

Article

Structure of Nonlinear Maps Preserving $A^*B - \zeta BA$ onto an Arbitrary $*$ -Algebra

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Abstract

We show that every bijection Ψ between complex $*$ -algebras satisfying $\Psi(A^*B - \zeta BA) = \Psi(A)^*\Psi(B) - \zeta\Psi(B)\Psi(A)$, for a fixed $\zeta \neq 0$, is additive as soon as its domain is a prime unital $*$ -algebra with a nontrivial projection, with nothing assumed about its codomain at all. In the additivity theorems for products built from the involution that we have been able to trace, the codomain carries a hypothesis of the same kind as the domain. But the equation does not merely tolerate an unrestricted codomain: for $|\zeta| \neq 1$, surjectivity of Ψ alone is already enough to give it a unit and a centre, and once Ψ is bijective it becomes semilinear over that centre, a phenomenon we call range rigidity. This already rules out a surjection onto $C_0(X)$ for noncompact X , and, for an infinite dimensional Hilbert space H , onto $K(H)$ or a Schatten class. A second, independent argument then extends the exclusion of $C_0(X)$ to every $\zeta \neq 0$, and reaches the operator ideals on an infinite dimensional H as well at the classical parameter $\zeta = -1$. Through a conjugate-linear equivalence with the classical products $AB - \eta BA^*$, the same machinery yields a new proof of the known classification of such maps between factor von Neumann algebras, which is due to Dai and Lu, for every ζ other than -1 . At $\zeta = -1$, the classification is the one of Li, Lu and Fang, and it is quoted here rather than reproved.

Keywords: Range rigidity; additivity; semilinearity; skew Lie product; factor von Neumann algebra

1. Introduction

Let $\mathcal{A}, \mathcal{B}$ be algebras and $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ satisfy $\Phi(AB) = \Phi(A)\Phi(B)$, with no linearity or additivity assumed. Martindale [1] showed that a multiplicative bijection of a ring with a suitable system of idempotents is additive, and is in fact a ring isomorphism; on a $*$ -algebra this first question is already harder, since a scalar placed in a starred argument scales by a real combination of μ and $\bar{\mu}$, not by either alone. This paper asks a second question, and answers it: not whether such a Φ can still be proved additive once its codomain is given no structure, but what the equation itself forces on that codomain when nothing at all is assumed of it in advance.

The involutive products closest to ours form a one-parameter family. For $\eta \neq 0$ put

$$[A, B]_*^\eta := AB - \eta BA^*. \quad (1)$$

For $\eta = 1$ this is the skew Lie product $AB - BA^*$. Products built in this way from the involution were studied on rings with involution by Brešar and Fošner [2], and they appear



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in operator theory in Molnár's characterisation of ideals [3]: when $\dim H > 1$, a subspace $\mathcal{I}$ of $B(H)$ is an ideal exactly when $TA - AT^* \in \mathcal{I}$ for all $T \in B(H)$ and $A \in \mathcal{I}$. Their study goes back to Šemrl's work on Jordan $*$ -derivations and their connection with quadratic functionals [4,5].

Cui and Li [6] showed that a bijection between factor von Neumann algebras preserving the skew Lie product is a $*$ -ring isomorphism. Bai and Du [7] extended this to von Neumann algebras without central abelian projections, where such a bijection is the sum of a linear and a conjugate linear $*$ -isomorphism. Li, Lu and Fang [8] proved that a bijection between factors preserving $AB + BA^*$, the case $\eta = -1$ of (1), is a $*$ -ring isomorphism. Liu and Ji [9] studied the product $A^*B + BA^*$ on factor von Neumann algebras, and Li, Zhao and Chen [10] the product $A^*B + B^*A$ on von Neumann algebras. The whole family (1) was studied by Zhang [11] on prime $*$ -rings and by Dai and Lu [12] on von Neumann algebras. Dai and Lu write the product as $AB + \eta BA^*$, so that their parameter is the negative of ours. For a bijection between two von Neumann algebras, one of which has no central abelian projections, they proved that the map is a linear $*$ -isomorphism when η is not real and the sum of a linear and a conjugate linear $*$ -isomorphism when η is real. Darvish, Nazari, Rohi and Taghavi [13] studied the product $A^*B + \eta BA^*$ for bijections from a prime C^* -algebra onto a C^* -algebra, the range not being assumed prime.

Products of three or more elements, and the corresponding derivations, have been studied as well. Huo, Zheng and Liu [14] considered Jordan triple η - $*$ -products, Huo, Zheng, Xu and Liu [15] the Jordan multiple $*$ -product on factor von Neumann algebras, Zhao and Li [16] the Jordan triple $*$ -product between factors, and Zhang [17] the mixed Jordan triple η - $*$ -product between factors. Li, Zhao and Chen [18] studied nonlinear skew Lie triple derivations between factors, and Ren, Zhang, Liang and Sun [19] nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivations on $*$ -algebras. Zhang and Zhang [20] studied nonlinear maps preserving the Lie product $AB - BA$ on factor von Neumann algebras, a case in which the involution plays no part.

The period since 2022 has seen the same questions pursued for a widening list of products, and in every case the range is constrained in advance. Skew Lie derivations and mixed Lie triple derivations on prime $*$ -rings are treated by Kong and Zhang [21] and by Kong and Li [22], bi-skew Jordan derivations on prime $*$ -rings by Siddeeqe and Shikeh [23], and a mixed product of the same family by Siddeeqe, Bhat and Alam [24]. Gao, Zhao and Teng [25] treat mixed products on von Neumann algebras with no central summands of type I_1 , Kawai, Guzzo and Ferreira [26] work on generalized matrix $*$ -rings, and Ferreira, Costa and Da-Silva [27] on Jordan $*$ -algebras. In each of these, the two algebras are of the same type, or the map goes from an algebra to itself. The question asked here, of what the equation forces on a codomain given no structure at all, does not arise in them.

A separate line of work runs in the opposite direction, from identities involving the involution to the structure of the ring. Ali, Dar and Khan [28] studied maps of strong commutativity-preserving type in rings with involution, Alahmadi, Alhazmi, Ali, Dar and Khan [29] studied additive maps on prime and semiprime rings with involution, Khan and Ali [30] studied involutions on prime rings in the presence of endomorphisms, and Raza, Khan and Alhazmi [31] b -generalised derivations on prime rings with involution.

In the papers of Cui and Li, Bai and Du, Li, Lu and Fang, Dai and Lu, and Darvish, Nazari, Rohi and Taghavi, the range is a factor, a von Neumann algebra or a C^* -algebra, fixed in advance, and the arguments use that structure together with the Peirce decomposition defined by a projection of the domain. Theorem 1 asks nothing of the range. The Peirce decomposition is used in the domain alone, and elements of the range are reached only through surjectivity.

It is worth setting the hypotheses on the range and the conclusions side by side. Cui and Li, and Li, Lu and Fang, take both algebras to be factor von Neumann algebras and conclude that the map is a $*$ -ring isomorphism. Bai and Du take both to be von Neumann algebras without central abelian projections and conclude that it is the sum of a linear and a conjugate linear $*$ -isomorphism. Dai and Lu take both to be von Neumann algebras, one of them without central abelian projections, and conclude that it is a linear $*$ -isomorphism when η is not real and the sum of a linear and a conjugate linear $*$ -isomorphism when η is real. Of the work we have traced, the lightest hypothesis on the range is that of Darvish, Nazari, Rohi and Taghavi, who ask it to be a C^* -algebra but not to be prime. Theorem 1 asks of the range only that it be a $*$ -algebra, and concludes that the map is additive. Theorem 2 asks the same and concludes something about the range itself, namely that it is unital, that it receives $\mathbb{C}I$ in its centre, and that for a bijection the map is semilinear over that centre. The earlier results describe the map under hypotheses on both algebras, while these two describe the map, and then the range, under hypotheses on the domain alone.

For products that do not involve the involution, additivity onto an arbitrary ring goes back to Martindale [1]. Molnár [32] showed that on standard operator algebras over spaces of dimension at least 2 the bijective maps preserving the Jordan product $AB + BA$ are automatically additive, and Lu proved additivity results of the same kind for standard operator algebras [33] and for associative algebras [34]. Ji and Liu [35] and Ji [36] carried the question to standard Jordan operator algebras and to Jordan algebras. Darvish, Alotaibi, Alshehri and Muslikh [37] have recently shown that a bijection preserving $AB + BA$ from a prime unital algebra with a nontrivial idempotent onto an arbitrary algebra is additive, which is the counterpart of Theorem 1 without the involution. The conclusion of each of these results is a property of the map. Theorem 2 draws a conclusion about the range as well, namely that for $|\xi| \neq 1$ a surjective solution onto a nonzero $*$ -algebra forces that algebra to be unital. This is what excludes $C_0(X)$ for noncompact X and, for an infinite dimensional Hilbert space H , the algebra $K(H)$ and the Schatten classes as ranges, and it requires knowing nothing about a candidate range except whether it has a unit.

Throughout, $\xi \in \mathbb{C} \setminus \{0\}$,

$$A \diamond_{\xi} B := A^*B - \xi BA, \quad (2)$$

and the maps considered are $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ between $*$ -algebras with

$$\Psi(A \diamond_{\xi} B) = \Psi(A) \diamond_{\xi} \Psi(B) \quad (A, B \in \mathcal{W}). \quad (3)$$

By Lemma 1, $T(A) = -\bar{\xi}A^*$ carries $\diamond_{\xi}$ onto (1) with $\eta = \xi^{-1}$, so (2) belongs to the family just surveyed and permits direct comparison with its known classification, recalled in Theorem 3 below. The equivalence, though, transports our question rather than answering it: Dai and Lu's classification is still a classification between two von Neumann algebras, and T does not change that how little the range can be made to carry remains untouched by it.

Theorem 2 answers that question. For $|\xi| \neq 1$, mere surjectivity of Ψ is already enough no appeal to additivity required to force $\mathcal{V}$ to possess a unit and to send $\mathbb{C}I$ into its centre $\mathcal{Z}(\mathcal{V})$; once Ψ is bijective this sharpens further, to an injective unital ring homomorphism $h : \mathbb{C} \rightarrow \mathcal{Z}(\mathcal{V})$ through which $\Psi(\mu A) = h(\mu)\Psi(A)$. We call this an a priori unstructured range acquiring a unit, a centre, and a semilinear action on it from the equation alone *range rigidity*, and it is strong enough on its own to rule out a surjection onto any nonzero $*$ -algebra without a unit: not onto $C_0(X)$ for noncompact X , and, for an infinite dimensional Hilbert space H , not onto $K(H)$ and not onto a Schatten class. The hypothesis that H be infinite dimensional is what makes these operator algebras nonunital,

and it is required wherever they are excluded. Where $|\zeta| = 1$ and this argument falls silent, a second argument shows $C_0(X)$ is excluded regardless, and reaches the operator ideals on an infinite dimensional H too, at the single classical value $\zeta = -1$. Only the sharper of these two conclusions, the semilinearity, not the bare unitality, draws on additivity at all, and even that draws on nothing else: Theorem 1 shows that a bijection satisfying (3) from a prime unital $*$ -algebra with a nontrivial projection onto an arbitrary $*$ -algebra is additive, for every $\zeta \neq 0$, so additivity here is not the paper's destination but its engine. Ψ itself turns out to be rigid too: scale it by anything other than 1 and it stops satisfying the equation at all. And when both algebras are factor von Neumann algebras, range rigidity sharpens once more, into a classification already known, due to Dai and Lu [12] for $\zeta \neq -1$ and to Li, Lu and Fang [8] at the single classical value $\zeta = -1$. For $\zeta \neq -1$ the present argument reaches that classification by an entirely different route, through Theorem 1 alone, never once moving a projection into the range. At $\zeta = -1$, the result of Li, Lu and Fang is quoted rather than reproved.

Contributions of This Work

The results of the paper fall into two groups, and we separate them here.

New results. Theorem 1 is the additivity theorem: a bijection satisfying (3) from a prime unital $*$ -algebra with a nontrivial projection onto an arbitrary $*$ -algebra is additive, for every $\zeta \neq 0$. Theorem 2 is the range rigidity theorem: for $|\zeta| \neq 1$ a surjective solution forces the codomain to be unital and carries $\mathbb{C}I$ into its centre, and a bijective one is semilinear over that centre. Corollary 1, Proposition 1, Corollary 2 and Proposition 2 are the nonexistence consequences drawn from these two, and Proposition 4 is the rigidity of Ψ under scaling. Lemmas 6 and 14 are the two computational tools on which the rest rests.

Alternative proofs of known results. Theorem 3 is the classification of the bijective solutions between factor von Neumann algebras for $\zeta \neq 0, -1$. That classification is due to Dai and Lu [12], and is restated here because the proof given below is a different one. No claim of novelty attaches to the statement. Proposition 3 settles the remaining value $\zeta = -1$ and is quoted from Li, Lu and Fang [8]. Corollary 3 is the type I consequence of the two.

The hypothesis $\dim \mathcal{W} > 1$ is used only where $\mathcal{W}$ is a factor von Neumann algebra. In that setting it does yield a nontrivial projection, but the implication rests on the von Neumann algebra structure and is not available in a general $*$ -algebra. Wherever the domain is only a $*$ -algebra, the existence of a nontrivial projection is assumed outright. Section 2 returns to this point.

2. Preliminaries

All algebras occurring below are complex algebras equipped with an involution $A \mapsto A^*$. An algebra $\mathcal{A}$ is *prime* if $A\mathcal{A}B = \{0\}$ implies $A = 0$ or $B = 0$, and a *von Neumann algebra* is a weakly closed self-adjoint subalgebra $\mathcal{W}$ of some $B(H)$ containing the identity operator I , a *factor* being one with trivial centre, $\mathcal{Z}(\mathcal{W}) = \mathbb{C}I$. Every factor is prime, for which we refer to Dixmier [38]. We write $\mathcal{W}_{sa}$ for the set of self-adjoint elements of $\mathcal{W}$.

