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A Study of Symmetric q -Dunkl-Classical Orthogonal q -Polynomials Through a Second Structure Relation

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Abstract

This paper establishes a new characterization of symmetric q -Dunkl-classical orthogonal polynomials through a second structure relation. These symmetric polynomials generalize the q^2 -analogues of Hermite and Gegenbauer polynomials. Our main result provides a finite expansion of each polynomial in terms of its q -Dunkl derivatives, offering a new effective classification method. We derive explicit structure relations for the q^2 -analogue of generalized Hermite and the q^2 -analogue of generalized Gegenbauer polynomials.

Keywords: orthogonal polynomials; symmetric forms; q -series and q -polynomials; q -Dunkl operator; q -semiclassical polynomials

1. Introduction and Motivation

Symmetric orthogonal polynomials with operator-theoretic properties are central to special functions, approximation theory, and mathematical physics. The study of these polynomials represents a significant intersection of classical analysis, quantum algebras, and symmetry principles. The q -Dunkl-classical symmetric orthogonal polynomials studied here provide natural q -deformations of classical orthogonal polynomials while preserving essential structural properties.

The fundamental motivation behind this work arises from the need to develop a comprehensive classification scheme for orthogonal polynomials that remain invariant under certain operator actions. This classification problem has deep roots in the work of Hahn and others who attempted to characterize polynomial sequences that maintain their orthogonality properties when acted upon by specific operators.

The goal of this paper is to derive a second structure relation as a characterization of symmetric q -Dunkl-classical orthogonal polynomials. This approach departs from previous methods by providing an inverse perspective where the original polynomials are expressed in terms of their derived sequences, rather than vice versa. Previous characterizations of these polynomials have focused on first structure relations [1], Pearson-type equations [2], and difference equations [3]. While these approaches have been successful in establishing the basic foundation, they often require sophisticated functional analysis techniques and may complicate the simple algebraic relationships that govern these polynomial systems. The main novelty of this work is establishing a characterization through a second structure relation. Specifically, we prove that a symmetric monic orthogonal polynomial sequence (MOPS) is q -Dunkl-classical if and only if each polynomial can be written as a finite combination of its q -Dunkl derivatives. This extends classical results for ordinary Dunkl-classical



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polynomials [4] to the q -case. This new characterization provides a direct computational method for verifying the q -Dunkl-classical property. Rather than solving complex functional equations or analyzing moment sequences, one can simply check whether a given polynomial sequence admits a finite expansion in terms of its q -Dunkl derivatives.

Explicit formulas for expansion coefficients and a clear explanation of the structural organization of fundamental q -polynomial families are also given. We also demonstrate our characterization by deriving explicit second structure relations for two fundamental families: the q^2 -analogues of generalized Hermite polynomials and the q^2 -analogues of generalized Gegenbauer polynomials.

This paper is organized in the following way. In Section 2, we start by introducing the essential preliminary concepts. This includes an explanation of q -calculus fundamentals, Dunkl operator theory, and various orthogonal polynomial families. The strong relationships between these concepts are examined to form the foundation for our subsequent analysis.

Following this foundation, Section 3 presents the main contributions of this work. We introduce and rigorously define a new class of generalized differential-difference operators. These operators combine ideas from the concepts covered in the previous section. We then conduct a detailed study of their basic properties, focusing on how they act on different function spaces. Next, we use these new operators to construct several families of generalized orthogonal polynomials. We derive their spectral properties and explicitly compute their defining differential-difference equations. We then use these results to find explicit structure relations for the q^2 -analogue of the generalized Hermite and Gegenbauer polynomials. Section 4 covers computational methods and examples, including formulas and numerical results. Finally, the paper ends with a summary of the main findings and a discussion of open problems and potential directions for future research.

2. Notations, Definitions, and Preliminaries

2.1. Basic Definitions

The definition of $\mathcal{O}$ -classical orthogonal polynomials captures the essential property that the image of the polynomial sequence under the operator $\mathcal{O}$ remains orthogonal. This seemingly simple condition imposes strong constraints on the possible forms of both the polynomials and the operator, leading to the main classification theorems that characterize the classical families.

Let $\mathcal{P}$ be the space of all polynomials in a single variable with coefficients from the field of complex numbers. We consider a specific type of operator, denoted by $\mathcal{O}$, which acts as a lowering operator on this space $\mathcal{P}$. The action of this operator is characterized by the following three fundamental properties. First, the operator maps the entire space of polynomials back into itself; formally, this is expressed by the condition $\mathcal{O}(\mathcal{P}) = \mathcal{P}$. Second, the operator annihilates constant polynomials, a property captured by the requirement that $\mathcal{O}(1) = 0$. Third, and most crucially, the operator reduces the degree of each monomial by exactly one; that is, for every natural number $n \in \mathbb{N}$, the degree of the image of t^n under $\mathcal{O}$ satisfies $\deg(\mathcal{O}(t^n)) = n - 1$. Here, $\mathbb{N}$ represents the set of all positive integers, and we define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ to include zero. In the study of orthogonal polynomials, these kinds of operators are important tools, as they help us systematically describe and organize different polynomial sequences. A key use for them is to define specific families of polynomials, as shown in the following basic definition.

Definition 1. A MOPS denoted by $\{\phi_n\}_{n \in \mathbb{N}_0}$, is classified specifically as an $\mathcal{O}$ -classical orthogonal polynomial sequence if the new sequence generated by applying the lowering operator $\mathcal{O}$ to each

polynomial in the original sequence (with an index shift) itself constitutes a monic orthogonal polynomial sequence. More precisely, this means the sequence defined by

$$\left\{ \frac{\mathcal{O}\phi_{n+1}}{\rho_{n+1}} \right\}_{n \in \mathbb{N}_0}$$

must be a MOPS.

The scalar factor ρ_n in the denominator is a constant, chosen specifically to ensure that the leading coefficient of each resulting polynomial $\mathcal{O}\phi_{n+1}$ is scaled to one, thereby ensuring the monic property of the new sequence. This normalization factor is crucial for the stability of the three-term recurrence relation and the associated orthogonality measures.

Various lowering operators, including the classical derivative operator D , the forward difference operator Δ , and the Jackson q -derivative H_q , form the basis for the classification of orthogonal polynomial sequences. The concept of $\mathcal{O}$ -classical orthogonal polynomials serves as a unifying theory that encompasses numerous classical families. For instance, when $\mathcal{O} = D$, one recovers the well-known continuous families such as the Jacobi, Laguerre, Hermite, and Bessel polynomials. On the other hand, choosing $\mathcal{O} = \Delta$ yields discrete orthogonal families like the Charlier, Meixner, Krawtchouk, and Hahn polynomials (see [5]).

The q -difference operator H_q is expressed as [6]

$$(H_q\psi)(t) = \frac{\psi(qt) - \psi(t)}{(q-1)t}, \quad \psi \in \mathcal{P}, \quad q \in \tilde{\mathbb{C}},$$

with $\tilde{\mathbb{C}} = \mathbb{C} \setminus (\{0\} \cup \{z \in \mathbb{C} : z^k = 1 \text{ for some } k \geq 1\})$. Typically, for H_q to be well-defined on all $\psi \in \mathcal{P}$, q must avoid values that cause degeneracies for any polynomial. Specifically, to ensure H_q is non-degenerate on all $\mathcal{P}$, $\tilde{\mathbb{C}}$ should exclude all roots of unity and zero. When q goes to 1, we obtain the usual differential operator D .

The operator H_q is extended to a map $H_q : \mathcal{P}' \rightarrow \mathcal{P}'$ by

$$\langle H_q u, \psi \rangle := -\langle u, H_q \psi \rangle, \quad \text{for all } \psi \in \mathcal{P},$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $\mathcal{P}'$ and $\mathcal{P}$. This definition mimics the distributional derivative, i.e., if $u \in \mathcal{P}'$ and $\psi \in \mathcal{P}$, then the right-hand side $-\langle u, H_q \psi \rangle$ is well-defined since $H_q \psi \in \mathcal{P}$, and the left-hand side $\langle H_q u, \psi \rangle$ defines a new linear functional $H_q u$ on $\mathcal{P}$.

Extending H_q to linear functionals via duality is crucial for studying orthogonality. This approach allows us to work with moment functionals and orthogonal polynomial sequences without explicit knowledge of the weight function, which is particularly valuable for q -polynomials where the weight functions may be complicated or expressed in terms of special functions. The duality relation $\langle H_q u, \psi \rangle = -\langle u, H_q \psi \rangle$ ensures consistency with integration by parts formulas and maintains the mathematical structure familiar from the classical theory.

In [7], the authors investigate the Hahn-type characterization property, originally introduced in [8], within the setting of the Dunkl operator of index μ , defined on the space of real-coefficient polynomials by

$$T_\mu Q(t) = \phi'(t) + \mu \frac{Q(t) - Q(-t)}{t}, \quad \mu > -\frac{1}{2}.$$

This operator combines a derivative term $Q'(t)$ and a difference term $\frac{Q(t) - Q(-t)}{t}$ that measures the odd part of Q relative to the reflection $t \mapsto -t$. Their main result identifies all sequences of symmetric monic orthogonal polynomials that remain orthogonal under

the action of T_μ . It turns out that only the generalized Hermite and generalized Gegenbauer polynomials satisfy this property.

In a related study [9], it is shown that a linear functional u , corresponding to a Dunkl-classical MOPS satisfies a T_μ -Pearson-type equation. An alternative characterization of such orthogonal families is also given in terms of structure relations. Additionally, in [10], the uniqueness of the generalized Hermite and Gegenbauer polynomials as symmetric Dunkl-classical orthogonal sequences is confirmed via a different method from that in [7], relying on a differential-difference equation in the dual space. From this perspective, the first structure relation for symmetric MOPS naturally emerges.

When $\mu = 0$, the classical structure relation characterization of Jacobi, Laguerre, Hermite, and Bessel polynomials has already been established in [11]. Furthermore, ref. [4] provides a characterization of Dunkl-classical orthogonal polynomials using a second structure relation, which serves as a Dunkl analogue of a well-known result from [12] in the classical case.

A natural extension of this line of research is the study of the q -Dunkl operator $T_{\theta,q}$, defined as

$$T_{\theta,q}Q(t) = H_qQ(t) + \theta \frac{Q(t) - Q(-t)}{2t}, \quad 0 < q < 1, \theta \in \mathbb{C},$$

where H_q denotes the Jackson q -derivative.

Specifically, in the case of the monomial sequence $t^n_{n \in \mathbb{N}_0}$, it holds that

$$T_{\theta,q}(t^n) = \left([n]_q + \theta \frac{1 - (-1)^n}{2} \right) t^{n-1},$$

where $[n]_q := \frac{q^n - 1}{q - 1}$ is the q -number.

This key q -Dunkl operator consists of the fundamental operator H_q (which is the q -derivative component serving as a q -analogue of derivative) and a θ -term that handles parity, i.e., equal to 0 for even n and θ for odd n . Clearly, the operator $T_{\theta,q}$ is indeed a lowering operator.

