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Singularity Formation of Classical Solutions to Euler–Boltzmann Equations with Damping in $\mathbb{R}^3$

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Abstract: The Euler–Boltzmann equations are an important class of mathematical models that describe the coupling between particle transport and macroscopic fluid dynamics. They find broad applications in plasma physics, rarefied gas dynamics, and astrophysics. In these fields, incorporating a time-dependent damping term is crucial for modeling real-world scenarios, as opposed to idealized inviscid conditions. In recent years, there has been growing interest in the long-time behavior of their solutions. This paper focuses on the initial value problem for the three-dimensional Euler–Boltzmann equations with time-dependent damping, aiming to investigate the finite-time blowup behavior of classical solutions. We use an integration method with general test function f and show that if the initial data are sufficiently large, classical solutions of the Euler–Boltzmann equations with time-dependent damping in $\mathbb{R}^3$ will blowup on or before the finite time $T^* > 0$.

Keywords: blowup; Euler–Boltzmann equations; time-dependent damping; initial value problem; test function

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Academic Editor: Octav Olteanu

Received: 16 April 2025
Revised: 16 May 2025
Accepted: 23 May 2025
Published: 27 May 2025

Citation: Liu, J.; Liu, M.; Yuen, M. Singularity Formation of Classical Solutions to Euler–Boltzmann Equations with Damping in $\mathbb{R}^3$. *Symmetry* **2025**, *17*, 835. https://doi.org/10.3390/sym17060835

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1. Introduction

Fluid mechanics that include the contributions of radiation energy and momentum are known as radiation hydrodynamics [1,2]. Radiation fluid dynamics theory has many applications, including various astrophysical phenomena, such as waves and oscillations in stellar atmospheres and envelopes, nonlinear stellar pulsations, supernova explosions, and stellar winds, see [1–3]. At high temperatures, the energy and momentum densities of the radiation field can match or even exceed those of the related fluid properties. In this condition, the radiation field remarkably influences fluid dynamics. In the field of radiation hydrodynamics, the Euler equations describe the macroscopic motion of the fluid, while the Boltzmann equation describes the evolution of the radiation field. Radiation hydrodynamics studies how radiation affects the motion and thermal state of a fluid, and in turn, how the state of the fluid affects the propagation of radiation. Therefore, the mathematical equations that govern radiation fluid dynamics are the coupling of the Euler equations for compressible fluids,

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \left(\rho \mathbf{u} + \frac{1}{c^2} F_r\right)_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + P_r) + \nabla p = 0, \end{cases} \tag{1}$$

where $\rho = \rho(\mathbf{x}, t)$, $\mathbf{u} = (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$, $p = p(\rho)$ represent the mass density, the fluid velocity, and the pressure, respectively, F_r and P_r represent the radiative flux and the radiative pressure tensor, respectively, defined by

$$F_r = \int_0^\infty \mathrm{d}\nu \int_{\mathbb{S}^2} \mathbf{\Omega} I(\nu, \mathbf{\Omega})\, \mathrm{d}\mathbf{\Omega}, \quad P_r = \frac{1}{c} \int_0^\infty \mathrm{d}\nu \int_{\mathbb{S}^2} \mathbf{\Omega} \otimes \mathbf{\Omega} I(\nu, \mathbf{\Omega}) \mathrm{d}\mathbf{\Omega}, \tag{2}$$

with the Boltzmann equation for particle transport, namely,

$$\begin{aligned} &\frac{1}{c}\frac{\partial I(\nu, \mathbf{\Omega})}{\partial t} + \mathbf{\Omega} \cdot \nabla I(\nu, \mathbf{\Omega}) = S(\nu, \rho) - \sigma_a(\nu, \rho) I(\nu, \mathbf{\Omega}) \\ &+ \int_0^\infty \mathrm{d}\nu' \int_{\mathbb{S}^2} \Big(\frac{\nu}{\nu'} \sigma_s(\nu' \to \nu, \mathbf{\Omega}' \cdot \mathbf{\Omega}, \rho) I(\nu', \mathbf{\Omega}') - \sigma_s(\nu \to \nu', \mathbf{\Omega} \cdot \mathbf{\Omega}', \rho) I(\nu, \mathbf{\Omega}) \Big) \mathrm{d}\mathbf{\Omega}', \end{aligned} \tag{3}$$