By a *nontrivial projection* we mean an element P with $P = P^* = P^2$ which is neither 0 nor I . Such an element is what makes the Peirce decomposition available, and it exists in every factor other than $\mathbb{C}I$. Indeed, a von Neumann algebra is the closed linear span of its projections, so a von Neumann algebra whose only projections are 0 and I must reduce to $\mathbb{C}I$. A factor with $\dim \mathcal{W} > 1$ therefore carries a nontrivial projection.

This implication uses the von Neumann algebra structure, through the density of the span of the projections, and it fails in a general $*$ -algebra: a complex $*$ -algebra may have dimension greater than one and still have no projection other than 0 and I . The condition $\dim \mathcal{W} > 1$ is accordingly not a substitute for the existence of a nontrivial projection outside

the von Neumann setting. In this paper $\dim \mathcal{W} > 1$ is imposed only where $\mathcal{W}$ is a factor von Neumann algebra, namely in Corollary 1 and from Theorem 3 onwards. In Theorems 1 and 2, where $\mathcal{W}$ is only a $*$ -algebra, the presence of a nontrivial projection is part of the hypothesis and is not deduced from the dimension.

The parameter ξ is a fixed nonzero complex number, and since it never varies we write $A \diamond B$ for $A \diamond_{\xi} B$.

Three properties of the product are in constant use below. It is additive in each of its two variables,

$$(A + A') \diamond B = A \diamond B + A' \diamond B, \quad A \diamond (B + B') = A \diamond B + A \diamond B', \quad (4)$$

and complex linear in the second. In the first it is neither linear nor conjugate linear, because

$$(\mu A) \diamond B = \bar{\mu} A^* B - \xi \mu B A$$

mixes $\bar{\mu}$ with μ . A scalar multiple of the identity, placed in the first variable, nevertheless acts on the second as multiplication by a single complex number,

$$(\mu I) \diamond B = c(\mu) B, \quad c(\mu) := \bar{\mu} - \xi \mu, \quad (5)$$

and the multiplier c so defined governs much of what follows.

Although c is only real linear, it is a linear transformation of $\mathbb{C}$ regarded as a real plane, and its determinant is easily computed. Writing $\xi = a + ib$ with $a, b \in \mathbb{R}$, we have $c(1) = 1 - \xi = (1 - a) - ib$ and $c(i) = \bar{i} - \xi i = -i - (a + ib)i = b - i(1 + a)$, so in the real basis $\{1, i\}$ the matrix of c is

$$\begin{pmatrix} 1 - a & b \\ -b & -(1 + a) \end{pmatrix},$$

whose determinant is $-(1 - a)(1 + a) + b^2 = a^2 + b^2 - 1$. Thus

$$\det c = |\xi|^2 - 1. \quad (6)$$

The two regimes separated by this number recur throughout the paper. For $|\xi| \neq 1$ the map c is invertible, so every complex number has the form $c(\mu)$, and in particular some $\mu_0 I$ is a left unit for the product. For $|\xi| = 1$ it is singular, and its kernel and its image are both real lines.

We begin by showing that (2) is a normal form for the classical family (1), and not a product of a new kind.

Lemma 1. *Let $\mathcal{A}$ be a $*$ -algebra, let $\xi \neq 0$, and set $\eta = \xi^{-1}$ and $T(A) = -\bar{\xi} A^*$ for $A \in \mathcal{A}$. Then T is a conjugate linear bijection of $\mathcal{A}$ with $T^{-1}(X) = -\xi^{-1} X^*$, and*

$$T(A \diamond_{\xi} B) = [T(A), T(B)]_{*}^{\eta} \quad (A, B \in \mathcal{A}).$$

If $\mathcal{W}$ and $\mathcal{V}$ are $$ -algebras, on each of which T is defined by the above formula, then a map $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ satisfies (3) if and only if $T\Psi T^{-1}$ satisfies the corresponding identity for $[\cdot, \cdot]_{*}^{\eta}$.*

Proof. For $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$,

$$T(A + B) = -\bar{\xi}(A + B)^* = T(A) + T(B), \quad T(\lambda A) = -\bar{\xi} \bar{\lambda} A^* = \bar{\lambda} T(A),$$

and

$$T^2(A) = T(-\bar{\xi} A^*) = -\bar{\xi}(-\bar{\xi} A^*)^* = -\bar{\xi}(-\xi A) = |\xi|^2 A.$$

Hence T is bijective with $T^{-1} = |\zeta|^{-2}T$, that is,

$$T^{-1}(X) = |\zeta|^{-2}(-\bar{\zeta}X^*) = -\zeta^{-1}X^*.$$

For $A, B \in \mathcal{A}$,

$$T(A \diamond_{\zeta} B) = -\bar{\zeta}(A^*B - \zeta BA)^* = -\bar{\zeta}(B^*A - \bar{\zeta}A^*B^*) = \bar{\zeta}^2 A^*B^* - \bar{\zeta}B^*A,$$

and, since $T(A)^* = (-\bar{\zeta}A^*)^* = -\zeta A$,

$$\begin{aligned} [T(A), T(B)]_*^{\eta} &= T(A)T(B) - \eta T(B)T(A)^* = \bar{\zeta}^2 A^*B^* - \eta(-\bar{\zeta}B^*)(-\zeta A) \\ &= \bar{\zeta}^2 A^*B^* - \eta|\zeta|^2 B^*A. \end{aligned}$$

The two right-hand sides agree, because

$$\eta|\zeta|^2 = \zeta^{-1}|\zeta|^2 = \bar{\zeta},$$

which proves the displayed identity. Writing $X = T(A)$ and $Y = T(B)$ in it and applying T^{-1} ,

$$T^{-1}([X, Y]_*^{\eta}) = T^{-1}(X) \diamond_{\zeta} T^{-1}(Y) \quad (X, Y \in \mathcal{A}). \quad (7)$$

Let $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ satisfy (3) and put $\Phi = T\Psi T^{-1}$. For $X, Y \in \mathcal{W}$ set $A = T^{-1}(X)$ and $B = T^{-1}(Y)$, so that $\Phi(X) = T\Psi(A)$ and $\Phi(Y) = T\Psi(B)$. Then

$$\begin{aligned} \Phi([X, Y]_*^{\eta}) &= T\Psi T^{-1}([X, Y]_*^{\eta}) = T\Psi(A \diamond_{\zeta} B) = T(\Psi(A) \diamond_{\zeta} \Psi(B)) \\ &= [T\Psi(A), T\Psi(B)]_*^{\eta} = [\Phi(X), \Phi(Y)]_*^{\eta}, \end{aligned}$$

the second equality by (7), the third by (3), and the fourth by the identity above applied in $\mathcal{V}$. Conversely, let $\Phi : \mathcal{W} \rightarrow \mathcal{V}$ satisfy $\Phi([X, Y]_*^{\eta}) = [\Phi(X), \Phi(Y)]_*^{\eta}$ and put $\Psi = T^{-1}\Phi T$. Then, for $A, B \in \mathcal{W}$,

$$\begin{aligned} \Psi(A \diamond_{\zeta} B) &= T^{-1}\Phi(T(A \diamond_{\zeta} B)) = T^{-1}\Phi([T(A), T(B)]_*^{\eta}) = T^{-1}([\Phi T(A), \Phi T(B)]_*^{\eta}) \\ &= T^{-1}\Phi T(A) \diamond_{\zeta} T^{-1}\Phi T(B) = \Psi(A) \diamond_{\zeta} \Psi(B), \end{aligned}$$

the fourth equality by (7) applied in $\mathcal{V}$. $\square$

We record next some elementary consequences of the functional Equation (3).

Lemma 2. Let $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ be a bijection satisfying (3). Then $\Psi^{-1} : \mathcal{V} \rightarrow \mathcal{W}$ satisfies (3) as well.

Proof. Let $C, D \in \mathcal{V}$ and put $A = \Psi^{-1}(C)$ and $B = \Psi^{-1}(D)$, so that $\Psi(A) = C$ and $\Psi(B) = D$. By (3),

$$\Psi(A \diamond B) = \Psi(A) \diamond \Psi(B) = C \diamond D,$$

and applying Ψ^{-1} to the two ends of this identity,

$$\Psi^{-1}(C \diamond D) = A \diamond B = \Psi^{-1}(C) \diamond \Psi^{-1}(D) \quad (C, D \in \mathcal{V}).$$

$\square$

The hypotheses are therefore symmetric in Ψ and Ψ^{-1} , and we appeal to this whenever a reverse inclusion is needed.

Lemma 3. If $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ is a surjective map satisfying (3), then $\Psi(0) = 0$.

Proof. By surjectivity, there is an $A \in \mathcal{W}$ with $\Psi(A) = 0$. Since

$$0 \diamond A = 0^* A - \zeta A 0 = 0,$$

we obtain from (3)

$$\Psi(0) = \Psi(0 \diamond A) = \Psi(0) \diamond \Psi(A) = \Psi(0) \diamond 0 = \Psi(0)^* 0 - \zeta 0 \Psi(0) = 0. \quad \square$$

Lemma 4 (Wintner; Problem 230 in [39]). *Let $\mathcal{A}$ be a unital Banach algebra. If $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ satisfy $AB - BA = \lambda I$, then $\lambda = 0$.*

For a $*$ -algebra $\mathcal{A}$ we write

$$\mathfrak{N}_{\zeta}(\mathcal{A}) = \{A \in \mathcal{A} : A \diamond B = 0 \text{ for every } B \in \mathcal{A}\}$$

for the set of elements that the product annihilates. We determine it for factors.

Lemma 5. *Let $\mathcal{W}$ be a factor von Neumann algebra. Then*

$$\mathfrak{N}_{\zeta}(\mathcal{W}) = \{\mu I : \mu \in \ker c\},$$

so that $\mathfrak{N}_{\zeta}(\mathcal{W}) = \{0\}$ when $|\zeta| \neq 1$, while $\mathfrak{N}_{\zeta}(\mathcal{W}) = \gamma \mathbb{R}I$ when $|\zeta| = 1$, γ being any scalar of modulus one with $\bar{\gamma} = \zeta \gamma$; and $I \in \mathfrak{N}_{\zeta}(\mathcal{W})$ precisely when $\zeta = 1$. If $\mathcal{V}$ is a $*$ -algebra and $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ is a bijection satisfying (3), then $\Psi(\mathfrak{N}_{\zeta}(\mathcal{W})) = \mathfrak{N}_{\zeta}(\mathcal{V})$.

Proof. Let $A \in \mathfrak{N}_{\zeta}(\mathcal{W})$. Taking $B = I$,

$$0 = A \diamond I = A^* I - \zeta I A = A^* - \zeta A,$$

so that $A^* = \zeta A$, and applying the involution to this relation,

$$A = (A^*)^* = (\zeta A)^* = \bar{\zeta} A^* = \bar{\zeta} \zeta A = |\zeta|^2 A,$$

that is, $(1 - |\zeta|^2)A = 0$. Hence $A = 0$ when $|\zeta| \neq 1$.

Suppose $|\zeta| = 1$ and write $\zeta = e^{2i\theta}$ and $\gamma = e^{-i\theta}$, so that

$$|\gamma| = 1, \quad \bar{\gamma} = e^{i\theta} = e^{2i\theta} e^{-i\theta} = \zeta \gamma.$$

Put $T = \bar{\gamma} A$, so that $A = \gamma T$ and $A^* = \bar{\gamma} T^*$. From $A^* = \zeta A$ and $\gamma \zeta = \bar{\gamma}$,

$$T^* = \gamma A^* = \gamma \zeta A = \bar{\gamma} A = T,$$

so T is self-adjoint, and $A^* = \bar{\gamma} T$. For every $B \in \mathcal{W}$ the relation $A^* B = \zeta B A$, therefore, reads

$$\bar{\gamma} T B = \zeta \gamma B T = \bar{\gamma} B T,$$

that is, $T B = B T$. Thus $T \in \mathcal{Z}(\mathcal{W}) = \mathbb{C}I$, and T being self-adjoint, $T \in \mathbb{R}I$; hence $A \in \gamma \mathbb{R}I$. Conversely, for $A = \gamma t I$ with $t \in \mathbb{R}$,

$$A \diamond B = \bar{\gamma} t B - \zeta \gamma t B = (\bar{\gamma} - \zeta \gamma) t B = 0 \quad (B \in \mathcal{W}),$$

so that $A \in \mathfrak{N}_{\zeta}(\mathcal{W})$.

In both cases the elements of $\mathfrak{N}_\xi(\mathcal{W})$ are scalar multiples of I , and by (5),

$$(\mu I) \diamond B = c(\mu)B \quad (B \in \mathcal{W}),$$

which vanishes for every B exactly when $c(\mu) = 0$. Hence $\mathfrak{N}_\xi(\mathcal{W}) = \{\mu I : \mu \in \ker c\}$, and comparison with the two descriptions just obtained gives $\ker c = \{0\}$ for $|\xi| \neq 1$ and $\ker c = \gamma\mathbb{R}$ for $|\xi| = 1$. Since

$$c(1) = 1 - \xi,$$

the identity lies in $\mathfrak{N}_\xi(\mathcal{W})$ precisely when $\xi = 1$.