By applying the formal operation of transposition to the q -Dunkl operator $T_{\theta,q}$, which maps the operator to its dual acting on the space of linear functionals, we derive a fundamental property of its adjoint. The resulting computation shows a specific antisymmetry inherent to the operator, which is concisely captured by the following identity:

$${}^tT_{\theta,q} = -T_{\theta,q}. \quad (1)$$

This relation signifies that the transposed operator is precisely the negative of the original operator, a property that influences the analysis of the structure of related orthogonal polynomial sequences. This antisymmetric behavior under transposition is a characteristic feature of this particular operator within the functional analytic framework being developed.

This operator preserves the polynomial space and lowers degrees by one, as in the classical derivative. The q -Dunkl operator $T_{\theta,q}$ represents a significant generalization that encompasses both the q -derivative and the Dunkl operator as special cases. When $\theta = 0$, we recover the pure q -derivative H_q , while if q approaches 1, we obtain the classical Dunkl operator T_θ [13].

In this direction, refs. [3,14] prove that, up to a dilation, the q^2 -analogues of the generalized Hermite and generalized Gegenbauer polynomials are the only symmetric OPS that are q -Dunkl-classical in the sense of Hahn.

2.2. Explicit Polynomial Definitions

The q^2 -analogue of generalized Hermite polynomials provides an example of how classical structures generalize to the q -setting. These polynomials combine the properties of Hermite polynomials with the rich structure of q -calculus. The even and odd degrees separate naturally, with the even-degree polynomials relating to q -Laguerre polynomials with parameter $\mu - \frac{1}{2}$ and the odd-degree polynomials involving an extra factor of t and q -Laguerre polynomials with parameter $\mu + \frac{1}{2}$. This separation reflects the underlying symmetry of the weight function and the action of the q -Dunkl operator. For clarity, we explicitly define the key polynomial families referenced throughout this work:

The q^2 -analogue of generalized Hermite polynomials $\{H_n^{(\mu, q^2)}\}$ are given by

$$\begin{aligned} H_{2n}^{(\mu, q^2)}(t) &= (-1)^n q^{n(n-1)} (q^2; q^2)_n L_n^{(\mu-\frac{1}{2})}(t^2; q^2) \\ H_{2n+1}^{(\mu, q^2)}(t) &= (-1)^n q^{n^2} (q^2; q^2)_n t L_n^{(\mu+\frac{1}{2})}(t^2; q^2) \end{aligned}$$

where $L_n^{(\alpha)}(t; q)$ are the q -Laguerre polynomials and $(a; q)_n$ is the q -Pochhammer symbol. The presence of the factors $(-1)^n q^{n(n-1)}$ and $(-1)^n q^{n^2}$ in the definition of the q^2 -Hermite polynomials ensures that the polynomials are monic and satisfy the appropriate orthogonality relations. These factors compensate for the normalization constants that arise in the q -Laguerre polynomials and maintain the consistency of the TTRR.

The q^2 -analogue of generalized Gegenbauer polynomials $\{S_n^{(\alpha, a, q^2)}\}$ are defined as

$$S_n^{(\alpha, a, q^2)}(t) = \frac{(q^{2\alpha+2}; q^2)_n}{(q^2; q^2)_n} q^{-\alpha n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, q^{n+2\alpha+2a+2}, q^{\alpha+1}e^{i\theta}, q^{\alpha+1}e^{-i\theta} \\ q^{2\alpha+2}, -q^{\alpha+a+2}, -q^{\alpha+a+1} \end{matrix}; q^2, q^2 \right)$$

where ${}_4\phi_3$ is the basic hypergeometric function and $t = \cos \theta$. The q^2 -analogue of generalized Gegenbauer polynomials demonstrates even more sophisticated structure, expressed as basic hypergeometric functions ${}_4\phi_3$. This representation highlights the connections between orthogonal polynomials and q -special functions. The parameters α and a control the analytical properties of the polynomials, while the base q^2 reflects the quadratic nature of the underlying symmetry. The appearance of $q^{\alpha+1}e^{i\theta}$ and $q^{\alpha+1}e^{-i\theta}$ ensures the polynomial nature of the resulting expression and maintains the appropriate symmetry properties.

The definition $t = \cos \theta$ connects these q -polynomials to the theory of q -trigonometric functions and q -Fourier analysis. This connection suggests potential applications in q -harmonic analysis and q -signal processing, where the orthogonality and completeness properties of these polynomials could be applied for series expansions and transform methods. The dense parameter structure allows these polynomials to adapt to various boundary conditions and symmetry requirements that arise in physical applications.

2.3. Basic Tools

Recall that $\mathcal{P}$ was defined to denote the vector space of all polynomials in one variable with complex coefficients. Its algebraic dual space, containing all linear functionals on $\mathcal{P}$, is denoted by $\mathcal{P}'$. For any linear functional $u \in \mathcal{P}'$ and polynomial $\psi \in \mathcal{P}$, the duality pairing is expressed as $\langle u, \psi \rangle$. The moments of the functional u are defined through evaluation on monomials, yielding the sequence

$$(u)_n := \langle u, t^n \rangle, \quad n \in \mathbb{N}_0.$$

The following operator identities are essential for subsequent analyses [6]:

$$H_q(\psi u) = (h_{q^{-1}}\psi)H_q u + q^{-1}(H_{q^{-1}}\psi)u, \quad \psi \in \mathcal{P}, u \in \mathcal{P}'. \quad (2)$$

Equation (2) decomposes the q -difference of a polynomial functional product into two terms: the first involves the scaled polynomial acting on the q -difference of the functional, while the second combines the inverse q -difference of the polynomial with the original functional. The composition of inverse q -difference and q -scaling can be expressed as a scaled q -difference operator:

$$(H_{q^{-1}} \circ h_q)(\psi) = qH_q(\psi), \quad \psi \in \mathcal{P}. \quad (3)$$

Also, note that scaling the product of a polynomial and functional is equivalent to multiplying the inversely scaled polynomial with the scaled functional:

$$h_a(gu) = (h_{a^{-1}}g)(h_a u), \quad g \in \mathcal{P}, u \in \mathcal{P}'. \quad (4)$$

Definition 2 (see [15]). A linear functional u acting on the vector space of polynomials is said to be regular, or alternatively quasi-definite, provided that there exists a sequence of monic polynomials $\{\phi_n\}_{n \geq 0}$ with the degree of each polynomial must be exactly equal to its index, meaning $\deg \phi_n = n$ for all $n \in \mathbb{N}_0$, satisfying the orthogonality relations:

$$\langle u, \phi_n \phi_m \rangle = r_n \delta_{n,m}, \quad n, m \in \mathbb{N}_0,$$

where the constants r_n are nonzero complex numbers whose values are permitted to depend on the index n , and $\delta_{n,m}$ denotes the Kronecker delta. Any polynomial sequence $\{\phi_n\}$ fulfilling these conditions is referred to as the monic orthogonal polynomial sequence (MOPS) corresponding to the regular functional u .

A fundamental characterization states that every MOPS satisfies a TTRR of the form

$$\phi_{n+2}(t) = (t - a_{n+1})\phi_{n+1}(t) - b_{n+1}\phi_n(t), \quad n \in \mathbb{N}_0, \quad (5)$$

with $\phi_0(t) = 1$ and $\phi_1(t) = t - a_0$, where $\{a_n\}$ are complex recurrence coefficients that determine linear shifts of the polynomial argument, and $\{b_{n+1}\}$ are non-zero complex coefficients that govern the coupling between consecutive polynomials, satisfying $b_{n+1} \neq 0$ for all $n \in \mathbb{N}_0$. This non-vanishing condition is equivalent to the non-degeneracy of the associated moment functional, while the recurrence relation provides a complete algebraic characterization of MOPS.

The functional u is normalized when its zeroth moment satisfies $(u)_0 = 1$. In practical applications, this normalization ensures consistency with probabilistic and physical systems where total measure is unitary. All functional forms considered hereafter will be assumed normalized.

The following proposition further investigates the relation between MOPS and TTRR.

Proposition 1 ([16,17]). Consider a monic polynomial sequence $\{\phi_n\}_{n \geq 0}$ where each ϕ_n has degree exactly n , along with its dual sequence $\{u_n\}_{n \geq 0}$ of linear functionals. The following conditions are equivalent:

- (i) The polynomial sequence $\{\phi_n\}$ is orthogonal relative to the initial functional u_0 , i.e., $\langle u_0, \phi_m \phi_n \rangle = 0$ for $m \neq n$ with nonzero norms.
- (ii) Each dual functional u_n is explicitly determined by the initial functional u_0 through the scaling relation:

$$u_n = \langle u_0, \phi_n^2 \rangle^{-1} \phi_n u_0. \quad (6)$$

(iii) The sequence $\{\phi_n\}$ satisfies the TTRR:

$$\begin{cases} \phi_0(t) = 1, \\ \phi_1(t) = t - a_0, \\ \phi_{n+2}(t) = (t - a_{n+1})\phi_{n+1}(t) - b_{n+1}\phi_n(t), \quad n \geq 0, \end{cases} \quad (7)$$

with coefficients a_n and b_{n+1} defined by the moment relations:

$$a_n = \langle u_0, t\phi_n^2 \rangle \langle u_0, \phi_n^2 \rangle^{-1}; \quad b_{n+1} = \langle u_0, \phi_{n+1}^2 \rangle \langle u_0, \phi_n^2 \rangle^{-1} \neq 0, \quad n \in \mathbb{N}_0. \quad (8)$$

This proposition shows that if the sequence $\{\phi_n\}_{n \in \mathbb{N}_0}$ is orthogonal with respect to u_0 , then the form u_n can be written, via dual basis expansion, as a normalized rescaling of the composition of ϕ_n and u_0 , ensuring the duality condition holds. To obtain orthogonality from TTRR, we apply mathematical induction on the recurrence to prove pairwise orthogonality, where the base case holds by a_0 's definition and the inductive step uses the recurrence structure with $b_{n+1} \neq 0$.

It is a well-established fact that the application of a dilation constitutes a symmetry operation for systems characterized by orthogonality relations. More precisely, consider a dilation operator E_a defined by its action on a function f as $(E_a f)(x) = f(ax)$ for some nonzero scaling parameter a . If a sequence of polynomials is orthogonal with respect to a given inner product, then the transformed sequence by E_a will also form an orthogonal set. This invariance arises because the dilation induces a change of variable that scales the underlying measure of orthogonality in a corresponding manner, thereby preserving the fundamental structure of the orthogonality relations. Consequently, the property of orthogonality remains unchanged under such a scaling transformation. The sequence $\{\tilde{\phi}_n(t)\}_{n \in \mathbb{N}_0}$ defined by

$$\tilde{\phi}_n(t) = a^{-n} \phi_n(at) \quad (n \in \mathbb{N}_0; a \neq 0)$$

satisfies the following relations:

$$\tilde{\phi}_0(t) = 1 \quad \text{and} \quad \tilde{\phi}_1(t) = t - \tilde{a}_0$$

and

$$\tilde{\phi}_{n+2}(t) = (t - \tilde{a}_{n+1})\tilde{\phi}_{n+1}(t) - \tilde{b}_{n+1}\tilde{\phi}_n(t),$$

with

$$\tilde{a}_n = a^{-1}a_n \quad \text{and} \quad \tilde{b}_{n+1} = a^{-2}b_{n+1} \quad (n \in \mathbb{N}_0).$$

The polynomial sequence $\{\tilde{\phi}_n(t)\}_{n \geq 0}$ is orthogonal with respect to the linear functional $\tilde{u}$ obtained through the scaling transformation

$$\tilde{u} = h_{a^{-1}}u,$$

where $h_{a^{-1}}$ denotes the argument scaling operator.

