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On the Conditions Under Which an Algebra over a Field Has a Multiplicative Identity Element

Małgorzata Jastrzębska 

Institute of Mathematics, University of Siedlce, 08-110 Siedlce, Poland; malgorzata.jastrzebska@uws.edu.pl

Abstract

The set of all annihilators in a ring forms a lattice whose structure reflects the internal properties of the ring. In contrast to the lattices of one-sided ideals, lattices of one-sided annihilators possess a particularly desirable property: symmetry. More precisely, the lattice of left annihilators in a given ring is anti-isomorphic to the lattice of right annihilators. In the literature, attention is primarily focused on rings with identity. In this work, we study lattices of one-sided annihilators in rings without a multiplicative identity element. The properties of rings with an identity element usually differ significantly from those of rings without an identity. We restrict our consideration to rings that are algebras over fields. We indicate certain connections between algebras A without identity and their extensions to algebras A^1 with identity (we consider the Dorroh extension). We highlight similarities and differences in the structure of the lattices of one-sided annihilators in A and those in A^1 . We also identify conditions on annihilators in semiprimary algebras that ensure the existence of an identity element. Furthermore, we show that the existence of an identity in an algebra A is not guaranteed even if A satisfies the condition $A^2 = A$ and possesses a one-sided identity.

Keywords: ring; algebra over a field; semiprimary algebra; annihilator in a ring; lattice of annihilators



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1. Introduction

Throughout this paper, all rings are assumed to be associative. The set $l(S) = \{r \in R : rS = 0\}$ is a left ideal of a ring R , called the left annihilator of S in R . In an analogous way, one can define a right annihilator of S in ring R . A lattice (complete lattice) is a poset P in which every pair (every subset) of P has both a least upper bound and a greatest lower bound in P . The set of annihilators in a ring forms a lattice whose structure reflects the internal properties of the ring.

In contrast to the lattices of one-sided ideals, lattices of one-sided annihilators possess a very desirable property: symmetry. More precisely, the lattice of left annihilators in a given ring is anti-isomorphic to the lattice of right annihilators. In the case of commutative rings, this implies that the lattice of annihilators is self-dual. Therefore, if we present Hasse diagrams of the lattices of left and right annihilators side by side, the overall structure exhibits central symmetry.

This symmetry, i.e., the anti-isomorphism—between the lattices of one-sided annihilators, allows us to study only the lattice of left (or right) annihilators, with the results being directly applicable to right (respectively left) annihilators as well.

In the literature, the lattices of annihilators of rings with a multiplicative identity element 1 are the primary focus. Among the characterizations of rings with identity using the concept of annihilators, the following can be distinguished:

1. Characterizations of rings based on the number of elements in the lattice of annihilators, e.g.,
 - Rings in which the lattice of one-sided annihilators is a two-element lattice, known as domains (see [1,2]).
 - Rings in which the annihilators of distinct elements are different, known as Boolean rings (see e.g., [3]).
2. Characterizations of rings in which the lattice of one-sided annihilators coincides (or almost coincides) with the lattice of one-sided ideals, e.g.,
 - Rings in which the lattices of one-sided annihilators coincide with the lattices of one-sided ideals, known as dual rings (see e.g., [4]).
 - Artinian rings, in which every left ideal is a left annihilator and every right ideal is a right annihilator, known as quasi-Frobenius rings (see e.g., [5,6]).
 - Rings in which the left annihilator of any single element (or any subset) of a ring is generated, as a left ideal, by an idempotent element, known as left Rickart rings (or left Baer rings, respectively) (see e.g., [5]).
 - Rings in which the right annihilator of any subset of the ring is comparable with every other right ideal, known as a right annihilator rings (see e.g., [7,8]).
3. Characterizations of rings in which the lattice of one-sided annihilators has a relatively simple structure, e.g.,
 - Rings in which the lattice of one-sided annihilators is a chain, known as lineal rings (see e.g., [9,10]).
 - Rings in which the lattice of one-sided (or two-sided) annihilators is a Boolean lattice (see e.g., [11,12]).

The assumption of the existence of a multiplicative identity element 1 in a ring significantly affects even the properties of lattices of one-sided ideals (see [2,13–15]). The lattice of left annihilators in a ring is a subsemilattice of the meet-semilattice of left ideals, so its properties must also depend on the presence of 1 in the ring. It is also well known that many characterizations of rings with 1 obtained using the notion of annihilator cannot be extended to the case of rings without identity. For example, the statement: “The lattice of one-sided annihilators in a ring with a 1 is a Boolean lattice if and only if the ring is reduced” is false for rings without identity (see [11]). The left (and right) annihilator of the entire ring with identity is the zero ideal, which need not be the case for rings without identity. Therefore, a natural approach to studying the existence of an identity element in a ring is through the investigation of annihilators.

