

Article

Symmetric Extensions of Separation Axioms via GP-Operators and Their Applications

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Abstract

This study is motivated by the need to investigate the largest possible collection of classical separation axioms within a newly introduced triple structure of generalized primal topological spaces, and to understand how primal collections influence these familiar notions. The purpose of the paper is to extend several classical concepts by introducing new classes of separation axioms, including $(g, \mathcal{P})-D_i$, $(g, \mathcal{P})-T_i$, and $(g, \mathcal{P})-R_i$ for $i = 0, 1, 2$. Within the same framework, we also define $(g, \mathcal{P})-G_\delta$ sets and $(g, \mathcal{P})-F_\sigma$ sets, which naturally lead to new symmetric variants of separation axioms such as $(g, \mathcal{P})-R_\delta$, $(g, \mathcal{P})$ -weakly regular, $(g, \mathcal{P})-R_{D_\delta}$, and $(g, \mathcal{P})-R_{D^*}$. The main contribution of this work lies in establishing the relationships among these newly introduced axioms and demonstrating how primal collections affect their behavior. Several illustrative examples based on simple graphs are provided to highlight the structure and significance of the results. Overall, the findings offer a broader perspective on separation phenomena in generalized primal settings and deepen the understanding of symmetry within these spaces.

Keywords: primal collection; generalized topology; generalized primal topology; separation axioms; GP-homeomorphism

MSC: 54A05; 54C10; 54A10

1. Introduction

Separation axioms in the theory of topology are mostly formulated to recognize non-homeomorphic topologies. If there are two topological spaces such that one satisfies a separation axiom but not the other, these two topologies are not homeomorphic. These axioms have received great interest from topologists for a long time. A topological space that satisfies T_i is called a T_i -space for short.

Recently, in [1], the author offered a special kind of structure called generalized primal topological spaces. This structure possesses all the properties of a generalized topology in the sense of a new topological structure called “primal collection.” Then, Al-Saadi and Al-Malki [2] studied some types of weak open sets based on an operator employing primal, which satisfied Kuraski’s closure axioms. The concept of “continuity” has gained a lot of attention for its importance in topology. Several types of generalized primal continuous functions are described in [3].

In this study, we continue our work to clarify the notion of “separation axioms” in this new structure. This paper contains five sections, presented as follows.



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Section 1 contains two sub-sections. The first deals with previous studies on which this scientific paper was based. The second includes a recall of the basic concepts and results. Section 2 presents several types of separation axioms, which are based on the concept of $D_{(\mathfrak{g}, \mathcal{P})}$ -set. Next, we study the relationship between them and the T_i axioms for $i = 0, 1, 2$. We also introduce R_i for the $i = 0, 1$ axioms, which are based on the concept of closure. In Section 3, we present the definition of $(\mathfrak{g}, \mathcal{P})$ - G_δ -set and $(\mathfrak{g}, \mathcal{P})$ - F_σ -set, which leads us to new and multiple types of axioms. In Section 4, we present the fundamental theorem and propositions that are connected between the GP-continuous function and the GP-homeomorphism function and the separation axioms. Finally, we discuss the conclusions of this research.

1.1. Literature Review

Over the years, various classical mathematical structures have appeared. The mathematical construction “grill” [4] was introduced as a useful tool for investigating topological concepts with significant applications in the field of topology. Building on this, Acharjee [5] introduced the dual structure, called a “primal”, in 2022. A collection $\mathcal{P}$ of $P(X)$ is named a primal over X when the following requirements hold for all $A, B \subseteq X$: (i) X never belongs to $\mathcal{P}$; (ii) if $A \in \mathcal{P}$ and $B \subseteq A$, then $B \in \mathcal{P}$; (iii) if $A \cap B \in \mathcal{P}$, then either A or B belongs to $\mathcal{P}$. A primal topological space (PT space) is defined as the triple $(X, \mathcal{T}, \mathcal{P})$.

In 1997, Császár [6] established the notion of generalized topological spaces, (GT spaces), where a family $\mathfrak{g}$ of subsets of X satisfies the following: (i) $\emptyset$ (the empty set) always belongs to $\mathfrak{g}$; (ii) any union of members of $\mathfrak{g}$ is also a member of $\mathfrak{g}$. $(X, \mathfrak{g})$ is considered a generalized topological space. According to [7], each element in X is called a $\mathfrak{g}$ -open set, and $\mathfrak{g}$ -closed refers to its complement. Respectively, $c_{\mathfrak{g}}(A)$ and $i_{\mathfrak{g}}(A)$ denote the closure and interior of A .

Various studies have investigated the topological properties of generalized spaces. In particular, separation axioms have been explored extensively. For example, ref. [8] introduced early forms of generalized separation axioms, ref. [9] examined their extensions in different contexts, ref. [10] focused on applications of these axioms in specific topological structures, and ref. [11] analyzed neighborhood structures under these axioms. Separation axioms have also been extended by substituting the concept of open sets with more generalized expressions [12,13]. Several forms of generalized continuity have been studied: contra generalized continuity [14], almost generalized continuity [15], and strong generalized continuity [16].

Although both PT and GT have been studied extensively, their integration has not been systematically explored. Generalized primal topological spaces (GPT spaces), introduced by [1], combine the structural constraints of PT with the flexibility of GT, resulting in a unified framework where classical concepts such as separation axioms and continuity can be independently developed and analyzed. This integration allows us to investigate how primal collections influence generalized notions, filling a gap in the literature. In particular, three previous studies [1–3] addressed foundational aspects of GPT spaces: ref. [2] studied a collection of new generalized structures, and ref. [3] focused on continuous functions and their generalizations in GPT spaces.

While these studies have significantly advanced the understanding of classical and generalized structures, they do not fully integrate the effects of primal collections. Therefore, GPT spaces offer a novel framework where separation axioms can be independently developed and analyzed. This work aims to fill this gap and highlight the influence of primal collections on classical topological notions.

1.2. Preliminaries

First, let us recall some basics about **GPT** spaces.

Definition 1 ([1]). A generalized primal topological space, or (**GPT** space), is a generalized topology $\mathfrak{g}$ and a primal $\mathcal{P}$ defined on X . The indication $(X, \mathfrak{g}, \mathcal{P})$ symbolizes the **GPT** space.

Definition 2 ([1]). Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space with $A \subseteq X$, and we have the following:

- (i) This structure's members are called $(\mathfrak{g}, \mathcal{P})$ -open sets. The complement of $(\mathfrak{g}, \mathcal{P})$ -open sets is called $(\mathfrak{g}, \mathcal{P})$ -closed sets.
- (ii) $C_{(\mathfrak{g}, \mathcal{P})}(X)$ symbolize the family of $(\mathfrak{g}, \mathcal{P})$ -closed sets.
- (iii) $cl_{(\mathfrak{g}, \mathcal{P})}(A)$ symbolize the closure of A . Hence, $cl_{(\mathfrak{g}, \mathcal{P})}(A)$ is given as the intersection of all $(\mathfrak{g}, \mathcal{P})$ -closed sets which contain A .
- (iv) A **GPT** space is named strong if $X = G_{(\mathfrak{g}, \mathcal{P})}$, where $G_{(\mathfrak{g}, \mathcal{P})} = \cup\{U : U \in \mathfrak{g}\}$.
- (v) An operator $\psi : X \rightarrow 2^{2^X}$ is called a generalized primal neighborhood of x , if $x \in U$ for each $U \in \psi(x)$.
- (vi) An operator $(\cdot)^\diamond : 2^X \rightarrow 2^X$ is defined by

$$A^\diamond(X, \mathfrak{g}, \mathcal{P}) = \{x \in X : A^c \cup U^c \in \mathcal{P}, \forall U \in \psi(x)\}.$$

Definition 3 ([3]). Let X and X^λ be **GPT** spaces. A function $f : X \rightarrow X^\lambda$ is known as GP-continuous if for each $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ -open set U , $f^{-1}(U)$ is $(\mathfrak{g}, \mathcal{P})$ -open.

Also, when f is GP-continuous, we get

$$f(X \setminus G_{(\mathfrak{g}, \mathcal{P})}) \subseteq (X^\lambda \setminus G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}).$$

Moreover, GP-continuity of f leads to $f(G_{(\mathfrak{g}, \mathcal{P})}) \subseteq G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}$, then

$$f^{-1}(X^\lambda \setminus G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}) = (X \setminus G_{(\mathfrak{g}, \mathcal{P})}) \text{ and } f^{-1}(G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}) = G_{(\mathfrak{g}, \mathcal{P})}.$$

Definition 4 ([3]). A function $f : X \rightarrow X^\lambda$ is known as GP-closed if for each $(\mathfrak{g}, \mathcal{P})$ -closed set U in X , the image $f(U)$ is $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ -closed in X^λ .

It follows from the above definition that f is GP-closed iff

$$cl_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}(f(A)) \subseteq f(cl_{(\mathfrak{g}, \mathcal{P})}(A)),$$

for all $A \subseteq X$.

2. Separation Axioms

Sarsak [17] studied the separation axioms: $\mathfrak{g}$ - D_0 , $\mathfrak{g}$ - D_1 , $\mathfrak{g}$ - D_2 , $\mathfrak{g}$ - T_0 , $\mathfrak{g}$ - T_1 , $\mathfrak{g}$ - T_2 , $\mathfrak{g}$ - R_0 , and $\mathfrak{g}$ - R_1 in generalized topological spaces. In this section, we study the alternative definitions of these kinds of separation axioms in a generalized primal topological space. Also, we investigate the main features of them.