A linear functional u is called symmetric when all its odd moments vanish identically:

$$(u)_{2n+1} = \langle u, t^{2n+1} \rangle = 0, \quad n \in \mathbb{N}_0.$$

This moment condition implies that the functional remains invariant under the reflection $t \mapsto -t$. When such a symmetric functional u is regular (quasi-definite), the associated MOPS is said to be symmetric.

If the polynomials alternate in parity, then the linear recurrence coefficients a_n vanish. Conversely, if the recurrence coefficients a_n are zero, then the polynomials must possess the alternating parity property and hence the sequence will be symmetric.

Proposition 2 (see [15]). Let $\{\phi_n(t)\}_{n \in \mathbb{N}_0}$ denote a MOPS which is defined by the recurrence relation (5) and which is orthogonal relative to a given linear functional u . Under these assumptions, the following three statements are equivalent:

- (a) The polynomial sequence $\{\phi_n(t)\}_{n \in \mathbb{N}_0}$ is symmetric.
- (b) $\phi_n(-t) = (-1)^n \phi_n(t)$ ($n \in \mathbb{N}_0$).
- (c) The recurrence coefficients a_n associated with the sequence, as they appear in the recurrence relation (5), are identically zero for all indices $n \in \mathbb{N}_0$.

Remark 1. The symmetric property implies that the sequences of even and odd parts are also orthogonal:

$$\begin{aligned}\phi_{2n}(t) &= Q_n(t^2) \quad (\text{even part}) \\ t^{-1}\phi_{2n+1}(t) &= R_n(t^2) \quad (\text{odd part})\end{aligned}$$

where $\{Q_n\}$ and $\{R_n\}$ are MOPS with respect to modified functionals. This decomposition is utilized in our analysis of q -Dunkl-classical polynomials.

The following lemma establishes an equivalence between two properties of a linear functional u acting on polynomials and its relationship with a polynomial sequence $\{\phi_n\}$.

Lemma 1 ([16,17]). Consider an arbitrary linear functional u belonging to the dual space $\mathcal{P}'$ of polynomials, and let m be any fixed positive integer, the following statements are equivalent:

- (1) Orthogonality condition: The functional u pairs non-zero with ϕ_{m-1} ($\langle u, \phi_{m-1} \rangle \neq 0$) but vanishes on all higher-degree polynomials in the sequence ($\langle u, \phi_n \rangle = 0$ for all $n \geq m$).
- (2) Basis representation: The functional u admits a representation as a finite linear combination of the functionals $\{u_v\}$ which constitute the dual basis. More precisely, it can be written in the form $u = \sum_{v=0}^{m-1} \lambda_v u_v$, where each coefficient λ_v is a uniquely determined complex number, and the last coefficient λ_{m-1} is non-zero, i.e.,

$$\exists \lambda_v \in \mathbb{C}, \quad 0 \leq v \leq m-1, \quad \lambda_{m-1} \neq 0 \quad \text{such that} \quad u = \sum_{v=0}^{m-1} \lambda_v u_v.$$

This equivalence means that the vanishing of u on higher-degree polynomials is equivalent to u being spanned by the first m dual functionals, with the representation necessarily including the $(m-1)$ -th functional. The non-vanishing condition $\lambda_{m-1} \neq 0$ corresponds directly to $\langle u, \phi_{m-1} \rangle \neq 0$, ensuring the representation does not terminate earlier than $m-1$.

Next, we recall the notion of an H_q -semiclassical functional, a concept that will be fundamental for our subsequent analysis. A linear form u is called H_q -semiclassical if the form is regular, meaning it generates a complete MOPS, and there must exist a pair of polynomials, denoted by Φ and Ψ , with the following properties: the polynomial Φ is monic, and its degree is a non-negative integer $\ell \in \mathbb{N}_0$, while the degree of Ψ is a positive integer $p \in \mathbb{N}$. Together, these polynomials are required to satisfy a specific q -deformation of the Pearson-type differential equation, expressed in the distributional sense as

$$(PE): H_q(\Phi u) + \Psi u = 0. \quad (9)$$

Furthermore, the pair (Φ, Ψ) must be admissible. This admissibility condition imposes a specific constraint on the leading coefficient of Ψ ; namely, in the case where the degrees are related by $p = \ell - 1$, if we express the polynomial Ψ in its expanded form as

$$\Psi(t) = a_p t^p + \dots,$$

then it is necessary that the leading coefficient a_p is not equal to $n + 1$ for any natural number $n \in \mathbb{N}$. When these conditions are met, the associated monic orthogonal polynomial sequence $\{\phi_n\}_{n \in \mathbb{N}_0}$ is itself classified as H_q -semiclassical [18,19]. For a semiclassical functional u fulfilling Equation (9), its class, denoted by the integer s , is formally defined as the smallest possible value of the quantity $\max(\deg \Phi - 2, \deg \Psi - 1)$. This class is thus given by

$$s = \min\{\max(\deg \Phi - 2, \deg \Psi - 1)\} \in \mathbb{N}_0$$

where the minimization is performed over all admissible polynomial pairs (Φ, Ψ) that satisfy the fundamental Equation (9). This definition determines s as the smallest possible value obtained by comparing the adjusted degrees of Φ and Ψ across all valid pairs, with $\deg \Phi - 2$ representing the degree reduction for Φ and $\deg \Psi - 1$ corresponding to the degree adjustment for Ψ . The existence of this minimum within the set of non-negative integers is assured due to the fact that the degrees of the polynomials Φ and Ψ are finite, which restricts the possible values of the expression $\max(\deg \Phi - 2, \deg \Psi - 1)$ to a discrete, well-ordered set. A particularly important special case arises when the class of the form is zero, that is, when $s = 0$. In this specific situation, the form u is conventionally referred to as H_q -classical, representing a natural q -generalization of the well-known classical orthogonal polynomials [6]. An essential and useful property of the H_q -semiclassical characterization is its invariance under the operation of a dilation. As established in [19], this characteristic is preserved when the functional is transformed in this manner. To be more precise, if the original functional u satisfies the Pearson-type Equation (9), then the transformed functional $h_{a^{-1}}u$ obtained through dilation will satisfy a corresponding, albeit transformed, Pearson equation. This new equation is given explicitly by

$$H_q(a^{-t}\Phi(at)h_{a^{-1}}u) + a^{1-t}\Psi(at)h_{a^{-1}}u = 0.$$

Furthermore, the recurrence coefficients $\tilde{a}_n$ and $\tilde{b}_{n+1}$ associated with the orthogonal polynomial sequence for this transformed functional maintain the same general form and relationship as those previously described for the original sequence.

Remark 2. The q -semiclassical forms provide a natural environment for studying q -Dunkl-classical polynomials, as they satisfy distributional equations that generalize the Pearson equation to the q -case.

This remark highlights the fundamental relationship between q -Dunkl-classical polynomials and the more general category of q -semiclassical orthogonal polynomials. The q -semiclassical framework offers a natural mathematical context for understanding these polynomials because they satisfy specialized equations that extend the classical Pearson equation into the q -calculus domain. This generalized Pearson equation serves as the q -analogue of the classical differential equation governing traditional orthogonal polynomials but adapted to accommodate both the q -difference operators and the reflection symmetries inherent in Dunkl operator theory. Consequently, the q -semiclassical scheme provides the necessary theoretical foundation for classifying q -Dunkl-classical polynomials and also explains their structural properties, such as the finite expansion observed in the second structure relation.

Definition 3. A MOPS $\{\phi_n\}_{n \in \mathbb{N}_0}$, which is orthogonal with respect to a given linear functional u_0 , is classified as $T_{\theta,q}$ -classical or, alternatively, as q -Dunkl-classical, provided there exists another MOPS, denoted by $\{\phi_n^{[1]}\}_{n \in \mathbb{N}_0}$, such that the action of the q -Dunkl operator $T_{\theta,q}$ on each polynomial ϕ_{n+1} yields a scalar multiple of the corresponding polynomial $\phi_n^{[1]}$ from the derived sequence. This relationship is formally expressed by the operator equation

$$T_{\theta,q}\phi_{n+1} = \theta_{n+1}\phi_n^{[1]} \quad \text{for all } n \in \mathbb{N}_0,$$

where θ_{n+1} is a nonzero complex constant whose value depends on the index n . In such a case, the original linear functional u_0 with respect to which the sequence $\{\phi_n\}$ is orthogonal is itself also termed a q -Dunkl-classical linear functional.

Remark 3. The sequence $\{\phi_n^{[1]}\}$ can be viewed as the q -Dunkl derivative of the original sequence. For q -Dunkl-classical polynomials, this derived sequence remains orthogonal, which is a highly restrictive property characterizing classical families.

For any MPS $\{\phi_n\}_{n \in \mathbb{N}_0}$ we define $\{\phi_n^{[1]}\}_{n \in \mathbb{N}_0}$, as

$$\theta_{n+1}\phi_n^{[1]}(t) := T_{\theta,q}\phi_{n+1}(t), \quad n \in \mathbb{N}_0$$

or equivalently

$$\theta_{n+1}\phi_n^{[1]}(t) := (H_q\phi_{n+1})(t) + \theta(H_{-1}\phi_{n+1})(t), \quad n \in \mathbb{N}_0, \quad (10)$$

where $\theta_{n+1} \neq 0$ is a normalization factor that never vanishes for regular sequences, given by

$$\theta_{n+1} := [n+1]_q + \theta \frac{1 - (-1)^{n+1}}{2}, \quad n \in \mathbb{N}_0.$$

Let $\{u_n\}_{n \in \mathbb{N}_0}$ and $\{u_n^{[1]}\}_{n \in \mathbb{N}_0}$ denote the dual bases in $\mathcal{P}'$ corresponding to the polynomial sequences $\{\phi_n\}_{n \in \mathbb{N}_0}$ and $\{\phi_n^{[1]}\}_{n \in \mathbb{N}_0}$, respectively. Then, by applying Lemma 1 together with Equation (1), we obtain the following fundamental relation:

$$T_{\theta,q}(u_n^{[1]}) = -\theta_{n+1}u_{n+1}, \quad n \in \mathbb{N}_0. \quad (11)$$

The primary objective of the present investigation is to derive a new characterization of symmetric orthogonal q -polynomials of the $T_{\theta,q}$ -classical type. This characterization will be formulated in terms of a distinctive algebraic identity known as a structure relation. More precisely, we aim to demonstrate that every MOPS $\{\phi_n\}_{n \in \mathbb{N}_0}$ which is symmetric and whose elements satisfy a specific TTRR of the form

$$\begin{cases} \phi_0(t) = 1, \\ \phi_1(t) = t, \\ \phi_{n+2}(t) = t\phi_{n+1}(t) - b_{n+1}\phi_n(t), \quad b_{n+1} \neq 0, \quad n \in \mathbb{N}_0, \end{cases} \quad (12)$$

such that the derived sequence $\{\phi_n^{[1]}\}_{n \in \mathbb{N}_0}$, given by (10), is also orthogonal. This structure relation will serve as a necessary and sufficient condition for the sequence to be q -Dunkl-classical, thereby providing a powerful tool for the classification and analysis of such polynomial sequences.

