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New Bayesian Estimation Method Based on Symmetric Projection Space and Particle Flow Velocity

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Abstract: Aiming at the state estimation problem of nonlinear systems (NLSs), the traditional typical nonlinear filtering methods (e.g., Particle Filter, PF) have large errors in system state, resulting in low accuracy and high computational speed. To perfect the imperfections, a new Bayesian estimation method based on particle flow velocity (PFV-BEM) is proposed in this paper. Firstly, a symmetrical projection space based on the state information is selected, the basis function is determined by a set of Fourier series with symmetric properties, the state update is carried out according to the projection principle to calculate the prior information of the state, and select its particle points. Secondly, the particle flow velocity is defined, which describes the evolution process of random samples from the prior distribution to the posterior distribution. The posterior information of the state is calculated by solving the parameters related to the particle flow velocity. Finally, the estimated mean and standard deviation of the state are solved. Simulation experiments are carried out based on two instances of one-dimensional general nonlinear examples and multi-target motion tracking, The newly proposed algorithm is compared with the Particle Filter (PF), and the simulation results clearly indicate the feasibility of this novel Bayesian estimation algorithm.

Keywords: particle flow velocity; nonlinear systems; Bayesian estimation; probability density function; Fokker–Planck equation

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1. Introduction

In recent years, numerous mathematical models of engineering systems have exhibited strong nonlinear characteristics, which has rendered traditional Kalman Filter (KF) methods relying on linear assumptions ineffective when dealing with state estimation problems [1–3]. To address this challenge, academia has conducted in-depth research into nonlinear filtering techniques. Among these studies, the Extended Kalman Filter (EKF) emerged as an early solution, which linearizes nonlinear systems through Taylor series expansion and then applies the KF algorithm [4]. However, truncation errors introduced during the linearization process limit the accuracy of state estimation.

To overcome this limitation, Julier, Uhlmann, and others proposed the Unscented Kalman Filter (UKF) [5]. UKF avoids errors resulting from linearization by selecting a specific set of sample points to approximate nonlinear transformations, significantly improving the accuracy of state estimation.

Recently, Bayesian-based nonlinear filtering methods have become a research hotspot [6–14]. These methods are mainly divided into two categories: one approximates the state distribution in state space, with Particle Filter (PF) being one of the most representative algorithms [15]. PF represents the required posterior probability density through a weighted sum of random samples to estimate the state. Although PF theoretically approximates optimal Bayesian estimation, it still faces challenges in terms of estimation accuracy, implementation difficulty, and computational complexity in practical applications [16–20]. The other category directly approximates the state probability density function in function space. For example, the nonlinear projection filter algorithm proposed by Randal Beard and John Kenney addresses state estimation problems for low-dimensional nonlinear Gaussian systems [21–23]. Subsequently, scholars such as Zhao Yuxin proposed projection filter algorithms for one-dimensional nonlinear non-Gaussian systems, greatly reducing computational time complexity and memory usage [3].

It is worth noting that both EKF and UKF aim to solve for the second-order moments of the system state probability distribution, but the actual distribution of the system state may not be normal. Therefore, Bayesian estimation methods have advantages in non-Gaussian nonlinear systems, as they can directly solve for the system state probability density function, better capturing possible abrupt changes in the target state [16–20].

The literature [23] introduced a projection filtering method limited to low-dimensional state estimation problems. To overcome this limitation, this paper proposes a new Bayesian estimation method based on particle flow velocity (PFV-BEM) and extends it to nonlinear systems. Through numerical approximation solutions, simulation results demonstrate that this method not only relaxes the restrictive condition that system noise must be Gaussian but also outperforms PF algorithms in estimation accuracy and has lower computational complexity than general Particle Filters. Additionally, this method exhibits stable performance and has certain practical application value.

In summary, this paper presents the development of the Particle Filter-based variational Bayesian estimation method (PFV-BEM). Unlike traditional Bayesian estimation algorithms such as the standard Particle Filter (PF) [15] and other variants [2,3], PFV-BEM integrates the advantages of variational Bayesian inference into the particle filtering framework. This integration allows for a more accurate approximation of the posterior probability density function (PDF) of the system state, especially in complex nonlinear and non-Gaussian scenarios. By leveraging variational Bayesian techniques, PFV-BEM can dynamically adjust the importance of the density function, thereby improving the particle weights' distribution and reducing the particle degeneracy problem. Moreover, the proposed algorithm demonstrates superior performance in terms of estimation accuracy and robustness when compared with traditional methods, as evidenced by the simulation results presented in this study.

This paper is organized as follows. In Section 2, a Bayesian estimation model in NLSs is given, and the Bayesian equation of the Bayesian estimation model is given. In Section 3, the (PFV-BEM) program is divided into two parts to solve the state solution: solving the prior probability for status update and using particle flow velocity for measurement update. In Section 4, two illustrative examples and their simulation results are given. Then, the conclusions of this paper are analyzed in detail in Section 5.

