. Let $n = 5$ and $Q_r = \text{circ}_r\left(\left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0, \dots, \left(\frac{b}{a}\right)^{\frac{\xi(4)}{2}} Q_4\right)$ be an r -circulant matrix. For the different values of a, b , and r the bounds for the spectral norm of Q_r are computed as follows by three methods (Lower Bound: L. B, Upper Bound: U. B.)

a	b	r	L. B. 1	U. B. 1	U. B. 2	L. B. 3	U. B. 3
1	1	1	37.363	83.546	58	34	340
2	4	2	1171.4	4829.8	2785.3	577	2885
2	4	1/2	585.7	2619.3	1393.6	577	2885

Example 2. Let $n = 5$ and $SQ_r = \text{circ}_r\left(\left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0, \dots, \left(\frac{b}{a}\right)^{\frac{\xi(4)}{2}} Q_4\right)$ be an symmetric r -circulant matrix. For the different values of a, b , and r the bounds for the spectral norm of SQ_r are computed as follows

$$\|SQ_r\|_2 \leq \begin{cases} 83.546, & \text{if } a = b = r = 1 \\ 4829.8, & \text{if } a = 2, b = 4, r = 2 \\ 2619.3, & \text{if } a = 2, b = 4, r = 1/2 \end{cases}$$

Example 3. Let $n = 5$ and $Q_{r^*} = \text{circ}_{r^*}\left(\left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0, \dots, \left(\frac{b}{a}\right)^{\frac{\xi(4)}{2}} Q_4\right)$ be a geometric circulant matrix. For the different values of a, b , and r , the bounds for the spectral norm of Q_{r^*} are computed as follows by three methods

a	b	r	L. B. 1	U. B. 1	U. B. 2	L. B. 3	U. B. 3
2	4	2	1171.4	21631	3530.9	577	2885
2	4	1/2	14.014	2619.3	1393.6	577	2885

Example 4. Let $n = 5$ and $Q_r = \text{circ}_r\left(\left(\frac{b}{a}\right)^{\frac{\xi(0)}{2}} Q_0, \dots, \left(\frac{b}{a}\right)^{\frac{\xi(4)}{2}} Q_4\right)$ be an r -circulant matrix with $a = 2, b = 4$, and $r = 2$. The eigenvalues are calculated as

$$\lambda_j(Q_r) = \frac{1}{2^{\frac{4}{5}} w^{4j} - 34 \cdot 2^{\frac{2}{5}} w^{2j} + 1} \left[\begin{aligned} &2^{\frac{8}{5}} w^{8j} 1154 - 2^{\frac{6}{5}} w^{6j} 39202 + \sqrt{2} \left(2^{\frac{7}{5}} w^{7j} 140 - 2 w^{5j} 4756 \right) \\ &+ 2 - 2^{\frac{2}{5}} (34) w^{2j} + 4 \left(2^{\frac{3}{5}} w^{3j} + 2^{\frac{1}{5}} w^j \right) \end{aligned} \right]$$

where $w = e^{-\frac{2\pi i}{5}}$, $i = \sqrt{-1}$, $j = 0, 1, \dots, 4$. For $j = 0$, we get $\lambda_0(Q_r) = 2362.7$.

Remark 3. (Comparative remark on the three methods) The first method, based on the Frobenius norm and symmetry-based decompositions, yields bounds that often sharp for moderate values of $|r|$. The second method, which relies on $\|\cdot\|_1$ and $\|\cdot\|_\infty$ norm estimates, produces simpler closed-form upper bounds that are easy to evaluate but may be less tight, especially for large n . The third method, derived from entrywise extremal bounds, gives the most straightforward estimates and clearly illustrates the dependence on the maximal matrix entries, although it generally leads to the loosest bounds. When these methods are compared between sharpness, analytical simplicity, and computational convenience, their comparison clarifies how the bi-periodic parameters and the circulant structure influence spectral norm estimates. The numerical results indicate that larger values of $|r|$ lead to wider gaps between upper and lower bounds, while changes in a and b primarily affect the overall growth rate.

Remark 4. (Effect of Parameter Variations on Growth, Bounds, and Spectral Behavior) The bi-periodic parameters a and b determine the alternating recurrence structure and directly control the exponential growth rate of the sequence. As the contrast between a and b increases, the asymptotic growth becomes more uneven between even and odd indices, which in turn influences the dominant terms in the generating polynomial and affects the leading eigenvalues. Variations of parameter r in the circulant and geometric circulant matrices affect the tightness of the norm bounds by enhancing or reducing eigenvalue separation, with stronger bi-periodicity typically yielding sharper estimates.

5. Conclusions

This study has examined several classes of symmetry-driven structured matrices constructed from the bi-periodic Pell–Lucas sequence. By incorporating alternating recurrence coefficients into circulant, r -circulant, symmetric r -circulant, and geometric circulant matri-

ces, we revealed how periodicity and matrix symmetry jointly influence spectral behavior. Explicit formulas for the Euclidean norms and sharp upper and lower bounds for the spectral norms were derived.

A key contribution of this work is the systematic comparison of different analytical approaches for estimating spectral norms. The results demonstrate that the interaction between bi-periodicity and circulant-type symmetry affects norm bounds. Moreover, the spectral behavior of these matrices under Kronecker and Hadamard products was analyzed, highlighting how symmetry-preserving matrix operations affect norm growth.

In addition, closed-form expressions for the eigenvalues of r -circulant matrices associated with the bi-periodic Pell–Lucas sequence were obtained, emphasizing the role of rotational symmetry in determining eigenvalue distributions. The presented results extend existing studies on symmetry-based matrix analysis by introducing a bi-periodic framework. These findings provide a foundation for further investigations of multi-periodic sequences and symmetry-driven matrices, with potential applications in spectral theory, numerical analysis, and related fields.

The comparative analysis shows that no single method uniformly dominates the others. Instead, the effectiveness of each bound depends on the interaction between bi-periodic parameters and the circulant structure, justifying the use of multiple complementary approaches. The analysis highlights how the parameters a , b and r influence both the magnitude and tightness of the derived bounds, providing qualitative insight into the spectral behavior of the considered matrices.

Future work may extend the present analysis to circulant-type matrices generated by other bi-periodic or multi-periodic recurrence sequences, allowing a broader class of symmetry-driven structures to be explored. The developed norm estimates may also find applications in numerical analysis and signal processing, particularly in the study of structured operators and stability properties. Another promising direction is the investigation of determinants, inverses, and pseudospectral behavior of geometric and block circulant matrices associated with bi-periodic sequences.

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