Remark 4. The three-term recurrence coefficients b_n can be computed efficiently using

$$b_n = \frac{\langle u_0, \phi_n^2 \rangle}{\langle u_0, \phi_{n-1}^2 \rangle}$$

The symmetry of $\{\phi_n\}_{n \in \mathbb{N}_0}$ implies the symmetry of $\{\phi_n^{[1]}\}_{n \in \mathbb{N}_0}$. Furthermore, ref. [10] establishes that

$$h_{-1}u_0 = u_0, \quad h_{-1}u_0^{[1]} = u_0^{[1]}, \quad tH_{-1}u_0 = -u_0, \quad tH_{-1}u_0^{[1]} = -u_0^{[1]}. \quad (13)$$

The product rule for $T_{\theta,q}$ is

$$T_{\theta,q}(\psi u) = (h_{q^{-1}}\psi)H_q u + q^{-1}(H_{q^{-1}}\psi)u + \theta\psi(H_{-1}u) + \theta(H_{-1}\psi)(h_{-1}u) \quad (14)$$

For symmetric forms, this simplifies to

$$\begin{aligned} T_{\theta,q}(\psi u) &= (h_{q^{-1}}\psi)T_{\theta,q}u + (T_{\theta,q}\psi)u \\ &\quad + q^{-1}(H_{q^{-1}}\psi)u + \theta\psi(H_{-1}u) - \theta(h_{q^{-1}}\psi)(H_{-1}u) \\ &\quad - (H_q\psi)u, \quad \psi \in \mathcal{P}, u \in \mathcal{P}'. \end{aligned} \quad (15)$$

It follows then that the derived MOPS $\{\phi_n^{[1]}\}$ satisfies the following TTRR:

$$\begin{cases} \phi_0^{[1]}(t) = 1, \\ \phi_1^{[1]}(t) = t, \\ \phi_{n+2}^{[1]}(t) = t\phi_{n+1}^{[1]}(t) - b_{n+1}^{[1]}\phi_n^{[1]}(t), \quad b_{n+1}^{[1]} \neq 0, n \in \mathbb{N}_0. \end{cases}$$

Any symmetric q -Dunkl-classical polynomial sequence $\{\phi_n\}_{n \geq 0}$ can be fully characterized by its orthogonality together with the associated difference equations, as established in the following theorem.

Theorem 1. For any symmetric MOPS $\{\phi_n\}_{n \geq 0}$, the following are equivalent.

- (i) The polynomial sequence $\{\phi_n\}_{n \geq 0}$ belongs to the q -Dunkl-classical family.
- (ii) One can find two specific polynomials, Φ and Ψ , and a sequence of nonzero complex numbers $\{\lambda_n\}_{n \geq 0}$ where the polynomial Φ is required to be an even, monic polynomial with a degree of at most two, while Ψ must be an odd polynomial of degree exactly one. With these polynomials, the sequence satisfies the following operator identity for all integers $n \geq 0$ [1]:

$$\begin{aligned} \Phi(t) \left(T_{\theta,q} \circ h_{q^{-1}} \circ T_{\theta,q} \right) \phi_{n+1}(t) - \Psi(t) \left(h_{q^{-1}} \circ T_{\theta,q} \right) \phi_{n+1}(t) \\ + \theta[n+1](q+1)\Phi(t) \left(H_{-q} \circ h_{q^{-1}} \circ T_{\theta,q} \right) \phi_{n+1}(t) = \lambda_n \phi_{n+1}(t), \end{aligned} \quad (16)$$

- (iii) There exist two polynomials, Φ and Ψ , with the same properties as in (ii) such that the normalized regular linear functional u_0 , with respect to which the sequence is orthogonal, satisfies the following distributional q -Dunkl Pearson equation and a non-degeneracy condition:

$$\begin{cases} T_{\theta,q}(\Phi u_0) + \Psi u_0 = 0 \\ q^{-n}\Psi'(0) - \frac{\Phi''(0)}{2}(\theta_n + q^{-1}[n]_{q^{-1}} - \theta + \theta q^{-n} - [n]_q) \neq 0, \quad n \geq 0. \end{cases} \quad (17)$$

The first equation governs the behavior of the functional under the combined action of the q -Dunkl operator and multiplication by polynomials, while the second, more technical

condition ensures the persistence of regularity for the associated sequence of polynomials. The result establishes a precise relationship between the action of various q -Dunkl and dilation operators on the polynomial ϕ_{n+1} and the polynomial itself, scaled by the complex parameter λ_n .

Remark 5 ([1]). Under conditions of relations (16) and (17), the linear form $u_0^{[1]}$, associated with $\{\phi_n^{[1]}\}_{n \geq 0}$, has the following form:

$$u_0^{[1]} = (1 + \theta)^{-1} b_1^{-1} K \Phi u_0, \quad (18)$$

where K is a nonzero normalization constant, specifically chosen to guarantee that the polynomial Φ possesses the required property of being monic. Furthermore, the polynomial Ψ , which acts as the counterpart to Φ in the governing equation, is explicitly defined by the expression

$$\Psi(t) = K^{-1} (1 + \theta)^2 \phi_1(t). \quad (19)$$

The previous expression for $u_0^{[1]}$ emerges from combining the orthogonality conditions with the recurrence structure. Starting from the relations (16) and (17), we consider the first associated polynomials $\{\phi_n^{[1]}\}$ which satisfy shifted orthogonality relative to $u_0^{[1]}$. Through the three-term recurrence, the functional equation

$$t\phi_n^{[1]}(t) = \phi_{n+1}^{[1]}(t) + a_n^{[1]} \phi_n^{[1]}(t) + b_n^{[1]} \phi_{n-1}^{[1]}(t)$$

must hold. Comparing initial terms with the original sequence shows that $\phi_0^{[1]} = 1$ and $\phi_1^{[1]}(t) = t - a_0^{[1]}$ necessitate the given connection between $u_0^{[1]}$ and Φu_0 . The $(1 + \theta)^{-1}$ factor originates from the perturbation parameter in conditions (16) and (17) that modifies the moment sequence.

3. Characterization of $T_{\theta,q}$ -Classical Symmetric Orthogonal q -Polynomials by a Second Structure Relation

Our main result provides a new theorem that characterizes the internal structure of q -Dunkl-classical polynomials. Theorem 2 represents the characterization of q -Dunkl-classical polynomials through a second structure relation. This theorem establishes that a symmetric MOPS is q -Dunkl-classical if and only if each polynomial $\phi_n(t)$ can be expressed as a finite linear combination of its q -Dunkl derivatives $\phi_k^{[1]}(t)$ for k from $n - \ell$ to n , with $\ell \leq 2$.

Theorem 2. (Second Structure Relation) The symmetric MOPS $\{\phi_n\}_{n \geq 0}$ is q -Dunkl-classical if and only if there exists a family of complex scalars $\{\lambda_{n,k}\}_{n \geq \ell, n-\ell \leq k \leq n}$ and an integer $\ell \in \mathbb{Z}$, $0 \leq \ell \leq 2$, such that

$$\phi_n(t) = \sum_{k=n-\ell}^n \lambda_{n,k} \phi_k^{[1]}(t), \quad n \geq \ell, \quad (20)$$

$$\lambda_{n,n} = 1, \quad n \geq \ell, \quad (21)$$

$$q^{-n} \frac{(1 + \theta)}{\lambda_{2,0}} b_2 - (\theta_n + q^{-1} [n]_{q^{-1}} - \theta + \theta q^{-n} - [n]_q) \neq 0, \quad \text{when } \lambda_{2,0} \neq 0, \quad n \geq 0. \quad (22)$$

with the convention that when $\lambda_{2,0} = 0$, the last condition simplifies to $b_n \neq 0$.

Proof. We provide a detailed proof with explanatory remarks:

Part 1: ($\implies$) Assume that the MOPS $\{\phi_n\}_{n \geq 0}$ is q -Dunkl-classical. Under this assumption, it follows from the definition that one can find two polynomials Φ and Ψ where the polynomial Φ is required to be monic and even, with its degree denoted by ℓ satisfying $0 \leq \ell \leq 2$, and the polynomial Ψ , must be an odd polynomial of degree exactly one. With these polynomials, the canonical regular linear functional u_0 , with respect to which the sequence is orthogonal, is constrained by the distributional Equation (17).

Furthermore, given that for $n \geq 0$, the polynomial ϕ_n is of precise degree n , then there exists a unique sequence of complex coefficients $\{\lambda_{n,k}\}$ for $n \geq \ell$ and $0 \leq k \leq n$. These coefficients allow for the representation of each polynomial ϕ_n as basis elements. Hence, each ϕ_n can be written as

$$\phi_n(t) = \sum_{k=0}^n \lambda_{n,k} \phi_k^{[1]}(t), \quad n \geq \ell. \quad (23)$$

This last expansion follows from fundamental properties of orthogonal polynomial sequences under the q -Dunkl-classical assumption. Since the derived sequence $\{\phi_k^{[1]}\}_{k \geq 0}$ consists of monic orthogonal polynomials, it forms a basis for the vector space of polynomials. Consequently, any monic polynomial of degree n can be uniquely expressed in this basis. The forward direction ($\implies$) assumes the q -Dunkl-classical property and derives the second structure relation. The key insight is that since both $\{\phi_n\}$ and $\{\phi_n^{[1]}\}$ are orthogonal sequences, they form bases for the space of polynomials, and thus each ϕ_n can be expanded in terms of the $\phi_k^{[1]}$. The orthogonality properties then force most of the coefficients $\lambda_{n,k}$ to vanish, leaving only those with k between $n - \ell$ and n .

Since both sequences ϕ_n and $\phi_n^{[1]}$ are monic polynomials of degree n , their leading coefficients coincide. Specifically,

$$\phi_n(t) = t^n + \text{lower-degree terms}, \quad \phi_n^{[1]}(t) = t^n + \text{lower-degree terms}.$$

In the expansion

$$\phi_n(t) = \sum_{k=0}^n \lambda_{n,k} \phi_k^{[1]}(t),$$

the leading term t^n must satisfy

$$t^n = \lambda_{n,n} \cdot t^n + \text{contributions from lower-degree polynomials}.$$

Since polynomials $\phi_k^{[1]}(t)$ for $k < n$ have degrees less than n , they contribute no t^n term. Equating coefficients of t^n gives

$$\lambda_{n,n} = 1.$$

Substituting into (23) yields

$$\phi_n(t) = \phi_n^{[1]}(t) + \sum_{k=0}^{n-1} \lambda_{n,k} \phi_k^{[1]}(t), \quad n \geq \ell. \quad (24)$$

which holds for $n \geq \ell$.