2. Bayesian Estimation Model

In nonlinear systems, the state estimation model consists of the state equation and measurement equation, formulated as follows [24]:

$$\begin{cases} dx_t = F(x_t, t)dt + G(t)d\beta_t \\ z_k = H(x_k, t_k) + e_k \end{cases} \tag{1}$$

where x_t denotes the state vector at time t, $F(x_t, t)$ is the state transition matrix, $G(t)$ represents the system noise driving matrix, β_t is the standard Brownian motion, and e_k represents measurement noise, respectively, with z_k denoting the measurement vector, and $H(x_k, t_k)$ being the measurement matrix. β_t and e_k are mutually independent and satisfy the following:

$$\begin{cases} \mathrm{E}\{\boldsymbol{\beta}_t\} = 0, \mathrm{E}\left\{\boldsymbol{\beta}_t \boldsymbol{\beta}_t^{\mathrm{T}}\right\} = \boldsymbol{Q}(t) \\ \mathrm{E}\{\mathrm{e}_k\} = 0, \mathrm{E}\{\mathrm{e}_k \mathrm{e}_k^{\mathrm{T}}\} = \boldsymbol{R}(k) \\ \mathrm{E}\left\{\mathrm{e}_k \boldsymbol{\beta}_{\mathrm{k}}^{\mathrm{T}}\right\} = \mathbf{0} \end{cases} \tag{2}$$

where $\boldsymbol{Q}(t)$ is given as the system noise covariance matrix; $\boldsymbol{R}(k)$ represents the measurement noise covariance matrix.

According to a theorem in ref. [25], the probability density function (PDF) of the state equation satisfies the Fokker–Planck equation (FPE):

$$\dot{p}\left(\mathbf{x}, t_k^- \middle| \mathbf{z}_{k-1}\right) = \frac{1}{2} \sum_{r,s=1}^{n} \frac{\partial^2 \left(G(t)Q(t)G^{\mathrm{T}}(t) \cdot p\left(\mathbf{x}, t_k^- \middle| \mathbf{z}_{k-1}\right)\right)}{\partial x_r \partial x_s} - \sum_{r=1}^{n} \frac{\partial \left(F(x_t, t) \cdot p\left(\mathbf{x}, t_k^- \middle| \mathbf{z}_{k-1}\right)\right)}{\partial x_r} \tag{3}$$

where $p\left(\mathbf{x}, t_k^- \middle| \mathbf{z}_{k-1}\right)$ is defined as prior probability density.

The likelihood probability density $p(\mathbf{z}_{k+1}|\mathbf{x})$ of the measurement equation is defined as follows:

$$p(\mathbf{z}_k|\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^m |\mathbf{R}_k|}} \exp\left\{-\frac{1}{2}(\mathbf{z}_k - h(\mathbf{x}_{k-1}, t_k))^{\mathrm{T}} \mathbf{R}_k^{-1} (\mathbf{z}_k - h(\mathbf{x}_{k-1}, t_k))\right\} \tag{4}$$

In ref. [17], the state estimate solution of Bayesian estimation models needs to calculate the probability density function $p(\mathbf{x}, t_k|\mathbf{z}_k)$, which is defined as the posterior PDF of the Bayesian estimation model. the posterior PDF $p(\mathbf{x}, t_k|\mathbf{z}_k)$ satisfies the Bayesian equation:

$$p(\mathbf{x}, t_k|\mathbf{z}_k) = \frac{p(\mathbf{z}_k|\mathbf{x}) p\left(\mathbf{x}, t_k^- \middle| \mathbf{z}_{k-1}\right)}{\int_\Omega p(\mathbf{z}_k|\mathbf{x}) p\left(\mathbf{x}, t_k^- \middle| \mathbf{z}_{k-1}\right) d\mathbf{x}} \tag{5}$$

3. Bayesian Filter Estimator Design-Based Projection Flow Velocity

The Bayesian estimation method based on particle flow velocity (PFV-BEM) is a state estimation method that approximates the PDF of the state. First, the prior PDF of the state is calculated within the projection space. Second, the particle points of the prior PDF are obtained through calculating the particle flow velocity, and the particle information of the posterior PDF can be obtained. Finally, the estimated solution and estimated variance of the system state can be calculated. Therefore, PFV-BEM is an effective method for solving Bayesian estimation problems. The flowchart of the PFV-BEM algorithm is shown in Figure 1.