In this paper, we restrict ourselves to the study of a particular class of rings, namely algebras over fields (briefly referred to as algebras). This class is very important because typical examples of rings—such as polynomial rings over fields, matrix rings over fields, and their subalgebras (both with and without a multiplicative identity element)—are algebras over fields. Moreover, many examples of rings in which the lattice of one-sided annihilators has a specific form (e.g., is not modular) are found within this class (see e.g., [11,15–19]).

In Section 2, we begin by recalling selected definitions and results concerning algebras, which are needed in the later parts of the paper. In some cases, the cited results are modified or even strengthened compared to their ring-theoretic versions known from the literature (cf. [1,5,20,21]). Examples include Theorems 2 and 3. Such generalizations are possible because they apply to algebras over fields rather than to arbitrary rings. Section 3 is devoted

to the properties of annihilators in algebras. We examine the relationship between the lattice of one-sided annihilators of an algebra A and the corresponding lattice of its unital extension A^1 , focusing primarily on the case where A is a semiprimary algebra. In Section 4, we provide equivalent conditions under which a semiprimary algebra possesses an identity element. We also show which natural conditions are insufficient to ensure the existence of an identity.

All information about lattices used here can be found, for example, in [15,22].

2. Introduction to Algebras over Fields

An algebra over a field $\mathbb{K}$, or short an algebra is a ring $(A, +, \cdot)$ having at the same time the structure of a linear space over $\mathbb{K}$ with the same addition, where for any $\lambda \in \mathbb{K}$ and for any $x, y \in A$ the following condition is satisfied:

$$\lambda(xy) = (\lambda x)y = x(\lambda y). \quad (1)$$

By A^{op} we denote an algebra with the same linear structure as A over $\mathbb{K}$, but with inverse multiplication, say $*$. Thus in A^{op} we have $a * b = ba$, for $a, b \in A$. By the dimension of the algebra A we mean its dimension as a linear space over $\mathbb{K}$. The set of all $\mathbb{K}$ linear combinations of elements $x_1, x_2, \dots, x_n$ is denoted by $\text{Lin}_{\mathbb{K}}(x_1, x_2, \dots, x_{n+1})$.

The concept of algebras over fields emerged in the algebraic literature in the late 19th century and was fully developed in the first half of the 20th century. Examples of algebras over fields include the quaternions, matrix algebras over fields and polynomial algebras over fields. Nowadays, the theory of algebras over fields is used, among other things, to study and construct error-correcting codes (see e.g., [23]). For more detailed information, history and examples we refer the reader to [20]. To make the article self-contained, in this section, we recall the key concepts related to algebras over fields, supplemented with additional information needed in the subsequent parts of the paper.

In a situation where this does not lead to any misunderstanding, we will assume that an algebra has a given property if it has this property as a ring. We will then use concepts and notations from ring theory. In this sense we understand algebras with an identity. If the algebra A has an identity, we will write it with the symbol 1_A , or in short 1, if it is known which algebra we mean. The left ideal of an algebra A is the left ideal of the ring A which is also a subspace in A . It may happen that for a given algebra A there exist ideals (and therefore one-sided ideals) of A as a ring which are not ideals of A as an algebra.

Example 1. Let $\mathbb{K}$ be an infinite field and A a nonzero $\mathbb{K}$ -algebra with zero multiplication. Let $0 \neq v$ be an arbitrary element in A . If the characteristic of $\mathbb{K}$ is equal to zero, then for any $n \in \mathbb{N}$, let $I_n = \{m2^n v : m \in \mathbb{Z}\}$.

If $\mathbb{K}$ has the characteristic $p > 0$, then, since $\mathbb{K}$ is an infinite field, there is an infinite sequence of elements $x_1, x_2, \dots$ linearly independent over a simple subfield $\mathbb{F} \subset \mathbb{K}$. In this case, for any $n \in \mathbb{N}$ let $I_n = \text{Lin}_{\mathbb{F}}(x_n, x_{n+1}, \dots)v$.

In both cases the sets I_n are ideals of the ring A which are not even one-sided ideals of the algebra A . The one-sided ideals of the algebra A are only $\mathbb{K}$ -subspaces in A . Therefore, if $A = \mathbb{K}$, then the only one-sided ideals in the algebra A are 0 and A .

If I is the left ideal (right ideal, two-sided ideal) of the algebra A , then we will write $I <_l A$ ($I <_r A$, $I \triangleleft A$). The set of all left (right, two-sided) ideals of the algebra A will be denoted $\mathcal{I}_l(A)$ ($\mathcal{I}_r(A)$, $\mathcal{I}(A)$).

Below we have a well-known fact (see e.g., [14,20,24]).