Now, multiply both sides of Equation (24) by $\phi_m^{[1]}(t)$ for $0 \leq m \leq n-1$ with $n \geq 1$:

$$\phi_m^{[1]}(t) \phi_n(t) = \phi_m^{[1]}(t) \phi_n^{[1]}(t) + \sum_{k=0}^{n-1} \lambda_{n,k} \phi_m^{[1]}(t) \phi_k^{[1]}(t).$$

Apply the functional Φu_0 .

$$\langle \Phi u_0, \phi_m^{[1]} \phi_n \rangle = \langle \Phi u_0, \phi_m^{[1]} \phi_n^{[1]} \rangle + \sum_{k=0}^{n-1} \lambda_{n,k} \langle \Phi u_0, \phi_m^{[1]} \phi_k^{[1]} \rangle. \quad (25)$$

For $m < n$, the sequence $\{\phi_k^{[1]}\}_{k \geq 0}$ is orthogonal relative to Φu_0 ; so,

$$\langle \Phi u_0, \phi_m^{[1]} \phi_k^{[1]} \rangle = 0 \quad \text{for } k \neq m, \quad \text{and} \quad \langle \Phi u_0, \phi_m^{[1]} \phi_n^{[1]} \rangle = 0.$$

Since $0 \leq m \leq n-1 < n$, both conditions hold. The sum in (25) simplifies to the single term where $k = m$:

$$\sum_{k=0}^{n-1} \lambda_{n,k} \langle \Phi u_0, \phi_m^{[1]} \phi_k^{[1]} \rangle = \lambda_{n,m} \langle \Phi u_0, (\phi_m^{[1]})^2 \rangle.$$

Substituting this into (25) yields

$$\langle \Phi u_0, \phi_m^{[1]} \phi_n \rangle = \lambda_{n,m} \langle \Phi u_0, (\phi_m^{[1]})^2 \rangle.$$

The last expression simplifies to

$$\lambda_{n,m} = \frac{\langle u_0, (\Phi \phi_m^{[1]}) \phi_n \rangle}{\langle \Phi u_0, (\phi_m^{[1]})^2 \rangle}. \quad (26)$$

From the expression (26), and since $\deg(\Phi \phi_m^{[1]}) = m + \ell$, the orthogonality of $\{\phi_n\}_{n \geq 0}$ forces the numerator to vanish when the denominator is non-zero

$$\langle \Phi u_0, \phi_m^{[1]} \phi_n \rangle = 0, \quad 0 \leq m \leq n - \ell - 1.$$

Using the functional property $\langle \Phi u_0, \psi \rangle = \langle u_0, \Phi \psi \rangle$,

$$\langle u_0, (\Phi \phi_m^{[1]}) \phi_n \rangle = 0, \quad 0 \leq m + \ell \leq n - 1,$$

Since polynomial multiplication is commutative,

$$\Phi(\phi_m^{[1]} \phi_n) = (\Phi \phi_m^{[1]}) \phi_n,$$

we obtain

$$\langle u_0, (\Phi \phi_m^{[1]}) \phi_n \rangle = 0, \quad 0 \leq m \leq n - \ell - 1.$$

Rewriting the constraint $0 \leq m \leq n - \ell - 1$ as $0 \leq m + \ell \leq n - 1$ yields

$$\langle u_0, (\Phi \phi_m^{[1]}) \phi_n \rangle = 0, \quad 0 \leq m + \ell \leq n - 1.$$

So,

$$\lambda_{n,m} = 0, \quad 0 \leq m \leq n - \ell - 1.$$

Consequently, (24) becomes

$$\phi_n(t) = \phi_n^{[1]}(t) + \sum_{k=n-\ell}^{n-1} \lambda_{n,k} \phi_k^{[1]}(t), \quad n \geq \ell. \quad (27)$$

The final step is the proof of (22). Under the assumption that $\lambda_{2,0} \neq 0$, the case $n = 2$ of (27) gives

$$\phi_2(t) = \phi_2^{[1]}(t) + \lambda_{2,0}\phi_0^{[1]}(t). \quad \left(\text{since } \{\phi_n\}_{n \geq 0} \text{ and } \{\phi_n^{[1]}\}_{n \geq 0} \text{ are symmetric.} \right)$$

Apply the functional $u_0^{[1]}$:

$$\langle u_0^{[1]}, \phi_2 \rangle = \langle u_0^{[1]}, \phi_2^{[1]} \rangle + \lambda_{2,0} \langle u_0^{[1]}, \phi_0^{[1]} \rangle.$$

The sequence $\{\phi_n^{[1]}\}_{n \geq 0}$ is orthogonal with respect to $u_0^{[1]}$, so by normalization we get $\langle u_0^{[1]}, \phi_0^{[1]} \rangle = 1$, and by orthogonality we have $\langle u_0^{[1]}, \phi_2^{[1]} \rangle = 0$. Substituting these values, we get

$$\langle u_0^{[1]}, \phi_2 \rangle = 0 + \lambda_{2,0} \cdot 1 = \lambda_{2,0}.$$

From (11) and the monicity of Φ , we obtain

$$\langle u_0^{[1]}, \phi_2 \rangle = \frac{\Phi''(0)}{2} (1 + \theta)^{-1} b_1^{-1} K r_2 = \frac{\Phi''(0)}{2} (1 + \theta)^{-1} b_2 K.$$

Then,

$$\frac{\Phi''(0)}{2} K = \frac{(1 + \theta) \lambda_{2,0}}{b_2}. \quad (28)$$

Substitution of (28) in (19) gives

$$\Psi(t) = \frac{\Phi''(0)}{2} \frac{(1 + \theta) b_2}{\lambda_{2,0}} \phi_1(t).$$

Therefore,

$$\Psi'(0) = \frac{\Phi''(0)}{2} \frac{(1 + \theta) b_2}{\lambda_{2,0}}.$$

Thus, (22) follows immediately from the second equality in (17), completing the proof of (20)–(22).

Part 2: ($\Leftarrow$) This backward direction uses the second structure relation to establish the q -Dunkl-classical property. The crucial step is showing that the existence of such a finite expansion implies that the derived functional $u_0^{[1]}$ can be written as a finite combination of the original dual functionals u_n , which in turn implies the existence of a polynomial Φ such that $u_0^{[1]} = k\Phi u_0$. This connection is fundamental to the q -Dunkl-classical property. Assume that there exist an integer ℓ , $0 \leq \ell \leq 2$ and a sequence $\{\lambda_{n,k}\}_{n \geq \ell, n-\ell \leq k \leq n}$ of complex numbers such that (20)–(22) hold.

Let $\{\phi_n\}_{n \geq 0}$ and $\{\phi_n^{[1]}\}_{n \geq 0}$ be sequences of monic polynomials with $\{u_n\}_{n \geq 0}$ and $\{u_n^{[1]}\}_{n \geq 0}$ be their respective dual sequences. Using (20) and (21) for $n \geq \ell + 1$, we have

$$\langle u_0^{[1]}, \phi_n \rangle = \langle u_0^{[1]}, \phi_n^{[1]} \rangle + \sum_{k=n-\ell}^{n-1} \lambda_{n,k} \langle u_0^{[1]}, \phi_k^{[1]} \rangle = 0.$$

It therefore follows from Lemma 1 that complex numbers α_i , $i \in 0, \dots, \ell$, exist for which

$$u_0^{[1]} = \sum_{i=0}^{\ell} \alpha_i u_i, \quad 0 \leq \ell \leq 2.$$

Or, equivalently,

$$u_0^{[1]} = \alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2. \quad (29)$$

By (6), the previous equation becomes

$$u_0^{[1]} = (\alpha_0 + \alpha_1 r_1^{-1} \phi_1 + \alpha_2 r_2^{-1} \phi_2) u_0.$$

Therefore, as a direct consequence of the preceding analysis, we deduce the existence of a polynomial function Φ such that its total degree does not exceed the second order, with

$$u_0^{[1]} = k\Phi u_0, \quad (30)$$

such that

$$k\Phi = \alpha_0 + \alpha_1 r_1^{-1} \phi_1 + \alpha_2 r_2^{-1} \phi_2, \quad (31)$$

where the non-zero constant k is chosen to make Φ monic.

The polynomial Φ has an even degree, and since we have $\phi_1(t) = \phi_1^{[1]}(t) = t$, then

$$0 = \langle u_0^{[1]}, \phi_1^{[1]} \rangle = k(\alpha_0 \langle u_0, \phi_1 \rangle + \alpha_1 r_1^{-1} \langle u_0, \phi_1^2 \rangle + \alpha_2 r_2^{-1} \langle u_0, \phi_1 \phi_2 \rangle) = k\alpha_1.$$

Hence, $\alpha_1 = 0$.

The explicit form of the polynomial ϕ_2 is given by the quadratic expression $\phi_2(t) = t^2 - b_1$, which implies using (31) that Φ is even.

On the contrary, by selecting the parameter value $n = 0$ and substituting it directly into Equation (11), we obtain the following identity:

$$T_{\theta,q} u_0^{[1]} = -(1 + \theta) u_1$$

Substitution of (30) in the previous equation gives (17), with

$$\Psi(t) = k^{-1} b_1^{-1} (1 + \theta) \phi_1(t).$$

To finalize the proof of the stated proposition, it remains to verify that the second identity contained within Equation (17) is indeed satisfied. From the application of Equation (29), it follows directly that

$$\alpha_2 = \langle u_0^{[1]}, \phi_2 \rangle.$$

But, from (20) and (21), where $n = 2$, we have

$$\phi_2(t) = \phi_2^{[1]}(t) + \lambda_{2,0} \phi_0^{[1]}(t).$$

Thus,

$$\alpha_2 = \lambda_{2,0}.$$

Building upon the relation provided in (29) and taking into account the normalization conditions imposed on the functions u_0 and $u_0^{[1]}$, we arrive at the consequent expression:

$$\alpha_0 = 1.$$

Therefore, (31) becomes

$$k\Phi(t) = 1 + \lambda_{2,0} r_2^{-1} \phi_2(t). \quad (32)$$

Therefore, the subsequent analysis splits into two distinct scenarios that must be considered separately: the situation where the parameter $\lambda_{2,0}$ is equal to zero, and the alternative situation where $\lambda_{2,0}$ is not equal to zero.

The first case: $\lambda_{2,0} = 0$. Under this condition, the degree of the polynomial Φ is necessarily zero. It follows that its second derivative at zero vanishes, meaning $\Phi''(0) = 0$. Furthermore, given that Φ is a monic polynomial, we conclude that the constant k must be equal to one. Consequently, the general expression simplifies to

$$q^{-n}\Psi'(0) - \frac{\Phi''(0)}{2}(\theta_n + q^{-1}[n]_{q^{-1}} - \theta + \theta q^{-n} - [n]_q) = q^{-n}\Psi'(0) = q^{-n}b_1^{-1}(1 + \theta) \neq 0, n \geq 0.$$

This expression remains non-zero for all non-negative integers n .

The second case: $\lambda_{2,0} \neq 0$. Here, the degree of Φ is two. Since Φ is monic, its second derivative at zero satisfies $\frac{\Phi''(0)}{2} = 1$. By comparing degrees in Equation (32), we determine that

$$k = \lambda_{2,0} r_2^{-1}.$$

Substituting these results gives the expression

$$\begin{aligned} q^{-n}\Psi'(0) - \frac{\Phi''(0)}{2}(\theta_n + q^{-1}[n]_{q^{-1}} - \theta + \theta q^{-n} - [n]_q) \\ = q^{-n} \frac{(1 + \theta)}{\lambda_{2,0}} b_2 - (\theta_n + q^{-1}[n]_{q^{-1}} - \theta + \theta q^{-n} - [n]_q) \neq 0, \quad n \geq 0 \end{aligned}$$

This quantity is non-zero for all $n \geq 0$, as guaranteed by (22).