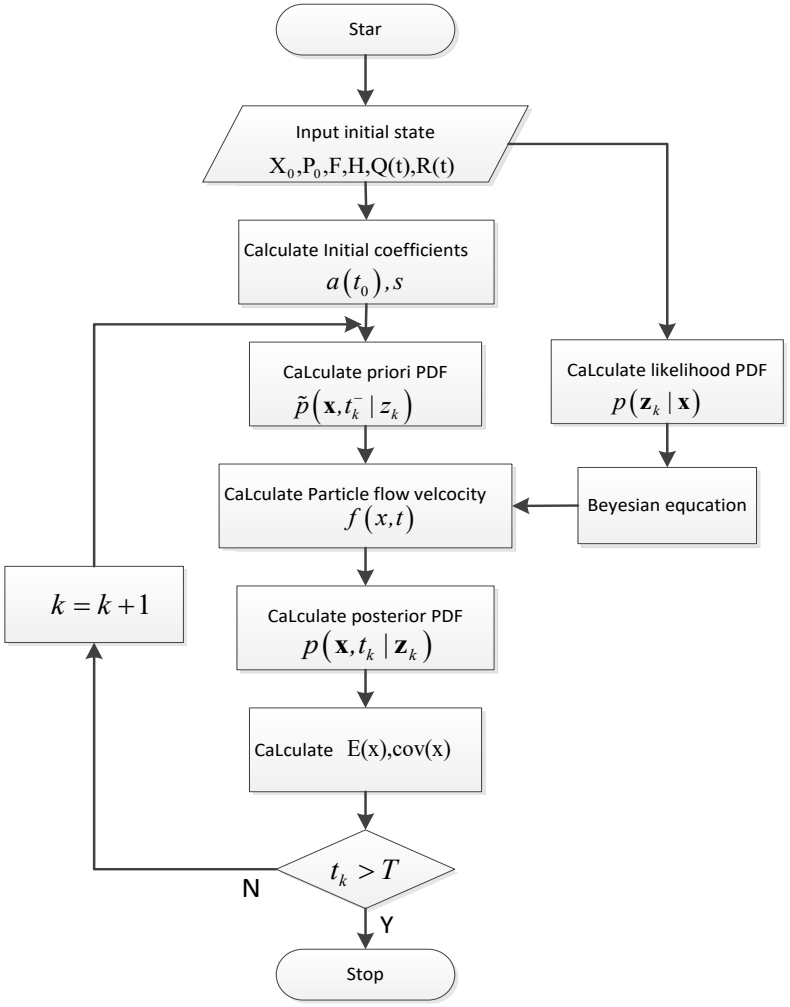


Figure 1. Flowchart of the estimator based on PFV-BEM.

3.1. Solving the Prior Probability for State Updates

In n-dimensional state space, t = t_k a symmetric projection space is given by $[-a_k, a_k]$, $a_k = \left(\mathbf{x}\big|_{\mathbf{x}=\mathbf{E}_{k-1}}\right) + 10\mathbf{Q}_{k-1}$, initial time t = t_0, and both the initial state x_0 and the initial distribution follows a normal distribution ~$(\mathbf{E}_0, \mathbf{Q}_0)$ are given.

The state vector of the state estimation model is given as $\mathbf{x} = (x_1, \ldots, x_n)$, and the PDF can be denoted as $p(\mathbf{x}, t)$. For $t \geq 0$, assume that the PDF of $\mathbf{x}$ can be expressed in the following form in ref. [24]

$$p(\mathbf{x}, t) = \sum_{i=0}^{\infty} a_i(t)\phi_i(\mathbf{x}) \tag{6}$$

where $\phi_i(x)$ is a Fourier series with symmetric properties, it denotes an orthogonal basis function in the projection subspace, $\phi_i(\mathbf{x})m_i(t)$ can be denoted as the coefficients of $p(\mathbf{x}, t)$. In Equation (1), we define $\langle \eta, \phi \rangle = \int_\Omega \eta(x)\phi(x)dx$, and $\Omega \subset R^n$ denotes the inner-product equation.

$$a_i(t) = \langle p(\mathbf{x}, t), \phi_i(\mathbf{x}) \rangle \tag{7}$$

where $V_N = span\{\phi_i\}_0^{N-1} \subset V$ denotes a complete set of basis functions for L_2.$p(\mathbf{x}, t)$, which could be denoted as the following form:

$$\widetilde{p}(\mathbf{x}, t) = \sum_{i=0}^{N-1} a_i(t)\phi_i(\mathbf{x}) \tag{8}$$

where coefficients $a_i(t)$ is denoted as $a_i(t) = \langle p(\mathbf{x}, t), \phi_i(\mathbf{x}) \rangle$. These basis functions $\phi_i(x)$ are given as follows:

$$[\mathbf{M}]_{q+1,i+1} = \langle \phi_i, \phi_q \rangle = \begin{cases} 0 & i \neq q \\ 1 & i = q \end{cases} \tag{9}$$

where $i, q = 0, 1, 2, \ldots, N-1$.

Simplify the above equation $\mathbf{M} = \mathbf{E}_{N\times N}$, $a(t) = [a_0(t)\, a_1(t) \ldots a_{N-1}(t)]^T$, $\Phi(\mathbf{x}) = [\phi_0(\mathbf{x})\, \phi_1(\mathbf{x}) \ldots \phi_{N-1}(\mathbf{x})]^T$.

Simplify Equation (8), $p(\mathbf{x}, t) = a(t)^T \Phi(\mathbf{x})$.

In ref. [24], the basic function $\phi_i(\mathbf{x})$ is given. To solve the coefficient matrix $a(t)$, the PDF solution of $\mathbf{x}$ must be solved.

Substitute Equation (7) into Equation (3), and Equation (3) can be rewritten as

$$\frac{\partial \sum_{i=0}^{N-1} a_i \phi_i}{\partial t} = \sum_{i=0}^{N-1} m_i \left\{ \frac{1}{2} \sum_{r,s=1}^{n} \frac{\partial^2 \left(g\mathbf{Q}(t) g^{\mathrm{T}} \cdot \phi_i \right)}{\partial x_r \partial x_s} \right\} - \sum_{i=0}^{N-1} m_i \left\{ \sum_{r=1}^{n} \frac{\partial (f(\mathbf{x}, t) \cdot \phi_i)}{\partial x_r} \right\} \tag{10}$$

Projecting Equation (9) onto the project space $V_N = span\{\phi_q\}_0^{N-1}$, Equation (9) can be rewritten as

$$\frac{\partial \sum_{i=0}^{N-1} \langle a_i \phi_i, \phi_q \rangle}{\partial t} = \sum_{i=0}^{N-1} a_i \left\{ \frac{1}{2} \sum_{r,s=1}^{n} \left\langle \frac{\partial^2 \left(g\mathbf{Q}(t) g^{\mathrm{T}} \cdot \phi_i \right)}{\partial x_r \partial x_s}, \phi_q \right\rangle \right\} - \sum_{i=0}^{N-1} a_i \left\{ \sum_{r=1}^{n} \left\langle \frac{\partial (f(\mathbf{x}, t) \cdot \phi_i)}{\partial x_r}, \phi_q \right\rangle \right\} \tag{11}$$