Definition 5. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space with $A \subseteq X$. A is known as a $D_{(\mathfrak{g}, \mathcal{P})}$ -set when we have $(\mathfrak{g}, \mathcal{P})$ -open sets U and V satisfies $U \neq X$ with $A = U \setminus V$.

Note that every $(\mathfrak{g}, \mathcal{P})$ -open set $A \neq X$ is $D_{(\mathfrak{g}, \mathcal{P})}$ -open. Moreover, a **GPT** space X is known as $(\mathfrak{g}, \mathcal{P})$ - D_0 if for each $x, y \in X$ with $x \neq y$, there exists a $D_{(\mathfrak{g}, \mathcal{P})}$ -set U of X that satisfies $x \in U$, $y \notin U$ or vice versa. However, since a $D_{(\mathfrak{g}, \mathcal{P})}$ -set is always contained in

$G_{(\mathfrak{g}, \mathcal{P})}$, a **GPT** space in which $X \setminus G_{(\mathfrak{g}, \mathcal{P})}$ contains more than one point trivially does not have the property $(\mathfrak{g}, \mathcal{P})$ - D_0 .

Example 1. Consider

$$X = \mathbb{N}, \quad \mathfrak{g} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}, \quad \mathcal{P} = P(X) \setminus \{X\}.$$

We have that $(X, \mathfrak{g}, \mathcal{P})$ is a **GPT** space. Then, the family of $(\mathfrak{g}, \mathcal{P})$ -open sets coincides with $\mathfrak{g}$, and hence

$$G_{(\mathfrak{g}, \mathcal{P})} = \{1, 2\}.$$

Consequently,

$$X \setminus G_{(\mathfrak{g}, \mathcal{P})} = \mathbb{N} \setminus \{1, 2\},$$

which is an infinite set. Since every $D_{(\mathfrak{g}, \mathcal{P})}$ -set is of the form $U \setminus V$, where U is a $(\mathfrak{g}, \mathcal{P})$ -open set with $U \neq X$, it follows that every $D_{(\mathfrak{g}, \mathcal{P})}$ -set is contained in $G_{(\mathfrak{g}, \mathcal{P})}$.

Hence, no $D_{(\mathfrak{g}, \mathcal{P})}$ -set can separate any two distinct points of $X \setminus G_{(\mathfrak{g}, \mathcal{P})}$. Therefore, the space $(X, \mathfrak{g}, \mathcal{P})$ does not satisfy the $(\mathfrak{g}, \mathcal{P})$ - D_0 property.

Hence, we present the notion of $(\mathfrak{g}, \mathcal{P})$ - D_i for $i = 0, 1, 2$ as follows:

Definition 6. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space; X is known as

- (i) $(\mathfrak{g}, \mathcal{P})$ - D_1 if for each $x, y \in G_{(\mathfrak{g}, \mathcal{P})} : x \neq y$, there exist $D_{(\mathfrak{g}, \mathcal{P})}$ -sets U and V satisfying $x \in U, y \notin U$ and $y \in V, x \notin V$.
- (ii) $(\mathfrak{g}, \mathcal{P})$ - D_2 if for each $x, y \in G_{(\mathfrak{g}, \mathcal{P})} : x \neq y$, there exist disjoint $D_{(\mathfrak{g}, \mathcal{P})}$ -sets U, V satisfying $x \in U$ and $y \in V$.

Definition 7. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. Then, X is known as

- (i) $(\mathfrak{g}, \mathcal{P})$ - T_0 if for each $x, y \in G_{(\mathfrak{g}, \mathcal{P})} : x \neq y$, there exists $(\mathfrak{g}, \mathcal{P})$ -open set containing precisely one element of x and y ;
- (ii) $(\mathfrak{g}, \mathcal{P})$ - T_1 if each $x, y \in G_{(\mathfrak{g}, \mathcal{P})} : x \neq y$, there are $(\mathfrak{g}, \mathcal{P})$ -open sets U and V satisfying $x \in U$ and $y \notin U$ and $y \in V$ and $x \notin V$;
- (iii) $(\mathfrak{g}, \mathcal{P})$ - T_2 if each $x, y \in G_{(\mathfrak{g}, \mathcal{P})} : x \neq y$, there are disjoint $(\mathfrak{g}, \mathcal{P})$ -open sets U and V satisfying $x \in U$ and $y \in V$;
- (iv) $(\mathfrak{g}, \mathcal{P})$ -regular if for each $(\mathfrak{g}, \mathcal{P})$ -closed set F and $x \notin F$, there are disjoint $(\mathfrak{g}, \mathcal{P})$ -open sets U and V satisfying $x \in U$ and $F \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq V$;
- (v) $(\mathfrak{g}, \mathcal{P})$ -normal if for all $(\mathfrak{g}, \mathcal{P})$ -closed sets A and B satisfying $A \cap B \cap G_{(\mathfrak{g}, \mathcal{P})} = \emptyset$, there are disjoint $(\mathfrak{g}, \mathcal{P})$ -open sets U and V satisfying $A \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq U$ and $B \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq V$.

Remark 1. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. When

- (i) X is $(\mathfrak{g}, \mathcal{P})$ - T_k , it is $(\mathfrak{g}, \mathcal{P})$ - T_{k-1} , $k = 1, 2$;
- (ii) X is $(\mathfrak{g}, \mathcal{P})$ - T_k , it is $(\mathfrak{g}, \mathcal{P})$ - D_k , $k = 0, 1, 2$;
- (iii) X is $(\mathfrak{g}, \mathcal{P})$ - D_k , it is $(\mathfrak{g}, \mathcal{P})$ - D_{k-1} , $k = 1, 2$.

Theorem 1. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space.

- (i) X is $(\mathfrak{g}, \mathcal{P})$ - D_0 iff X has the $(\mathfrak{g}, \mathcal{P})$ - T_0 property;
- (ii) X is $(\mathfrak{g}, \mathcal{P})$ - D_1 iff X has the $(\mathfrak{g}, \mathcal{P})$ - D_2 property.

Proof. (i) Consider X as $(\mathfrak{g}, \mathcal{P})$ - D_0 and $x, y \in G_{(\mathfrak{g}, \mathcal{P})} : x \neq y$. Thus, there is a $D_{(\mathfrak{g}, \mathcal{P})}$ -set G satisfying $x \in G, y \notin G$. Thus, there are $(\mathfrak{g}, \mathcal{P})$ -open sets U_1 and U_2 satisfying $G = U_1 \setminus U_2$. Thus, $x \in U_1$ and $x \notin U_2$. If $y \in U_2$, then U_2 is the required set. If $y \notin U_2$, then $y \notin U_1$ and U_1 satisfies the matter.

The opposite direction follows from Remark 1.

(ii) Consider X as $(\mathfrak{g}, \mathcal{P})$ - D_1 and $x, y \in G_{(\mathfrak{g}, \mathcal{P})}$ with $x \neq y$. Then, there exist $D_{(\mathfrak{g}, \mathcal{P})}$ -sets G_1 and G_2 satisfies $x \in G_1, y \notin G_1$, and $x \notin G_2, y \in G_2$. Consider

$$G_1 = V_1 \setminus V_2 \text{ and } G_2 = V_3 \setminus V_4,$$

where V_1, V_2, V_3 and V_4 are $D_{(\mathfrak{g}, \mathcal{P})}$ -sets. Now, $x \notin G_2$ implies $x \notin V_3$ or $x \in V_3$ and $x \in V_4$. Now, if $x \notin V_3$, from $y \notin G_1$ we get the next two cases:

(1) $y \notin V_1$. From $x \in V_1 \setminus V_2$, we get $x \in V_1 \setminus (V_2 \cup V_3)$. From $y \in V_3 \setminus V_4$, we get $y \in V_3 \setminus (V_1 \cup V_4)$. Also,

$$(V_1 \setminus (V_2 \cup V_3)) \cap (V_3 \setminus (V_1 \cup V_4)) = \emptyset.$$

(2) $y \in V_1$ and $y \in V_2$. We get $x \in V_1 \setminus V_2$ and $y \in V_2$ and

$$(V_1 \setminus V_2) \cap V_2 = \emptyset.$$

Moreover, if $x \in V_3$ and $x \in V_4$, we get $y \in V_3 \setminus V_4$ and $x \in V_4$ and

$$(V_3 \setminus V_4) \cap V_4 = \emptyset.$$

The opposite direction follows from Remark 1. $\square$

Corollary 1. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. When X is $(\mathfrak{g}, \mathcal{P})$ - D_1 , then it is $(\mathfrak{g}, \mathcal{P})$ - T_0 .

Theorem 2. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. Then, X is $(\mathfrak{g}, \mathcal{P})$ - T_0 iff for every $x, y \in G_{(\mathfrak{g}, \mathcal{P})}$ with $x \neq y$, we have

$$cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \neq cl_{(\mathfrak{g}, \mathcal{P})}(\{y\}).$$

Proof. Let **GPT** space be $(\mathfrak{g}, \mathcal{P})$ - T_0 . Hence, for every $x, y \in G_{(\mathfrak{g}, \mathcal{P})}$ with $x \neq y$, $\exists$ $(\mathfrak{g}, \mathcal{P})$ -open set U satisfies $x \in U, y \notin U$. Thus, $(X \setminus U)$ is $(\mathfrak{g}, \mathcal{P})$ -closed, with $x \notin (X \setminus U)$, which implies $y \in (X \setminus U)$. Hence,

$$cl_{(\mathfrak{g}, \mathcal{P})}(\{y\}) \subseteq (X \setminus U)$$

Note that $x \notin cl_{(\mathfrak{g}, \mathcal{P})}(\{y\})$. Therefore, $cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \neq cl_{(\mathfrak{g}, \mathcal{P})}(\{y\})$. $\square$

Theorem 3. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. X is $(\mathfrak{g}, \mathcal{P})$ - T_1 iff for every $x \in G_{(\mathfrak{g}, \mathcal{P})}$, $\{x\} \cup (X \setminus G_{(\mathfrak{g}, \mathcal{P})})$ is a $(\mathfrak{g}, \mathcal{P})$ -closed set.