Having shown that the relevant expression is non-vanishing in both cases, and by reference to relation (17), we conclude that the polynomial sequence $\{\phi_n\}_{n \geq 0}$ is q -Dunkl-classical. $\square$

Building upon the foundational results established in the preceding theorem, we will derive the second structure relation for two significant classes of orthogonal polynomials: the q^2 -analogue of generalized Hermite polynomials and the corresponding q^2 -analogue of generalized Gegenbauer polynomials. This derivation will use the operator formulation and recurrence properties developed earlier to characterize the algebraic relationships governing these q -deformed polynomials. The second structure relation provides essential understanding of the connection coefficients between consecutive polynomials and shows structural symmetries particular to these q -analogues. Our method applies the operator commutation relations from the main results to the weight functions and moment conditions to characterize these two polynomial families.

We first parameterize the even polynomial Φ with curvature term and write $\Phi(t) = \frac{\Phi''(0)}{2}t^2 + \Phi(0)$. Consider the linear odd polynomial $\Psi(t) = \Psi'(0)t$ and let $\{\phi_n\}_{n \geq 0}$ be a symmetric q -Dunkl-classical MOPS, such that its associated regular form u_0 satisfies (17). So, from (20) and (21) we have the following general expansion:

$$\phi_n(t) = \phi_n^{[1]}(t) + \lambda_{n,n-1}\phi_{n-1}^{[1]}(t) + \lambda_{n,n-2}\phi_{n-2}^{[1]}(t), \quad n \geq \ell, \quad (33)$$

where $\ell = \deg \Phi \leq 2$ determines the number of terms in the expansion.

Since the sequences $\{\phi_n\}_{n \geq 0}$ and $\{\phi_n^{[1]}\}_{n \geq 0}$ are symmetric, the odd-index coefficients must vanish due to parity conservation:

$$\lambda_{n,n-1} = 0, \quad n \geq \ell. \quad (34)$$

leaving only the $n - 2$ term in the expansion beyond the leading coefficient.

The coefficient $\lambda_{n,n-2}$ is given by the following rational expression derived from orthogonality conditions:

$$\lambda_{n,n-2} = \frac{\frac{\Phi''(0)}{2} \theta_{n-1}}{q^{-n+2} \Psi'(0) - \frac{\Phi''(0)}{2} (\theta_{n-2} + q^{-1}[n-2]_{q-1} - \theta + \theta q^{-n+2} - [n-2]_q)} b_n, \quad n \geq \ell, \quad (35)$$

where the denominator contains the critical non-degeneracy condition from Theorem 1, with the convention $\lambda_{0,n-2} = 0$.

Indeed, from (26), we have the general coefficient formula:

$$\lambda_{n,n-2} = \frac{\langle u_0, (\Phi \phi_{n-2}^{[1]}) \phi_n \rangle}{\langle \Phi u_0, (\phi_{n-2}^{[1]})^2 \rangle}, \quad n \geq \ell.$$

The numerator simplifies because $\Phi \phi_{n-2}^{[1]}$ has leading term proportional to t^n :

$$\Phi(t) \phi_{n-2}^{[1]}(t) = \frac{\Phi''(0)}{2} t^n + \text{lower degree terms.}$$

and by orthogonality only the highest degree term contributes. From the orthogonality of $\{\phi_n\}_{n \geq 0}$ with respect to u_0 , we get

$$\langle u_0, (\Phi \phi_{n-2}^{[1]}) \phi_n \rangle = \frac{\Phi''(0)}{2} \langle u_0, t^n \phi_n \rangle = \frac{\Phi''(0)}{2} \langle u_0, \phi_n^2 \rangle, \quad n \geq \ell.$$

From the identity provided in (15) and the established symmetry of the function Φu_0 , we proceed by applying the operator calculus which gives the relation:

$$\begin{aligned} \langle \Phi u_0, (\phi_{n-2}^{[1]})^2 \rangle &= -\frac{1}{\theta_{n-1}} \langle T_{\theta,q}(\phi_{n-2}^{[1]} \Phi u_0), \phi_{n-1} \rangle = \\ &= -\frac{1}{\theta_{n-1}} \langle (h_{q^{-1}} \phi_{n-2}^{[1]}) T_{\theta,q}(\Phi u_0) + (T_{\theta,q} \phi_{n-2}^{[1]}) \Phi u_0 + q^{-1} (H_{q^{-1}} \phi_{n-2}^{[1]}) \Phi u_0 \\ &\quad + \theta \phi_{n-2}^{[1]} (H_{-1} \Phi u_0) - \theta (h_{q^{-1}} \phi_{n-2}^{[1]}) (H_{-1} \Phi u_0) - (H_q \phi_{n-2}^{[1]}) \Phi u_0, \phi_{n-1} \rangle. \end{aligned}$$

Substituting the Pearson equation $T_{\theta,q}(\Phi u_0) = \Psi u_0$ transforms the first operator term to

$$\begin{aligned} \langle \Phi u_0, (\phi_{n-2}^{[1]})^2 \rangle &= \frac{1}{\theta_{n-1}} \langle (h_{q^{-1}} \phi_{n-2}^{[1]}) \Psi u_0 - T_{\theta,q}(\phi_{n-2}^{[1]}) \Phi u_0 - q^{-1} (H_{q^{-1}} \phi_{n-2}^{[1]}) \Phi u_0 \\ &\quad - \theta \phi_{n-2}^{[1]} (H_{-1} \Phi u_0) + \theta (h_{q^{-1}} \phi_{n-2}^{[1]}) (H_{-1} \Phi u_0) + (H_q \phi_{n-2}^{[1]}) \Phi u_0, \phi_{n-1} \rangle. \end{aligned}$$

Orthogonality of $\{\phi_n\}_{n \geq 0}$ with respect to u_0 eliminates all terms except those proportional to t^{n-1} , leaving

$$\begin{aligned} &\langle \Phi u_0, (\phi_{n-2}^{[1]})^2 \rangle \\ &= \frac{q^{-n+2} \Psi'(0) - \frac{\Phi''(0)}{2} (\theta_{n-2} + q^{-1}[n-2]_{q-1} - \theta + \theta q^{-n+2} - [n-2]_q)}{\theta_{n-1}} \langle u_0, t^{n-1} \phi_{n-1} \rangle \\ &= \frac{q^{-n+2} \Psi'(0) - \frac{\Phi''(0)}{2} (\theta_{n-2} + q^{-1}[n-2]_{q-1} - \theta + \theta q^{-n+2} - [n-2]_q)}{\theta_{n-1}} \langle u_0, \phi_{n-1}^2 \rangle, \quad n \geq \ell. \end{aligned}$$

Combining numerator and denominator through the squared norm ratio $b_n = \langle u_0, \phi_n^2 \rangle / \langle u_0, \phi_{n-1}^2 \rangle$ yields Equation (35).

Substituting the zero coefficient (34) and the expression for $\lambda_{n,n-2}$ (35) into the expansion (33) gives the final simplified structure relation:

$$\phi_n(t) = \phi_n^{[1]}(t) + \frac{\frac{\Phi''(0)}{2}\theta_{n-1}}{q^{-n+2}\Psi'(0) - \frac{\Phi''(0)}{2}(\theta_{n-2} + q^{-1}[n-2]_{q^{-1}} - \theta + \theta q^{-n+2} - [n-2]_q)} b_n \phi_{n-2}^{[1]}(t), \quad n \geq \ell. \quad (36)$$

Now, we consider the following corollary which shows that the q^2 -analogue of generalized Hermite polynomials are invariant under the q -Dunkl derivative operation, a remarkable property unique to the q^2 -analogue of generalized Hermite case.

Corollary 1. (1) The family of polynomials $\{H_n^{(\mu, q^2)}\}_{n \geq 0}$, which constitutes a q^2 -analogue of the generalized Hermite polynomials, can be uniquely characterized by a specific structural relation of the second kind, given by

$$H_n^{(\mu, q^2)}(t) = \left(H_n^{(\mu, q^2)}\right)^{[1]}(t), \quad n \geq 0. \quad (37)$$

(2) The family of polynomials $\{S_n^{(\alpha, \mu - \frac{1}{2}, q^2)}\}_{n \geq 0}$, which constitutes a q^2 -analogue of the generalized Gegenbauer polynomials, can be uniquely characterized by a specific structural relation of the second kind, given by

$$S_n^{(\alpha, \mu - \frac{1}{2}, q^2)}(t) = \left(S_n^{(\alpha, \mu - \frac{1}{2}, q^2)}\right)^{[1]}(t) - \frac{\theta_{n-1}}{q^{-n+3}(q+1)(\alpha+1) + \theta_{n-2} + q^{-1}[n-2]_{q^{-1}} - \theta + \theta q^{-n+2} - [n-2]_q} b_n \left(S_{n-2}^{(\alpha, \mu - \frac{1}{2}, q^2)}\right)^{[1]}(t), \quad n \geq 2, \quad (38)$$

where

$$b_{2n+1} = q^{2n} \frac{(\alpha + \mu + \frac{3}{2} + [n-1]_{q^2})(\mu + \frac{1}{2} + [n]_{q^2})}{(\alpha + \mu + \frac{3}{2} + [2n-1]_{q^2})(\alpha + \mu + \frac{3}{2} + [2n]_{q^2})},$$

$$b_{2n+2} = q^{2n} [n+1]_{q^2} \frac{\alpha + \mu + \frac{3}{2} - (\mu + \frac{1}{2})q^{2n} + [n]_{q^2}}{(\alpha + \mu + \frac{3}{2} + [2n]_{q^2})(\alpha + \mu + \frac{3}{2} + [2n+1]_{q^2})}, \quad n \in \mathbb{N}_0.$$

Proof. (1) The q^2 -analogue of generalized Hermite polynomials $\{H_n^{(\mu, q^2)}\}_{n \geq 0}$ satisfies (7) with (see [18]):

$$\begin{cases} a_n = 0, \\ b_{2n+1} = q^{2n} \left([n]_{q^2} + \mu + \frac{1}{2}\right), \quad n \geq 0, \\ b_{2n+2} = q^{2n} [n+1]_{q^2}. \end{cases} \quad (39)$$

showing the recurrence coefficients alternate between linear and q -integer forms based on parity.

Orthogonality is ensured by the following regularity condition:

$$\mu \neq -[n]_{q^2} - \frac{1}{2}, \quad n \geq 0.$$

This sequence of polynomials is known to be q -Dunkl-classical. Its associated form, denoted $\mathcal{H}(\mu, q^2)$, satisfies the functional Equation (17) with specific parameters as established in the literature (see [1,3])

$$\Phi(t) = 1, \Psi(t) = q(q+1)t. \quad (40)$$

where $\Phi = 1$ (degree 0) explains the absence of lower terms in the expansion. So, using (39) and (40) the proof of (37) is an immediate consequence of (36).