Proposition 1. *The sets $\mathcal{I}_l(A)$, $\mathcal{I}_r(A)$ and $\mathcal{I}(A)$, ordered by the inclusion relation are modular and complete lattices. They are sublattices in the lattice of the subspaces $\text{Sub}(A)$.*

By standard convention, if V_1 and V_2 are subspaces of a linear space V and $V_1 \cap V_2 = 0$, then the sum of these subspaces is called a direct sum and denoted by $V_1 \oplus V_2$.

Now let A and B be algebras. Let us take the linear space $A \times B$. It is naturally the direct sum of the linear subspaces $A = A \times 0$ and $B = 0 \times B$, so $A \times B = A \oplus B$. Let us extend the multiplications in A and in B to the multiplication in $A \oplus B$ assuming $ab = ba = 0$ for any $a \in A$, $b \in B$. Then we get an algebra called *direct sum of algebras* A and B , which we still denote by $A \oplus B$. In this situation, A and B are ideals in the algebra $A \oplus B$.

If the algebra A can be written as a direct sum of its ideals I and J , then we get a decomposition of A into a direct sum of algebras $A = I \oplus J$ in the sense of the previous definition.

Let B, C be nonempty subsets of A and at least one of them is not single-element. Then we denote by BC the subspace in A generated by all products bc where $b \in B$, $c \in C$. If at least one of these sets is a subspace in A , then it is clear that the subspace BC is given by the formula

$$BC = \left\{ \sum_{i=1}^n b_i c_i : n \in \mathbb{N}, b_i \in B, c_i \in C \right\}.$$

If B and C are subalgebras in A , then BC need not be a subalgebra. However, if B is a left ideal (C is a right ideal) of A , then their product BC is also a left ideal (right ideal) in A . So if B and C are ideals in A , then obviously BC is also an ideal in A , and $BC \subseteq B \cap C$.

Since multiplication on A is associative, the multiplication of subsets of A defined above is also associative. Therefore, this product can be extended to any number of $n \in \mathbb{N}$ factors, without specifying the parentheses. We assume that $B^1 = B$. The product of $n > 0$ factors, each of which is equal to B , will be written as B^n .

A subset B of A is called *nilpotent* if $B^n = 0$ for some $n \geq 1$. If moreover $n \geq 2$ and $B^{n-1} \neq 0$, then we say that B has *degree of nilpotence* n . It is known that if $I \subseteq A$ is a one-sided nilpotent ideal, then the ideal generated by I in A is also nilpotent, of the same degree of nilpotence. Algebras without nonzero nilpotent ideals are called *semiprime*.

We will now distinguish certain elements of algebras that have important properties. By *center of algebra* A we will understand the set

$$Z(A) = \{a \in A : \forall b \in A, ab = ba\}.$$

The element $a \in Z(A)$ will be called central element. If all elements in A are central, then A is a commutative algebra. It is obvious that a ring with identity is a $\mathbb{K}$ -algebra if and only if we are given a homomorphism of the field $\mathbb{K}$ into $Z(A)$ that transforms $1_{\mathbb{K}}$ into 1_A .

An important type of element in algebras is an idempotent. An element $e \in A$ is called an idempotent element (or, more briefly, idempotent) if it satisfies the condition $e = e^2$. Obviously, 0 and 1 are idempotents in A . An idempotent e of A is called a left (right) identity if $ea = a(ae = a)$ for any a of A . Of course, the identity is always a left and right identity. On the other hand, if an algebra has a left identity and a right identity, they are the same and are the identity of the algebra.

In the literature, rings in which the set of left ideals with order determined by inclusion satisfies the Descending Chain Condition (DCC) are known as left Artinian rings. Right Artinian rings are defined analogously. Of course, every left (right) Artinian algebra as a ring is left (right) Artinian as an algebra. From the Example 1 it follows that the converse implication need not be true, even when the algebra has dimension 1 and only two ideals. The situation is simple if the algebra A has an identity because then $\mathbb{K} \subseteq A$ naturally as a subalgebra. Then, all one-sided ring ideals in A are one-sided ideals of A as an algebra, of course from the same side.

The set of all elements in A that are simultaneously left- and right-invertible will be denoted by $U(A)$. The right ideal J of the algebra A is called maximal ideal, if it is a coatom in the lattice $\mathcal{I}_r(A)$. The right ideal $J <_r A$ is called regular, if there exists an element $a \in A$ such that $x - ax \in J$ for all $x \in A$. One of the most important ideals in the algebra A is the ideal that is the intersection of all right maximal ideals, which are also regular ideals. This ideal is known as Jacobson's radical or for short radical of algebra A and is usually denoted by the symbol $J(A)$. Various characterizations of the radical can be found e.g., in [21]. For us, the following information will be important: for any algebra A , its radical is an ideal of A as an algebra, contains all one-sided nilpotent ideals, and $J(I) = I \cap J(A)$ for any ideal $I \triangleleft A$. Moreover, if A is an algebra with identity, then for any ideal $I \triangleleft A$ we have $1 + I \subseteq U(A)$ if and only if $I \subseteq J(A)$.