where c, $I\ (\mathbf{x}, t, \nu, \mathbf{\Omega})$ represent the light speed, and the specific intensity of radiation at space point $\mathbf{x} \in \mathbb{R}^3$, with the radiation frequency ν in a direction $\mathbf{\Omega} \in \mathbb{S}^2$ ($\mathbb{S}^2$ is the unit sphere of $\mathbb{R}^3$), respectively. $S(\nu, \rho) = S(x, t, \nu, \mathbf{\Omega}, \rho)$ is the rate of energy emission due to spontaneous processes. $\sigma_a(\nu, \rho) = \sigma_a(x, t, \nu, \mathbf{\Omega}, \rho)$ is the absorption coefficient.

Similar to the way photons interact with matter through absorption, photons can interact with matter through scattering, and the scattering interaction serves to change the photon's characteristics $(\nu', \mathbf{\Omega}')$ to a new set of photon characteristics $(\nu, \mathbf{\Omega})$. This leads to the definition of the 'differential scattering coefficient' $\sigma_s(\nu' \to \nu, \mathbf{\Omega}' \cdot \mathbf{\Omega}, \rho) = \sigma_s(x, t, \nu' \to \nu, \mathbf{\Omega}' \cdot \mathbf{\Omega}, \rho)$.

Due to its complexity and mathematical difficulty, the Euler–Boltzmann system is a challenging subject. In [4], under appropriate assumptions on the transport coefficients and data, the authors proved the existence of weak solutions to the Cauchy problem for one-dimensional equations of motion of a compressible inviscid gas coupled with radiation. Subsequently, in [5], Pu and Zhang proved the existence of global smooth solutions to the isentropic Euler–Boltzmann equations in R^d with a certain class of small initial data. In [6], Li and Xi proved that the Cauchy problem of three-dimensional compressible radiation fluids with vacuum shares the same BKM-type blowup criterion as the compressible Navier–Stokes equations, while the Serrin-type criterion should involve the L^p $(p \in [2, 3])$ norm of the density gradient.

We examine the manifestation in the equation of transfer of the quantum statistics (i.e., (3)) obeyed by photons. From the 'induced processes' and the local thermodynamics equilibrium (LTE) assumption together, $S(\nu, \rho)$ and $\sigma_a(\nu, \rho)$ in (3) can be written as

$$S(\nu, \rho) = a(\nu, \rho)\overline{B}(\nu)\left(1 + \frac{c^2 I(\nu, \mathbf{\Omega})}{2h\nu^3}\right), \quad \sigma_a(\nu, \rho) = a(\nu, \rho)\left(1 + \frac{c^2 \overline{B}(\nu)}{2h\nu^3}\right), \tag{4}$$

where $a(\nu, \rho)$ is the absorption coefficient, we assume throughout this paper that $a(\nu, \rho) > 0$, $\overline{B}(\nu)$ is a simplification of the Planck function, which represents the energy density of blackbody radiation, and h is the Planck constant. By using Equation (4) and assuming $\sigma_s = 0$ in (3), we can rewrite the Equations (1) and (3) as follows

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = -\frac{1}{c} \int_0^\infty d\nu \int_{\mathbb{S}^2} \mathbf{\Omega}(a(\nu, \rho)(\overline{B}(\nu) - I)) d\mathbf{\Omega}, \\ \frac{1}{c} I_t + \mathbf{\Omega} \cdot \nabla I = a(\nu, \rho)(\overline{B}(\nu) - I). \end{cases} \tag{5}$$

For system (5) in [7], Zhong and Jiang obtained the local existence of C^1 solutions to the Cauchy problem for the equations of multidimensional radiation hydrodynamics, and used the energy method to prove that C^1 solutions to the Cauchy problem in $\mathbb{R}^3$ will blowup in finite time when the initial data are sufficiently large. In [8,9], the authors constructed

global weak entropy solutions using the Godunov finite difference scheme, establishing the global existence of weak entropy solutions for the one-dimensional Euler–Boltzmann equations in L^∞ with the aid of the compensated compactness method. In [10], Jiang obtained the finite-time blowup of C^1 solutions to the Cauchy problem in $\mathbb{R}^3$ by utilizing the weighted functional

$$P(r,t) = \int_{|\mathbf{x}|>r} \omega(\mathbf{x},r)(\rho(\mathbf{x},t) - \overline{\rho})\, \mathrm{d}\mathbf{x}, \quad r > 0, \tag{6}$$