- (2) The q^2 -analogue of generalized Gegenbauer polynomials $\left\{ S_n^{(\alpha, \mu - \frac{1}{2}, q^2)} \right\}_{n \geq 0}$ satisfies (7) with (see [18])

$$\begin{cases} a_n = 0, \\ b_{2n+1} = q^{2n} \frac{(\alpha + \mu + \frac{3}{2} + [n-1]_{q^2})(\mu + \frac{1}{2} + [n]_{q^2})}{(\alpha + \mu + \frac{3}{2} + [2n-1]_{q^2})(\alpha + \mu + \frac{3}{2} + [2n]_{q^2})}, & n \geq 0, \\ b_{2n+2} = q^{2n} [n+1]_{q^2} \frac{\alpha + \mu + \frac{3}{2} - (\mu + \frac{1}{2})q^{2n} + [n]_{q^2}}{(\alpha + \mu + \frac{3}{2} + [2n]_{q^2})(\alpha + \mu + \frac{3}{2} + [2n+1]_{q^2})}, \end{cases} \quad (41)$$

where the recurrence coefficients are rational functions of q -integers with parameter dependencies.

Multiple regularity conditions that ensure the denominator never vanishes are given by

$$\begin{cases} \alpha + \mu - \frac{1}{2} \neq \frac{3-2q^2}{q^2-1}, & \alpha + \mu \neq -[n]_{q^2} - \frac{3}{2}, \\ \alpha + \mu + \frac{3}{2} - (\mu + \frac{1}{2})q^{2n} + [n]_{q^2} \neq 0, & n \geq 0, \\ \mu \neq -[n]_{q^2} - \frac{1}{2}, & \mu \neq \frac{1}{q(q+1)} - \frac{1}{2}. \end{cases}$$

This sequence is q -Dunkl-classical and its associated form $\mathcal{G}(\alpha, \mu - \frac{1}{2}, q^2)$ satisfies (17) (see [1,3]), with

$$\Phi(t) = t^2 - 1, \Psi(t) = -q(q+1)(\alpha+1)t, \quad (42)$$

explaining the two-term expansion in the structure relation.

Then, by direct substitution of $\Phi''(0) = 2$, $\Psi'(0) = -q(q+1)(\alpha+1)$, and the recurrence coefficients, Equation (38) is deduced from (36). $\square$

This work distinguishes itself from earlier characterizations of symmetric q -Dunkl-classical orthogonal polynomials by establishing a second structure relation as a defining property. Previous research had successfully characterized these polynomials through several other approaches: a first structure relation [1], which expresses the q -Dunkl derivative of a polynomial as a linear combination of lower-degree polynomials in the same sequence; a Pearson-type equation for the associated linear functional [2], drawing a direct analogy to the classical continuous case; and a second-order q -difference-differential equation that the polynomials satisfy [1]. While these approaches are fundamental to the theory, the novel second structure relation derived in this paper, given by

$$\phi_n(t) = \sum_{k=n-\ell}^n \lambda_{n,k} \phi_k^{[1]}(t),$$

operates in the converse direction. It expresses the original polynomial itself as a finite linear combination of its own q -Dunkl derivatives. This provides a powerful inversion of the problem. Rather than analyzing the action of the operator on the polynomials, this new method allows one to reconstruct the polynomials from their images under the operator, offering a perspective for verification and classification. The efficacy of this new characterization is demonstrated through its application to derive explicit, closed-form

second structure relations for the fundamental examples of the q^2 -analogue of generalized Hermite and the q^2 -analogue of generalized Gegenbauer polynomials, thereby offering a complete characterization and uniqueness result; meaning any polynomial sequence satisfying the standard three-term recurrence alongside this specific structure must be exactly the identified q^2 -analogues of the generalized Hermite or Gegenbauer polynomials. This shifts the definition from recurrences or Rodrigues formulas to a more powerful, operator-based definition centered on their behavior under the q -derivative. This allows mathematicians to definitively classify these polynomials and apply them to advanced problems in quantum calculus and mathematical physics such as the representation theory of quantum groups like $SU_q(1, 1)$.

Remark 6. We conclude the section with the following observations.

Part I. In Case 1 of Corollary 1, the second structure relation implies that the q^2 -analogue $\mathcal{H}_n^{(\mu, q^2)}$ of the generalized Hermite q -polynomials forms an Appell sequence with respect to the q -Dunkl operator (see [20]).

Part II. The general problem of characterizing the q -Dunkl-classical MOPS remains open. To the best of our knowledge, only the symmetric case has been investigated and fully resolved thus far (see [3]).

4. Computational Feasibility and Examples

This section demonstrates the practical computability of the recurrence and structure coefficients. While the preceding sections establish the theoretical results and explicit formulas, here we validate their feasibility by showing how these coefficients can be calculated efficiently for both small and large indices n . The importance of this computational perspective lies in bridging the abstract structural results with concrete numerical implementations, confirming that the theory can be used in real-world applications such as spectral methods, q -difference equations, and problems in mathematical physics. We begin by describing the computational steps for b_n and $\lambda_{n,n-2}$, and then provide explicit Python 3.10.12 code together with numerical tables to illustrate the procedure through the q^2 -analogue of generalized Gegenbauer polynomials. This highlights both the stability and the efficiency of the approach.

4.1. Efficient Computation of b_n and $\lambda_{n,n-2}$

For any integer $m \geq 0$ and base $b \in \{q, q^2, q^{-1}\}$, the q -numbers are defined by

$$[m]_b = \frac{b^m - 1}{b - 1}. \quad (43)$$

The recurrence coefficients b_n are obtained from the explicit formulas in Corollary 1, which are rational functions of q -numbers and the model parameters. The Appell factors take the form

$$\theta_n = [n+1]_q + \theta \frac{1 - (-1)^{n+1}}{2}, \quad (44)$$

while the structure coefficient for $n \geq 2$ is

$$\lambda_{n,n-2} = - \frac{q^{-n+3}(q+1)(\alpha+1) + \theta_{n-2} + q^{-1}[n-2]_{q^{-1}} - \theta + \theta q^{-n+2} - [n-2]_q}{\theta_{n-1} b_n}. \quad (45)$$

These computations require only basic arithmetic with q -numbers and they remain numerically stable.

4.2. Illustrative Python Code and Results

The Python code below implements these formulas to compute b_n , θ_{n-1} , and $\lambda_{n,n-2}$ for the q^2 -analogue of generalized Gegenbauer polynomials and prints values for $n = 2, \dots, 10$.

```
import numpy as np

def q_number(n, q):
    if q == 1:
        return n
    return (q**n - 1) / (q - 1)

def compute_b_n(n, q, alpha, mu):
    q2 = q**2
    if n % 2 == 1:
        k = (n - 1) // 2
        num = (alpha + mu + 1.5 + q_number(k-1, q2))
            * (mu + 0.5 + q_number(k, q2))
        den = (alpha + mu + 1.5 + q_number(2*k-1, q2))
            * (alpha + mu + 1.5 + q_number(2*k, q2))
        return (q2**k) * num / den
    else:
        k = n // 2 - 1
        term1 = alpha + mu + 1.5 - (mu + 0.5)*(q2**k) + q_number(k, q2)
        num = q_number(k+1, q2) * term1
        den = (alpha + mu + 1.5 + q_number(2*k, q2))
            * (alpha + mu + 1.5 + q_number(2*k+1, q2))
        return (q2**k) * num / den

def compute_lambda_n(n, q, alpha, mu, theta):
    b_n = compute_b_n(n, q, alpha, mu)
    theta_n1 = q_number(n, q) + theta*(1 - (-1)**n)/2
    theta_n2 = q_number(n-1, q) + theta*(1 - (-1)**(n-1))/2
    term1 = q**(-n+3)*(q+1)*(alpha+1)
    term2 = theta_n2
    term3 = q**(-1) * q_number(n-2, 1/q)
    term4 = -theta + theta*q**(-n+2) - q_number(n-2, q)
    den = term1 + term2 + term3 + term4
    return - (theta_n1 * b_n) / den

q, alpha, mu, theta = 0.8, 2, 1, 0.5
for n in range(2, 11):
    print(n, compute_b_n(n,q,alpha,mu),
          q_number(n,q)+theta*(1-(-1)**n)/2,
          compute_lambda_n(n,q,alpha,mu,theta))
```

This code implements the explicit formulas derived in Theorem 2 and Corollary 1 to compute recurrence coefficients b_n and structure relation coefficients $\lambda_{n,n-2}$ for the q^2 -analogue of generalized Gegenbauer polynomials. It first defines a function for q -numbers, then uses them to evaluate b_n through separate expressions for even and odd indices, and then determines $\lambda_{n,n-2}$ by combining b_n with the Appell-type factors θ_{n-1} and θ_{n-2} . For given parameters (q, α, μ, θ) , the code prints a table of values for $n = 2$

to $n = 10$, showing b_n , θ_{n-1} , and $\lambda_{n,n-2}$. The output confirms the decay of b_n . It also shows that the coefficients $\lambda_{n,n-2}$ are negative and diminish in magnitude, indicating the decreasing contribution of lower-degree terms in the structure relation, which confirms the computational feasibility and numerical stability of the theoretical results. Since each coefficient is computable in constant time, the approach scales efficiently even for large n (Table 1).

Table 1. Computed values for $q = 0.8$, $\alpha = 2$, $\mu = 1$, $\theta = 0.5$.

n	b_n	θ_{n-1}	$\lambda_{n,n-2}$
2	0.121212	1.8000	−0.03749
3	0.213207	2.9400	−0.08275
4	0.118494	2.9520	−0.03185
5	0.158555	3.8616	−0.04314
6	0.097572	3.6893	−0.01857
7	0.115507	4.4514	−0.02074
8	0.073570	4.1611	−0.00936
9	0.081303	4.8289	−0.00945
10	0.052443	4.4631	−0.00436
100	1.2148×10^{-10}	5.0000	-1.9688×10^{-20}

This computational approach enables efficient construction of orthogonal polynomial bases for spectral methods, numerical solutions of q -difference equations, and approximation schemes in analysis. In particular, one important application arises in quantum algebra computations and in q -deformed quantum systems, where explicit polynomial coefficients are essential for studying quantum oscillators, energy spectra, and coherent states. The explicit computability of these coefficients thus provides a direct bridge between abstract theory and concrete models in mathematical physics.

Now, through constructed concrete examples, we demonstrate that the sequence of monomials satisfies the Hermite-type structure relation under a common definition of the $[1]$ operation, and for the Gegenbauer-type polynomials, select specific parameter values and determine appropriate values for the undefined parameters to show that the polynomials satisfy the structure relation for $n = 2$. This verification serves to illustrate the practical implications of these characterizing relations and provides insight into the structure of q -deformed orthogonal polynomial systems.

1. q^2 -analogue of Generalized Hermite Polynomials. In the theory of orthogonal polynomials, the $[1]$ operation typically denotes a transformation that generates a related sequence of polynomials, often called the co-recursive or first-associated sequence. A common definition for a monic polynomial sequence $\{P_n(x)\}$ is

$$(P_n)^{[1]}(x) = \frac{P_{n+1}(x) + a_n P_n(x)}{x - c}$$

for specific constants a_n and c derived from the recurrence coefficients. The corollary states that for this specific q^2 -analogue, the $[1]$ operation acts as the identity operator. This is an important property that characterizes the family.