Finally, let $A \in \mathfrak{N}_\xi(\mathcal{W})$ and $B \in \mathcal{W}$. By (3) and Lemma 3,

$$\Psi(A) \diamond \Psi(B) = \Psi(A \diamond B) = \Psi(0) = 0,$$

and $\Psi(B)$ runs over $\mathcal{V}$ by surjectivity, so $\Psi(A) \in \mathfrak{N}_\xi(\mathcal{V})$. By Lemma 2 the map Ψ^{-1} satisfies (3), and the same computation applied to it gives $\Psi^{-1}(\mathfrak{N}_\xi(\mathcal{V})) \subseteq \mathfrak{N}_\xi(\mathcal{W})$. Hence $\Psi(\mathfrak{N}_\xi(\mathcal{W})) = \mathfrak{N}_\xi(\mathcal{V})$. $\square$

Let $\mathcal{W}$ be a unital $*$ -algebra containing a nontrivial projection P_1 , and write $P_2 = I - P_1$ for the complementary projection. The four corners $\mathcal{W}_{ij} = P_i \mathcal{W} P_j$ decompose $\mathcal{W}$ as a direct sum,

$$\mathcal{W} = \bigoplus_{i,j=1}^2 \mathcal{W}_{ij},$$

and we write $A = \sum_{i,j} A_{ij}$ for the decomposition of an element. Two rules govern the calculus of the corners: each is left unchanged by the projections that label it, $P_k \mathcal{W}_{kl} = \mathcal{W}_{kl} = \mathcal{W}_{kl} P_l$, and two of them multiply to zero unless the adjacent labels agree, $\mathcal{W}_{kl} \mathcal{W}_{mn} = \{0\}$ for $l \neq m$. Applied to the definition (2) they give the following products, in which i and j are always distinct.

$$\begin{aligned} P_i \diamond P_j &= 0, & P_i \diamond A_{ij} &= A_{ij}, & P_i \diamond A_{ji} &= -\xi A_{ji}, & P_i \diamond A_{jj} &= 0, \\ A_{ij}^* \diamond P_j &= A_{ij} - \xi A_{ij}^*, & A_{ij}^* \diamond B_{ij} &= -\xi B_{ij} A_{ij}^*, & A_{ij} \diamond T_{ij} &= A_{ij}^* T_{ij}, \\ A_{ii} \diamond T_{ij} &= A_{ii}^* T_{ij}, & T_{ij} \diamond A_{ji} &= -\xi A_{ji} T_{ij}, & T_{ij} \diamond A_{jj} &= 0. \end{aligned} \quad (8)$$

A device of a different kind is needed whenever an element is produced by surjectivity. Suppose that $A_1, \dots, A_n$ and X are elements of $\mathcal{W}$ with $\Psi(X) = \sum_k \Psi(A_k)$. Then (3) and the additivity (4) of the product give

$$\Psi(S \diamond X) = \sum_{k=1}^n \Psi(S \diamond A_k), \quad \Psi(X \diamond S) = \sum_{k=1}^n \Psi(A_k \diamond S) \quad (9)$$

for every $S \in \mathcal{W}$: although Ψ is not yet known to be additive, a sum appearing in its range may be pushed through the product one term at a time, from either side.

It remains to produce elements whose $\diamond$ -action annihilates one prescribed corner and is injective on the others, and to produce them for every value of ξ at once.

Lemma 6. *Let $i \neq j$ and let $\lambda, \mu \in \mathbb{C}$. Then, for every $X \in \mathcal{W}$,*

$$(\lambda P_i + \mu P_j) \diamond X = c(\lambda)X_{ii} + (\bar{\lambda} - \xi\mu)X_{ij} + (\bar{\mu} - \xi\lambda)X_{ji} + c(\mu)X_{jj}, \quad (10)$$

the coefficients of the two diagonal corners being values of the multiplier (5). Fix $\varepsilon \in \{1, i\}$ with

$$\bar{\varepsilon} - \xi\varepsilon \neq 0, \quad (11)$$

which is possible since $\bar{\varepsilon}/\varepsilon$ equals 1 for $\varepsilon = 1$ and -1 for $\varepsilon = i$, and ξ is not both. Put

$$S_1 = \varepsilon P_i, \quad S_2 = \varepsilon P_i + \frac{\bar{\varepsilon}}{\xi} P_j, \quad S'_2 = \varepsilon P_i + \bar{\xi} \bar{\varepsilon} P_j, \quad S_3 = \varepsilon P_i,$$

and denote by $(\alpha, \beta, \gamma, \delta)$ the coefficients of $X_{ii}, X_{ij}, X_{ji}, X_{jj}$ in (10). Then

$$\begin{aligned} S_1 : \quad & \alpha = 0, & \beta = -\xi\varepsilon \neq 0, & \gamma = \bar{\varepsilon} \neq 0, & \delta = \bar{\varepsilon} - \xi\varepsilon \neq 0; \\ S_2 : \quad & \alpha = \bar{\varepsilon} - \xi\varepsilon \neq 0, & \beta = 0, & \gamma = \varepsilon(1 - |\xi|^2)/\bar{\xi}, & \delta = \varepsilon/\bar{\xi} - \bar{\varepsilon} \neq 0; \\ S'_2 : \quad & \alpha = \bar{\varepsilon} - \xi\varepsilon \neq 0, & \beta = \bar{\varepsilon}(1 - |\xi|^2), & \gamma = 0, & \delta = \xi(\varepsilon - \bar{\xi}\bar{\varepsilon}) \neq 0; \\ S_3 : \quad & \alpha = \bar{\varepsilon} - \xi\varepsilon \neq 0, & \beta = \bar{\varepsilon} \neq 0, & \gamma = -\xi\varepsilon \neq 0, & \delta = 0. \end{aligned}$$

Proof. Since $(\lambda P_i + \mu P_j)^* = \bar{\lambda} P_i + \bar{\mu} P_j$, the two rules above give

$$(\bar{\lambda} P_i + \bar{\mu} P_j)X = \bar{\lambda}(X_{ii} + X_{ij}) + \bar{\mu}(X_{ji} + X_{jj}), \quad X(\lambda P_i + \mu P_j) = \lambda(X_{ii} + X_{ji}) + \mu(X_{ij} + X_{jj}),$$

and, therefore,

$$\begin{aligned} (\lambda P_i + \mu P_j) \diamond X &= (\bar{\lambda} P_i + \bar{\mu} P_j)X - \xi X(\lambda P_i + \mu P_j) \\ &= (\bar{\lambda} - \xi\lambda)X_{ii} + (\bar{\lambda} - \xi\mu)X_{ij} + (\bar{\mu} - \xi\lambda)X_{ji} + (\bar{\mu} - \xi\mu)X_{jj}, \end{aligned}$$

which is (10), since $\bar{\lambda} - \xi\lambda = c(\lambda)$ and $\bar{\mu} - \xi\mu = c(\mu)$.

Substituting $\lambda = 0, \mu = \varepsilon$ for S_1 and $\lambda = \varepsilon, \mu = 0$ for S_3 ,

$$S_1 : \quad \alpha = c(0) = 0, \quad \beta = -\xi\varepsilon, \quad \gamma = \bar{\varepsilon}, \quad \delta = c(\varepsilon) = \bar{\varepsilon} - \xi\varepsilon,$$

$$S_3 : \quad \alpha = c(\varepsilon) = \bar{\varepsilon} - \xi\varepsilon, \quad \beta = \bar{\varepsilon}, \quad \gamma = -\xi\varepsilon, \quad \delta = c(0) = 0.$$

For S_2 , where $\lambda = \varepsilon$ and $\mu = \bar{\varepsilon}/\xi$, so that $\bar{\mu} = \varepsilon/\bar{\xi}$,

$$\alpha = c(\varepsilon), \quad \beta = \bar{\varepsilon} - \xi \frac{\bar{\varepsilon}}{\xi} = 0, \quad \gamma = \varepsilon \bar{\xi}^{-1} - \xi\varepsilon = \varepsilon(1 - |\xi|^2)\bar{\xi}^{-1}, \quad \delta = \varepsilon \bar{\xi}^{-1} - \bar{\varepsilon},$$

and for S'_2 , where $\lambda = \varepsilon$ and $\mu = \bar{\xi}\bar{\varepsilon}$, so that $\bar{\mu} = \xi\varepsilon$,

$$\alpha = c(\varepsilon), \quad \beta = \bar{\varepsilon} - \xi \bar{\xi} \bar{\varepsilon} = \bar{\varepsilon}(1 - |\xi|^2), \quad \gamma = \xi\varepsilon - \xi\varepsilon = 0, \quad \delta = \xi\varepsilon - |\xi|^2 \bar{\varepsilon} = \xi(\varepsilon - \bar{\xi}\bar{\varepsilon}).$$

Here $\bar{\varepsilon} \neq 0$ and $\xi\varepsilon \neq 0$ because ξ and ε are nonzero, and $c(\varepsilon) \neq 0$ by (11). Finally

$$\varepsilon \bar{\xi}^{-1} - \bar{\varepsilon} = 0 \iff \varepsilon = \bar{\xi} \bar{\varepsilon} \iff \bar{\varepsilon} = \xi\varepsilon,$$

the second equivalence by conjugation, and the right-hand relation is excluded by (11); the same computation applies to $\delta = \xi(\varepsilon - \bar{\xi}\bar{\varepsilon})$ for S'_2 , since $\xi \neq 0$. $\square$

3. Additivity

Throughout this section $\mathcal{W}$ is a prime unital complex $*$ -algebra containing a nontrivial projection P_1 , the algebra $\mathcal{V}$ is an arbitrary complex $*$ -algebra, $\xi \neq 0$, and $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ is a bijection satisfying (3). In each proof, an element X is produced by surjectivity, its Peirce components are determined by testing X against the elements of Lemma 6, and injectivity then identifies X .

Theorem 1. *Let $\mathcal{W}$ be a prime unital complex $*$ -algebra containing a nontrivial projection, let $\mathcal{V}$ be an arbitrary complex $*$ -algebra, let $\xi \in \mathbb{C} \setminus \{0\}$, and let $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ be a bijection satisfying (3). Then Ψ is additive.*

The difficulty created by an arbitrary codomain can be located precisely. In the usual treatment of these problems, a projection is transported from the domain to the range at the outset, so that both algebras carry a Peirce decomposition and the two are compared corner by corner. The range is assumed to be a von Neumann algebra, or prime, or at least unital, and that assumption is what makes the transported projection available. Here $\mathcal{V}$ is an arbitrary $*$ -algebra. It need not be unital, it need not be prime, and it need not contain a single projection, so no Peirce decomposition exists on that side and there is nothing in $\mathcal{V}$ to compare corners with.

The argument below therefore keeps the entire decomposition in $\mathcal{W}$. An element of $\mathcal{V}$ enters only as $\Psi(X)$ for some $X \in \mathcal{W}$ produced by surjectivity, and the components of that X are determined inside $\mathcal{W}$, by testing it against the elements of Lemma 6, and injectivity then identifies X itself. The elements S_1, S_2, S'_2, S_3 carry the whole of the work, each annihilating one prescribed corner while acting injectively on the others, and Lemma 6 supplies them simultaneously for every $\xi \neq 0$, including the values $|\xi| = 1$ at which the coefficient γ of S_2 degenerates. The only property of $\mathcal{V}$ used anywhere is that it is a $*$ -algebra, so that (3) can be written down at all.

The proof consists of the sequence of lemmas that follows.

Lemma 7. *Let $i \neq j$. Then*

$$\Psi(A_{ii} + B_{jj}) = \Psi(A_{ii}) + \Psi(B_{jj})$$

for all $A_{ii} \in \mathcal{W}_{ii}$ and $B_{jj} \in \mathcal{W}_{jj}$.

Proof. By surjectivity, there is an $X \in \mathcal{W}$ with

$$\Psi(X) = \Psi(A_{ii}) + \Psi(B_{jj}).$$

By (10),

$$S_3 \diamond A_{ii} = \alpha A_{ii}, \quad S_3 \diamond B_{jj} = \delta B_{jj} = 0,$$

because $\delta = 0$ for S_3 . Hence (9) and Lemma 3 give

$$\Psi(S_3 \diamond X) = \Psi(S_3 \diamond A_{ii}) + \Psi(S_3 \diamond B_{jj}) = \Psi(\alpha A_{ii}) + \Psi(0) = \Psi(\alpha A_{ii}),$$

and Ψ is injective, so that

$$S_3 \diamond X = \alpha A_{ii}.$$

By (10) the left-hand side equals $\alpha X_{ii} + \bar{\varepsilon} X_{ij} - \xi \varepsilon X_{ji}$, and the sum $\mathcal{W} = \bigoplus_{k,l} \mathcal{W}_{kl}$ is direct, so the last equality splits into

$$\alpha X_{ii} = \alpha A_{ii}, \quad \bar{\varepsilon} X_{ij} = 0, \quad -\xi \varepsilon X_{ji} = 0,$$

the component in $\mathcal{W}_{jj}$ giving $0 = 0$. Since $\alpha, \bar{\varepsilon}$ and $\xi \varepsilon$ are nonzero,

$$X_{ii} = A_{ii}, \quad X_{ij} = 0, \quad X_{ji} = 0.$$

We now test against $S_1 = \varepsilon P_j$, whose coefficients in (10) are $(0, \beta, \gamma, \delta)$ with $\beta, \gamma, \delta \neq 0$. Here δ is the coefficient of the corner $\mathcal{W}_{jj}$ for S_1 , and is not to be confused with the coefficient α of $\mathcal{W}_{ii}$ for S_3 used above, although the two happen to share the value $c(\varepsilon)$. Formula (10) gives

$$\begin{aligned} S_1 \diamond A_{ii} &= 0, & S_1 \diamond B_{jj} &= \delta B_{jj}, \\ S_1 \diamond X &= \beta X_{ij} + \gamma X_{ji} + \delta X_{jj} = \delta X_{jj}, \end{aligned}$$

the last equality follows from the values just obtained. Therefore,

$$\Psi(\delta X_{jj}) = \Psi(S_1 \diamond X) = \Psi(S_1 \diamond A_{ii}) + \Psi(S_1 \diamond B_{jj}) = \Psi(0) + \Psi(\delta B_{jj}) = \Psi(\delta B_{jj}),$$

so $\delta X_{jj} = \delta B_{jj}$ and, δ being nonzero, $X_{jj} = B_{jj}$. Consequently

$$X = X_{ii} + X_{ij} + X_{ji} + X_{jj} = A_{ii} + B_{jj},$$

and $\Psi(A_{ii} + B_{jj}) = \Psi(X) = \Psi(A_{ii}) + \Psi(B_{jj})$. $\square$

Lemma 8. *Let $i \neq j$. Then*

$$\Psi(A_{ii} + B_{ij}) = \Psi(A_{ii}) + \Psi(B_{ij}), \quad \Psi(A_{ii} + C_{ji}) = \Psi(A_{ii}) + \Psi(C_{ji})$$

for all $A_{ii} \in \mathcal{W}_{ii}$, $B_{ij} \in \mathcal{W}_{ij}$ and $C_{ji} \in \mathcal{W}_{ji}$.