The following theorem is crucial in the theory of noncommutative rings (see e.g., [1]).

Theorem 1 (Wedderburn–Artin Theorem). *If R is a semiprime left (or right) Artinian ring, then*

$$R \simeq M_{n_1}(D_1) \times M_{n_2}(D_2) \times \dots \times M_{n_r}(D_r),$$

where each D_i is a division ring and $M_{n_i}(D_i)$ denotes the ring of $n_i \times n_i$ matrices over D_i .

Algebras that satisfy the conditions of the above theorem are called semisimple algebras. It should be noted that a semisimple algebra A has an identity element and every one-sided ideal in A is generated by an idempotent element.

Extending the notion of Artinian algebras, an algebra A is called semiprimary if its radical $J(A)$ is nilpotent and the quotient algebra $A/J(A)$ is a semisimple algebra. Among other things, all finite-dimensional algebras that are not nilpotent are semiprimary algebras.

Theorem 2. *Let A be a semiprimary algebra with radical J .*

- (1) *If $0 \neq a \in A$ is not nilpotent, but $(a^2 - a)^n = 0$ for some $n \geq 1$, then there exists an idempotent $e \in \text{Lin}(a, a^2, \dots)$ such that $a^n = a^n e$.*
- (2) *If $a \in A$ and $a^2 - a \in J$, then there exists an idempotent $e \in A$ such that $a - e \in J$, i.e., the idempotent $a + J \in A/J$ can be raised to the idempotent $e \in A$.*
- (3) *If $I \subseteq A$ is a one-sided ideal, then it is either nilpotent or contains a nontrivial idempotent element.*

Proof. (1) Let $a \in A$ be non-nilpotent, but $(a^2 - a)^n = 0$ for some $n \geq 1$. Then, we can rewrite this condition in the form $a^n = a^{n+1}f(a)$ for some polynomial $f(x) \in \mathbb{K}[x]$. Hence, by induction, for any $k \geq 1$ we have $a^n = a^{n+k}f(a)^k$. In particular, $a^n = a^{2n}f(a)^n$. Let us put $e = a^n f(a)^n$. Then,

$$e^2 = a^{2n}f(a)^{2n} = a^{2n}f(a)^n f(a)^n = a^n f(a)^n = e,$$

so e is idempotent.

Since a is not nilpotent and $a^n = a^n e$, then $e \neq 0$. It is also clear from the construction that $e \in \text{Lin}(a, a^2, \dots)$.

(2) If $a \in J$, then it is enough to assume $e = 0$. Let $a \in A$ be such an element that $a \notin J$, but $a^2 - a \in J$. Therefore, it is easy to see that a is not a nilpotent element. There also exists $n \geq 1$, such that $(a^2 - a)^n = 0$. If $n = 1$, then it is enough to assume $e = a$. Let $n > 1$. So since a is not a nilpotent element, let us take $f(a)$ and e as in the proof of point (1). So we

have $a^n = a^{2n}f(a)^n$ and $e = a^n f(a)^n$. Let $\equiv$ denote congruence modulo J in A . Since J is an ideal, it follows by assumption that $a \equiv a^k$ for any $k \geq 1$. Therefore,

$$a \equiv a^n = a^{2n}f(a)^n \equiv a^n f(a)^n = e,$$

i.e., $a \equiv e$.

(3) Let $I \subseteq A$ be a one-sided ideal. If $I \subseteq J(A)$, then I is nilpotent, since A is a semiprimary algebra. If, on the other hand, $I \not\subseteq J(A)$, then in the semisimple algebra $A/J(A)$ we have a nonzero one-sided ideal $(I + J(A))/J(A)$. By Theorem 1 it contains a nonzero idempotent element, the image of some element $a \in I$. Now we see that a is not a nilpotent element, but $a^2 - a \in J(A)$ is a nilpotent element. The thesis follows from point (1). $\square$

We now present a theorem, known in a slightly weaker form for rings, which allows us to take advantage of the properties of algebras with identity.

Theorem 3. Let A be a $\mathbb{K}$ -algebra. On the direct sum of linear spaces $A^1 = A \oplus \mathbb{K}$ let us determine the multiplication by the formula:

$$(a_1, k_1)(a_2, k_2) = (a_1a_2 + a_1k_2 + k_1a_2, k_1k_2), \quad (2)$$

where $a_1, a_2 \in A$ and $k_1, k_2 \in \mathbb{K}$. Then, A^1 is an algebra with identity $1_{A^1} = (0, 1)$. Using natural identification of A and $\mathbb{K}$ with the corresponding subspaces in A^1 , the following conditions are satisfied:

- (1) A is an ideal in A^1 , $A^1/A = \mathbb{K}$, $J(A^1) = J(A)$ and $(A^1)^{op} = (A^{op})^1$;
- (2) Every left (right) ideal in A is a left (right) ideal in A^1 as a ring;
- (3) If I is a one-sided ideal in the algebra A^1 , and $I \not\subseteq A$ then there exists an element $a \in A$ such that $I = (I \cap A) \oplus \mathbb{K}(1 + a)$;
- (4) If $I \subseteq J$ are left (right) ideals in A^1 such that $I \not\subseteq A$ and $I \cap A = J \cap A$, then $I = J$;
- (5) A is a left (right) Artinian algebra if and only if A^1 is a left (right) Artinian ring.