The monic q^2 -analogue of generalized Hermite polynomials is defined by the following three-term recurrence relation:

$$xH_n^{(\mu, q^2)}(x) = H_{n+1}^{(\mu, q^2)}(x) + \gamma_n^{(\mu, q)} H_{n-1}^{(\mu, q^2)}(x)$$

with initial conditions

$$\begin{aligned} H_{-1}^{(\mu, q^2)}(x) &= 0 \\ H_0^{(\mu, q^2)}(x) &= 1 \end{aligned}$$

The recurrence coefficients are given by

$$\gamma_{2m}^{(\mu, q)} = q^{-2m+1} [2m]_{q^2}, \quad \gamma_{2m+1}^{(\mu, q)} = q^{-2m} [2\mu + 1 + 2m]_{q^2}$$

where $[k]_{q^2} = \frac{1 - q^{2k}}{1 - q^2}$ is the q^2 -number. For this specific family of polynomials, the precise definition of the $^{[1]}$ operation that satisfies the invariance property is

$$\left(H_n^{(\mu, q^2)} \right)^{[1]}(x) = \frac{1}{1 + q^{2n}} \left(H_n^{(\mu, q^2)}(x) + q^{2n} H_n^{(\mu, q^2)}(qx) \right)$$

Let us verify this with specific parameter values using exact fractions:

$$q = \frac{7}{10}, \quad \mu = 1, \quad q^2 = \left(\frac{7}{10} \right)^2 = \frac{49}{100}, \quad 1 - q^2 = 1 - \frac{49}{100} = \frac{51}{100}$$

Step 1: Compute Recurrence Coefficients. First, compute the relevant q^2 -numbers:

$$[2]_{q^2} = \frac{1 - q^4}{1 - q^2} = \frac{1 - \left(\frac{7}{10} \right)^4}{\frac{51}{100}} = \frac{1 - \frac{2401}{10000}}{\frac{51}{100}} = \frac{\frac{7599}{10000}}{\frac{51}{100}} = \frac{7599}{10000} \cdot \frac{100}{51} = \frac{7599}{5100} = \frac{2533}{1700}$$

$$[4]_{q^2} = \frac{1 - q^8}{1 - q^2} = \frac{1 - \left(\frac{7}{10} \right)^8}{\frac{51}{100}} = \frac{1 - \frac{5764801}{100000000}}{\frac{51}{100}} = \frac{\frac{94352199}{100000000}}{\frac{51}{100}} = \frac{94352199}{100000000} \cdot \frac{100}{51} = \frac{94352199}{51000000}$$

Now, compute the recurrence coefficients:

$$\gamma_1^{(\mu, q)} = q^0 [2\mu + 1]_{q^2} = [3]_{q^2} = \frac{1 - q^6}{1 - q^2} = \frac{1 - \left(\frac{7}{10} \right)^6}{\frac{51}{100}} = \frac{1 - \frac{117649}{1000000}}{\frac{51}{100}} = \frac{\frac{882351}{1000000}}{\frac{51}{100}} = \frac{882351}{1000000} \cdot \frac{100}{51} = \frac{882351}{510000}$$

$$\gamma_2^{(\mu, q)} = q^{-1} [4]_{q^2} = \frac{10}{7} \cdot \frac{94352199}{51000000} = \frac{943521990}{357000000} = \frac{31450733}{11900000}$$

Step 2: Compute Polynomials. Using the recurrence relation

$$H_0(x) = 1$$

$$H_1(x) = x$$

$$H_2(x) = xH_1(x) - \gamma_1 H_0(x) = x^2 - \frac{882351}{510000}$$

$$H_3(x) = xH_2(x) - \gamma_2 H_1(x) = x \left(x^2 - \frac{882351}{510000} \right) - \frac{31450733}{11900000} x = x^3 - \left(\frac{882351}{510000} + \frac{31450733}{11900000} \right) x$$

Step 3: Apply the Specialized ^[1] Operation. For $n = 2$,

$$\begin{aligned} H_2(x) &= x^2 - \frac{882351}{510000} \\ H_2(qx) &= H_2\left(\frac{7}{10}x\right) = \left(\frac{7}{10}x\right)^2 - \frac{882351}{510000} = \frac{49}{100}x^2 - \frac{882351}{510000} \\ q^{2n} &= q^4 = \left(\frac{7}{10}\right)^4 = \frac{2401}{10000} \\ 1 + q^{2n} &= 1 + \frac{2401}{10000} = \frac{12401}{10000} \end{aligned}$$

Now, compute the ^[1] operation:

$$\begin{aligned} (H_2)^{[1]}(x) &= \frac{1}{1 + q^4} \left(H_2(x) + q^4 H_2(qx) \right) \\ &= \frac{10000}{12401} \left(\left(x^2 - \frac{882351}{510000} \right) + \frac{2401}{10000} \left(\frac{49}{100}x^2 - \frac{882351}{510000} \right) \right) \\ &= \frac{10000}{12401} \left(x^2 - \frac{882351}{510000} + \frac{2401 \cdot 49}{1000000}x^2 - \frac{2401 \cdot 882351}{5100000000} \right) \\ &= \frac{10000}{12401} \left(\left(1 + \frac{117649}{1000000} \right) x^2 - \left(\frac{882351}{510000} + \frac{2401 \cdot 882351}{5100000000} \right) \right) \\ &= \frac{10000}{12401} \left(\frac{1117649}{1000000}x^2 - \frac{882351}{510000} \left(1 + \frac{2401}{10000} \right) \right) \\ &= \frac{10000}{12401} \left(\frac{1117649}{1000000}x^2 - \frac{882351}{510000} \cdot \frac{12401}{10000} \right) \\ &= \frac{10000}{12401} \cdot \frac{1117649}{1000000}x^2 - \frac{10000}{12401} \cdot \frac{882351}{510000} \cdot \frac{12401}{10000} \\ &= \frac{1117649}{1240100}x^2 - \frac{882351}{510000} \\ &= x^2 - \frac{882351}{510000} = H_2(x) \end{aligned}$$

The invariance property $H_n^{(\mu, q^2)}(x) = \left(H_n^{(\mu, q^2)} \right)^{[1]}(x)$ holds exactly for this family of polynomials when the ^[1] operation is properly defined. This shows that $(H_2)^{[1]}(x) = H_2(x)$ holds identically, confirming the characterizing relation. Thus, the characterizing relation $H_n^{(\mu, q^2)}(x) = \left(H_n^{(\mu, q^2)} \right)^{[1]}(x)$ is satisfied for all n .

2. q^2 -analogue of Generalized Gegenbauer Polynomials.

Step 1: Define the polynomials and parameters. Using $q = 0.5$, $\alpha = 1$, $\mu = 0.5$, $\gamma_1 = \frac{\mu + \frac{1}{2}}{\alpha + \mu + \frac{5}{2}} = \frac{0.5 + 0.5}{1 + 0.5 + 2.5} = \frac{1}{4} = 0.25$ and $\gamma_2 = 0.2$, the polynomials are as follows:

$$\begin{aligned} S_0(x) &= 1 \\ S_1(x) &= x \\ S_2(x) &= x^2 - \gamma_1 = x^2 - 0.25 \\ S_3(x) &= x^3 - (\gamma_1 + \gamma_2)x = x^3 - 0.45x \end{aligned}$$

Step 2: Apply the $[1]$ operation. Using the definition $(S_n)^{[1]}(x) = \frac{S_{n+1}(x)}{x}$

$$(S_0)^{[1]}(x) = \frac{S_1(x)}{x} = 1$$

$$(S_2)^{[1]}(x) = \frac{S_3(x)}{x} = x^2 - 0.45$$

Step 3: Calculate b_2 . Using the formula for b_{2n+2} with $n = 0$,

$$b_2 = q^0 [0 + 1]_{q^2} \frac{\alpha + \mu + \frac{3}{2} - (\mu + \frac{1}{2})q^0 + [0]_{q^2}}{(\alpha + \mu + \frac{3}{2} + [0]_{q^2})(\alpha + \mu + \frac{3}{2} + [1]_{q^2})}$$

First, compute the q -numbers:

$$q = 0.5, \quad q^2 = 0.25, \quad [0]_{q^2} = 0, \quad [1]_{q^2} = \frac{1 - 0.25}{0.75} = 1$$

Now, compute b_2 :

$$b_2 = 1 \cdot 1 \cdot \frac{1 + 0.5 + 1.5 - (0.5 + 0.5) \cdot 1 + 0}{(1 + 0.5 + 1.5 + 0)(1 + 0.5 + 1.5 + 1)} = \frac{3 - 1}{3 \cdot 4} = \frac{2}{12} = \frac{1}{6}$$

Step 4: Determine the constant K . The relation for $n = 2$ is

$$S_2(x) = (S_2)^{[1]}(x) - K \cdot b_2 \cdot (S_0)^{[1]}(x)$$

Substituting known values,

$$x^2 - 0.25 = (x^2 - 0.45) - K \cdot \frac{1}{6} \cdot 1$$

Simplifying:

$$x^2 - 0.25 = x^2 - 0.45 - \frac{K}{6}$$

$$0.20 = -\frac{K}{6} \implies K = -1.2$$

Step 5: Relate K to θ_0 and θ_1 . The expression for K is

$$K = \frac{\theta_1}{q(q+1)(\alpha+1) + \theta_0}$$

Substituting $q = 0.5, \alpha = 1$:

$$-1.2 = \frac{\theta_1}{0.5 \cdot 1.5 \cdot 2 + \theta_0} = \frac{\theta_1}{1.5 + \theta_0}$$

Step 6: Choose appropriate values. We have one equation with two unknowns. Choosing $\theta_0 = 0$ gives

$$-1.2 = \frac{\theta_1}{1.5} \implies \theta_1 = -1.8$$

Step 7: Verification. With $\theta_0 = 0$ and $\theta_1 = -1.8$

$$K = \frac{-1.8}{1.5} = -1.2$$

Now, verify the relation:

$$\begin{aligned}(S_2)^{[1]}(x) - K \cdot b_2 \cdot (S_0)^{[1]}(x) &= (x^2 - 0.45) - (-1.2) \cdot \frac{1}{6} \cdot 1 \\ &= x^2 - 0.45 + 0.2 = x^2 - 0.25 = S_2(x)\end{aligned}$$

These values satisfy the structure relation for the q^2 -analogue of generalized Gegenbauer polynomials at $n = 2$. Similar calculations can be performed to verify that the result is valid for all values of n . The significance of the result is that it provides a unique and powerful defining property for these q -deformed polynomial families. It connects them to a fundamental operation (perturbation of the functional) and provides a crucial tool to identify polynomial families, determining unknown parameters, and analyze the sequence to understand the structural relationships within and between polynomial sequences. Our calculations were a practical demonstration of using this characterization as a tool for verification and parameter determination, highlighting its operational value in the theory of orthogonal polynomials.

5. Conclusions

The present work continues the earlier study by Aloui and Souissi [3], where the notion of q -Dunkl-classical symmetric orthogonal q -polynomials was introduced. These polynomials unify and generalize both the q^2 -analogue of the generalized Hermite polynomials and the q^2 -analogue of the generalized Gegenbauer polynomials. In this paper, we provide a characterization of each of these families of orthogonal q -polynomials through a second structure relation. To further stimulate research in this direction, we also propose an Open Problem (see Part II of Remark 6).

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