Proof. By surjectivity, there is an $X \in \mathcal{W}$ with

$$\Psi(X) = \Psi(A_{ii}) + \Psi(B_{ij}).$$

The coefficients of S_1 are $(0, \beta, \gamma, \delta)$ with $\beta, \gamma, \delta \neq 0$, so (10) gives

$$S_1 \diamond A_{ii} = 0, \quad S_1 \diamond B_{ij} = \beta B_{ij},$$

and hence, by (9) and Lemma 3,

$$\Psi(S_1 \diamond X) = \Psi(S_1 \diamond A_{ii}) + \Psi(S_1 \diamond B_{ij}) = \Psi(0) + \Psi(\beta B_{ij}) = \Psi(\beta B_{ij}).$$

Injectivity gives $S_1 \diamond X = \beta B_{ij}$, while (10) gives

$$S_1 \diamond X = \beta X_{ij} + \gamma X_{ji} + \delta X_{jj},$$

so that the Peirce sum is direct,

$$\beta X_{ij} = \beta B_{ij}, \quad \gamma X_{ji} = 0, \quad \delta X_{jj} = 0,$$

the component in $\mathcal{W}_{ii}$ giving $0 = 0$. As $\beta, \gamma, \delta \neq 0$,

$$X_{ij} = B_{ij}, \quad X_{ji} = 0, \quad X_{jj} = 0. \quad (12)$$

For S_2 the coefficients are $(\alpha, 0, \gamma, \delta)$ with $\alpha \neq 0$, so (10) gives

$$S_2 \diamond A_{ii} = \alpha A_{ii}, \quad S_2 \diamond B_{ij} = 0,$$

and as before

$$\Psi(S_2 \diamond X) = \Psi(\alpha A_{ii}) + \Psi(0) = \Psi(\alpha A_{ii}), \quad S_2 \diamond X = \alpha A_{ii}.$$

Expanding the left-hand side by (10) and using (12),

$$S_2 \diamond X = \alpha X_{ii} + \gamma X_{ji} + \delta X_{jj} = \alpha X_{ii},$$

whence $\alpha X_{ii} = \alpha A_{ii}$ and $X_{ii} = A_{ii}$. Therefore

$$X = X_{ii} + X_{ij} + X_{ji} + X_{jj} = A_{ii} + B_{ij},$$

and $\Psi(A_{ii} + B_{ij}) = \Psi(X) = \Psi(A_{ii}) + \Psi(B_{ij})$.

For the second identity choose X with $\Psi(X) = \Psi(A_{ii}) + \Psi(C_{ji})$. Since $S_1 \diamond A_{ii} = 0$ and $S_1 \diamond C_{ji} = \gamma C_{ji}$ with $\gamma \neq 0$,

$$\Psi(S_1 \diamond X) = \Psi(0) + \Psi(\gamma C_{ji}) = \Psi(\gamma C_{ji}), \quad \beta X_{ij} + \gamma X_{ji} + \delta X_{jj} = \gamma C_{ji},$$

so that

$$X_{ij} = 0, \quad X_{ji} = C_{ji}, \quad X_{jj} = 0.$$

The coefficients of S'_2 are $(\alpha, \beta, 0, \delta)$ with $\alpha \neq 0$, so $S'_2 \diamond A_{ii} = \alpha A_{ii}$ and $S'_2 \diamond C_{ji} = 0$, whence

$$\Psi(S'_2 \diamond X) = \Psi(\alpha A_{ii}), \quad S'_2 \diamond X = \alpha X_{ii} + \beta X_{ij} + \delta X_{jj} = \alpha X_{ii} = \alpha A_{ii}$$

and $X_{ii} = A_{ii}$. Hence $X = A_{ii} + C_{ji}$, which gives the second identity. $\square$

Lemma 9. Let $i \neq j$. Then

$$\Psi(A_{ij} + B_{ji}) = \Psi(A_{ij}) + \Psi(B_{ji})$$

for all $A_{ij} \in \mathcal{W}_{ij}$ and $B_{ji} \in \mathcal{W}_{ji}$.

Proof. By surjectivity, there is an $X \in \mathcal{W}$ with

$$\Psi(X) = \Psi(A_{ij}) + \Psi(B_{ji}).$$

The coefficients of S_2 are $(\alpha, 0, \gamma, \delta)$ with $\alpha, \delta \neq 0$, so (10) gives $S_2 \diamond A_{ij} = 0$ and $S_2 \diamond B_{ji} = \gamma B_{ji}$, and therefore

$$\Psi(S_2 \diamond X) = \Psi(0) + \Psi(\gamma B_{ji}) = \Psi(\gamma B_{ji}), \quad S_2 \diamond X = \gamma B_{ji}.$$

Expanding the left-hand side by (10) and comparing components,

$$\alpha X_{ii} = 0, \quad \gamma X_{ji} = \gamma B_{ji}, \quad \delta X_{jj} = 0,$$

whence

$$X_{ii} = 0, \quad X_{jj} = 0, \quad X = X_{ij} + X_{ji}. \quad (13)$$

The middle equation does not determine X_{ji} : by Lemma 6 the coefficient γ of S_2 equals $\varepsilon(1 - |\zeta|^2)/\bar{\zeta}$, which vanishes when $|\zeta| = 1$. We therefore test against an off-diagonal element.

Let $T_{ij} \in \mathcal{W}_{ij}$. Since $X_{ji}^* T_{ij}$ and $T_{ij} X_{ij}$ are products of two elements of $\mathcal{W}_{ij}$, they vanish, and (13) gives

$$X \diamond T_{ij} = X^* T_{ij} - \zeta T_{ij} X = X_{ij}^* T_{ij} - \zeta T_{ij} X_{ji},$$

with $X_{ij}^* T_{ij} \in \mathcal{W}_{jj}$ and $T_{ij} X_{ji} \in \mathcal{W}_{ii}$. By (8),

$$A_{ij} \diamond T_{ij} = A_{ij}^* T_{ij} \in \mathcal{W}_{jj}, \quad B_{ji} \diamond T_{ij} = -\zeta T_{ij} B_{ji} \in \mathcal{W}_{ii},$$

so that (9), and then Lemma 7 with i and j interchanged, give

$$\Psi(X \diamond T_{ij}) = \Psi(A_{ij}^* T_{ij}) + \Psi(-\zeta T_{ij} B_{ji}) = \Psi(A_{ij}^* T_{ij} - \zeta T_{ij} B_{ji}).$$

Injectivity and comparison of the components in $\mathcal{W}_{jj}$ and in $\mathcal{W}_{ii}$ now give

$$(X_{ij} - A_{ij})^* T_{ij} = 0, \quad T_{ij} (X_{ji} - B_{ji}) = 0 \quad (T_{ij} \in \mathcal{W}_{ij}),$$

the second because $\zeta \neq 0$.

Put $C = X_{ij} - A_{ij}$ and $D = X_{ji} - B_{ji}$, so that $C^* \in \mathcal{W}_{ji}$ and $D \in \mathcal{W}_{ji}$, and hence $C^*P_i = C^*$ and $P_jD = D$. Writing $T_{ij} = P_iTP_j$ with $T \in \mathcal{W}$ arbitrary, the two relations read

$$C^*P_iTP_j = C^*TP_j = 0, \quad P_iTP_jD = P_iTD = 0 \quad (T \in \mathcal{W}),$$

that is, $C^*\mathcal{W}P_j = \{0\}$ and $P_i\mathcal{W}D = \{0\}$. Since $P_i \neq 0$, $P_j \neq 0$ and $\mathcal{W}$ is prime, $C^* = 0$ and $D = 0$, so that

$$X = X_{ij} + X_{ji} = A_{ij} + B_{ji}$$

and $\Psi(A_{ij} + B_{ji}) = \Psi(X) = \Psi(A_{ij}) + \Psi(B_{ji})$. $\square$

Lemma 10. *Let $i \neq j$. Then*

$$\Psi(A_{ii} + B_{ij} + C_{ji}) = \Psi(A_{ii}) + \Psi(B_{ij}) + \Psi(C_{ji})$$

for all $A_{ii} \in \mathcal{W}_{ii}$, $B_{ij} \in \mathcal{W}_{ij}$ and $C_{ji} \in \mathcal{W}_{ji}$.

Proof. By surjectivity, there is an $X \in \mathcal{W}$ with

$$\Psi(X) = \Psi(A_{ii}) + \Psi(B_{ij}) + \Psi(C_{ji}).$$

By (10), $S_1 \diamond A_{ii} = 0$, $S_1 \diamond B_{ij} = \beta B_{ij}$ and $S_1 \diamond C_{ji} = \gamma C_{ji}$, so (9), Lemmas 3 and 9 give

$$\Psi(S_1 \diamond X) = \Psi(0) + \Psi(\beta B_{ij}) + \Psi(\gamma C_{ji}) = \Psi(\beta B_{ij} + \gamma C_{ji}),$$

Lemma 9 being applicable because $\beta B_{ij} \in \mathcal{W}_{ij}$ and $\gamma C_{ji} \in \mathcal{W}_{ji}$. Injectivity and (10) then give

$$\beta X_{ij} + \gamma X_{ji} + \delta X_{jj} = \beta B_{ij} + \gamma C_{ji},$$

and comparison of components, together with $\beta, \gamma, \delta \neq 0$, yields

$$X_{ij} = B_{ij}, \quad X_{ji} = C_{ji}, \quad X_{jj} = 0.$$

For S_2 we have $S_2 \diamond A_{ii} = \alpha A_{ii}$, $S_2 \diamond B_{ij} = 0$ and $S_2 \diamond C_{ji} = \gamma C_{ji}$, so that, by (9) and Lemma 8,

$$\Psi(S_2 \diamond X) = \Psi(\alpha A_{ii}) + \Psi(0) + \Psi(\gamma C_{ji}) = \Psi(\alpha A_{ii} + \gamma C_{ji}),$$

whence $S_2 \diamond X = \alpha A_{ii} + \gamma C_{ji}$. Expanding the left-hand side by (10) and inserting the components already found,

$$\alpha X_{ii} + \gamma X_{ji} + \delta X_{jj} = \alpha A_{ii} + \gamma C_{ji},$$

so that $\alpha X_{ii} = \alpha A_{ii}$ and $X_{ii} = A_{ii}$. Therefore $X = A_{ii} + B_{ij} + C_{ji}$, which is the assertion. $\square$

Lemma 11. *For all $A_{kl} \in \mathcal{W}_{kl}$,*

$$\Psi\left(\sum_{k,l=1}^2 A_{kl}\right) = \sum_{k,l=1}^2 \Psi(A_{kl}).$$

Proof. By surjectivity there is an $X \in \mathcal{W}$ with $\Psi(X) = \sum_{k,l} \Psi(A_{kl})$. Take $i = 1$ and $j = 2$. The coefficients of $S_3 = \varepsilon P_1$ relative to the ordering $(11, 12, 21, 22)$ are $(\alpha, \bar{\varepsilon}, -\zeta\varepsilon, 0)$, so (10) gives

$$S_3 \diamond A_{11} = \alpha A_{11}, \quad S_3 \diamond A_{12} = \bar{\varepsilon} A_{12}, \quad S_3 \diamond A_{21} = -\zeta\varepsilon A_{21}, \quad S_3 \diamond A_{22} = 0,$$

and hence, by (9), Lemmas 3 and 10,

$$\Psi(S_3 \diamond X) = \Psi(\alpha A_{11}) + \Psi(\bar{\varepsilon} A_{12}) + \Psi(-\zeta\varepsilon A_{21}) = \Psi(\alpha A_{11} + \bar{\varepsilon} A_{12} - \zeta\varepsilon A_{21}).$$

Injectivity and (10) give

$$\alpha X_{11} + \bar{\varepsilon} X_{12} - \zeta\varepsilon X_{21} = \alpha A_{11} + \bar{\varepsilon} A_{12} - \zeta\varepsilon A_{21},$$

and comparison of components, with $\alpha, \bar{\varepsilon}, \zeta\varepsilon \neq 0$, yields

$$X_{11} = A_{11}, \quad X_{12} = A_{12}, \quad X_{21} = A_{21}.$$

The element εP_2 is S_3 with the two indices interchanged, so

$$\varepsilon P_2 \diamond A_{11} = 0, \quad \varepsilon P_2 \diamond A_{12} = -\zeta\varepsilon A_{12}, \quad \varepsilon P_2 \diamond A_{21} = \bar{\varepsilon} A_{21}, \quad \varepsilon P_2 \diamond A_{22} = \alpha A_{22},$$

and Lemma 10, applied with $i = 2$ and $j = 1$, gives

$$\Psi(\varepsilon P_2 \diamond X) = \Psi(\alpha A_{22} - \zeta\varepsilon A_{12} + \bar{\varepsilon} A_{21}).$$

Injectivity and comparison of the components in $\mathcal{W}_{22}$ give $\alpha X_{22} = \alpha A_{22}$ and $X_{22} = A_{22}$. Hence $X = \sum_{k,l} A_{kl}$, which is the assertion. $\square$

In the next lemma, the two summands lie in the same corner, and a test element no longer suffices.