Proof. It is easy to check that the multiplication given in the Formula (2) transforms A^1 into a $\mathbb{K}$ -algebra with identity equal to $(0, 1)$.

The proofs of points (1) and (2) follow from direct calculations. As an example, we compute that every left ideal I in A is a left ideal in A^1 as a ring. Obviously, $I = I \oplus 0$ is closed under addition. It follows from Formula (2) applied to $(a, k) \in A \oplus \mathbb{K}$, $(i, 0) \in I \oplus 0$, that $(a, k)(i, 0) = (ai + 0 + 0, 0) = (ai, 0) \in I \oplus 0 = I$. Hence, I is a left ideal in A^1 .

(3) Let $x \in I \setminus A$. Since I is a subspace in A^1 , we can assume $x = 1 + a$ for some $a \in A$. We will show that $I = (I \cap A) + \mathbb{K}(1 + a)$. Let us take any $y \in I \setminus A$. We can write it in the form $y = \lambda 1 + b$, for some $\lambda \in \mathbb{K}$ and $b \in A$. Of course $y = y - \lambda x + \lambda x$ and $y - \lambda x \in I \cap A$, which completes the proof.

(4) From the construction of A^1 , it follows that A is both a left and a right maximal ideal in A^1 . Therefore, by assumption, it follows that $I + A = J + A = A^1$. Hence, $I = J$, since the lattices of one-sided ideals in A^1 are modular according to Proposition 1.

(5) Since A^1 is an algebra with identity, then A^1 is Artinian as a ring if and only if it is Artinian as an algebra. Now, point (5) easily follows from (4). $\square$

Remark 1. One can also obtain (4) as a direct consequence of (3).

3. Lattices of Annihilators of A and A^1

We begin this section by recalling the definitions of annihilators and lattices of annihilators known from the literature, and we present the most important properties of

annihilators. We provide an example of an algebra whose annihilators possess significant properties. The chapter concludes with a proof of a new result stating that in a semiprimary algebra with right identity the lattice of left annihilators in A is a sublattice in the lattice of left annihilators in A^1 .

Definition 1. Let A be an algebra and $S \subseteq A$ a nonempty subset.
The left annihilator of S in A is the set:

$$l_A(S) = l(S) = \{a \in A : aS = 0\}.$$

Similarly, the right annihilator of S in A , is the set:

$$r_A(S) = r(S) = \{a \in A : Sa = 0\}.$$

We now give some properties of annihilators, which follow directly from the definition.

Lemma 1. Let A be an algebra and let S, T be subsets of A . Let I be an arbitrary set and let S_i , for any $i \in I$, be a subset of the algebra A . Then:

- (1) If $S \subseteq T$, then $l(S) \supseteq l(T)$ and $r(S) \supseteq r(T)$;
- (2) $l(S) = l(r(l(S)))$ and $r(S) = r(l(r(S)))$;
- (3) If $l(r(S)) = l(r(T))$, then $r(S) = r(T)$;
- (4) If $r(l(S)) = r(l(T))$, then $l(S) = l(T)$;
- (5) $l(\bigcup_{i \in I} S_i) = \bigcap_{i \in I} l(S_i)$ i $r(\bigcup_{i \in I} S_i) = \bigcap_{i \in I} r(S_i)$;
- (6) If S_i are subspaces of A , then $l(\sum_{i \in I} S_i) = \bigcap_{i \in I} l(S_i)$ and $r(\sum_{i \in I} S_i) = \bigcap_{i \in I} r(S_i)$;
- (7) If $S \triangleleft A$, then $l(S) \triangleleft A$ i $r(S) \triangleleft A$.

In the following, we will be interested in the set of all left annihilators in the algebra A , which we will denote by the symbol $\mathcal{A}_l(A)$. We will also consider the set $\mathcal{A}_r(A)$, of all right annihilators in the algebra A .

Considering the associative property of multiplication in A , we note that every left annihilator in A is a left ideal in A and similarly, every right annihilator in A is a right ideal in A .