Proof. Consider $\{x\}$ as $(\mathfrak{g}, \mathcal{P})$ -closed. Also, for each $x \in G_{(\mathfrak{g}, \mathcal{P})}$ consider $x, y \in G_{(\mathfrak{g}, \mathcal{P})} : x \neq y$. Thus, the complements of $\{x\}$ and $\{y\}$ are $(\mathfrak{g}, \mathcal{P})$ -open sets with $x \in (X \setminus \{y\})$, $y \notin (X \setminus \{y\})$ and $x \notin (X \setminus \{x\})$, $y \in (X \setminus \{x\})$. Therefore, the **GPT** space is $(\mathfrak{g}, \mathcal{P})$ - T_1 . $\square$

Definition 8. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. A point $x \in G_{(\mathfrak{g}, \mathcal{P})}$ is called $(\mathfrak{g}, \mathcal{P})$ -neat point if $x \in U \in \mathfrak{g}$, leads to $U = G_{(\mathfrak{g}, \mathcal{P})}$.

Remark 2. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. If X is $(\mathfrak{g}, \mathcal{P})$ - T_0 and has $(\mathfrak{g}, \mathcal{P})$ -neat point, then it is unique.

Theorem 4. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. If X is $(\mathfrak{g}, \mathcal{P})$ - T_0 and has no $(\mathfrak{g}, \mathcal{P})$ -neat point, then it is $(\mathfrak{g}, \mathcal{P})$ - D_1 .

In Figure 1, we present the relationship among the above notions in a simple way.

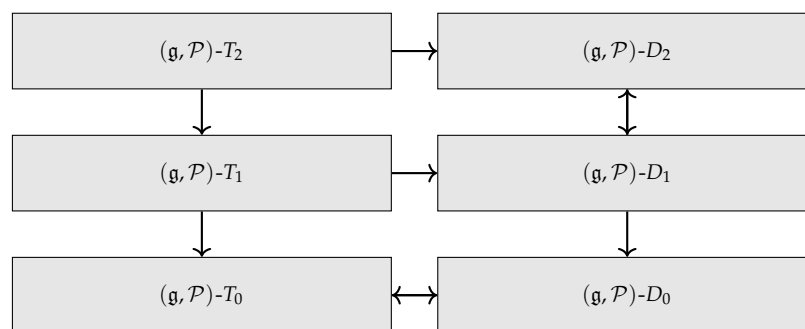


Figure 1. The relationship among the separation axioms in **GPT** spaces.

Example 2. Consider the infinite **GPT** space defined as follows:

$$X = \mathbb{N} = \{1, 2, 3, \dots\}, \quad \mathcal{P} = \{A \subseteq \mathbb{N} \mid 1 \notin A\}, \quad \mathfrak{g} = \{\emptyset\} \cup \{U \subseteq \mathbb{N} : 1 \in U\} \cup \{\mathbb{N}\}.$$

- (i) We show that for each pair of distinct elements $x \neq y$ in $G_{(\mathfrak{g}, \mathcal{P})}$, there exists a $(\mathfrak{g}, \mathcal{P})$ -open set that contains exactly one of them. This proves that the space satisfies the $(\mathfrak{g}, \mathcal{P})$ - T_0 separation axiom.

Consider any two distinct points $x, y \in \mathbb{N}$.

Case 1: One of the points is 1. Assume without loss of generality that $x = 1$ and $y \neq 1$. Since any subset of $\mathbb{N}$ containing 1 is $(\mathfrak{g}, \mathcal{P})$ -open and satisfies

$$1 \in \{1\}, \quad y \notin \{1\}.$$

Thus, 1 is distinguished from any other point in X .

Case 2: Both points are greater than 1. Suppose $x = n > 1$ and $y = m > 1$ with $n \neq m$. The set $\{1, n\}$ is $(\mathfrak{g}, \mathcal{P})$ -open because it contains the element 1. Moreover,

$$n \in \{1, n\}, \quad m \notin \{1, n\}.$$

Hence, n is distinguished from m by a $(\mathfrak{g}, \mathcal{P})$ -open set.

Since in both cases we can find a $(\mathfrak{g}, \mathcal{P})$ -open set containing exactly one of x and y , the space satisfies the $(\mathfrak{g}, \mathcal{P})$ - T_0 property.

- (ii) The space is not $(\mathfrak{g}, \mathcal{P})$ - T_1 . Since every nonempty $(\mathfrak{g}, \mathcal{P})$ -open set must contain the element 1, no $(\mathfrak{g}, \mathcal{P})$ -open set can isolate any $n > 1$ from 1. Thus, the $(\mathfrak{g}, \mathcal{P})$ - T_1 requirement fails for pairs $(1, n)$. Consequently, the space cannot satisfy the stronger $(\mathfrak{g}, \mathcal{P})$ - T_2 property.
- (iii) Since the space represents $(\mathfrak{g}, \mathcal{P})$ - T_0 , it also represents the $(\mathfrak{g}, \mathcal{P})$ - D_0 -property, Theorem 1. However, no pair of disjoint $(\mathfrak{g}, \mathcal{P})$ -open sets can separate all distinct elements of the space. Therefore, the space does not satisfy the $(\mathfrak{g}, \mathcal{P})$ - D_1 property, and consequently, it also does not satisfy the $(\mathfrak{g}, \mathcal{P})$ - D_2 property.

Definition 9. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. Then, X is known as

- (i) $(\mathfrak{g}, \mathcal{P})$ - R_0 when $x \in U \in \mathfrak{g}$ implies

$$cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq U.$$

- (ii) $(\mathfrak{g}, \mathcal{P})$ - R_1 when for every $x, y \in G_{(\mathfrak{g}, \mathcal{P})}$ satisfies

$$cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \neq cl_{(\mathfrak{g}, \mathcal{P})}(\{y\}),$$

there exist disjoint $(\mathfrak{g}, \mathcal{P})$ -open sets U and V satisfying $x \in U, y \in V$.

In a **GPT** space that is not strong, it is impossible for the closure of any set to be contained in $(\mathfrak{g}, \mathcal{P})$ -open set, as is required in definitions of $(\mathfrak{g}, \mathcal{P})$ - R_0 and $(\mathfrak{g}, \mathcal{P})$ - R_1 . Then, a **GPT**, that is not strong is not a $(\mathfrak{g}, \mathcal{P})$ - R_0 unless $G_{(\mathfrak{g}, \mathcal{P})} = \emptyset$. Similarly, it is not a $(\mathfrak{g}, \mathcal{P})$ - R_1 .

Theorem 5. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. Then,

- (i) X is $(\mathfrak{g}, \mathcal{P})$ - T_1 iff it is $(\mathfrak{g}, \mathcal{P})$ - T_0 and $(\mathfrak{g}, \mathcal{P})$ - R_0 ;
- (ii) When X is $(\mathfrak{g}, \mathcal{P})$ - R_1 , then it is $(\mathfrak{g}, \mathcal{P})$ - R_0 .

Proof. The result is obtained as a direct consequence of Definition 7 together with Theorems 2 and 3. $\square$

Definition 10. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. Then, X is called $(\mathfrak{g}, \mathcal{P})$ -symmetric if for each $x, y \in G_{(\mathfrak{g}, \mathcal{P})}$ satisfying $x \in cl_{(\mathfrak{g}, \mathcal{P})}(\{y\})$, we get $y \in cl_{(\mathfrak{g}, \mathcal{P})}(\{x\})$.

Theorem 6. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. Then, X is $(\mathfrak{g}, \mathcal{P})$ - R_0 iff it is $(\mathfrak{g}, \mathcal{P})$ -symmetric.

Theorem 7. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. Then, X is $(\mathfrak{g}, \mathcal{P})$ - R_1 iff for every $x, y \in G_{(\mathfrak{g}, \mathcal{P})}$, with $y \notin cl_{(\mathfrak{g}, \mathcal{P})}(\{x\})$, there exist disjoint $(\mathfrak{g}, \mathcal{P})$ -open sets U, V satisfying $x \in U, y \in V$.

Corollary 2. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. Then, X is $(\mathfrak{g}, \mathcal{P})$ - R_1 iff for every $x, y \in G_{(\mathfrak{g}, \mathcal{P})}$ satisfying $cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \neq cl_{(\mathfrak{g}, \mathcal{P})}(\{y\})$, there exist disjoint $(\mathfrak{g}, \mathcal{P})$ -open sets U and V satisfying

$$cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq U$$

and

$$cl_{(\mathfrak{g}, \mathcal{P})}(\{y\}) \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq V.$$

Corollary 3. In a $(\mathfrak{g}, \mathcal{P})$ - R_0 **GPT** space X , the following are identical:

- (i) X is $(\mathfrak{g}, \mathcal{P})$ - T_0 .
- (ii) X is $(\mathfrak{g}, \mathcal{P})$ - T_1 .
- (iii) X is $(\mathfrak{g}, \mathcal{P})$ - D_1 .