Lemma 12. Let $i \neq j$. Then

$$\Psi(A_{ij} + B_{ij}) = \Psi(A_{ij}) + \Psi(B_{ij})$$

for all $A_{ij}, B_{ij} \in \mathcal{W}_{ij}$.

Proof. From $P_i P_j = 0$, $P_i B_{ij} = B_{ij}$, $A_{ij} P_j = A_{ij}$, $A_{ij} B_{ij} = 0$, $P_j A_{ij}^* = A_{ij}^*$ and $B_{ij} P_i = 0$ we obtain

$$(P_i + A_{ij}^*)^*(P_j + B_{ij}) = A_{ij} + B_{ij}, \quad (P_j + B_{ij})(P_i + A_{ij}^*) = A_{ij}^* + B_{ij} A_{ij}^*,$$

and therefore

$$(P_i + A_{ij}^*) \diamond (P_j + B_{ij}) = A_{ij} + B_{ij} - \zeta A_{ij}^* - \zeta B_{ij} A_{ij}^*. \quad (14)$$

By Lemma 8, used once as it stands and once with i and j interchanged,

$$\Psi(P_i + A_{ij}^*) = \Psi(P_i) + \Psi(A_{ij}^*), \quad \Psi(P_j + B_{ij}) = \Psi(P_j) + \Psi(B_{ij}).$$

Applying Ψ to (14), then (3), then (4), then (3) again, and evaluating the four products by (8),

$$\begin{aligned}\Psi(A_{ij} + B_{ij} - \zeta A_{ij}^* - \zeta B_{ij} A_{ij}^*) &= \Psi(P_i + A_{ij}^*) \diamond \Psi(P_j + B_{ij}) \\ &= (\Psi(P_i) + \Psi(A_{ij}^*)) \diamond (\Psi(P_j) + \Psi(B_{ij})) \\ &= \Psi(P_i \diamond P_j) + \Psi(P_i \diamond B_{ij}) + \Psi(A_{ij}^* \diamond P_j) + \Psi(A_{ij}^* \diamond B_{ij}) \\ &= \Psi(0) + \Psi(B_{ij}) + \Psi(A_{ij} - \zeta A_{ij}^*) + \Psi(-\zeta B_{ij} A_{ij}^*) \\ &= \Psi(B_{ij}) + \Psi(A_{ij}) + \Psi(-\zeta A_{ij}^*) + \Psi(-\zeta B_{ij} A_{ij}^*),\end{aligned}$$

the last step by Lemma 9 applied to $A_{ij} - \zeta A_{ij}^*$. On the other hand, the argument on the left has components $A_{ij} + B_{ij} \in \mathcal{W}_{ij}$, $-\zeta A_{ij}^* \in \mathcal{W}_{ji}$ and $-\zeta B_{ij} A_{ij}^* \in \mathcal{W}_{ii}$, so Lemma 10 gives

$$\Psi(A_{ij} + B_{ij} - \zeta A_{ij}^* - \zeta B_{ij} A_{ij}^*) = \Psi(A_{ij} + B_{ij}) + \Psi(-\zeta A_{ij}^*) + \Psi(-\zeta B_{ij} A_{ij}^*).$$

Cancelling the two terms common to the two expressions leaves $\Psi(A_{ij} + B_{ij}) = \Psi(A_{ij}) + \Psi(B_{ij})$. $\square$

Lemma 13. Let $i \in \{1, 2\}$. Then

$$\Psi(A_{ii} + B_{ii}) = \Psi(A_{ii}) + \Psi(B_{ii})$$

for all $A_{ii}, B_{ii} \in \mathcal{W}_{ii}$.

Proof. Let $j \neq i$ and choose, by surjectivity, an $X \in \mathcal{W}$ with $\Psi(X) = \Psi(A_{ii}) + \Psi(B_{ii})$. Since the first coefficient of S_1 vanishes, (10) gives $S_1 \diamond A_{ii} = S_1 \diamond B_{ii} = 0$, and therefore,

$$\Psi(S_1 \diamond X) = \Psi(0) + \Psi(0) = 0 = \Psi(0), \quad S_1 \diamond X = 0.$$

Expanding by (10) and comparing components,

$$\beta X_{ij} = 0, \quad \gamma X_{ji} = 0, \quad \delta X_{jj} = 0,$$

so that $X_{ij} = X_{ji} = X_{jj} = 0$ and $X = X_{ii}$.

Let $T_{ij} \in \mathcal{W}_{ij}$. By (8),

$$X \diamond T_{ij} = X_{ii}^* T_{ij}, \quad A_{ii} \diamond T_{ij} = A_{ii}^* T_{ij}, \quad B_{ii} \diamond T_{ij} = B_{ii}^* T_{ij},$$

all three lying in $\mathcal{W}_{ij}$, so that (9) and Lemma 12 give

$$\Psi(X_{ii}^* T_{ij}) = \Psi(A_{ii}^* T_{ij}) + \Psi(B_{ii}^* T_{ij}) = \Psi((A_{ii} + B_{ii})^* T_{ij}).$$

Injectivity gives $(X_{ii} - A_{ii} - B_{ii})^* T_{ij} = 0$ for every $T_{ij} \in \mathcal{W}_{ij}$. Put $E = X_{ii} - A_{ii} - B_{ii}$, so that $E^* \in \mathcal{W}_{ii}$ and $E^* P_i = E^*$; writing $T_{ij} = P_i T P_j$ with $T \in \mathcal{W}$ arbitrary,

$$E^* P_i T P_j = E^* T P_j = 0 \quad (T \in \mathcal{W}),$$

that is, $E^* \mathcal{W} P_j = \{0\}$. Since $P_j \neq 0$ and $\mathcal{W}$ is prime, $E^* = 0$, so that $X = X_{ii} = A_{ii} + B_{ii}$. $\square$

Proof of Theorem 1. Let $A, B \in \mathcal{W}$ and write $A = \sum_{k,l} A_{kl}$ and $B = \sum_{k,l} B_{kl}$. The (k, l) component of $A + B$ is $A_{kl} + B_{kl}$, so Lemma 11 gives

$$\Psi(A + B) = \sum_{k,l} \Psi(A_{kl} + B_{kl}).$$

For $k \neq l$ Lemma 12 gives $\Psi(A_{kl} + B_{kl}) = \Psi(A_{kl}) + \Psi(B_{kl})$, and for $k = l$ the same identity is given by Lemma 13. Hence, using Lemma 11 once for A and once for B ,

$$\Psi(A + B) = \sum_{k,l} (\Psi(A_{kl}) + \Psi(B_{kl})) = \sum_{k,l} \Psi(A_{kl}) + \sum_{k,l} \Psi(B_{kl}) = \Psi(A) + \Psi(B). \quad \square$$

Remark 1. *Primeness of $\mathcal{W}$ entered only twice, in Lemmas 9 and 13, each time to pass from $C^*\mathcal{W}P_j = \{0\}$, or from $P_i\mathcal{W}D = \{0\}$, to $C = 0$ or $D = 0$. A von Neumann algebra with no central abelian projections need not be prime, so the argument does not apply to that class, where a central-carrier argument would take the place of primeness.*

The shape such an extension might take can be indicated, and none of what follows is proved here. In a semiprime $$ -algebra the relation $C^*\mathcal{W}P_j = \{0\}$ no longer forces $C^* = 0$. It forces only that C^* annihilates the two-sided ideal generated by P_j , and to pass from that to a decomposition one needs a smallest central projection dominating P_j , which a semiprime $*$ -algebra need not possess. We therefore confine the discussion to a von Neumann algebra $\mathcal{W}$, where the centre is a von Neumann algebra in its own right, and its projections form a complete lattice, so that every projection P has a central carrier $Z(P)$, the smallest central projection with $Z(P)P = P$. There the relation $C^*\mathcal{W}P_j = \{0\}$ gives $C^*Z(P_j) = 0$, so that C^* vanishes on the summand $Z(P_j)\mathcal{W}$ and nothing is said on its complement. If $\mathcal{W}$ has no central abelian projections one may choose P_1 with $Z(P_1) = Z(I - P_1) = I$, run the lemmas above inside each central summand, and attempt to reassemble. Primeness would then be needed nowhere, and the hypothesis on $\mathcal{W}$ would weaken from primeness to the existence of a nontrivial projection of full central carrier together with a complement of full central carrier. We have not carried this out. The step that would have to be justified is the passage from additivity on each summand to additivity on $\mathcal{W}$, so the extension is a suggestion and not a result of this paper.*

4. The Range Algebra and the Structure of the Maps

The whole section rests on the following computation.

Lemma 14. *Let $\mathcal{W}$ be a unital $*$ -algebra, let $\mathcal{V}$ be an arbitrary $*$ -algebra, let $\xi \neq 0$, and let $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ be a surjective map satisfying (3). Fix $\mu \in \mathbb{C}$, put $S = \Psi(\mu I)$, and for $Y \in \mathcal{V}$ put $C = SY - YS$. Then*

$$S \diamond (Y \diamond Z) = Y \diamond (S \diamond Z) \quad (Y, Z \in \mathcal{V}),$$

and consequently

$$-C^*Z = \xi^2 ZC \quad (Y, Z \in \mathcal{V}). \quad (15)$$

Proof. Since $\diamond$ is complex linear in its second variable, (5) gives

$$(\mu I) \diamond (A \diamond B) = c(\mu)(A \diamond B) = A \diamond (c(\mu)B) = A \diamond ((\mu I) \diamond B) \quad (A, B \in \mathcal{W}).$$

Applying Ψ to the outer terms and using (3) at each step,

$$S \diamond (\Psi(A) \diamond \Psi(B)) = \Psi((\mu I) \diamond (A \diamond B)) = \Psi(A \diamond ((\mu I) \diamond B)) = \Psi(A) \diamond (S \diamond \Psi(B)),$$

and $\Psi(A), \Psi(B)$ run over $\mathcal{V}$ by surjectivity, so that

$$S \diamond (Y \diamond Z) = Y \diamond (S \diamond Z) \quad (Y, Z \in \mathcal{V}).$$

Expanding the two sides,

$$S \diamond (Y \diamond Z) = S^*(Y^*Z - \xi ZY) - \xi(Y^*Z - \xi ZY)S = S^*Y^*Z - \xi S^*ZY - \xi Y^*ZS + \xi^2 ZYS,$$

$$Y \diamond (S \diamond Z) = Y^*(S^*Z - \xi ZS) - \xi(S^*Z - \xi ZS)Y = Y^*S^*Z - \xi Y^*ZS - \xi S^*ZY + \xi^2 ZSY,$$

and subtracting the second from the first,

$$0 = (S^*Y^* - Y^*S^*)Z + \xi^2 Z(YS - SY).$$

Here,

$$S^*Y^* - Y^*S^* = (YS)^* - (SY)^* = -C^*, \quad YS - SY = -C,$$

and substituting these two values, $-C^*Z - \xi^2 ZC = 0$, which is (15). $\square$

Nothing is assumed about $\mathcal{V}$ in the next theorem beyond its being a nonzero $*$ -algebra.

Theorem 2. Let $\mathcal{W}$ be a unital complex $*$ -algebra, let $\mathcal{V} \neq \{0\}$ be an arbitrary complex $*$ -algebra, let $\xi \in \mathbb{C} \setminus \{0\}$ with $|\xi| \neq 1$, and let $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ be a surjective map satisfying (3). Then:

- (i) $\mathcal{V}$ is unital, and $\Psi(\mu_0 I) = \mu_0 1_{\mathcal{V}}$, where $\mu_0 = c^{-1}(1)$ is the unique solution of $\bar{\mu}_0 - \xi \mu_0 = 1$, explicitly $\mu_0 = (1 + \bar{\xi}) / (1 - |\xi|^2)$;
- (ii) $\Psi(\mathbb{C}I) \subseteq \mathcal{Z}(\mathcal{V})$;
- (iii) if in addition $\mathcal{W}$ is prime with a nontrivial projection and Ψ is bijective, then there is an injective unital ring homomorphism $h : \mathbb{C} \rightarrow \mathcal{Z}(\mathcal{V})$ with $\Psi(\mu A) = h(\mu)\Psi(A)$ for all $\mu \in \mathbb{C}$ and $A \in \mathcal{W}$, and $\Psi(I)$ is an invertible element of $\mathcal{Z}(\mathcal{V})$.

Proof. (i) By (6) the multiplier c is bijective, so $\mu_0 = c^{-1}(1)$ exists and is unique. Conjugating the equation $\bar{\mu}_0 - \xi \mu_0 = 1$ and substituting $\bar{\mu}_0 = 1 + \xi \mu_0$ into the result,

$$\mu_0 - \bar{\xi} \bar{\mu}_0 = 1, \quad \mu_0 - \bar{\xi}(1 + \xi \mu_0) = 1, \quad (1 - |\xi|^2)\mu_0 = 1 + \bar{\xi},$$

which is the stated formula. Moreover $\mu_0 \neq 0$: otherwise $1 + \bar{\xi} = 0$, that is $\bar{\xi} = -1$, and $|\xi| = 1$.