By the Formula (1), every left annihilator and every right annihilator in A is a subspace over $\mathbb{K}$, and thus a one-sided ideal of A . This implies that $\mathcal{A}_l(A) \subseteq \mathcal{I}_l(A)$ and $\mathcal{A}_r(A) \subseteq \mathcal{I}_r(A)$.

Let $I = l(S)$ and $J = l(T)$ be left annihilators in the algebra A . Then, by the Lemma 1, it follows that the set $I \cap J = l(S) \cap l(T) = l(S \cup T)$ is a left ideal and a left annihilator in A . On the other hand, the set $I + J$ is a left ideal, but need not be a left annihilator. However, the sets $\mathcal{A}_l(A)$ and $\mathcal{A}_r(A)$ have a lattice structure.

The set $\mathcal{A}_l(A)$ with order given by inclusion is a complete lattice with operations given for any family of left annihilators $\{I_t : t \in T\}$ by the condition:

$$\bigwedge_{t \in T} I_t = \bigcap_{t \in T} I_t \quad \text{and} \quad \bigvee_{t \in T} I_t = l(r(\bigcup_{t \in T} I_t)) = l(r(\sum_{t \in T} I_t)). \quad (3)$$

Analogously, the set $\mathcal{A}_r(A)$ with order given by inclusion is a complete lattice with operations given for any family of right annihilators $\{I_t : t \in T\}$ by the condition:

$$\bigwedge_{t \in T} I_t = \bigcap_{t \in T} I_t \quad \text{oraz} \quad \bigvee_{t \in T} I_t = r(l(\bigcup_{t \in T} I_t)) = r(l(\sum_{t \in T} I_t)). \quad (4)$$

It is well known that in any ring (and algebra) there is a Galois correspondence of the lattices of right annihilators and left annihilators. The lattice $\mathcal{A}_l(A)$ is anti-isomorphic to

the lattice $\mathcal{A}_r(A)$ under the mapping $l : I \rightarrow r(I)$, for $I \in \mathcal{A}_l(A)$. The inverse of l is the mapping $r : J \rightarrow l(J)$, for $J \in \mathcal{A}_r(A)$. This often allows us to restrict our considerations to the lattice of left annihilators of a given algebra, and to interpret the results in terms of the lattice of right annihilators of this algebra. It is also known that there is no nontrivial identity satisfied by lattices of one-sided annihilators in rings (even if ring has an identity element). In particular, every finite lattice is a sublattice of the lattice of annihilators of a certain ring with an identity element (see [16], for commutative rings see [18]).

Proposition 2. *Let A be a semiprimary algebra. Then, in the lattice $\mathcal{A}_l(A^1)$ there exists a proper left annihilator I such that $I \notin \mathcal{A}_l(A)$. Similarly, in the lattice $\mathcal{A}_r(A^1)$ there exists a proper right annihilator J such that $J \notin \mathcal{A}_r(A)$.*

Proof. Since A is semiprimary, then by Theorem 2 (3) in A there exists a nontrivial idempotent e . By point (3) of Theorem 3 we have $r_{A^1}(e) = r_A(e) + \mathbb{K}(-e + 1) \not\subseteq A$ and $l_{A^1}(e) = l_A(e) + \mathbb{K}(-e + 1) \not\subseteq A$. $\square$

We now give an example of an algebra A and a left annihilator $I \in \mathcal{A}_l(A)$, such that $I \notin \mathcal{A}_l(A^1)$.

Example 2. *Let A be an algebra of dimension 2 with basis $\{e, x\}$. Let multiplication in the basis be of the form: $e^2 = e, x^2 = 0, ex = x, xe = 0$, so the table of the multiplication is*

$\cdot$	e	x
e	e	x
x	0	0

Then $x \in l_A(A)$. Hence $\mathcal{A}_l(A)$ is a chain with two elements: $\mathbb{K}x, A$. Then $r_{A^1}(\mathbb{K}x) = A$, because A is a maximal ideal in A^1 . It is obvious that $l_{A^1}(A) = l_A(A) + \mathbb{K}(-e + 1)$; therefore $\mathbb{K}x \notin \mathcal{A}_l(A^1)$. Let us additionally note that $r_{A^1}(A) = 0$ and $l_{A^1}(0) = A^1$. Using (2) of Lemma 1 we have checked that no left annihilator in A is a left annihilator in A^1 .