where $\omega(\mathbf{x},r) = |\mathbf{x}|^{-1}(|\mathbf{x}| - r)^2$. Subsequently, in [11], Jiang and Yin used the same weighted functional, and studied the finite-time blowup of C^1 solutions to the Cauchy problem to the three-dimensional non-relativistic radiation hydrodynamic equations in $\mathbb{R}^3$. In [12], Li and Zhu proved that no matter how small the initial data are, if the initial mass density has compact support, the classical solutions to the Cauchy problem for multi-dimensional compressible radiation hydrodynamics equations with a vacuum will blowup within a finite time. In [13], Cao and Li investigated the formation of singularities in regular solutions to the Cauchy problem for n-dimensional isentropic Euler equations and the Euler–Boltzmann equations with vacuum. Specifically, it was shown that regardless of how small and smooth the initial data are, if the initial velocity satisfies certain conditions on the integral J in the "isolated mass group", then regular solutions to the Euler system (for $J \leq 0$, $n \geq 1$) and the Euler–Boltzmann system (for $J < 0, n \geq 1$, and $J = 0, n \geq 2$) will blowup in finite time. Some studies of simplified versions of radiative hydrodynamic model systems are referred to in [14,15].

It is important to recognize that, although the Euler–Boltzmann system (5) is valuable for modeling ideal radiative fluids, it inherently neglects viscous effects and other physical mechanisms present in realistic scenarios. To more accurately capture the dynamics of real-world radiative fluids, it is essential to develop and employ more sophisticated models that incorporate these effects. For example, damping terms can be introduced to improve the model's capacity to describe practical situations. In particular, when considering the motion of a radiative fluid through a porous medium, the Euler–Boltzmann equation with a time-dependent damping term can be used. In this paper, we consider the following three-dimensional Euler–Boltzmann equations with time-dependent damping of a polytropic and isentropic fluid in radiation hydrodynamics

$$\begin{cases} \frac{1}{c} I_t + \mathbf{\Omega} \cdot \nabla I = a(\nu,\rho)(\overline{B}(\nu) - I), \\ \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = -\frac{1}{c} \int_0^\infty d\nu \int_{\mathbb{S}^2} \mathbf{\Omega}(a(\nu,\rho)(\overline{B}(\nu) - I)) d\mathbf{\Omega} - \frac{\mu}{(1+t)^\lambda}\, \rho \mathbf{u}, \end{cases} \tag{7}$$

where $\frac{\mu}{(1+t)^\lambda}$ is the frictional coefficient with $\lambda \in (-\infty, +\infty)$ and $\mu \geq 0$ represents the damping coefficient and decay rate, repectively. For the polytropic, ideal, and isentropic fluid, the pressure p is $p = A\rho^\gamma$, where $A > 0$, and $\gamma > 1$ is the adiabatic index.

In the following, we consider the initial value problem of the system (7) with the initial data

$$\begin{cases} I(\mathbf{x},0,\nu,\Omega) = \overline{B}(\nu) + I_0(\mathbf{x},\nu,\Omega), \\ \rho(\mathbf{x},0) = \overline{\rho} + \rho_0(\mathbf{x}) > 0, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \\ \mathrm{supp}(I_0,\rho_0,\mathbf{u}_0) \subseteq \{\mathbf{x} : |\mathbf{x}| \leq R\}. \end{cases} \tag{8}$$

Define the total mass function by

$$m(t) = \int_{\mathbb{R}^3} (\rho(\mathbf{x},t) - \overline{\rho}) d\mathbf{x}, \quad t \in [0, T^*]. \tag{9}$$

For completeness, we provide the related results for the Euler system (1). The compressible Euler equations, serving as a fundamental model for the study of fluid dynamics, plasma, and atmospheric dynamics, have been extensively researched. Due to their physical significance and mathematical challenges, many scholars have carried out in-depth analyses of the blowup phenomena, which can be found in [16–21].