If Q, f, and g are given in Equations (1) and (2). So Equation (11) can be rewritten as

$$\sum_{i=0}^{N-1} \dot{a}_i \cdot \langle \phi_i, \phi_q \rangle = \sum_{i=0}^{N-1} a_i \left\{ \frac{1}{2} \sum_{r,s=1}^{n} \left\langle \frac{\partial^2 \left(gQ(t) g^{\mathrm{T}} \cdot \phi_i \right)}{\partial x_r \partial x_s}, \phi_q \right\rangle \right\} - \sum_{i=0}^{N-1} a_i \left\{ \sum_{r=1}^{n} \left\langle \frac{\partial (f(\mathbf{x}, t) \cdot \phi_i)}{\partial x_r}, \phi_q \right\rangle \right\} \tag{12}$$

Simplify Equation (12), we defined

$$\begin{cases} gQ(t)g^{\mathrm{T}} = B \\ [A]_{q,i} = \langle \phi_{i-1}, \phi_{q-1} \rangle = \begin{cases} 1 & i = q \\ 0 & i \neq q \end{cases} \end{cases} \tag{13}$$

Equation (12) can be rewritten as

$$[\mathbf{A}]_{q+1,i+1} = \left\langle \left\{ \frac{1}{2} \sum_{r,s=1}^{n} \frac{B \partial^2 (\phi_i)}{\partial x_r \partial x_s} - \sum_{r=1}^{n} \frac{\partial (f(\mathbf{x}, t) \cdot \phi_i)}{\partial x_r} \right\}, \phi_q \right\rangle \tag{14}$$

where $i, q = 0, 1, \ldots N-1$, $[A]_{qi}$ is the element of matrix $\mathbf{A}$ in Equation (14).

When $\mathbf{M} = \mathbf{E}_{N\times N}$, Equation (12) can be rewritten as

$$\mathbf{M}\dot{a}(t) = \mathbf{A}a(t) \tag{15}$$

Equation (15) can be rewritten as

$$\dot{a}(t) = \mathbf{A}a(t) \tag{16}$$

Set initial time $t = t_0$, and the initial information $p_0 = p(x_0, t_0 | z_0)$ and the initial coefficients $\mathbf{a}(t_0)$ of PDF can be obtained

$$\langle p_0, \phi_j \rangle = \left\langle \sum_{i=0}^{N-1} a_i(t_0) \phi_i, \phi_j \right\rangle = \sum_{i=0}^{N-1} a_i(t_0) \cdot \langle \phi_i, \phi_j \rangle \tag{17}$$

$$\mathbf{s} = [\langle p_0, \phi_0 \rangle \ \ldots \langle p_0, \phi_{N-1} \rangle]^T \tag{18}$$

$$a(t_0) = \mathbf{M}^{-1}\mathbf{s} = \mathbf{s} \tag{19}$$

Substitute Equation (19) into Equation (16), and $a(t)$ can be obtained by

$$a(t) = exp\left\{ \int_{t_0}^{t} \mathbf{A} d\tau \right\} \mathbf{s} \tag{20}$$

If t $\in [t_{k-1}, t_k)$, Equation (20) have the following simple solution

$$a(t) = exp(\mathbf{A}(t - t_{k-1}))a(t_{k-1}) \tag{21}$$

The priori coefficients $a^-(t_k)$ can be obtained by

$$a^-(t_k) = exp(\mathbf{A}(t_k - t_{k-1}))a(t_{k-1}) \tag{22}$$

where $a(t_{k-1})$ are posterior coefficients of state **x** at the time t $= t_{k-1}$.

A priori PDF of state **x** is given as

$$\widetilde{p}\left(\mathbf{x}, t_k^- \middle| z_k\right) = \sum_{i=0}^{N-1} a_i^-(t_k)\phi_i = a^-(t_k)^T \Phi(\mathbf{x}) \tag{23}$$

3.2. Using Particle Flow Velocity for Measurement Update

Randomly extract particle points based on the prior PDF at time t_k. The posterior PDF of the state estimation model at time can be obtained by the Bayesian formula Equation (4). The homotopy function is expressed as

$$p(x, \lambda) = \frac{g(x)h^\lambda(x)}{\int\limits_x g(x)h^\lambda(x)dx} \tag{24}$$

where $p(x, \lambda)$ is the conditional PDF, $g(x)$ represents the prior PDF, and $h(x)$ represents the likelihood PDF.