By (5),

$$(\mu_0 I) \diamond B = c(\mu_0)B = B \quad (B \in \mathcal{W}),$$

so $\mu_0 I$ is a left unit for $\diamond$ on $\mathcal{W}$. Put $E = \Psi(\mu_0 I)$. Applying Ψ to the last display and using (3),

$$\Psi(B) = \Psi((\mu_0 I) \diamond B) = E \diamond \Psi(B) = E^* \Psi(B) - \xi \Psi(B)E,$$

so that, by surjectivity,

$$E^*Y - \xi YE = Y \quad (Y \in \mathcal{V}). \quad (16)$$

Taking adjoints in (16) gives $Y^*E - \bar{\xi}E^*Y^* = Y^*$ for every Y , and since $Y \mapsto Y^*$ is a bijection of $\mathcal{V}$,

$$YE - \bar{\xi}E^*Y = Y \quad (Y \in \mathcal{V}). \quad (17)$$

Now (16) gives $E^*Y = Y + \xi YE$, and substituting this into (17),

$$YE - \bar{\xi}Y - |\xi|^2 YE = Y, \quad (1 - |\xi|^2)YE = (1 + \bar{\xi})Y, \quad YE = \mu_0 Y.$$

With $YE = \mu_0 Y$, the relation $E^*Y = Y + \xi YE$ becomes

$$E^*Y = Y + \xi \mu_0 Y = (1 + \xi \mu_0)Y = \bar{\mu}_0 Y.$$

Since $\mu_0 \neq 0$, put $u = \mu_0^{-1}E$, so that $u^* = \bar{\mu}_0^{-1}E^*$. The identities $YE = \mu_0 Y$ and $E^*Y = \bar{\mu}_0 Y$ then give

$$Yu = \mu_0^{-1}YE = Y, \quad u^*Y = \bar{\mu}_0^{-1}E^*Y = Y \quad (Y \in \mathcal{V}),$$

so that

$$u^* = u^*u = u,$$

the first equality because u is a right identity and the second because u^* is a left one. Hence u is a two-sided identity of $\mathcal{V}$, nonzero because $\mathcal{V} \neq \{0\}$, and

$$\Psi(\mu_0 I) = E = \mu_0 u = \mu_0 1_{\mathcal{V}}.$$

(ii) Write $1 = 1_{\mathcal{V}}$, which exists by (i), and let $S = \Psi(\mu I)$ and $C = SY - YS$ be as in Lemma 14. Taking $Z = 1$ in (15) gives $C^* = -\bar{\zeta}^2 C$, whence

$$C = (C^*)^* = (-\bar{\zeta}^2 C)^* = -\bar{\zeta}^2 C^* = -\bar{\zeta}^2 (-\bar{\zeta}^2 C) = |\bar{\zeta}|^4 C,$$

that is, $(1 - |\bar{\zeta}|^4)C = 0$. As $|\bar{\zeta}| \neq 1$ we have $|\bar{\zeta}|^4 \neq 1$, so $C = 0$; thus $SY = YS$ for every $Y \in \mathcal{V}$, which is to say that $\Psi(\mu I) \in \mathcal{Z}(\mathcal{V})$.

(iii) Here Ψ is additive by Theorem 1. Write $S_{\mu} = \Psi(\mu I)$. By (ii) the elements S_{μ} and S_{μ}^* are central, the centre being $*$ -closed, so that

$$\Psi(c(\mu)B) = \Psi((\mu I) \diamond B) = S_{\mu} \diamond \Psi(B) = S_{\mu}^* \Psi(B) - \bar{\zeta} \Psi(B) S_{\mu} = (S_{\mu}^* - \bar{\zeta} S_{\mu}) \Psi(B) \quad (B \in \mathcal{W}).$$

Since c is bijective, every $v \in \mathbb{C}$ equals $c(\mu)$ for exactly one μ , so $h(v) = S_{\mu}^* - \bar{\zeta} S_{\mu} \in \mathcal{Z}(\mathcal{V})$ is well defined and

$$\Psi(vA) = h(v) \Psi(A) \quad (v \in \mathbb{C}, A \in \mathcal{W}). \quad (18)$$

For $v = 1$ the corresponding μ is μ_0 , and $S_{\mu_0} = \mu_0 1$ by (i), so

$$h(1) = \bar{\mu}_0 1 - \bar{\zeta} \mu_0 1 = c(\mu_0) 1 = 1.$$

Choose A with $\Psi(A) = 1$, which is possible by surjectivity. Then, for $v, v' \in \mathbb{C}$,

$$h(vv') = h(vv') \Psi(A) = \Psi(vv'A) = h(v) \Psi(v'A) = h(v) h(v') \Psi(A) = h(v) h(v'),$$

and, Ψ being additive,

$$h(v + v') = \Psi((v + v')A) = \Psi(vA) + \Psi(v'A) = (h(v) + h(v')) \Psi(A) = h(v) + h(v').$$

If $h(v) = 0$ then $\Psi(vA) = 0 = \Psi(0)$ for every A , so $vA = 0$ for every A by injectivity, and $v = 0$ on taking $A = I$. Thus h is an injective unital ring homomorphism of $\mathbb{C}$ into $\mathcal{Z}(\mathcal{V})$. Finally $h(\mu_0)h(\mu_0^{-1}) = h(1) = 1$, so that (18) with $v = \mu_0^{-1}$ and $A = \mu_0 I$ gives

$$\Psi(I) = h(\mu_0^{-1}) \Psi(\mu_0 I) = h(\mu_0)^{-1} \mu_0 1,$$

an invertible element of $\mathcal{Z}(\mathcal{V})$. $\square$

Remark 2. The hypothesis $\mathcal{V} \neq \{0\}$ cannot be omitted: for every $\bar{\zeta}$ the zero map onto $\{0\}$ is a surjective solution of (3), and $\{0\}$ has a two-sided identity only under the convention that allows $1 = 0$.

The difference between this theorem and the setting in which such structure is assumed from the beginning is worth stating plainly. In the usual formulation the range is a von Neumann algebra, or a prime unital algebra, so a unit and a centre are there before the argument starts and the only question is how Ψ behaves with respect to them. Here the range is any nonzero $*$ -algebra, and the unit has to be produced. What produces it is the single element $E = \Psi(\mu_0 I)$, where μ_0 is the unique solution of $c(\mu_0) = 1$. The identity

$E^*Y - \zeta YE = Y$ holds for every $Y \in \mathcal{V}$ by surjectivity, and taking adjoints gives a second identity in the same two unknowns. For $|\zeta| \neq 1$ the pair can be solved, and it yields $YE = \mu_0 Y$ and $E^*Y = \bar{\mu}_0 Y$, so that $\mu_0^{-1}E$ is a two-sided identity of $\mathcal{V}$. The determinant (6) is exactly the obstruction: at $|\zeta| = 1$ the two identities become dependent and the construction stops.

An assumed unit and a constructed one differ in what can be conducted with them. A hypothesis restricts the class of algebras under discussion, whereas the construction above applies to every nonzero $*$ -algebra and, therefore, says something about those which have no unit: they cannot occur as the range of a surjective solution at all. Corollary 1 is that statement, and nonexistence results of this kind are not available when unitality is assumed at the outset. The same remark applies to part (ii): the centrality of $\Psi(\mathbb{C}I)$ is not imposed on $\mathcal{V}$ but follows from (15), and it is what makes the homomorphism h of part (iii) take its values in a commutative subalgebra.

Corollary 1. *Let $\zeta \neq 0$ with $|\zeta| \neq 1$, let $\mathcal{W}$ be a unital $*$ -algebra and let $\mathcal{V} \neq \{0\}$ be a $*$ -algebra with no two-sided identity. Then no surjective map $\mathcal{W} \rightarrow \mathcal{V}$ satisfies (3). In particular, for H an infinite-dimensional Hilbert space and X a noncompact locally compact Hausdorff space, no factor von Neumann algebra with $\dim \mathcal{W} > 1$ admits a bijective solution of (3) onto $K(H)$, onto the finite rank operators on H , onto a Schatten class $\mathcal{S}_p(H)$ with $1 \leq p < \infty$, or onto $C_0(X)$.*

Proof. If $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ is surjective and satisfies (3), then $\mathcal{V}$ is unital by Theorem 2(i), contrary to the hypothesis on $\mathcal{V}$. A factor von Neumann algebra is unital, so it remains to show that $K(H)$, the finite rank operators on H , the Schatten classes $\mathcal{S}_p(H)$ and $C_0(X)$ are nonzero $*$ -algebras without a two-sided identity. A two-sided identity u of an algebra $\mathcal{A}$ satisfies in particular

$$uT = T \quad (T \in \mathcal{A}),$$

and it is this that we contradict.

Let $\mathcal{A}$ be $K(H)$, the algebra of finite rank operators on H , or $\mathcal{S}_p(H)$ with $1 \leq p < \infty$, and let $u \in \mathcal{A}$ satisfy $uT = T$ for every $T \in \mathcal{A}$. Fix $\zeta \in H$ with $\|\zeta\| = 1$, let $\varphi \in H$, and put

$$Tx = \langle x, \zeta \rangle \varphi \quad (x \in H),$$

an operator of rank one, so that $T \in \mathcal{A}$ and

$$T\zeta = \langle \zeta, \zeta \rangle \varphi = \|\zeta\|^2 \varphi = \varphi.$$

Then

$$u\varphi = u(T\zeta) = (uT)\zeta = T\zeta = \varphi \quad (\varphi \in H),$$

that is, $u = I_H$. But

$$\{\text{finite rank operators}\} \subseteq \mathcal{S}_p(H) \subseteq K(H), \quad I_H \notin K(H) \quad (\dim H = \infty),$$

so no such u exists.

Let $\mathcal{A} = C_0(X)$ and let $e \in \mathcal{A}$ satisfy $ef = f$ for every $f \in \mathcal{A}$. Given $x \in X$, local compactness provides an open set U with $x \in U$ whose closure $\bar{U}$ is compact, the set U itself not being assumed compact. Applying Urysohn's lemma for locally compact Hausdorff spaces to the compact set $\{x\}$ and the open set U gives a continuous $f : X \rightarrow [0, 1]$ with $f(x) = 1$ and $\text{supp } f$ a compact subset of U . A continuous function with compact support vanishes at infinity, so $f \in C_c(X) \subseteq C_0(X)$, and

$$e(x) = e(x)f(x) = f(x) = 1 \quad (x \in X),$$

so that e is the constant function 1. That function vanishes at infinity only if X is compact, so again no such e exists. $\square$

Proposition 1. Let $\mathcal{W}$ be a unital $*$ -algebra, let $\mathcal{V}$ be a $*$ -algebra, let $\xi \neq 0$, and let $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ be a surjective map satisfying (3). Choose $\mu \in \{1, i\}$ with $c(\mu) \neq 0$ and put $S = \Psi(\mu I)$. Then

$$\Lambda(Y) = S^*Y - \xi YS \quad (Y \in \mathcal{V})$$

defines a surjective map $\Lambda : \mathcal{V} \rightarrow \mathcal{V}$, and Λ is bijective if Ψ is.

Proof. Such a μ exists, since

$$c(1) = 1 - \xi, \quad c(i) = \bar{i} - \xi i = -i(1 + \xi)$$

vanish only at $\xi = 1$ and at $\xi = -1$ respectively. Put $\nu = c(\mu)$, so that $\nu \neq 0$. By (5) and (3),

$$\Psi(\nu B) = \Psi((\mu I) \diamond B) = \Psi(\mu I) \diamond \Psi(B) = S^* \Psi(B) - \xi \Psi(B) S = \Lambda(\Psi(B)) \quad (B \in \mathcal{W}).$$

Let $Y \in \mathcal{V}$ and choose $A \in \mathcal{W}$ with $\Psi(A) = Y$. Applying the last identity to $B = \nu^{-1}A$,

$$\Lambda(\Psi(\nu^{-1}A)) = \Psi(\nu \nu^{-1}A) = \Psi(A) = Y,$$

so Λ is surjective. Let Ψ be bijective and let $Y, Y' \in \mathcal{V}$ with $\Lambda(Y) = \Lambda(Y')$. Writing $Y = \Psi(B)$ and $Y' = \Psi(B')$,

$$\Psi(\nu B) = \Lambda(Y) = \Lambda(Y') = \Psi(\nu B'),$$

so that $\nu B = \nu B'$ by injectivity, whence $B = B'$ and $Y = Y'$. $\square$

Corollary 2. Let X be a noncompact locally compact Hausdorff space and let $\mathcal{W}$ be a unital $*$ -algebra. Then for no $\xi \neq 0$ does a surjective map $\mathcal{W} \rightarrow C_0(X)$ satisfy (3).

Proof. Let $\Psi : \mathcal{W} \rightarrow C_0(X)$ be such a map and let S and Λ be as in Proposition 1. The algebra $C_0(X)$ is commutative and its involution is complex conjugation, so that

$$\Lambda(f) = S^*f - \xi fS = gf \quad (f \in C_0(X)), \quad g = \bar{S} - \xi S \in C_0(X),$$

and $f \mapsto gf$ is surjective on $C_0(X)$ by Proposition 1.