2. Materials and Methods

In the blowup analysis for solutions, the integral functional method is often used with the aim of proving that if the initial data of the functional are large enough, the singularity of the solution will occur in a finite time. Sideris [17] initially introduced the integral functional

$$F(t) = \int_{\mathbb{R}^3} \rho \mathbf{x} \cdot \mathbf{u} \mathrm{d}V, \tag{10}$$

and demonstrated that for three-dimensional non-isentropic compressible Euler equations, if the initial value $F(0)$ is sufficiently large, the C^1 solutions will develop singularities within a finite time. It is worth mentioning that the weighted functional (6) was originally used by Sideris in this work. In [22], Lei et al., by using test functions $\frac{1}{re^r}$ and the modified Bessel function associated with the radius r, proved that under the assumptions of symmetric initial velocity and initial sound speed vanishing at the origin, the solutions for Euler equations in $\mathbb{R}^2$ and $\mathbb{R}^3$ will blowup in finite time. Subsequently, Zhu, Tu, and Fu in [23] considered variations of (10), given by

$$F_1(t) = \int_{\mathbb{R}^3} |\mathbf{x}|^n \rho \mathbf{x} \cdot \mathbf{u} \, d\mathbf{x}, \tag{11}$$

$$F_2(t) = \int_{\mathbb{R}^3} \left(1 - e^{-n|\mathbf{x}|}\right) \rho \mathbf{x} \cdot \mathbf{u} \, d\mathbf{x}. \tag{12}$$

Using these functionals, they were able to derive finite-time blowup results for the three-dimensional non-isentropic compressible Euler equations, under the conditions that $F_i(0)$ are sufficiently large. In [24], Yuen obtained the blowup results for the irrotational C^1 solution by introducing the new density-independent functional

$$H(t) = \int_{\mathbb{R}^N} \mathbf{x} \cdot \mathbf{u} \, d\mathbf{x}, \tag{13}$$

In addition, Cheung, Wong, and Yuen, in their work [25], formulated a test function $f = f(r)$ that shows an increasing C^1 property. They applied the functional

$$F(t) = \int_{\Omega} f \rho \mathbf{x} \cdot \mathbf{u} \mathrm{d}\mathbf{x}, \tag{14}$$

to investigate the blowup results of the initial-boundary value problem for the three-dimensional non-isentropic Euler equations. In [26], Dong presented the blowup results of the three-dimensional compressible isothermal Euler equations in both radial symmetry and non-radial symmetry by using the integral functional method. Recently, in [27], Liu, Qin, and Yuen used the density-independent functionals

$$F_1(t) = \int_{\mathbb{R}^n} f_1 \mathbf{x} \cdot \mathbf{u} \, d\mathbf{x}, \tag{15}$$

and

$$F_2(t) = \int_{\mathbb{R}^n} f_2 \mathbf{x} \cdot \mathbf{u} \, d\mathbf{x}, \tag{16}$$

with the general test function $f = f(r)$, and demonstrated the corresponding blowup results of C^1 irrotational solutions for Euler equations with time-dependent damping in $\mathbb{R}^n (n \geq 2)$, provided that the density-independent initial functional is sufficiently large.

In this article, we find general test functions $f = f(r)$ with $r = |\mathbf{x}|$, where $f(r)$ is a strictly increasing C^1 function with $r = |\mathbf{x}|$ satisfying $f(0) = 0$. By using the weighted functional

$$F(t) = \int_{\mathbb{R}^3} f\mathbf{x} \cdot \rho \mathbf{u} d\mathbf{x} + \frac{1}{c^2} \int_{\mathbb{R}^3} d\mathbf{x} \int_0^\infty d\nu \int_{\mathbb{S}^2} f\mathbf{x} \cdot \mathbf{\Omega} U \, d\mathbf{\Omega}, \tag{17}$$

where $U := I(\mathbf{x}, t, \nu, \mathbf{\Omega}) - \overline{B}(\nu)$, we investigate the blowup phenomenon of classical solutions for Euler–Boltzmann equations with time-dependent damping in $\mathbb{R}^3$.

3. Results

In this section, we present the main results along with the detailed proof.

3.1. Existence of Classical Solutions

We first note that system (7) can be cast into a first-order symmetric hyperbolic form. According to the theories established by Kato [28] and Majda [29], for smooth initial data, classical solutions to such systems exist locally in time. To establish the existence of classical solutions to system (7), we begin with the following lemma.