Corollary 4. In a **GPT** space X , the following are identical:

- (i) X is $(\mathfrak{g}, \mathcal{P})$ - T_2 .
- (ii) X is $(\mathfrak{g}, \mathcal{P})$ - T_1 and $(\mathfrak{g}, \mathcal{P})$ - R_1 .
- (iii) X is $(\mathfrak{g}, \mathcal{P})$ - T_0 and $(\mathfrak{g}, \mathcal{P})$ - R_1 .

Corollary 5. In a $(\mathfrak{g}, \mathcal{P})$ - R_1 **GPT** space X , the following are identical:

- (i) X is $(\mathfrak{g}, \mathcal{P})$ - T_2 .
- (ii) X is $(\mathfrak{g}, \mathcal{P})$ - T_1 .
- (iii) X is $(\mathfrak{g}, \mathcal{P})$ - T_0 .

Definition 11. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space with $A \subseteq G_{(\mathfrak{g}, \mathcal{P})}$. The $(\mathfrak{g}, \mathcal{P})$ -kernel of A , denoted by $(\mathfrak{g}, \mathcal{P})$ -ker(A), is given by

$$(\mathfrak{g}, \mathcal{P})\text{-ker}(A) = \cap \{U \in \mathfrak{g} : A \subseteq U\}.$$

Lemma 1. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space with $A \subseteq G_{(\mathfrak{g}, \mathcal{P})}$. Thus,

$$(\mathfrak{g}, \mathcal{P})\text{-ker}(A) = \{x \in G_{(\mathfrak{g}, \mathcal{P})} : cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \cap A \neq \emptyset\}.$$

Lemma 2. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space with $x \in G_{(\mathfrak{g}, \mathcal{P})}$. Thus,

$$y \in (\mathfrak{g}, \mathcal{P})\text{-ker}(\{x\}) \iff x \in cl_{(\mathfrak{g}, \mathcal{P})}(\{y\}).$$

Lemma 3. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space with $x, y \in G_{(\mathfrak{g}, \mathcal{P})}$. Then,

$$(\mathfrak{g}, \mathcal{P})\text{-ker}(\{x\}) \neq (\mathfrak{g}, \mathcal{P})\text{-ker}(\{y\}) \iff cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \neq cl_{(\mathfrak{g}, \mathcal{P})}(\{y\}).$$

Corollary 6. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. Then, X is $(\mathfrak{g}, \mathcal{P})\text{-}R_1$ iff for every $x, y \in G_{(\mathfrak{g}, \mathcal{P})}$ satisfying

$$(\mathfrak{g}, \mathcal{P})\text{-ker}(\{x\}) \neq (\mathfrak{g}, \mathcal{P})\text{-ker}(\{y\}),$$

there are disjoint $(\mathfrak{g}, \mathcal{P})$ -open sets U, V satisfying $x \in U, y \in V$.

Theorem 8. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. Then, X is $(\mathfrak{g}, \mathcal{P})\text{-}R_0$ iff every $x, y \in G_{(\mathfrak{g}, \mathcal{P})}$ satisfying

$$cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \neq cl_{(\mathfrak{g}, \mathcal{P})}(\{y\})$$

leads to

$$cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \cap cl_{(\mathfrak{g}, \mathcal{P})}(\{y\}) \cap G_{(\mathfrak{g}, \mathcal{P})} = \emptyset.$$

The following counterexamples demonstrate **GPT**-spaces where the $(\mathfrak{g}, \mathcal{P})\text{-}R_0$ and $(\mathfrak{g}, \mathcal{P})\text{-}R_1$ properties fail, highlighting the role of the $(\mathfrak{g}, \mathcal{P})$ -kernel in understanding closure and separability.

Example 3. Consider the infinite **GPT**-space defined by

$$X = \mathbb{N}, \quad \mathcal{P} = \{A \subseteq \mathbb{N} \mid 1 \notin A\}, \quad \mathfrak{g} = \{\emptyset\} \cup \{U \subseteq \mathbb{N} : 1 \in U\} \cup \{\mathbb{N}\}.$$

The closures of singletons are

$$cl_{(\mathfrak{g}, \mathcal{P})}(\{1\}) = \{1\}, \quad cl_{(\mathfrak{g}, \mathcal{P})}(\{n\}) = \mathbb{N} \text{ for all } n > 1.$$

The $(\mathfrak{g}, \mathcal{P})$ -kernels are

$$(\mathfrak{g}, \mathcal{P})\text{-ker}(\{1\}) = \mathbb{N}, \quad (\mathfrak{g}, \mathcal{P})\text{-ker}(\{n\}) = \mathbb{N} \text{ for all } n > 1.$$

$$cl_{(\mathfrak{g}, \mathcal{P})}(\{1\}) \cap cl_{(\mathfrak{g}, \mathcal{P})}(\{2\}) \cap G_{(\mathfrak{g}, \mathcal{P})} = \{1\} \cap \mathbb{N} = \{1\} \neq \emptyset.$$

Therefore, this space does not represent the $(\mathfrak{g}, \mathcal{P})\text{-}R_0$ property.

Also, since

$$(\mathfrak{g}, \mathcal{P})\text{-ker}(\{1\}) = \mathbb{N} = (\mathfrak{g}, \mathcal{P})\text{-ker}(\{n\}) \text{ for all } n > 1,$$

this space does not represent the $(\mathfrak{g}, \mathcal{P})\text{-}R_1$ property.

Theorem 9. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. Then, X is $(\mathfrak{g}, \mathcal{P})\text{-}R_0$ iff for each $x, y \in G_{(\mathfrak{g}, \mathcal{P})}$, we get

$$(\mathfrak{g}, \mathcal{P})\text{-ker}(\{x\}) \neq (\mathfrak{g}, \mathcal{P})\text{-ker}(\{y\})$$

leads to

$$(\mathfrak{g}, \mathcal{P})\text{-ker}(\{x\}) \cap (\mathfrak{g}, \mathcal{P})\text{-ker}(\{y\}) = \emptyset.$$

Theorem 10. In a **GPT** space X , the following are identical:

- (i) X is $(\mathfrak{g}, \mathcal{P})\text{-}R_0$.

(ii) For every $\emptyset \neq A \subseteq X$, $G \in \mathfrak{g}$ satisfying $A \cap G \neq \emptyset$, there is a $(\mathfrak{g}, \mathcal{P})$ -closed set F satisfying

$$A \cap F \neq \emptyset \text{ with } F \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq G.$$

(iii) Every $G \in \mathfrak{g}$ has the property

$$G = \cup \{F \cap G_{(\mathfrak{g}, \mathcal{P})} : F \text{ is } (\mathfrak{g}, \mathcal{P})\text{-closed and } F \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq G\}.$$

(iv) Every $(\mathfrak{g}, \mathcal{P})$ -closed set F has the property

$$F \cap G_{(\mathfrak{g}, \mathcal{P})} = (\mathfrak{g}, \mathcal{P})\text{-ker}(F \cap G_{(\mathfrak{g}, \mathcal{P})}).$$

(v) Every $x \in G_{(\mathfrak{g}, \mathcal{P})}$ has the property

$$cl_{(\mathfrak{g}, \mathcal{P})}\{x\} \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq (\mathfrak{g}, \mathcal{P})\text{-ker}(\{x\}).$$

Theorem 11. In a **GPT** space, the following are identical:

- (i) X is $(\mathfrak{g}, \mathcal{P})$ - R_0 ;
- (ii) When F is $(\mathfrak{g}, \mathcal{P})$ -closed, $x \in F \cap G_{(\mathfrak{g}, \mathcal{P})}$, we get $(\mathfrak{g}, \mathcal{P})\text{-ker}(\{x\}) \subseteq F$;
- (iii) When $x \in G_{(\mathfrak{g}, \mathcal{P})}$, thus $(\mathfrak{g}, \mathcal{P})\text{-ker}(\{x\}) \subseteq cl_{(\mathfrak{g}, \mathcal{P})}(\{x\})$.

Theorem 12. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. Then, X is $(\mathfrak{g}, \mathcal{P})$ - R_0 iff for each $x \in G_{(\mathfrak{g}, \mathcal{P})}$, we get

$$cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \cap G_{(\mathfrak{g}, \mathcal{P})} = (\mathfrak{g}, \mathcal{P})\text{-ker}(\{x\}).$$

3. More Types of Separation Axioms

Following the same approach, some researchers present different types of separation axioms in generalized topology, for example, R_δ , weakly regular, R_{D_δ} , and R_{D^*} in [18,19]. Here, we will study them on **GPT** spaces.

Definition 12. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. A subset $A \subseteq G_{(\mathfrak{g}, \mathcal{P})}$ is known as a $(\mathfrak{g}, \mathcal{P})$ - G_δ -set when A represents a countable intersection of $(\mathfrak{g}, \mathcal{P})$ -open sets. Moreover, its complement is known as a $(\mathfrak{g}, \mathcal{P})$ - F_σ set.

Definition 13. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. A subset $A \subseteq G_{(\mathfrak{g}, \mathcal{P})}$ is known as a regular $(\mathfrak{g}, \mathcal{P})$ - G_δ -set if

$$A = \bigcap_{i=1}^{\infty} F_i \cap G_{(\mathfrak{g}, \mathcal{P})} = \bigcap_{i=1}^{\infty} i_{\mathfrak{g}}(F_i),$$

where each F_i in the sequence $\{F_i\}$ is a $(\mathfrak{g}, \mathcal{P})$ -closed set.

Moreover, its complement known as a regular $(\mathfrak{g}, \mathcal{P})$ - F_σ -set, which means that A is a regular $(\mathfrak{g}, \mathcal{P})$ - F_σ -set if

$$A = \bigcup_{i=1}^{\infty} O_i \cup (X \setminus G_{(\mathfrak{g}, \mathcal{P})}) = \bigcup_{i=1}^{\infty} cl_{(\mathfrak{g}, \mathcal{P})}(O_i),$$

where each O_i is $(\mathfrak{g}, \mathcal{P})$ -open.