Taking the logarithm of the above equation yields:

$$logp(x, \lambda) = logg(x) + \lambda logh(x) - logK(\lambda) \tag{25}$$

In the equation, $K(\lambda) = \int\limits_x g(x)h^\lambda(x)dx$ is the normalization constant independent of x. Let λ change continuously from 0 to 1, and the homotopy function $p(x, \lambda)$ defines the probability distribution of the change from the prior distribution ($\lambda = 0$) to the posterior distribution ($\lambda = 1$). The flow velocity of the state particle x during this process is described by the following differential equation.

$$\frac{\partial x}{\partial \lambda} = f(x, \lambda) + \frac{\mathrm{d}\omega}{\partial \lambda} \tag{26}$$

The particle flow velocity $f(x, \lambda)$ satisfies FPE, which is the process noise. It can be obtained that

$$\frac{\partial p}{\partial \lambda} = -div(fp) + \frac{1}{2} div\left(Q\frac{\partial p}{\partial x}\right) \tag{27}$$

Simplify the above equation:

$$\frac{\partial logp(x, \lambda)}{\partial \lambda} p(x, \lambda) = -div(fp) + \frac{1}{2} div\left(Q\frac{\partial p}{\partial x}\right) \tag{28}$$

Substituting Equation (25) into Equation (28) yields the following:

$$\left[logh(x) - \frac{dlogK(\lambda)}{d\lambda}\right] p(x,\lambda) = -div(fp) + \frac{1}{2} div\left(Q\frac{\partial p}{\partial x}\right) \tag{29}$$

Simplify the above equation

$$logh - \frac{dlogK}{d\lambda} = -div(f) - \frac{\partial logp}{\partial x} f + \frac{1}{2p} div\left(Q\frac{\partial p}{\partial x}\right) \tag{30}$$

The particle flow velocity field can be obtained as follows:

$$f = -\left[\frac{\partial^2 logp}{\partial x^2}\right]^{-1} \left(\frac{\partial logh}{\partial x}\right)^T \tag{31}$$

The variance Q is

$$Q = -\left[\frac{\partial^2 logp}{\partial x^2}\right]^{-1} \frac{\partial^2 logh}{\partial x^2} \left[\frac{\partial^2 logp}{\partial x^2}\right]^{-1} \tag{32}$$

By using the velocity field calculation Equation (31), the numerical integration of the prior particles $\mathbf{x}_k^-$ with the posterior particles $\mathbf{x}_k^+$ can be obtained as follows:

$$\mathbf{x}_k^+ = \mathbf{x}_k^- + \int_0^1 f(x,\lambda) d\lambda \tag{33}$$

The estimated value $\widehat{x}$ of the state can be denoted as

$$\widehat{x} = \frac{1}{N} \sum_{k=1}^{N} \mathbf{x}_k^+ \tag{34}$$

4. Illustrative Simulation Examples and Analysis

4.1. Example 1 and Analysis

For the first simulation example, in ref. [26] considered a scalar signal process x_t is observed through a scalar measurement process y_t, and the NLSs model is given as

$$\begin{cases} \dot{x}_t = sin(x_t/2) + \mathrm{w}_t \\ \mathrm{y}_t = x_t + v_t \end{cases} \tag{35}$$

where w_t and v_t are mutually independent, and the initial state is $x_0 = 0$, $\mathrm{y}_0 = 0$.

The simulation time is T = 2 s, and the number of particles is 1000. The estimated results are shown in Figures 2 and 3. The priori PDF and posterior PDFs are plotted in Figure 2. The normalized density functions are plotted every other 0.1 s. The partial results have been shown in Figure 2, where the actual position information (represented by black stars), PF estimated position information (represented by green squares), and PFV-BEM estimated position information (represented by red circles) are all displayed in Figure 2. The approximate particle flow in the PDF of the state is shown through the four graphs in Figure 2. Figure 3 provides a comparison chart of the estimated mean and variance of the two algorithms, where the PF estimation error is represented by a green line and the PFV-BEM estimation error is marked by a red line.