Lemma 1. *Consider system (7) expressed in the first-order symmetric hyperbolic form*

$$A_0 \frac{\partial \mathbf{W}}{\partial t} + \sum_{k=1}^{n} A_k \frac{\partial \mathbf{W}}{\partial x_k} = \mathbf{D}, \tag{18}$$

where $\mathbf{W} := (I, p, u_1, u_2, u_3)^T$, $A_0, A_1, \ldots, A_n$ *are symmetric matrices with* A_0 *being positive definite, and* $\mathbf{D}$ *is a 5-dimensional column vector. Then, there exists a time interval* $[0, T^*)$ *such that system (7) has a unique classical solution. Assume the initial condition* $\mathbf{W}(\mathbf{x}, 0) = \mathbf{W}_0(\mathbf{x})$ *belongs to the Sobolev space* $H^s(\mathbb{R}^n)$ *(with* $s > \frac{n}{2} + 1$*). Then, there exists a time interval* $[0, T^*)$ *such that system (7) has a unique classical solution* $\mathbf{W} \in C([0, T^*); H^s(\mathbb{R}^n)) \cap C^1([0, T^*); H^{s-1}(\mathbb{R}^n))$.

Proof. We denote the direction vector $\mathbf{\Omega}$ as

$$\mathbf{\Omega} := (\Omega_1, \Omega_2, \Omega_3),$$

where Ω_1, Ω_2, and Ω_3 are its components in the x_1, x_2, and x_3, directions, respectively. Applying the chain rule, we obtain

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial p} \frac{\partial p}{\partial t}.$$

Using this relation, system (7) can be rewritten in the following form

$$\begin{cases} \frac{1}{c} \frac{\partial I}{\partial t} + \sum\limits_{k=1}^{3} \Omega_k \frac{\partial I}{\partial x_k} = a(\nu, \rho) U, \\ \frac{\partial \rho}{\partial p} \frac{\partial p}{\partial t} + \sum\limits_{k=1}^{3} u_k \frac{\partial \rho}{\partial p} \frac{\partial p}{\partial x_k} + \rho \sum\limits_{k=1}^{3} \frac{\partial u_k}{\partial x_k} = 0, \\ \rho \frac{\partial u_i}{\partial t} + \sum\limits_{k=1}^{3} \rho u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} = -\frac{1}{c} \int_0^\infty d\nu \int \Omega_i a(\nu, \rho) U d\Omega_i - \frac{\mu}{(1+t)^\lambda} \rho u_i, \quad (i = 1, 2, 3). \end{cases} \tag{19}$$

Next, we explicitly write out the coefficient matrices A_0, A_1, A_2, A_3 of system (19). The specific forms of these matrices are as follows

$$A_0 = \begin{pmatrix} \frac{1}{c} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial \rho}{\partial p} & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & \rho \end{pmatrix}, \quad A_1 = \begin{pmatrix} \Omega_1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial \rho}{\partial p} u_1 & \rho & 0 & 0 \\ 0 & 1 & \rho u_1 & 0 & 0 \\ 0 & 0 & 0 & \rho u_1 & 0 \\ 0 & 0 & 0 & 0 & \rho u_1 \end{pmatrix}, \tag{20}$$

$$A_2 = \begin{pmatrix} \Omega_2 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial \rho}{\partial p} u_2 & 0 & \rho & 0 \\ 0 & 0 & \rho u_2 & 0 & 0 \\ 0 & 1 & 0 & \rho u_2 & 0 \\ 0 & 0 & 0 & 0 & \rho u_2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} \Omega_3 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial \rho}{\partial p} u_3 & 0 & 0 & \rho \\ 0 & 0 & \rho u_3 & 0 & 0 \\ 0 & 0 & 0 & \rho u_3 & 0 \\ 0 & 1 & 0 & 0 & \rho u_3 \end{pmatrix}. \tag{21}$$

It is clear that matrix A_0 is positive definite. Although the matrices A_1, A_2, A_3 exhibit a symmetric structure, the entries at corresponding symmetric positions are not exactly equal. In the following, we will address this issue by adjusting the coefficients of system (19) to ensure full symmetry.