Definition 14. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. A subset $A \subseteq G_{(\mathfrak{g}, \mathcal{P})}$ is called a $(\mathfrak{g}, \mathcal{P})$ - d_δ -open set if

$$A = \bigcup_i F_i \cap G_{(\mathfrak{g}, \mathcal{P})},$$

where each F_i is a regular $(\mathfrak{g}, \mathcal{P})$ - F_σ set.

Moreover, its complement is called $(\mathfrak{g}, \mathcal{P})$ - d_δ -closed, which means that A is $(\mathfrak{g}, \mathcal{P})$ - d_δ -closed when

$$A = \bigcap_i G_i \cup (X \setminus G_{(\mathfrak{g}, \mathcal{P})}),$$

where G_i is a regular $(\mathfrak{g}, \mathcal{P})$ - G_δ -set.

Definition 15. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space. A subset A is known as $(\mathfrak{g}, \mathcal{P})$ - d -closed if

$$A = \bigcap_i \{F_i : F_i \text{ is } (\mathfrak{g}, \mathcal{P})\text{-closed set and } F_i \cap G_{(\mathfrak{g}, \mathcal{P})} \text{ is } (\mathfrak{g}, \mathcal{P})\text{-}G_\delta\text{-set}\}.$$

Lemma 4. Let $(X, \mathfrak{g}, \mathcal{P})$ be a **GPT** space with $A \subseteq X$. The following are true:

- (i) When A is a regular $(\mathfrak{g}, \mathcal{P})$ - G_δ -set, thus $A \cup (X \setminus G_{(\mathfrak{g}, \mathcal{P})})$, consider as $(\mathfrak{g}, \mathcal{P})$ - d_δ -closed;
- (ii) Each regular $(\mathfrak{g}, \mathcal{P})$ - G_δ -set consider as a $(\mathfrak{g}, \mathcal{P})$ - G_δ -set;
- (iii) When A is a regular $(\mathfrak{g}, \mathcal{P})$ - G_δ -set, thus $A \cup (X \setminus G_{(\mathfrak{g}, \mathcal{P})})$, consider as $(\mathfrak{g}, \mathcal{P})$ -closed;
- (iv) Each $(\mathfrak{g}, \mathcal{P})$ -closed set F satisfying $F \cap G_{(\mathfrak{g}, \mathcal{P})}$, consider as a $(\mathfrak{g}, \mathcal{P})$ - G_δ -set;
- (v) Each $(\mathfrak{g}, \mathcal{P})$ - d -closed set consider as $(\mathfrak{g}, \mathcal{P})$ -closed;
- (vi) Each $(\mathfrak{g}, \mathcal{P})$ - d_δ -closed set consider as $(\mathfrak{g}, \mathcal{P})$ - d -closed.

Proof. The proof comes immediately from the definitions. $\square$

Definition 16. For a **GPT** space X with $A \subseteq X$.

- (i) A collection $\mathfrak{S}$ contained in $\mathfrak{g}$ is called a $(\mathfrak{g}, \mathcal{P})$ -open complementary system if for every $S \in \mathfrak{S}$, there exists a sequence $\{S_n\}$ in $\mathfrak{S}$ satisfying

$$S = \bigcup_n (X \setminus S_n) \cap G_{(\mathfrak{g}, \mathcal{P})}.$$

- (ii) A subset A is known as strongly open $(\mathfrak{g}, \mathcal{P})$ - F_σ -set when we have a countable $(\mathfrak{g}, \mathcal{P})$ -open complementary system $\mathfrak{S}$ satisfying $A \in \mathfrak{S}$. Moreover, its complement is known as a strongly closed $(\mathfrak{g}, \mathcal{P})$ - G_δ -set.
- (iii) Any intersection of a strongly closed $(\mathfrak{g}, \mathcal{P})$ - G_δ -set is a $(\mathfrak{g}, \mathcal{P})$ - d^* -closed set. Then, each strongly closed $(\mathfrak{g}, \mathcal{P})$ - G_δ -set is considered as $(\mathfrak{g}, \mathcal{P})$ - d^* -closed.
- (iv) Each $(\mathfrak{g}, \mathcal{P})$ - d^* -closed set is considered as $(\mathfrak{g}, \mathcal{P})$ - d -closed.

The following examples illustrate infinite **GPT**-spaces where certain new separation axioms hold, highlighting the role of $(\mathfrak{g}, \mathcal{P})$ -closed sets, $(\mathfrak{g}, \mathcal{P})$ - G_δ sets, and $(\mathfrak{g}, \mathcal{P})$ -kernels.

Definition 17. A **GPT** space is called weakly $(\mathfrak{g}, \mathcal{P})$ - D_1 if

$$\bigcap_{x \in G_{(\mathfrak{g}, \mathcal{P})}} cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) = X \setminus G_{(\mathfrak{g}, \mathcal{P})}.$$

Theorem 13. A **GPT** space X is weakly $(\mathfrak{g}, \mathcal{P})$ - D_1 iff X has no $(\mathfrak{g}, \mathcal{P})$ -neat points.

Proof. Consider X as a **GPT** space that is weakly $(\mathfrak{g}, \mathcal{P})$ - D_1 and X has a $(\mathfrak{g}, \mathcal{P})$ -neat point y . Thus, $y \in cl_{(\mathfrak{g}, \mathcal{P})}(\{x\})$, for every $x \in G_{(\mathfrak{g}, \mathcal{P})}$, is a contradiction. Conversely, let the **GPT** space have no $(\mathfrak{g}, \mathcal{P})$ -neat points and not be weakly $(\mathfrak{g}, \mathcal{P})$ - D_1 . Then, there exists a

$$y \in \left(\bigcap_{x \in G_{(\mathfrak{g}, \mathcal{P})}} cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \right) \cap G_{(\mathfrak{g}, \mathcal{P})}.$$

Then, any $(\mathfrak{g}, \mathcal{P})$ -open set containing y must be $G_{(\mathfrak{g}, \mathcal{P})}$. Thus, y is $(\mathfrak{g}, \mathcal{P})$ -neat point of X , is a contradiction. $\square$

Theorem 14. When a GPT space X is $(\mathfrak{g}, \mathcal{P})$ - T_0 and weakly $(\mathfrak{g}, \mathcal{P})$ - D_1 , then it is $(\mathfrak{g}, \mathcal{P})$ - D_1 .

Theorem 15. A GPT space is weakly $(\mathfrak{g}, \mathcal{P})$ - D_1 iff for each $x \in G_{(\mathfrak{g}, \mathcal{P})}$

$$(\mathfrak{g}, \mathcal{P})\text{-ker}(\{x\}) \neq G_{(\mathfrak{g}, \mathcal{P})}$$

Proof. Consider X as weakly $(\mathfrak{g}, \mathcal{P})$ - D_1 . Consider $y \in G_{(\mathfrak{g}, \mathcal{P})}$ as satisfying

$$(\mathfrak{g}, \mathcal{P})\text{-ker}(\{y\}) = G_{(\mathfrak{g}, \mathcal{P})}.$$

According to Lemma 2, $y \in cl_{(\mathfrak{g}, \mathcal{P})}(\{x\})$, $\forall x \in G_{(\mathfrak{g}, \mathcal{P})}$, is a contradiction.

Conversely, let $(\mathfrak{g}, \mathcal{P})\text{-ker}(\{x\}) \neq G_{(\mathfrak{g}, \mathcal{P})}$ for every x that belongs to $G_{(\mathfrak{g}, \mathcal{P})}$. If X is not weakly $(\mathfrak{g}, \mathcal{P})$ - D_1 , then according to Theorem 13, it has a $(\mathfrak{g}, \mathcal{P})$ -neat point y . Therefore,

$$(\mathfrak{g}, \mathcal{P})\text{-ker}(\{x\}) = G_{(\mathfrak{g}, \mathcal{P})},$$

is a contradiction. $\square$

Definition 18. A GPT space is called

(i) $(\mathfrak{g}, \mathcal{P})$ - π_0 if for every $(\mathfrak{g}, \mathcal{P})$ -open set $V \neq \emptyset$, there exists $(\mathfrak{g}, \mathcal{P})$ -closed set F satisfying

$$\emptyset \neq F \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq V.$$

(ii) $(\mathfrak{g}, \mathcal{P})$ -weakly regular if it has a base $\{F \cap G_{(\mathfrak{g}, \mathcal{P})} : F \text{ is regular } (\mathfrak{g}, \mathcal{P})\text{-}F_\sigma\text{-set}\}$.