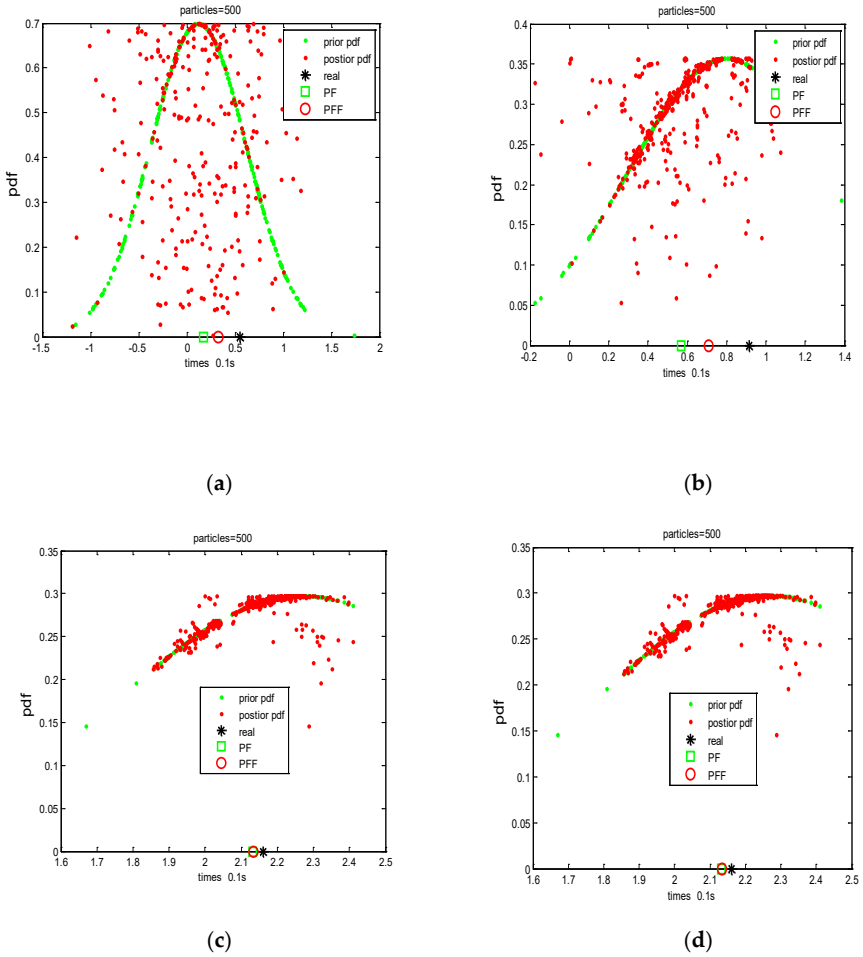


Figure 2. The comparison results of prior and posterior probability density functions under different time intervals, as well as the actual information and the estimation results obtained from the two filtering algorithms. The black asterisks represent the actual position information, the green squares denote the estimation results of the Particle Filter (PF) algorithm, and the red circles indicate the estimation results of the PFV−BEM filtering algorithm. (**a**) Display state estimation information result at t = 0.1 s, (**b**) display system state estimation information result at t = 1 s, (**c**) display system state estimation information result at t = 1.5 s, (**d**) display system state estimation information result at t = 2 s.

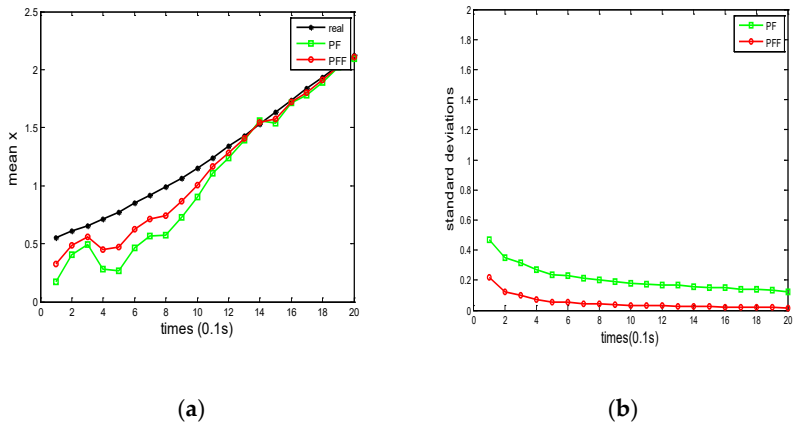


Figure 3. The estimated mean and standard deviation of PF and PFV-BEM should be listed as (**a**) the first section contains the comparison results of estimation errors between the two algorithms; (**b**) the second section contains the comparison results of estimating mean square error using the two algorithms.

Figure 2a–d display partial estimation results of both particle filtering (PF) and the filtering algorithm proposed in this paper, along with the temporal evolution of particles. As illustrated in Figure 2, compared to the PF algorithm, the proposed algorithm demonstrates higher estimation accuracy, faster convergence, improved precision, and stable performance in system state estimation. The simulation results clearly indicate its capability to effectively track the system state. The feasibility of the proposed algorithm has been validated, suggesting that PFV-BEM is an effective Bayesian estimation method for addressing state estimation problems in one-dimensional nonlinear systems.

4.2. Example 2 and Analysis

The continuous time stochastic differential equation interpretation of the dynamic tracking model is given by refs. [27], which is described as follows:

The state vector is defined as $X(t) = \left[x(t)\ v_x(t)\ y(t)\ v_y(t)\right]^T$

The measurement vector is defined as $z_{\mathrm{k}} = \left[\mathrm{r_k}\ \theta_{\mathrm{k}}\right]^T$

The state estimation model is given as

$$\dot{X}(t) = F_k X(t) + Q_k \tag{36}$$

The measurement model is denoted as

$$z_k = \begin{bmatrix} \sqrt{\mathrm{x_2} + \mathrm{y_2}} \\ \arctan(\frac{y}{x}) \end{bmatrix} + R_{\mathrm{k}} \tag{37}$$

where Q_{k}, R_k are, respectively, described as process noise of the state equation and measurement noise of the measurement equation, and the covariance matrix, respectively, are Q and R. The state transition matrix is given as

$$F_k = [[1,\ 1,\ 0,\ 0], [0,\ 1,\ 0,\ 0], [0,\ 0,\ 1,\ 1], [0,\ 0,\ 0,\ 1]]$$

The number of targets is four, the time steps are 400 s, the filter sampling time interval is $\Delta T = 1s$, the running time T = 400 s, and the number of particles is 2000.

For the trajectory tracking model of four targets, the state estimation results based on the Particle Filter (PF) and the Particle Filter-based variational Bayesian estimation method (PFV-BEM) are presented in Figures 4–6. Figure 4 displays the state estimation results of the four targets based on PFV-BEM, including a comparison between the actual position information of the four targets and the position estimates obtained using the proposed algorithm, as well as a comparison between the actual velocity information of the targets and the velocity estimates derived from the PFV-BEM algorithm. Figure 5 shows a comparison of the position estimates of the four targets obtained using both the PF and PFV-BEM algorithms. Figure 5 also presents the velocity estimation results of the four targets based on PF and PFV-BEM. Additionally, Figure 6 illustrates the position and velocity error estimation results of the four targets tracked using the two algorithms. Here, the estimation errors of PF are represented by blue lines, while those of PFV-BEM are represented by orange lines.