By multiplying both sides of the momentum equation in system (19) by ρ, we obtain the following form

$$\begin{cases} \frac{1}{c}\frac{\partial I}{\partial t} + \sum\limits_{k=1}^{3} \Omega_k \frac{\partial I}{\partial x_k} = a(\nu, \rho) U, \\ \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial t} + \sum\limits_{k=1}^{3} u_k \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial x_k} + \rho \sum\limits_{k=1}^{3} \frac{\partial u_k}{\partial x_k} = 0, \\ \rho^2 \frac{\partial u_i}{\partial t} + \sum\limits_{k=1}^{3} \rho^2 u_k \frac{\partial u_i}{\partial x_k} + \rho \frac{\partial p}{\partial x_i} = -\frac{\rho}{c} \int_0^\infty d\nu \int \Omega_i a(\nu, \rho) U d\Omega_i - \frac{\mu}{(1+t)^\lambda} \rho^2 u_i, \ \ (i = 1, 2, 3). \end{cases} \tag{22}$$

The corresponding coefficient matrices now take the following explicit forms

$$A_0 = \begin{pmatrix} \frac{1}{c} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial \rho}{\partial p} & 0 & 0 & 0 \\ 0 & 0 & \rho^2 & 0 & 0 \\ 0 & 0 & 0 & \rho^2 & 0 \\ 0 & 0 & 0 & 0 & \rho^2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} \Omega_1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial \rho}{\partial p} u_1 & \rho & 0 & 0 \\ 0 & \rho & \rho^2 u_1 & 0 & 0 \\ 0 & 0 & 0 & \rho^2 u_1 & 0 \\ 0 & 0 & 0 & 0 & \rho^2 u_1 \end{pmatrix}, \tag{23}$$

$$A_2 = \begin{pmatrix} \Omega_2 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial \rho}{\partial p} u_2 & 0 & \rho & 0 \\ 0 & 0 & \rho^2 u_2 & 0 & 0 \\ 0 & \rho & 0 & \rho^2 u_2 & 0 \\ 0 & 0 & 0 & 0 & \rho^2 u_2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} \Omega_3 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial \rho}{\partial p} u_3 & 0 & 0 & \rho \\ 0 & 0 & \rho^2 u_3 & 0 & 0 \\ 0 & 0 & 0 & \rho^2 u_3 & 0 \\ 0 & \rho & 0 & 0 & \rho^2 u_3 \end{pmatrix}, \tag{24}$$

and $\mathbf{D}$ satisfies

$$\mathbf{D} = \begin{pmatrix} a(\nu, \rho) U \\ 0 \\ -\frac{\rho}{c} \int_0^\infty d\nu \int \Omega_1 a(\nu, \rho) U d\Omega_1 - \frac{\mu}{(1+t)^\lambda} \rho^2 u_1 \\ -\frac{\rho}{c} \int_0^\infty d\nu \int \Omega_2 a(\nu, \rho) U d\Omega_2 - \frac{\mu}{(1+t)^\lambda} \rho^2 u_2 \\ -\frac{\rho}{c} \int_0^\infty d\nu \int \Omega_3 a(\nu, \rho) U d\Omega_3 - \frac{\mu}{(1+t)^\lambda} \rho^2 u_3 \end{pmatrix}. \tag{25}$$

So far, we have transformed system (7) into a first-order symmetric hyperbolic form. The proof of Lemma 1 is completed. □

In the previous section, we established the existence of a local classical solution to system (7). We now present the main result of this paper.

3.2. Main Result

Theorem 1. *Fix $T^* > 0$, $\mu > 0$, and $a \in (0,1)$. Let $(I, \rho, \mathbf{u})$ be a classical solution of the initial value problem (7) and (8) for $0 \le t < \tau$ and $U_0 =: I_0(\mathbf{x}, \nu, \mathbf{\Omega}) - \overline{B}(\nu) \ge 0$.*
1. For the case that decay rate $\mu > 0$ and the damping coefficient $\lambda \in [0, +\infty)$, if

$$F(0) \ge \frac{\mu + \sqrt{\mu^2 + 4I_1 I_2}}{2I_1}, \tag{26}$$

and

$$F(0) \ge \frac{1}{1-a}\left[\int_0^{T^*} \frac{dt}{G(t)+Q(t)}\right]^{-1}, \tag{27}$$

then $\tau < T^*$, where

$$I_1 =: \frac{a}{G(T^*)+Q(T^*)}, \tag{28}$$

$$I_2 =: 4\pi\overline{p}(R+\sigma T^*)^3 f(R+\sigma T^*), \tag{29}$$

$$G(t) = 2(R+\sigma t)^2 f(R+\sigma t)[m(0) + \overline{\rho}|B(t)|], \tag{30}$$

$$Q(t) = \frac{2}{3c^3}(R+\sigma t)^2 f(R+\sigma t)|B(t)| \cdot \max\left(\int_0^{\infty} d\nu \int_{\mathbb{S}^2} U_0 \, d\Omega\right), \tag{31}$$