Theorem 4. *Let A be a semiprimary algebra. Then, if A has a right identity, then the lattice $\mathcal{A}_l(A)$ is a sublattice of the lattice $\mathcal{A}_l(A^1)$. Similarly, if A has a left identity, then the lattice $\mathcal{A}_r(A)$ is a sublattice of the lattice $\mathcal{A}_r(A^1)$.*

Proof. If $X \subseteq A$ and e_r is a right identity of an algebra A , then by Theorem 3 (3) we have $r_{A^1}(X) = r_A(X) + \mathbb{K}(-e_r + 1)$. It is easy to check that if J_1, J_2 are subspaces in A^1 such that $J_1 J_2 = 0$, then at least one of the subspaces J_1, J_2 is contained in A . Therefore $l_{A^1}(r_{A^1}(X)) \subseteq A$. Additionally, using Lemma 1 (1), we have $l_{A^1}(r_{A^1}(X)) \subseteq l_{A^1}(r_A(X)) \cap A = l_A(r_A(X))$. Obviously, $l_A(r_A(X)) \subseteq l_{A^1}(r_{A^1}(X))$ and hence $\mathcal{A}_l(A) \subseteq \mathcal{A}_l(A^1)$.

Now let $I_1, I_2 \in \mathcal{A}_l(A)$. Of course, $I_1 \cap I_2$ is a lower bound of I_1, I_2 in the lattice $\mathcal{A}_l(A^1)$. Substituting $I_1 + I_2$ into the above considerations, we obtain $l_{A^1}(r_{A^1}(I_1 + I_2)) = l_A(r_A(I_1 + I_2))$. From this it is already clear that the lattice $\mathcal{A}_l(A)$ is a sublattice of the lattice $\mathcal{A}_l(A^1)$.

The proof of the statement for right annihilators proceeds analogously. $\square$

4. Some Conditions Suspected of Implying the Existence of an Identity Element in an Algebra

It is obvious that if the algebra A has identity, then the following conditions are satisfied:

- (i) A has a nonzero idempotent;
- (ii) $A^2 = A$;
- (iii) $l(A) = r(A) = 0$, i.e., A has zero annihilators on both sides.

We now discuss which of the above conditions, possibly under additional assumptions, ensure that A has an identity element.

The following fact follows directly from the definition of one-sided identities and one-sided annihilators.

Proposition 3. *If A has a right identity, then $l_A(A) = 0$. Similarly, if A has a left identity, then $r_A(A) = 0$.*

The example below shows that conditions (i) and (ii) do not imply the existence of an identity element in the algebra.

Example 3. *Let A be an algebra with a basis e, x as linear space and with the multiplication given by $e^2 = e, x^2 = 0, ex = x$ and $xe = 0$ (see Example 2). Then A does not have an identity element, although it has the left identity. Furthermore, $A^2 = A$.*

Now let us check how the condition $l(A) = r(A) = 0$ affects the existence of identity element in A .

For semiprimary algebras we have the following result (cf. [21] (Theorem 1.4.3.)).

Theorem 5. *Let A be a semiprimary algebra and J its radical. The following conditions are equivalent:*

- (1) *Algebra A has a left identity;*
- (2) *Every homomorphic image of A has a zero right annihilator;*
- (3) *For any $k \geq 1$ the algebra A/J^k has a zero right annihilator.*

Proof. (1) $\Rightarrow$ (2). Let e be a left identity in A , and let $I \triangleleft A$ be an arbitrary ideal. Then, of course, $e + I$ is a left identity in A/I . Therefore, by Proposition 3, we have $r_{A/I}((A/I)) = 0$.

Obviously, (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). Let $J^n = 0$ for some $n \geq 1$. By assumption it follows that $J \neq A$ and the algebra A/J has identity. Let $e \in A$ be an element such that $e + J$ is identity in $A/J = A/J^1$. By Theorem 2 (2) we can assume that e is an idempotent. Assume that for $k \geq 1$ we already know that $e + J^k$ is a left identity in A/J^k . So $(1 - e)A \subseteq J^k$. Furthermore, by the choice of e we know that $A(1 - e) \subseteq J$. So

$$A(1 - e)(1 - e)A \subseteq J^{k+1}.$$

The assumption of the value zero of the right annihilator A/J^{k+1} gives that $(1 - e)(1 - e)A \subseteq J^{k+1}$. Therefore for $e_{k+1} = e + e - e^2 = e$ we have $(1 - e_{k+1})A \subseteq J^{k+1}$, so $e + J^{k+1}$ is the left identity of A/J^{k+1} . Now, substituting $k = n$ we get that $e_n = e$ is the left identity modulo $J^n = 0$, so e is the left identity in A . $\square$

Corollary 1. *Let $\mathbb{K}$ be an infinite field. If the nontrivial $\mathbb{K}$ -algebra A is left-Artinian as a ring, then A has a left identity.*

Proof. Suppose that $r(A) \neq 0$. Then, every subspace (and even a subgroup of the additive group) in $r(A)$ is a left ideal in A . From the Example 1 and the left Artinianity of A it follows that this is impossible. Therefore, it must be that $r(A) = 0$. A similar observation holds for any homomorphic image of the algebra A . Therefore, from the theorem above it follows that A has a left identity. $\square$

Theorem 6. *Let A be a semiprimary algebra and J its radical. The following conditions are equivalent:*

- (1) Algebra A has an identity;
- (2) Every homomorphic image of A has a zero right and left annihilators;
- (3) For any $k \geq 1$ the algebra A/J^k has a zero right and left annihilators.