Figures 4–6 present partial experimental results obtained using the Particle Filter (PF) algorithm and the PFV-BEM algorithm. From Figure 4, it can be seen that the PFV-BEM algorithm can acquire the position and velocity tracking information of four targets. When compared with the actual information of the targets, it shows favorable tracking results. The experimental results indicate that the PFV-BEM algorithm can effectively track the changes in the system state, fully verifying the feasibility of the algorithm. Figure 5 displays the estimation results of the position and velocity information of the four targets, respectively. The experimental results show that, compared with the PF algorithm, the PFV-BEM method

exhibits higher accuracy and can well address the state estimation problem in nonlinear systems. Figure 6 provides the error estimation results for tracking the four targets. The experimental results demonstrate that the PFV-BEM method has high estimation accuracy. Evidently, for high-dimensional nonlinear systems, the PFV-BEM algorithm functions as an effective Bayesian estimation method for tackling the state estimation problem.

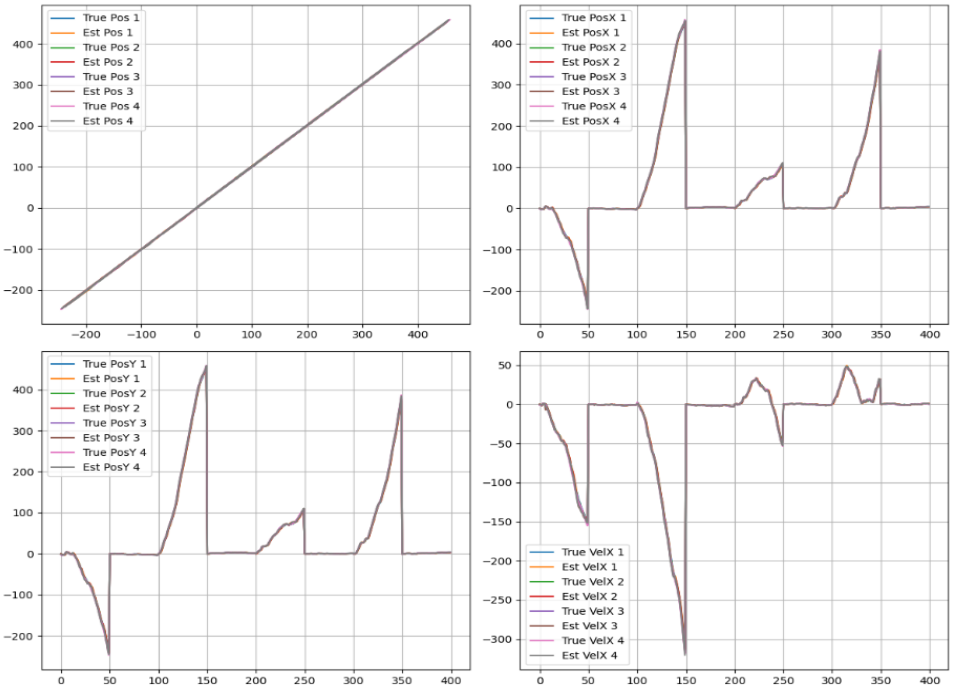


Figure 4. The state estimation results of 4 targets based on PFV-BEM.