$\overline{p}$ is the pressure at the background density $\overline{\rho}$ and is constant, $B(t)$ is a moving region that satisfies

$$B(t) = \{\mathbf{x} : |\mathbf{x}| \le R + \sigma t\}, \tag{32}$$

and its volume satisfies the following formula

$$|B(t)| = \frac{4}{3}\pi(R+\sigma t)^3. \tag{33}$$

It should be noted that $G(T^*)$, $Q(T^*)$ and $|B(T^*)|$ are the maximum values of the functions $G(t)$, $Q(t)$ and $|B(t)|$, respectively, for $t \in [0, T^*]$, satisfying

$$G(T^*) = 2(R+\sigma T^*)^2 f(R+\sigma T^*)[m(0) + \overline{\rho}|B(T^*)|], \tag{34}$$

$$Q(T^*) = \frac{2}{3c^3}(R+\sigma T^*)^2 f(R+\sigma T^*)|B(T^*)| \cdot \max\left(\int_0^{\infty} d\nu \int_{\mathbb{S}^2} U_0 \, d\Omega\right), \tag{35}$$

and

$$|B(T^*)| = \frac{4}{3}\pi(R+\sigma T^*)^3. \tag{36}$$

2. For the case that decay rate $\mu > 0$ and the damping coefficient $\lambda \in (-\infty, 0)$, if

$$F(0) \ge \frac{I_3 + \sqrt{I_3^2 + 4I_1 I_2}}{2I_1}, \tag{37}$$

and inequality (27) is satisfied, where $I_3 =: \frac{\mu}{(1+T^*)^\lambda}$, then $\tau < T^*$.

Remark 1. *As an application, if one chooses f to be $1-e^{-|\mathbf{x}|}$, where $r=|\mathbf{x}|$, the solution of systems (7) and (8) will blowup on or before time T^* if the initial data are sufficiently large. Some test functions f that have the same effects as follows: $f=|\mathbf{x}|$, $f=|\mathbf{x}|^n$, $f=1-e^{-n|\mathbf{x}|}$, etc.*

Remark 2. *When decay rate $\mu=0$, the system (7) is the system (5). In this case, provided that the initial value satisfies*

$$F(0)\geq\frac{\sqrt{4I_1I_2}}{2I_1}, \tag{38}$$

and inequality (27) also holds, we conclude that $\tau<T^*$. This case is clearly a simplified version of Theorem 1, and the proof is therefore omitted.

3.3. Preliminaries

In order to prove Theorem 1, we first give the following preliminaries. Since $\bar{u}=0$, the maximum speed of propagation of the front of a smooth disturbance is governed by the sound speed

$$\sigma:=\sqrt{\partial_\rho p(\bar{\rho})}=\sqrt{A\gamma\bar{\rho}^{\gamma-1}}>0. \tag{39}$$

We will give the preliminary of finite propagation speed for the system (7). For the proof, we refer to Proposition 3.1 in [7].

Lemma 2. *Let $(I,\rho,\mathbf{u})$ be a classical solution of the initial value problem (7) and (8), then, we have*

$$(I,\rho,\mathbf{u})=(\overline{B}(\nu),\bar{\rho},0) \tag{40}$$

for $|\mathbf{x}|\geq R+\sigma t$, $t\in[0,T^*]$.

Assume the density ρ has compact support in the region $B(t)$ for a classical solution of system (7) in $\mathbb{R}^3$. We can also give the total mass conservation result.

Lemma 3. *Suppose $(I,\rho,\mathbf{u})$ is a classical solution of the initial problem (7)–(8) for $0\leq t<T^*$. Then, we have,*

$$m(t)=m(0),\ 0\leq t\leq T^*, \tag{41}$$

where $m(t)$ is the total mass function.