Proof. If A has a left identity e and a right identity f , then $e = ef = f$ and this is an identity of the algebra A . Now the result is a direct consequence of Theorem 5 and its analogous version for algebras with a right identity. $\square$

Corollary 2. Let A be a semiprimary algebra and J its radical. If $l(A) = r(A) = 0$ and $J^2 = 0$, then A has identity.

Proof. Note that A/J as a semisimple algebra has an identity, so both of its annihilators are zero. Since $A/J^2 \simeq A$, the thesis follows directly from Theorem 6. $\square$

It turns out that already in algebras of dimension 4, the condition $l(A) = r(A) = 0$ does not guarantee the existence of even a one-sided identity in A .

Example 4. Let A be an algebra of dimension 4 over the field $\mathbb{K}$, whose basis is the elements x, y, z, t . Let the multiplication table of the elements of the basis have the following form:

$\cdot$	x	y	z	t
x	0	0	0	x
y	z	0	0	0
z	0	0	0	z
t	0	y	z	t

It is easy to check that multiplication in A is associative. Directly from the table of multiplication we can see that every element of A is of the form $k_1x + k_2y + k_3z + k_4t$, where $k_1, k_2, k_3, k_4 \in \mathbb{K}$. Moreover, t is idempotent in A . Furthermore, $A = A^2$.

We first show that $l(t) = \mathbb{K}y$. If $(k_1x + k_2y + k_3z + k_4t)t = 0$, for some $k_1, k_2, k_3, k_4 \in \mathbb{K}$, then $k_1xt + k_2yt + k_3zt + k_4t^2 = 0$. From the table of multiplication, it follows that $k_1x + k_3z + k_4t = 0$. Linear independence of elements x, z, t gives $k_1 = k_3 = k_4 = 0$. Then,

$$l(t) = \{k_2y : k_2 \in \mathbb{K}\} = \mathbb{K}y.$$

Now we show that $r(t) = \mathbb{K}x$. Let $t(k_1x + k_2y + k_3z + k_4t) = 0$ for some $k_1, k_2, k_3, k_4 \in \mathbb{K}$. Then $k_2y + k_3z + k_4t = 0$, and linear independence of elements y, z, t gives $k_2 = k_3 = k_4 = 0$. This shows that

$$r(t) = \{k_1x : k_1 \in \mathbb{K}\} = \mathbb{K}x.$$

From Lemma 1 (1) and $\{t\} \subseteq A$, follows $l(A) \subseteq l(t) = \mathbb{K}y$ and $r(A) \subseteq r(t) = \mathbb{K}x$. Therefore, $l(A)$ as a subspace in $\mathbb{K}y$ satisfies either $l(A) = \mathbb{K}y$ or $l(A) = 0$. Analogously, $r(A) = \mathbb{K}x$ or $r(A) = 0$.

From

$$yx = z \neq 0,$$

follows:

- (a) $y \notin l(A)$ and then $l(A) = 0$;
- (b) $x \notin r(A)$ and then $r(A) = 0$.

Therefore, A has zero annihilators on both sides.

Since in any algebra the left (right) identity is an idempotent, let us determine the idempotents in A . Let $k_1x + k_2y + k_3z + k_4t$ be an idempotent in A . Hence, $(k_1x + k_2y + k_3z + k_4t)(k_1x + k_2y + k_3z + k_4t) = k_1x + k_2y + k_3z + k_4t$. Then, $k_1k_4x + k_2k_1z + k_3k_4z + k_4k_2y + k_4k_3z + k_4^2t = k_1x + k_2y + k_3z + k_4t$ and we obtain system of equations $k_1k_4 = 0, k_4k_2 = k_2, k_2k_1 +$

$k_3k_4 + k_4k_3 = k_3, k_4^2 = k_4$. Hence $k_4 = 0$ or $k_4 = 1$. If $k_4 = 0$ then $k_1 = k_2 = k_3 = 0$. If $k_4 = 1$ then $k_3 = -k_1k_2$ for any $k_2, k_3 \in \mathbb{K}$.

In consequence, $\{k_1x + k_2y - k_1k_2z + t : k_1, k_2 \in \mathbb{K}\}$ is the set of all nonzero idempotents in A . However, $(k_1x + k_2y - k_1k_2z + t)x = k_2z \neq x$ and $y(k_1x + k_2y - k_1k_2z + t) = k_1z \neq y$. That means that there is no idempotent which is left identity in A , and analogously there is no idempotent which is right identity in A .

The results obtained appear to provide a promising starting point for further investigation of lattices of one-sided annihilators in rings without identity. It is possible that simple characterizations of such rings may be found in terms of their lattices of annihilators.

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