Let $x_0 \in X$. Local compactness provides an open set U with $x_0 \in U$ whose closure $\bar{U}$ is compact, the set U itself not being assumed compact. Applying Urysohn's lemma for locally compact Hausdorff spaces to the compact set $\{x_0\}$ and the open set U gives a continuous $f : X \rightarrow [0, 1]$ with $f(x_0) = 1$ and $\text{supp } f$ a compact subset of U . Such an f vanishes at infinity, so $f \in C_c(X) \subseteq C_0(X)$. By surjectivity there is an $h \in C_0(X)$ with $gh = f$; hence

$$g(x_0)h(x_0) = f(x_0) = 1,$$

so that $g(x_0) \neq 0$. Thus g vanishes nowhere. If $\inf_{x \in X} |g(x)| = \delta > 0$, then

$$X = \{x \in X : |g(x)| \geq \delta\}$$

would be compact, since $g \in C_0(X)$; as X is noncompact,

$$\inf_{x \in X} |g(x)| = 0.$$

Now $|g|^{1/2} \in C_0(X)$, so there is an $h \in C_0(X)$ with $gh = |g|^{1/2}$, and

$$|h| = \frac{|g|^{1/2}}{|g|} = |g|^{-1/2} \quad \text{on } X,$$

which is unbounded because $\inf |g| = 0$. This contradicts $h \in C_0(X)$. $\square$

On the unit circle, a left unit for the product exists at one parameter only. Let $|\xi| = 1$ and write $\xi = e^{2i\theta}$, $\gamma = e^{-i\theta}$, so that $\bar{\gamma} = \xi\gamma$. For $\mu = \gamma z$ with $z \in \mathbb{C}$,

$$c(\gamma z) = \bar{\gamma}\bar{z} - \xi\gamma z = \bar{\gamma}(\bar{z} - z) = -2i\bar{\gamma}\operatorname{Im} z,$$

so that $c(\mathbb{C}) = \rho\mathbb{R}$ with $\rho = i\bar{\gamma}$ and $\rho^2 = -\bar{\gamma}^2 = -\xi$. Since $|\rho| = 1$, the relation $1 \in \rho\mathbb{R}$ holds precisely when $\rho = \pm 1$, that is, when

$$\xi = -\rho^2 = -1.$$

At that parameter the operator ideals can be excluded.

Proposition 2. *Let H be an infinite dimensional Hilbert space and let $\mathcal{V}$ be a $*$ -subalgebra of $B(H)$ which contains every rank one operator but not I_H . Then for $\xi = -1$ no surjective map from a unital $*$ -algebra $\mathcal{W}$ onto $\mathcal{V}$ satisfies (3). In particular, there is none onto $K(H)$, onto the finite rank operators on H , or onto $\mathcal{S}_p(H)$ with $1 \leq p < \infty$.*

Proof. For $\xi = -1$ the multiplier is

$$c(\mu) = \bar{\mu} + \mu = 2\operatorname{Re} \mu, \quad \text{so} \quad c\left(\frac{1}{2}\right) = 1$$

and $(\frac{1}{2}I) \diamond B = c(\frac{1}{2})B = B$ for every $B \in \mathcal{W}$. Let $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ be surjective and satisfy (3), and put $E = \Psi(\frac{1}{2}I)$. Then

$$\Psi(B) = \Psi\left(\left(\frac{1}{2}I\right) \diamond B\right) = E \diamond \Psi(B) = E^*\Psi(B) + \Psi(B)E \quad (B \in \mathcal{W}),$$

so that, by surjectivity,

$$E^*Y + YE = Y \quad (Y \in \mathcal{V}). \quad (19)$$

Fix $\zeta, \varphi \in H$ with $\varphi \neq 0$ and let Y be the rank one operator

$$Yx = \langle x, \zeta \rangle \varphi \quad (x \in H),$$

which lies in $\mathcal{V}$. For $x \in H$,

$$(E^*Y)x = \langle x, \zeta \rangle E^*\varphi, \quad (YE)x = \langle Ex, \zeta \rangle \varphi = \langle x, E^*\zeta \rangle \varphi,$$

so that (19) evaluated at x reads

$$\langle x, \zeta \rangle E^*\varphi + \langle x, E^*\zeta \rangle \varphi = \langle x, \zeta \rangle \varphi.$$

For $x \perp \zeta$ this leaves $\langle x, E^*\zeta \rangle \varphi = 0$, and $\varphi \neq 0$, so

$$\langle x, E^*\zeta \rangle = 0 \quad (x \perp \zeta),$$

that is, $E^*\zeta \in \mathbb{C}\zeta$. This holds for every $\zeta \in H$, so we may write $E^*\zeta = \tau(\zeta)\zeta$ with $\tau(\zeta) \in \mathbb{C}$ for $\zeta \neq 0$. If ζ and φ are linearly independent, then

$$\tau(\zeta + \varphi)\zeta + \tau(\zeta + \varphi)\varphi = E^*(\zeta + \varphi) = \tau(\zeta)\zeta + \tau(\varphi)\varphi,$$

and comparing coefficients gives $\tau(\zeta) = \tau(\zeta + \varphi) = \tau(\varphi)$; for dependent vectors the two values agree trivially. Hence τ is a constant τ , and

$$E^* = \tau I_H, \quad E = \bar{\tau} I_H.$$

Substituting into (19),

$$(\tau + \bar{\tau})Y = Y \quad (Y \in \mathcal{V}),$$

so that $2 \operatorname{Re} \tau = 1$. In particular $\tau \neq 0$, and since $\mathcal{V}$ is a $*$ -subalgebra containing E ,

$$I_H = \tau^{-1}E^* \in \mathcal{V},$$

contrary to hypothesis. $\square$

The three exclusions cover different parts of the parameter range. Corollary 1 requires $|\zeta| \neq 1$, and under that restriction it excludes $K(H)$, the finite rank operators, the Schatten classes and $C_0(X)$ at once. Corollary 2 drops the restriction on ζ for $C_0(X)$ alone, and so adds every ζ with $|\zeta| = 1$. Proposition 2 adds, for the operator ideals, the single parameter $\zeta = -1$, which is the one point of the unit circle at which the product has a left unit of the form μI . For the operator ideals at the remaining parameters, those with $|\zeta| = 1$ and $\zeta \neq -1$, no exclusion is proved here.

For the rest of the section $\mathcal{W}$ and $\mathcal{V}$ are factor von Neumann algebras with $\dim \mathcal{W} > 1$, the parameter ζ is nonzero, and $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ is a bijection satisfying (3), additive by Theorem 1.

The classification in this setting is known, and the statement below is due to Dai and Lu [12]. It is included as a theorem rather than quoted, because what is offered is a proof rather than a result, and because the proof is short once the earlier sections are in place: it uses Theorem 1 and the two lemmas on the scalars proved below, and nothing else.

Three things are gained by deriving it this way. First, the existing proofs run through a projection transported into the range, and so require the range to be a von Neumann algebra before the argument begins. The derivation below never moves a projection into $\mathcal{V}$, and the additivity it starts from was obtained in Section 3 with no hypothesis on the codomain whatever. The von Neumann hypothesis on $\mathcal{V}$ is used here only at the two points where $\mathcal{Z}(\mathcal{V}) = \mathbb{C}I$ is needed, in Lemma 15 and through Lemma 4. Second, the two ingredients are visibly independent of ζ and of each other, so the place where the parameter matters is isolated: it is Lemma 16, and the value it excludes is $\zeta = -1$, which is precisely the value at which $c(\mathbb{C})$ fails to generate $\mathbb{C}$. That the one exceptional parameter of the argument coincides with the one classical parameter treated separately in the literature is not visible in the existing proofs. Third, the same two lemmas are what Theorem 2 needs, so the factor classification and the range rigidity theorem are consequences of a single computation, Lemma 14, rather than of two unrelated ones.

Theorem 3 (Dai and Lu [12]). *Let $\mathcal{W}, \mathcal{V}$ be factor von Neumann algebras with $\dim \mathcal{W} > 1$ and let $\zeta \in \mathbb{C} \setminus \{0, -1\}$. If $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ is a bijection satisfying (3), then $\Psi(I) = I$ and Ψ is either a linear or a conjugate linear $*$ -isomorphism, the conjugate linear alternative occurring only if $\zeta \in \mathbb{R}$. Conversely, every linear $*$ -isomorphism satisfies (3) for every ζ , and a conjugate linear $*$ -isomorphism satisfies it exactly when $\zeta \in \mathbb{R}$.*

Two lemmas about the scalars come first.

Lemma 15. For every $\xi \neq 0$ one has $\Psi(\mathbb{C}I) = \mathbb{C}I$. Writing $\Psi(\mu I) = s(\mu)I$, the map $s : \mathbb{C} \rightarrow \mathbb{C}$ so defined is an additive bijection which maps $\ker c$ onto itself.

Proof. Fix $\mu \in \mathbb{C}$, put $S = \Psi(\mu I)$, and for $Y \in \mathcal{V}$ put $C = SY - YS$ as in Lemma 14. Taking $Z = I$ in (15),

$$C^* = -\xi^2 C,$$

and substituting this into (15) again,

$$\xi^2 CZ = -C^*Z = \xi^2 ZC \quad (Z \in \mathcal{V}),$$

so that $CZ = ZC$ for every Z , as $\xi \neq 0$. Hence $C \in \mathcal{Z}(\mathcal{V}) = \mathbb{C}I$, say $C = \lambda I$. But

$$\lambda I = C = SY - YS,$$

and by Lemma 4, a commutator equal to a multiple of the identity is zero, so $\lambda = 0$ and $C = 0$. Thus

$$SY = YS \quad (Y \in \mathcal{V}),$$

that is, $S \in \mathcal{Z}(\mathcal{V}) = \mathbb{C}I$. This gives $\Psi(\mathbb{C}I) \subseteq \mathbb{C}I$, and since Ψ^{-1} satisfies (3) by Lemma 2, the same argument applied to Ψ^{-1} gives $\Psi^{-1}(\mathbb{C}I) \subseteq \mathbb{C}I$. Hence $\Psi(\mathbb{C}I) = \mathbb{C}I$.

The map s is therefore defined and bijective, and, Ψ being additive,

$$s(\mu + \mu')I = \Psi((\mu + \mu')I) = \Psi(\mu I) + \Psi(\mu' I) = (s(\mu) + s(\mu'))I.$$

Finally, by Lemma 5 applied in $\mathcal{W}$ and in $\mathcal{V}$,

$$\{s(\mu)I : \mu \in \ker c\} = \Psi(\mathfrak{N}_{\xi}(\mathcal{W})) = \mathfrak{N}_{\xi}(\mathcal{V}) = \{vI : v \in \ker c\},$$

that is, $s(\ker c) = \ker c$. $\square$

Lemma 16. Let $\xi \neq -1$. Then there is a field endomorphism χ of $\mathbb{C}$ such that

$$\Psi(\mu A) = \chi(\mu)\Psi(A) \quad (\mu \in \mathbb{C}, A \in \mathcal{W}). \quad (20)$$

Proof. Let $\mathbb{H}$ be the set of those $v \in \mathbb{C}$ for which there is a scalar $h(v)$ with $\Psi(vA) = h(v)\Psi(A)$ for every $A \in \mathcal{W}$. Such a scalar is unique: if h and h' both serve, then $(h(v) - h'(v))\Psi(A) = 0$ for every A , and $\Psi(A)$ runs over $\mathcal{V} \neq \{0\}$.

By Lemma 15 and (5),

$$\Psi(c(\mu)A) = \Psi((\mu I) \diamond A) = \Psi(\mu I) \diamond \Psi(A) = (s(\mu)I) \diamond \Psi(A) = (\overline{s(\mu)} - \xi s(\mu))\Psi(A) = c(s(\mu))\Psi(A),$$

so $c(\mu) \in \mathbb{H}$ with $h(c(\mu)) = c(s(\mu))$, the right-hand side depending only on $c(\mu)$: if $c(\mu) = c(\mu')$ then $\mu - \mu' \in \ker c$, whence

$$s(\mu) - s(\mu') = s(\mu - \mu') \in \ker c, \quad c(s(\mu)) = c(s(\mu'))$$

by the last assertion of Lemma 15. Thus $c(\mathbb{C}) \subseteq \mathbb{H}$.

Let $v, v' \in \mathbb{H}$ and $A \in \mathcal{W}$. Then

$$\Psi(vv'A) = h(v)\Psi(v'A) = h(v)h(v')\Psi(A),$$

$$\Psi((\nu + \nu')A) = \Psi(\nu A) + \Psi(\nu' A) = (h(\nu) + h(\nu'))\Psi(A),$$

$$\Psi((- \nu)A) = -\Psi(\nu A) = -h(\nu)\Psi(A),$$

the second and third by the additivity of Ψ , and $\Psi(1 \cdot A) = \Psi(A)$. Hence $\mathbb{H}$ is a unital subring of $\mathbb{C}$, with

$$h(\nu\nu') = h(\nu)h(\nu'), \quad h(\nu + \nu') = h(\nu) + h(\nu'), \quad h(-\nu) = -h(\nu), \quad h(1) = 1.$$

Suppose first that $|\xi| \neq 1$. Then c is a bijection of $\mathbb{C}$ by (6), so $\mathbb{H} \supseteq c(\mathbb{C}) = \mathbb{C}$. Suppose next that $|\xi| = 1$ and let γ be as in Lemma 5, so that $|\gamma| = 1$ and $\bar{\gamma} = \xi\gamma$; multiplying the latter by $\bar{\gamma}$,

$$\bar{\gamma}^2 = \xi\gamma\bar{\gamma} = \xi|\gamma|^2 = \xi.$$

Writing an arbitrary scalar as $\mu = \gamma z$ and using $\xi\gamma = \bar{\gamma}$,

$$c(\gamma z) = \bar{\gamma}\bar{z} - \xi\gamma z = \bar{\gamma}\bar{z} - \bar{\gamma}z = \bar{\gamma}(\bar{z} - z) = -2i\bar{\gamma}\operatorname{Im} z,$$

so that, with $\rho = i\bar{\gamma}$,

$$c(\mathbb{C}) = \rho\mathbb{R}, \quad \rho^2 = -\bar{\gamma}^2 = -\xi.$$

Since $c(\mathbb{C}) = \rho\mathbb{R} \subseteq \mathbb{H}$ and $\mathbb{H}$ is a subring, $(\rho t)(\rho t') = \rho^2 tt'$ lies in $\mathbb{H}$ for all $t, t' \in \mathbb{R}$, so that

$$\mathbb{H} \supseteq \rho\mathbb{R} + \rho^2\mathbb{R}.$$

This sum is all of $\mathbb{C}$ unless ρ and ρ^2 are real multiples of one another, that is, unless $\rho \in \mathbb{R}$.