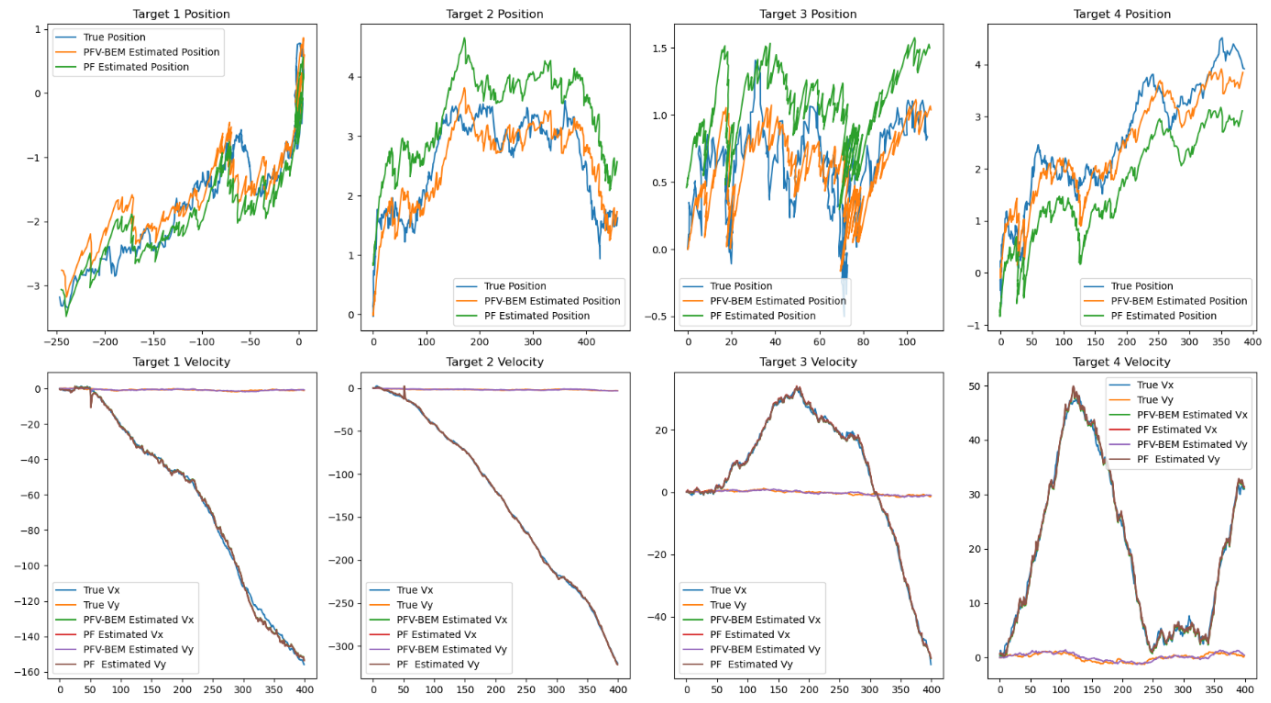


Figure 5. The position estimation and velocity estimation results of 4 targets based on PF and PFV-BEM.

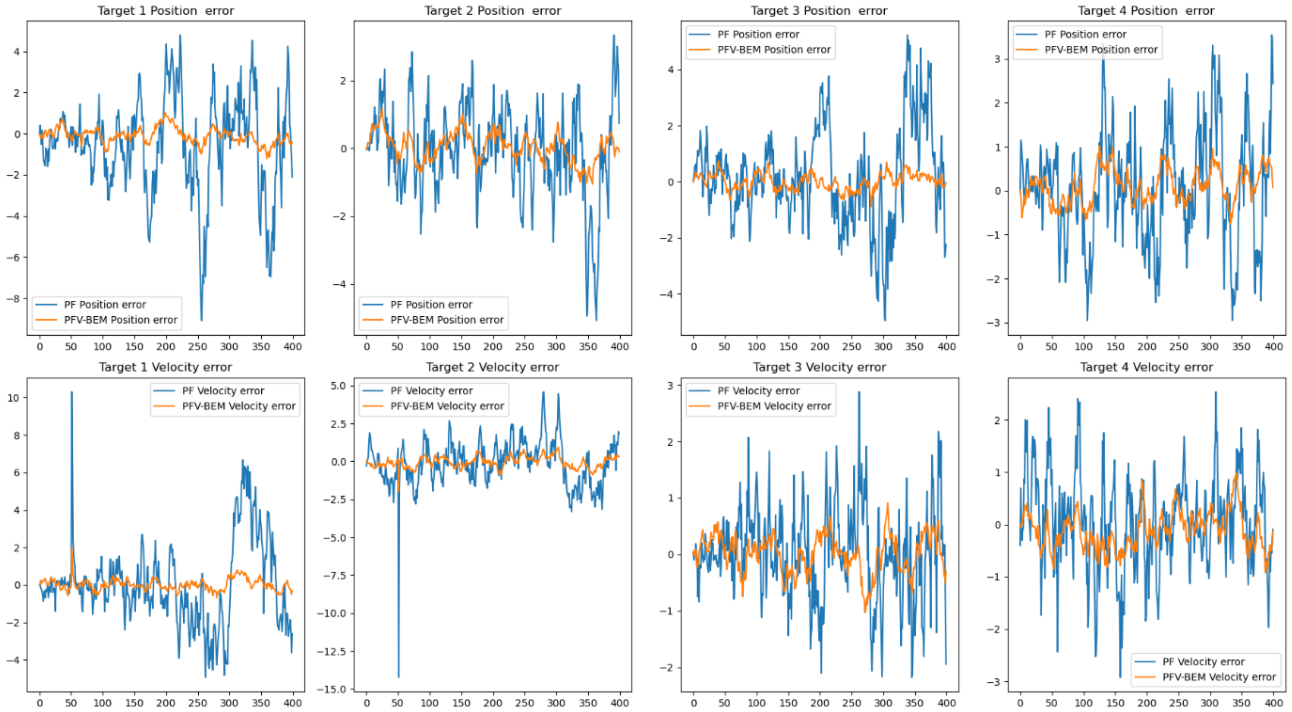


Figure 6. The position and velocity estimation error results of 4 targets based on PF and PFV-BEM.

5. Conclusions

This article validates the effectiveness of the PFV-BEM algorithm within Bayesian estimation methods. The core of this algorithm lies in the filter estimator's ability to accurately calculate the prior and posterior probability density functions (PDFs) through Bayesian iteration. Subsequently, it estimates the mean and standard deviation of the system state. The article elaborates on the theoretical derivation in detail, primarily focusing on two aspects: obtaining prior PDFs through state updates and obtaining posterior PDFs through measurement updates.

Simulation experiments demonstrate that the PFV-BEM filter outperforms the particle filter (PF) method in terms of estimation accuracy. It exhibits significant advantages in processing sensor data from complex nonlinear systems, particularly in high-dimensional state estimation problems. Not only does it improve performance, but it also accelerates the computation speed and reduces computational complexity.

Although the algorithm provides high-precision approximate solutions through iterative processes over time, selecting appropriate iteration time intervals remains challenging. Moreover, the convergence and robustness of the algorithm require further investigation. Therefore, we are committed to exploring more robust Bayesian estimation methods. These unresolved issues will become the focus of our future research. Although the PFV-BEM method demonstrates higher accuracy, its computational efficiency in high-dimensional spaces still requires further exploration.

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