(iii) $(\mathfrak{g}, \mathcal{P})$ - R_{D_δ} if for every $(\mathfrak{g}, \mathcal{P})$ -open V and for every $x \in V$, there exists a regular $(\mathfrak{g}, \mathcal{P})$ - G_δ -set F satisfying

$$x \in F \subseteq V.$$

(iv) $(\mathfrak{g}, \mathcal{P})$ - R_{d_δ} if for all $(\mathfrak{g}, \mathcal{P})$ -open V and for all $x \in V$, there exists a $(\mathfrak{g}, \mathcal{P})$ - d_δ -closed set F satisfying

$$x \in F \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq V.$$

(v) $(\mathfrak{g}, \mathcal{P})$ - R_D if for all $x \in V \in \mathfrak{g}$, there exists a $(\mathfrak{g}, \mathcal{P})$ -closed set F satisfying $F \cap G_{(\mathfrak{g}, \mathcal{P})}$ is a $(\mathfrak{g}, \mathcal{P})$ - G_δ -set with

$$x \in F \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq V.$$

(vi) $(\mathfrak{g}, \mathcal{P})$ - R_d if for all $x \in V \in \mathfrak{g}$, there exists a $(\mathfrak{g}, \mathcal{P})$ - d -closed set F satisfying

$$x \in F \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq V.$$

(vii) $(\mathfrak{g}, \mathcal{P})$ - R_D^* if for all $(\mathfrak{g}, \mathcal{P})$ -open set V and for all $x \in V$, there exists a strongly closed $(\mathfrak{g}, \mathcal{P})$ - G_δ -set F satisfying

$$x \in F \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq V.$$

(viii) $(\mathfrak{g}, \mathcal{P})$ - R_d^* if for all $(\mathfrak{g}, \mathcal{P})$ -open set V and for all $x \in V$, there exists a $(\mathfrak{g}, \mathcal{P})$ - d^* -closed set F satisfying

$$x \in F \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq V.$$

Theorem 16. When a GPT space X is $(\mathfrak{g}, \mathcal{P})$ - R_{d_δ} , then it is $(\mathfrak{g}, \mathcal{P})$ - R_1 .

Proof. Consider that $x, y \in G_{(\mathfrak{g}, \mathcal{P})}$ satisfies $y \notin cl_{(\mathfrak{g}, \mathcal{P})}(\{x\})$. Thus,

$$y \in X \setminus cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \in \mathfrak{g}.$$

As the space is $(\mathfrak{g}, \mathcal{P})$ - R_{d_δ} , thus $\exists (\mathfrak{g}, \mathcal{P})$ - d_δ -closed set

$$H = \bigcap_i O_i \cup (X \setminus G_{(\mathfrak{g}, \mathcal{P})})$$

satisfies

$$y \in H \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq X \setminus cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}).$$

Thus, $y \in O_i$ and $x \notin O_i$ for some i . Now

$$O_i = \bigcap_{n=1}^{\infty} F_n \cap G_{(\mathfrak{g}, \mathcal{P})} = \bigcap i_{\mathfrak{g}}(F_n),$$

where $\{F_n\}$ is a sequence of $(\mathfrak{g}, \mathcal{P})$ -closed sets. Then, there exists F_n such that $y \in F_n$ and $x \notin F_n$. Then, $i_{\mathfrak{g}}(F_n)$ and $X \setminus F_n$ are required $(\mathfrak{g}, \mathcal{P})$ -open sets. $\square$

Example 4. Consider the **GPT** space defined as

$$X = \mathbb{N}, \quad \mathcal{P} = \{A \subseteq \mathbb{N} \mid 1 \notin A\}, \quad \mathfrak{g} = \{\emptyset\} \cup \{U \subseteq \mathbb{N} : 1 \in U\} \cup \{\mathbb{N}\}.$$

For each $x \in X$, define a descending sequence of $(\mathfrak{g}, \mathcal{P})$ -open sets:

$$V_n(x) = \{1, x, x+1, x+2, \dots\}, \quad n \geq 1.$$

The corresponding $(\mathfrak{g}, \mathcal{P})$ -closures:

$$cl_{(\mathfrak{g}, \mathcal{P})}(V_n(x)) = V_n(x), \quad \bigcap_{n=1}^{\infty} V_n(x) = \{1, x, x+1, \dots\} \subseteq X$$

Define the $(\mathfrak{g}, \mathcal{P})$ - d_δ -closed set associated with x :

$$F_x = \bigcup_{n=1}^{\infty} cl_{(\mathfrak{g}, \mathcal{P})}(V_n(x)) = \{1, x, x+1, \dots\}.$$

So, for any $(\mathfrak{g}, \mathcal{P})$ -open set U with $x \in U$, there exists $(\mathfrak{g}, \mathcal{P})$ - d_δ -closed F_x satisfying

$$x \in F_x \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq U.$$

Hence, this space represents $(\mathfrak{g}, \mathcal{P})$ - R_{d_δ} and, by Theorem 16, is also $(\mathfrak{g}, \mathcal{P})$ - R_1 .

Theorem 17. A $(\mathfrak{g}, \mathcal{P})$ -weakly regular **GPT** space is $(\mathfrak{g}, \mathcal{P})$ - R_0 .

Proof. Consider $x \in U \in \mathfrak{g}$ and $y \in G_{(\mathfrak{g}, \mathcal{P})} \setminus U$. Thus, $x \notin cl_{(\mathfrak{g}, \mathcal{P})}(\{y\})$. Hence, there exists a regular $(\mathfrak{g}, \mathcal{P})$ - F_σ -set F satisfying

$$x \in F \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq X \setminus cl_{(\mathfrak{g}, \mathcal{P})}(\{y\}).$$

Let

$$F = \bigcup_{n=1}^{\infty} O_n \cup (X \setminus G_{(\mathfrak{g}, \mathcal{P})}) = \bigcup_{n=1}^{\infty} cl_{(\mathfrak{g}, \mathcal{P})}(O_n),$$

where $O_n \in \mathfrak{g}$ and $cl_{(\mathfrak{g}, \mathcal{P})}(O_n) \subseteq F$, $x \in O_n$ for some n . Then,

$$cl_{(\mathfrak{g}, \mathcal{P})}(O_n) \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq F \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq X \setminus cl_{(\mathfrak{g}, \mathcal{P})}(\{y\}).$$

Hence, $y \in X \setminus cl_{(\mathfrak{g}, \mathcal{P})}(O_n)$. Therefore, $y \notin cl_{(\mathfrak{g}, \mathcal{P})}(\{x\})$. $\square$

Theorem 18. When a GPT space X is $(\mathfrak{g}, \mathcal{P})$ -regular, hence it is $(\mathfrak{g}, \mathcal{P})$ - R_{D_δ} .

Proof. Consider X is as $(\mathfrak{g}, \mathcal{P})$ -regular with $x \in U \in \mathfrak{g}$. Hence, $\exists (\mathfrak{g}, \mathcal{P})$ -open set V_1 satisfies

$$x \in V_1 \subseteq cl_{(\mathfrak{g}, \mathcal{P})}(V_1) \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq U.$$

Again $(\mathfrak{g}, \mathcal{P})$ -regularity yields a $(\mathfrak{g}, \mathcal{P})$ -open set V_2 with

$$x \in V_2 \subseteq cl_{(\mathfrak{g}, \mathcal{P})}(V_2) \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq V_1.$$

By induction we get a sequence of $(\mathfrak{g}, \mathcal{P})$ -open sets $\{V_n\}$ satisfying

$$V_{n+1} \subseteq cl_{(\mathfrak{g}, \mathcal{P})}(V_{n+1}) \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq V_n.$$

Then, the set

$$A_x = \bigcap_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty} cl_{(\mathfrak{g}, \mathcal{P})}(V_n) \cap G_{(\mathfrak{g}, \mathcal{P})}$$

is a regular $(\mathfrak{g}, \mathcal{P})$ - G_δ -set satisfying $x \in A_x \subseteq U$. $\square$

We can summarize the previous results in the following diagram (Figure 2).

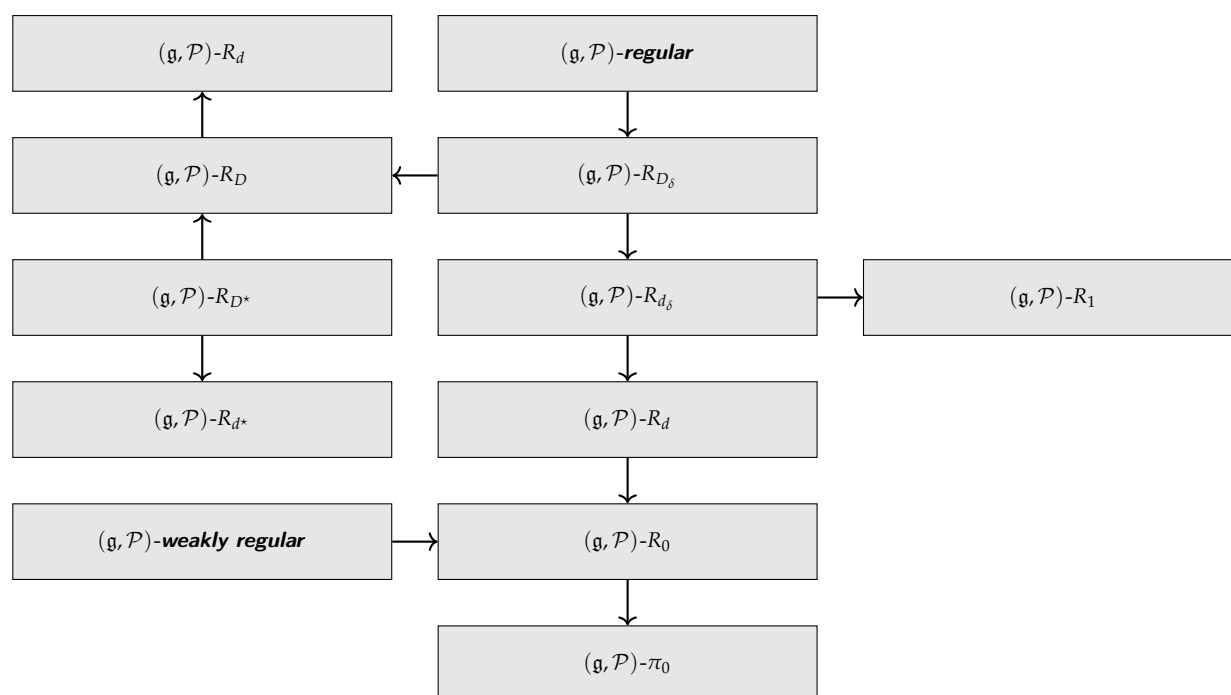


Figure 2. The relationship among new kinds of separation axioms in a GPT space.