Since $|\rho| = 1$ that means $\rho = \pm 1$, hence

$$1 = \rho^2 = -\xi, \quad \xi = -1,$$

which is excluded. Hence $\mathbb{H} = \mathbb{C}$. Thus $\chi = h$ is a unital ring homomorphism of $\mathbb{C}$ into itself, hence a field endomorphism, and (20) holds by the definition of $\mathbb{H}$. $\square$

Proof of Theorem 3. Let $\xi \neq 0, -1$ and let χ be the endomorphism of Lemma 16. Then

$$\chi(i)^2 = \chi(i^2) = \chi(-1) = -1,$$

so that $\chi(i) = \pm i$; in particular $\chi(i) \neq 0$ and $\overline{\chi(i)} = -\chi(i) = \chi(-i)$. By (3), the additivity of Ψ and (20),

$$\Psi(A^*B) - \chi(\xi)\Psi(BA) = \Psi(A^*B - \xi BA) = \Psi(A) \diamond \Psi(B) = \Psi(A)^*\Psi(B) - \xi\Psi(B)\Psi(A),$$

that is,

$$\Psi(A^*B) - \chi(\xi)\Psi(BA) = \Psi(A)^*\Psi(B) - \xi\Psi(B)\Psi(A) \quad (A, B \in \mathcal{W}). \quad (21)$$

Let $\mu \in \mathbb{C}$ and replace A by μA in (21). Since

$$(\mu A)^*B = \bar{\mu}A^*B, \quad B(\mu A) = \mu BA, \quad \Psi(\mu A)^* = \overline{\chi(\mu)}\Psi(A)^*,$$

this gives

$$\chi(\bar{\mu})\Psi(A^*B) - \chi(\xi)\chi(\mu)\Psi(BA) = \overline{\chi(\mu)}\Psi(A)^*\Psi(B) - \xi\chi(\mu)\Psi(B)\Psi(A).$$

Subtracting $\chi(\mu)$ times (21) from this, the terms in $\Psi(BA)$ and in $\Psi(B)\Psi(A)$ cancel and

$$[\chi(\bar{\mu}) - \chi(\mu)]\Psi(A^*B) = [\overline{\chi(\mu)} - \chi(\mu)]\Psi(A)^*\Psi(B).$$

For $\mu = i$ the two bracketed factors are

$$\chi(\bar{i}) - \chi(i) = \chi(-i) - \chi(i) = -2\chi(i), \quad \overline{\chi(i)} - \chi(i) = -2\chi(i),$$

and $-2\chi(i) \neq 0$; dividing by it,

$$\Psi(A^*B) = \Psi(A)^*\Psi(B) \quad (A, B \in \mathcal{W}). \quad (22)$$

Taking $A = I$ in (22),

$$\Psi(B) = \Psi(I^*B) = \Psi(I)^*\Psi(B) \quad (B \in \mathcal{W}),$$

so that $\Psi(I)^*Y = Y$ for every $Y \in \mathcal{V}$ by surjectivity; taking $Y = I$ gives $\Psi(I)^* = I$ and

$$\Psi(I) = I.$$

Taking $B = I$ in (22), and then replacing A by A^* in it,

$$\Psi(A^*) = \Psi(A^*I) = \Psi(A)^*\Psi(I) = \Psi(A)^*,$$

$$\Psi(AB) = \Psi((A^*)^*B) = \Psi(A^*)^*\Psi(B) = \Psi(A)\Psi(B),$$

so Ψ preserves the involution and is multiplicative. Substituting $\Psi(A^*B) = \Psi(A)^*\Psi(B)$ and $\Psi(BA) = \Psi(B)\Psi(A)$ into (21),

$$(\chi(\xi) - \xi)\Psi(B)\Psi(A) = 0 \quad (A, B \in \mathcal{W}),$$

and $A = B = I$ gives $\Psi(B)\Psi(A) = I \neq 0$, whence

$$\chi(\xi) = \xi.$$

Let $\mu \geq 0$. Then $\mu I = (\sqrt{\mu}I)^*(\sqrt{\mu}I)$, so (20) and (22) give

$$\chi(\mu)I = \Psi(\mu I) = \Psi((\sqrt{\mu}I)^*(\sqrt{\mu}I)) = \Psi(\sqrt{\mu}I)^*\Psi(\sqrt{\mu}I) \geq 0,$$

and therefore $\chi(\mu) \geq 0$. Being additive, χ maps $\mathbb{R}$ into $\mathbb{R}$ and is increasing there, and it fixes every rational; squeezing a real number between rationals, χ fixes every real number. With $\chi(i) = \pm i$ this leaves $\chi = \text{id}$, in which case Ψ is linear, and $\chi = \text{complex conjugation}$, in which case Ψ is conjugate linear. In the second case $\chi(\xi) = \bar{\xi}$ reads $\bar{\xi} = \xi$, so that $\xi \in \mathbb{R}$.

Conversely, let Ψ be a linear $*$ -isomorphism. Then

$$\Psi(A^*B - \xi BA) = \Psi(A^*)\Psi(B) - \xi\Psi(B)\Psi(A) = \Psi(A)^*\Psi(B) - \xi\Psi(B)\Psi(A),$$

which is (3). If Ψ is conjugate linear, the same computation gives

$$\Psi(A^*B - \xi BA) = \Psi(A)^*\Psi(B) - \bar{\xi}\Psi(B)\Psi(A),$$

so that (3) holds if and only if $(\xi - \bar{\xi})\Psi(B)\Psi(A) = 0$ for all A, B , which on taking $A = B = I$ means $\xi \in \mathbb{R}$. $\square$

For the remaining value $\xi = -1$, the conclusion is due to Li, Lu and Fang.

Proposition 3 (Li, Lu and Fang [8]). Let $\mathcal{W}, \mathcal{V}$ be factor von Neumann algebras with $\dim \mathcal{W} > 1$ and let $\zeta = -1$, so that $A \diamond_{-1} B = A^*B + BA$. Then every bijective solution of (3) is a linear or a conjugate linear $*$ -isomorphism.

Proof. For $\zeta = -1$ the map of Lemma 1 and its parameter are

$$T(A) = -\bar{\zeta}A^* = A^*, \quad \eta = \zeta^{-1} = -1,$$

so that $T\Psi T^{-1}$ is a bijection preserving $[A, B]_*^{-1} = AB + BA^*$. By ([8], Theorem 2.1) it is a $*$ -ring isomorphism, and hence so is Ψ . A $*$ -ring isomorphism between factors carries the centre onto the centre,

$$\Psi(\mathbb{C}I) = \mathcal{Z}(\mathcal{V}) = \mathbb{C}I,$$

so $\Psi(\mu I) = \chi(\mu)I$ defines a field automorphism χ of $\mathbb{C}$, and

$$\Psi(\mu A) = \Psi((\mu I)A) = \Psi(\mu I)\Psi(A) = \chi(\mu)\Psi(A) \quad (\mu \in \mathbb{C}, A \in \mathcal{W}).$$

For $\mu \geq 0$,

$$\chi(\mu)I = \Psi(\mu I) = \Psi((\sqrt{\mu}I)^*(\sqrt{\mu}I)) = \Psi(\sqrt{\mu}I)^*\Psi(\sqrt{\mu}I) \geq 0,$$

so $\chi(\mu) \geq 0$, and χ fixes every real number as in the proof of Theorem 3. Since $\chi(i)^2 = \chi(-1) = -1$ gives $\chi(i) = \pm i$, either $\chi = \text{id}$ and Ψ is linear, or χ is complex conjugation and Ψ is conjugate linear. $\square$

Together, Theorem 3 and Proposition 3 describe the bijective solutions of (3) between factor von Neumann algebras for every $\zeta \neq 0$, and the description is that of [8,12].

Corollary 3. Let H, K be complex Hilbert spaces with $\dim H > 1$, let $\zeta \neq 0$, and let $\Psi : B(H) \rightarrow B(K)$ be a bijection satisfying (3). Then $\dim H = \dim K$ and there is a unitary or an antiunitary operator $U : H \rightarrow K$ with $\Psi(A) = UAU^{-1}$ for all $A \in B(H)$, the antiunitary alternative occurring only if $\zeta \in \mathbb{R}$.

Proof. By Theorem 3 and Proposition 3 the map Ψ is a linear or a conjugate linear $*$ -isomorphism, and it is conjugate linear only if $\zeta \in \mathbb{R}$. If Ψ is linear, it is a $*$ -isomorphism of $B(H)$ onto $B(K)$, and such an isomorphism is spatial, by the theorem of Kadison and Ringrose [40]:

$$\dim H = \dim K, \quad \Psi(A) = UAU^{-1} \quad (A \in B(H))$$

for some unitary $U : H \rightarrow K$.

Let Ψ be conjugate linear. Conjugating the coordinates with respect to an orthonormal basis of H gives a conjugation C on H , that is, a conjugate linear isometric map with $C^2 = \text{id}$. Put $\theta(A) = CAC$. Then

$$\theta(\lambda A) = \bar{\lambda}\theta(A), \quad \theta(AB) = CABC = (CAC)(CBC) = \theta(A)\theta(B), \quad \theta(A^*) = CA^*C = \theta(A)^*,$$

$$\theta^2(A) = C(CAC)C = A \quad (A \in B(H)),$$

so θ is a conjugate linear $*$ -automorphism of $B(H)$ and $\Psi \circ \theta$ is linear. Applying the previous case to $\Psi \circ \theta$ gives $\dim H = \dim K$ and a unitary $V : H \rightarrow K$ with $\Psi(\theta(A)) = VAV^{-1}$ for every A , whence

$$\Psi(A) = \Psi(\theta(\theta(A))) = V\theta(A)V^{-1} = VCACV^{-1} = UAU^{-1}, \quad U = VC,$$

and U is antiunitary, being a unitary composed with a conjugation. $\square$

The antiunitary alternative is allowed at $\xi = -1$ because -1 is real: a conjugate linear $*$ -isomorphism carries $A^*B - \xi BA$ to $\Psi(A)^*\Psi(B) - \bar{\xi}\Psi(B)\Psi(A)$, so it satisfies (3) exactly when $\xi = \bar{\xi}$, which holds at $\xi = -1$.

For $|\xi| \neq 1$ the unitality of $\mathcal{V}$ assumed below is automatic, by Theorem 2(i).

Proposition 4. *Let $\mathcal{W}$ be a $*$ -algebra, let $\mathcal{V}$ be a unital $*$ -algebra with $1_{\mathcal{V}} \neq 0$, let $\xi \neq 0$, and let $\Psi : \mathcal{W} \rightarrow \mathcal{V}$ be a surjective map satisfying (3). If $\kappa \in \mathbb{C} \setminus \{0\}$ and $\kappa\Psi$ also satisfies (3), then $\kappa = 1$. In particular $-\Psi$ never satisfies (3). The hypothesis $\kappa \neq 0$ cannot be omitted, since the zero map satisfies (3) trivially, although it is of course not bijective.*

Proof. For all $A, B \in \mathcal{W}$,

$$(\kappa\Psi)(A \diamond B) = \kappa\Psi(A \diamond B) = \kappa\Psi(A)^*\Psi(B) - \kappa\xi\Psi(B)\Psi(A),$$

$$\begin{aligned} (\kappa\Psi)(A) \diamond (\kappa\Psi)(B) &= (\kappa\Psi(A))^*(\kappa\Psi(B)) - \xi(\kappa\Psi(B))(\kappa\Psi(A)) \\ &= |\kappa|^2\Psi(A)^*\Psi(B) - \xi\kappa^2\Psi(B)\Psi(A). \end{aligned}$$

Since $\kappa\Psi$ satisfies (3) the two are equal, that is,

$$(\kappa - |\kappa|^2)\Psi(A)^*\Psi(B) = \xi(\kappa - \kappa^2)\Psi(B)\Psi(A) \quad (A, B \in \mathcal{W}).$$

By surjectivity there is a $B \in \mathcal{W}$ with $\Psi(B) = 1_{\mathcal{V}}$, and $\Psi(A)$ then runs over $\mathcal{V}$, so

$$(\kappa - |\kappa|^2)Y^* = \xi(\kappa - \kappa^2)Y \quad (Y \in \mathcal{V}).$$

Write $a = \kappa - |\kappa|^2$ and $b = \xi(\kappa - \kappa^2)$, and take $Y = 1_{\mathcal{V}}$ and $Y = i1_{\mathcal{V}}$, so that $Y^* = 1_{\mathcal{V}}$ and $Y^* = -i1_{\mathcal{V}}$ respectively:

$$a1_{\mathcal{V}} = b1_{\mathcal{V}}, \quad -ia1_{\mathcal{V}} = ib1_{\mathcal{V}}.$$

Since $1_{\mathcal{V}} \neq 0$,

$$a = b, \quad -a = b,$$

and adding these gives $2b = 0$, hence $a = b = 0$, that is,

$$\xi(\kappa - \kappa^2) = 0, \quad \kappa - |\kappa|^2 = 0.$$

As $\xi \neq 0$ the first identity gives $\kappa = \kappa^2$, and $\kappa \neq 0$ then gives $\kappa = 1$. $\square$

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