Example 5. Consider the GPT space:

$$X = \mathbb{Z}, \quad \mathcal{P} = \{A \subseteq \mathbb{Z} \mid 0 \notin A\}, \quad \mathfrak{g} = \{\emptyset\} \cup \{U \subseteq \mathbb{Z} : 0 \in U\} \cup \{\mathbb{Z}\}.$$

- (i) Let the base of $(\mathfrak{g}, \mathcal{P})$ -weakly regular sets be

$$\mathcal{B} = \{F \cap G_{(\mathfrak{g}, \mathcal{P})} : F = \{0, n, n+1, \dots\}, n \in \mathbb{Z}\}.$$

Each F is a regular $(\mathfrak{g}, \mathcal{P})$ - F_σ -set (countable union of $(\mathfrak{g}, \mathcal{P})$ -closed sets). Every $(\mathfrak{g}, \mathcal{P})$ -open set U contains a set $F \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq U$ from the base $\mathcal{B}$. Thus, this space represents $(\mathfrak{g}, \mathcal{P})$ -weakly regular and, by Theorem 17, is also $(\mathfrak{g}, \mathcal{P})$ - R_0 , hence $(\mathfrak{g}, \mathcal{P})$ - π_0 .

- (ii) Define the closures of singletons as

$$\text{cl}_{(\mathfrak{g}, \mathcal{P})}(\{0\}) = \{0\}, \quad \text{cl}_{(\mathfrak{g}, \mathcal{P})}(\{n\}) = \{n, n+1, n+2, \dots\}, \text{ for all } n > 0.$$

The corresponding $(\mathfrak{g}, \mathcal{P})$ -kernels are

$$(\mathfrak{g}, \mathcal{P})\text{-ker}(\{0\}) = \mathbb{Z}, \quad (\mathfrak{g}, \mathcal{P})\text{-ker}(\{n\}) = \{n, n+1, n+2, \dots\}, \text{ for all } n > 0.$$

$(\mathfrak{g}, \mathcal{P})$ - R_{d_δ} does not hold for $n > 1$, since no $(\mathfrak{g}, \mathcal{P})$ - d_δ closed set F_n satisfies $n \in F_n \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq U$ for arbitrary U containing n , showing that $(\mathfrak{g}, \mathcal{P})$ -weakly regular $\nRightarrow (\mathfrak{g}, \mathcal{P})$ - R_{d_δ} in this space.

- (iii) Similarly, $(\mathfrak{g}, \mathcal{P})$ - R_1 fails for $n > 1$, showing that $(\mathfrak{g}, \mathcal{P})$ -weakly regular $\nRightarrow R_1$ in this space.

It is worth mentioning that focusing the study on special classes of sets whose definitions rely on the primal collection often leads to results that are particularly interesting. This is exactly the approach we shall adopt next. To begin, we first define the notion of a $\mathcal{P}^\circ$ -open set within a GPT-space.

4. GP-Homeomorphism Mapping

As we said previously, separation axioms in the theory of topology are very important for the rule that plays to recognize non-homeomorphic topologies. In this section, we present the fundamental theorem and propositions about the GP-continuous functions and the GP-homeomorphism functions.

Definition 19. Assuming X and X^λ are two **GPT** spaces and h is a function from X into X^λ . Then, h is called GP-homeomorphism if it is bijective, GP-continuous, and its inverse is GP-continuous too.

Theorem 19. Let X and X^λ be two **GPT** spaces. When a GP-continuous function $h : X \rightarrow X^\lambda$ satisfies

$$h(G_{(\mathfrak{g}, \mathcal{P})}) \subseteq G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)},$$

the following are true:

- (i) The inverse image or (for short h^{-1}) of $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - G_δ -sets is $(\mathfrak{g}, \mathcal{P})$ - G_δ -set;
- (ii) h^{-1} of a strongly closed $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - G_δ -set is a strongly closed $(\mathfrak{g}, \mathcal{P})$ - G_δ -set;
- (iii) h^{-1} of a $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - d^* -closed set is $(\mathfrak{g}, \mathcal{P})$ - d^* -closed;
- (iv) h^{-1} of a regular $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - G_δ -set is regular $(\mathfrak{g}, \mathcal{P})$ - G_δ -set;
- (v) h^{-1} of a $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - d_δ -closed set is $(\mathfrak{g}, \mathcal{P})$ - d_δ -closed;
- (vi) h^{-1} of a $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - d -closed is $(\mathfrak{g}, \mathcal{P})$ - d -closed.

Proof. (i) and (iii) are clear.

(ii) Is true since

$$h^{-1}\left(\bigcap_k B_k \cup (X^\lambda \setminus G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)})\right) = \bigcap_k h^{-1}(B_k) \cup (X \setminus G_{(\mathfrak{g}, \mathcal{P})}).$$

(iv) Consider a regular $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - G_δ -set H given by

$$H = \bigcap_{k=1}^{\infty} F_k \cap G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)} = \bigcap_{k=1}^{\infty} i_{\mathfrak{g}^\lambda}(F_k)$$

Then,

$$\begin{aligned} h^{-1}(H) &= \bigcap_{k=1}^{\infty} h^{-1}(F_k) \cap G_{(\mathfrak{g}, \mathcal{P})}, \\ &= \bigcap_{k=1}^{\infty} h^{-1}(i_{\mathfrak{g}^\lambda}(F_k)). \end{aligned}$$

Now,

$$\begin{aligned} h^{-1}(i_{\mathfrak{g}^\lambda}(F_k)) &= h^{-1}(X^\lambda \setminus cl_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}(X^\lambda \setminus F_k)), \\ &= X \setminus h^{-1}(cl_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}(X^\lambda \setminus F_k)), \\ &\subseteq X \setminus cl_{(\mathfrak{g}, \mathcal{P})}(h^{-1}(X^\lambda \setminus F_k)), \\ &= i_{\mathfrak{g}}(h^{-1}(F_k)). \end{aligned}$$

Since

$$i_{\mathfrak{g}}(h^{-1}(F_k)) \subseteq h^{-1}(F_k) \cap G_{(\mathfrak{g}, \mathcal{P})},$$

the result follows.

(v) Let

$$H = \bigcap_i O_i \cup (X^\lambda \setminus G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}),$$

where O_i is regular $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - G_δ -set, be $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - d_δ -closed. Now, apply (iv). Then,

$$h^{-1}(H) = \bigcap_i h^{-1}(O_i) \cup (X \setminus G_{(\mathfrak{g}, \mathcal{P})}).$$

(vi) Comes directly from (i). $\square$

Theorem 20. Let X and X^λ be two **GPT** spaces. When X is finer than X^λ and X is $(\mathfrak{g}, \mathcal{P})$ - D_0 (resp. $(\mathfrak{g}, \mathcal{P})$ - D_1 , $(\mathfrak{g}, \mathcal{P})$ - D_2 , $(\mathfrak{g}, \mathcal{P})$ - T_0 , $(\mathfrak{g}, \mathcal{P})$ - T_1 , $(\mathfrak{g}, \mathcal{P})$ - T_2), then X^λ is $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - D_0 (resp. $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - D_1 , $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - D_2 , $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - T_0 , $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - T_1 , $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - T_2).

Theorem 21. Let X and X^λ be two **GPT** spaces. When a GP-continuous function $h : X \rightarrow X^\lambda$ satisfies

$$G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)} \subseteq h(G_{(\mathfrak{g}, \mathcal{P})}),$$

then $h^{-1}(E)$ is $D_{(\mathfrak{g}, \mathcal{P})}$ -set for each $D_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}$ -set E .

Theorem 22. Let X and X^λ be two **GPT** spaces. When X^λ is $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - D_1 and $h : X \rightarrow X^\lambda$ is GP-continuous such that h is injective on $G_{(\mathfrak{g}, \mathcal{P})}$ and satisfies

$$h(G_{(\mathfrak{g}, \mathcal{P})}) = G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)},$$

then X is $(\mathfrak{g}, \mathcal{P})$ - D_1 .

Theorem 23. A **GPT** space X is $(\mathfrak{g}, \mathcal{P})$ - D_1 iff for all $x, y \in X$ with $x \neq y$, there exists a **GPT** space X^λ and GP-continuous mapping $h : X \rightarrow X^\lambda$, where X^λ is $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - D_1 with

$$G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)} \subseteq h(G_{(\mathfrak{g}, \mathcal{P})}) \text{ and } h(x) \neq h(y).$$

Theorem 24. Let X and X^λ be two **GPT** spaces. Assuming $h : X \rightarrow X^\lambda$ as a GP-continuous and GP-closed function satisfying

$$h^{-1}(G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}) = G_{(\mathfrak{g}, \mathcal{P})}.$$

When X is $(\mathfrak{g}, \mathcal{P})$ - R_0 , then X^λ is $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - R_0 .

Proof. Consider $y \in V \in \mathfrak{g}^\lambda$. Thus, there exists $x \in G_{(\mathfrak{g}, \mathcal{P})}$ satisfying $h(x) = y$. Then, $x \in h^{-1}(V)$. Since X is $(\mathfrak{g}, \mathcal{P})$ - R_0 ,

$$cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \cap G_{(\mathfrak{g}, \mathcal{P})} \subseteq h^{-1}(V).$$

As h is GP-closed,

$$\begin{aligned} cl_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}(h(x)) &\subseteq h(cl_{(\mathfrak{g}, \mathcal{P})}(\{x\})), \\ &= h((cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \cap G_{(\mathfrak{g}, \mathcal{P})}) \cup (X \setminus G_{(\mathfrak{g}, \mathcal{P})})), \\ &\subseteq h(cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \cap G_{(\mathfrak{g}, \mathcal{P})}) \cup (X^\lambda \setminus G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}). \end{aligned}$$

as $h^{-1}(G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}) = G_{(\mathfrak{g}, \mathcal{P})}$. Therefore,

$$\begin{aligned} cl_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}(\{y\}) \cap G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)} &\subseteq h(cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) \cap G_{(\mathfrak{g}, \mathcal{P})}) \\ &\subseteq hh^{-1}(V) \\ &\subseteq V. \end{aligned}$$

Thus, X^λ is $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - R_0 . $\square$

Theorem 25. Let X and X^λ be two **GPT** spaces. When $h : X \rightarrow X^\lambda$, consider as GP-homeomorphism and X as $(\mathfrak{g}, \mathcal{P})$ - R_1 , thus X^λ is $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - R_1 .

Proof. Consider $x, y \in G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}$ and $y \notin cl_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}(\{x\})$. Then,

$$y_0 = h^{-1}(y) \notin h^{-1}(cl_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}(\{x\})).$$

Since

$$cl_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}(h^{-1}(x)) \subseteq h^{-1}(cl_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}(\{x\})).$$

Therefore,

$$y_0 \notin cl_{(\mathfrak{g}, \mathcal{P})}(h^{-1}(x)) = cl_{(\mathfrak{g}, \mathcal{P})}(\{x_0\}),$$

where $x_0 = h^{-1}(x) \in G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}$. As X is $(\mathfrak{g}, \mathcal{P})$ - R_1 , $\exists (\mathfrak{g}, \mathcal{P})$ -open sets U, V satisfying $x_0 \in U$ and $y_0 \in V$. Then, $h(U), h(V)$ are disjoint $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ -open with $x \in h(U), y \in h(V)$. Thus, X^λ is $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - R_1 . $\square$

Theorem 26. Let X and X^λ be two **GPT** spaces. When $h : X \rightarrow X^\lambda$ is GP-closed injection function satisfying

$$h(G_{(\mathfrak{g}, \mathcal{P})}) \subseteq G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}$$

implies

$$h(X \setminus G_{(\mathfrak{g}, \mathcal{P})}) \subseteq (X^\lambda \setminus G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}),$$

X is weakly $(\mathfrak{g}, \mathcal{P})$ - D_1 , thus X^λ is weakly $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - D_1 .

Proof. Since X is weakly $(\mathfrak{g}, \mathcal{P})$ - D_1 , then

$$\bigcap_{x \in G_{(\mathfrak{g}, \mathcal{P})}} cl_{(\mathfrak{g}, \mathcal{P})}(\{x\}) = X \setminus G_{(\mathfrak{g}, \mathcal{P})}.$$

Since h is an injection,

$$\begin{aligned} h\left(\bigcap_{x \in G_{(\mathfrak{g}, \mathcal{P})}} cl_{(\mathfrak{g}, \mathcal{P})}(\{x\})\right) &= \bigcap_{x \in G_{(\mathfrak{g}, \mathcal{P})}} h(cl_{(\mathfrak{g}, \mathcal{P})}(\{x\})), \\ &= h(X \setminus G_{(\mathfrak{g}, \mathcal{P})}). \end{aligned}$$

As h is a GP-closed function,

$$\begin{aligned} \bigcap_{y \in G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}} cl_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}(\{y\}) &\subseteq \bigcap_{x \in G_{(\mathfrak{g}, \mathcal{P})}} cl_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}(h(\{x\})), \\ &\subseteq \bigcap_{x \in G_{(\mathfrak{g}, \mathcal{P})}} h(cl_{(\mathfrak{g}, \mathcal{P})}(\{x\})), \\ &= h(X \setminus G_{(\mathfrak{g}, \mathcal{P})}), \\ &\subseteq X^\lambda \setminus G_{(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)}. \end{aligned}$$

Therefore, X^λ is weakly $(\mathfrak{g}^\lambda, \mathcal{P}^\lambda)$ - D_1 . $\square$

5. Discussion

In this section, we provide a brief discussion highlighting that certain classical axioms in **GPT** spaces may differ when restricted to a primal collection. Based on this observation, we introduce relevant definitions and preliminary concepts to clarify the structural differences. This framework will allow us to systematically analyze the consequences of these restrictions.

Definition 20. In a **GPT** space X , a subset $A \subseteq X$ is called $\mathcal{P}^\diamond$ -open if $A \subseteq i_{\mathfrak{g}}(A)^\diamond$.

Definition 21. A **GPT** space X is said to be $\mathcal{P}^\diamond$ -Hausdorff if for any two distinct points $x, y \in X$, there exist two disjoint $\mathcal{P}^\diamond$ -open sets U and V such that $x \in U$ and $y \in V$.

It is important to note that the relationships between separation axioms may differ when we restrict attention to $\mathcal{P}^\diamond$ -open sets. In particular, one property does not necessarily imply the other without specific assumptions on the primal collection.

Remark 3. A $(\mathfrak{g}, \mathcal{P})$ -Hausdorff **GPT** space does not necessarily imply that it is $\mathcal{P}^\diamond$ -Hausdorff, and vice versa.

To illustrate, consider a subset $A \subseteq X$. If $A^c \subseteq \mathcal{P}$, then $A^\diamond = \emptyset$ (see Theorem 3.1 [1]), meaning that no set other than the empty set is $\mathcal{P}^\diamond$ -open. Consequently, it is impossible to find two disjoint $\mathcal{P}^\diamond$ -open sets containing two distinct points.

Furthermore, consider $X = \{x_1, x_2\}$ with $\mathfrak{g} = \{\emptyset, \{x_1\}, X\}$ and primal collection $\mathcal{P} = \{\emptyset, \{x_1\}, \{x_2\}\}$. This example demonstrates that a $\mathcal{P}^\diamond$ -Hausdorff space may fail to be $(\mathfrak{g}, \mathcal{P})$ -Hausdorff.

Within the framework developed in this paper, however, every $(\mathfrak{g}, \mathcal{P})$ -Hausdorff **GPT** space is indeed a $\mathcal{P}^\diamond$ -Hausdorff space, while the converse does not always hold. To avoid ambiguity, we shall adopt the term $\mathcal{P}$ -Hausdorff to refer to the new notion.

Definition 22. A **GPT** space X is called $\mathcal{P}$ -Hausdorff if for each pair of distinct points $x, y \in X$, there exist two $(\mathfrak{g}, \mathcal{P})$ -open sets U_1 and U_2 such that $x \in U_1$, $y \in U_2$, and $U_1 \cap U_2 \in \mathcal{P}$.

From Definition 22, it follows that every $(\mathfrak{g}, \mathcal{P})$ -Hausdorff **GPT** space is $\mathcal{P}$ -Hausdorff, regardless of the specific primal collection.

Consider the infinite **GPT** space

$$X = \mathbb{N}, \quad \mathfrak{g} = \{\emptyset\} \cup \{U \subseteq \mathbb{N} : 1 \in U\} \cup \{\mathbb{N}\}, \quad \mathcal{P} = \{A \subseteq \mathbb{N} : 1 \notin A\}.$$

Observe that without considering the primal collection, the space fails to be Hausdorff because it is impossible to find disjoint $(\mathfrak{g}, \mathcal{P})$ -open sets separating points $n > 1$. However, with the primal collection included, we can define $\mathcal{P}^\circ$ -open sets that separate each point of X , ensuring that the space is $\mathcal{P}$ -Hausdorff. This example clearly shows that the primal collection has a direct effect on separation properties, even in infinite spaces.

6. Conclusions

In this paper, we introduced new classes of separation axioms within the framework of generalized primal topological (**GPT**) spaces and analyzed their interrelations. These axioms extend classical concepts and provide a deeper understanding of the structural properties of **GPT** spaces. Our results demonstrate how primal collections influence generalized notions of separation and continuity, thereby enriching the theoretical foundation of **GPT** spaces.

For future research, several specific directions are suggested:

1. **Extension to higher-order GPT spaces:** Investigate how the newly introduced separation axioms behave in more complex or higher-dimensional **GPT** spaces, possibly with additional algebraic or combinatorial structures.
2. **Applications in functional analysis and continuity:** Study how generalized continuous functions and other types of function mappings behave under these new axioms, and whether new classes of continuous functions emerge.
3. **Connections with other generalized topologies:** Explore relationships between **GPT** spaces and other generalized or hybrid topological structures, such as soft topologies, fuzzy topologies, or closure spaces, to identify common properties and potential unifying frameworks.
4. **Algorithmic and computational aspects:** Develop algorithms to construct **GPT** spaces satisfying certain separation axioms, and investigate computational complexity of checking these properties in finite or discrete settings.
5. **Topological invariants and classification:** Examine whether these new axioms lead to new invariants or classifications within **GPT** spaces, aiding in the systematic categorization of different types of generalized primal topologies.

These directions aim to deepen the understanding of **GPT** spaces and extend their theoretical and practical applications in mathematics.

In summary, this work provides a deeper insight into the structure and properties of **GPT** spaces through the introduction of new separation axioms. The results not only extend classical topological concepts but also offer a foundation for further theoretical and practical developments. Future studies may build on these findings to explore more complex **GPT** structures, applications, and connections with other generalized topologies.

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