Proof. We have

$$m'(t)=\int_{\mathbb{R}^3}\rho_t\,d\mathbf{x}=-\int_{B(t)}\nabla\cdot(\rho\mathbf{u})\,d\mathbf{x}=-\int_{\partial B(t)}\rho\mathbf{u}\cdot\mathbf{n}\,dS=0, \tag{42}$$

with the preliminary of finite propagation speed, where $\mathbf{n}$ is the unit outward normal vector to $\partial B(t):=\{\mathbf{x}:|\mathbf{x}|=R+\sigma t\}$.
Then, $m(t)=m(0)$, $0\leq t\leq T^*$. □

Next, we present a lemma related to the specific intensity of radiation I. Lemma 4 states that the second term of functional (17) is non-negative, i.e., we have $F(t)\geq\int_{\mathbb{R}^3}f\mathbf{x}\cdot\rho\mathbf{u}d\mathbf{x}$, which is needed in the proof of the blowup result later.

Lemma 4. *If $U_0\geq 0$, then we have*

$$U\geq 0, \tag{43}$$

and

$$\int_{B(t)}d\mathbf{x}\int_0^\infty d\nu\int_{\mathbb{S}^2}U\,d\mathbf{\Omega}\leq\int_{B(t)}d\mathbf{x}\int_0^\infty d\nu\int_{\mathbb{S}^2}U_0\,d\mathbf{\Omega},\ \text{ for all t }\geq 0. \tag{44}$$

If $U_0 = 0$ for $\mathbf{x} \cdot \mathbf{\Omega} < 0$, then we have

$$U = 0 \quad for \quad \mathbf{x} \cdot \mathbf{\Omega} < 0. \tag{45}$$

Proof. We note that $\overline{B}(\nu)$ is independent of $\mathbf{x}$ and t. Then, the first equation of system (7) can be rewritten as

$$\frac{1}{c}\frac{\partial U}{\partial t} + \mathbf{\Omega} \cdot \nabla U + a(\nu, \rho)U = 0. \tag{46}$$

By using the method of characteristics [30–33], we have

$$\frac{\partial U}{\partial t} + c\mathbf{\Omega} \cdot \nabla U + ca(\nu, \rho)U = 0. \tag{47}$$

Along the characteristic curve $\frac{d\mathbf{x}}{dt} = c\mathbf{\Omega}$ from the point $(\mathbf{x}_0, 0)$, we have

$$\frac{dU}{dt} + ca(\nu, \rho)U = 0. \tag{48}$$

Integrating the above equation (48) along the characteristic line, we obtain

$$U = U_0 e^{-\int_0^t ca(\nu,\rho)dt}. \tag{49}$$

Hence, U has the same sign as U_0. If $U_0 \geq 0$, then we have $U \geq 0$.

On the other hand, it follows from (49) that

$$0 \leq U \leq U_0. \tag{50}$$

The handling of the second term of the functional $F(t)$, in the special case where $\mathbf{x} \cdot \mathbf{\Omega} < 0$, is as follows.

By equation (49), we have $U = 0$ for $\mathbf{x} \cdot \mathbf{\Omega} < 0$, provided that $U_0 = 0$ for $\mathbf{x} \cdot \mathbf{\Omega} < 0$. The proof of Lemma 4 is completed. □

3.4. Integration Method by Test Function

In this part, we shall provide the proof of the blowup result of the classical solutions to the Euler–Boltzmann equations with time-dependent damping in $\mathbb{R}^3$.

Proof of Theorem 1. It is observed that the first and third equations in system (7) share the common term $a(\nu, \rho)(\overline{B}(\nu) - I)$. Therefore, the third equation can be rewritten in the following form

$$\begin{aligned}(\rho\mathbf{u})_t + \frac{1}{c^2}\int_0^\infty d\nu \int_{\mathbb{S}^2} \mathbf{\Omega} I_t d\mathbf{\Omega} + \nabla p + \nabla \cdot (\rho\mathbf{u} \otimes \mathbf{u}) \\ = -\frac{1}{c}\int_0^\infty d\nu \int_{\mathbb{S}^2} |\mathbf{\Omega}|^2 \cdot \nabla I d\mathbf{\Omega} - \frac{\mu}{(1+t)^\lambda}\rho\mathbf{u}.\end{aligned} \tag{51}$$

Noting the definition of $F(t)$ in (17), by integration by parts and the finite propagation speed property, we have

$$
\begin{aligned}
F^{'}(t) &= \int_{\mathbb{R}^3} f\mathbf{x}\cdot(\rho\mathbf{u})_t d\mathbf{x} + \frac{1}{c^2}\int_{\mathbb{R}^3} d\mathbf{x}\int_0^\infty d\nu \int_{\mathbb{S}^2} f\mathbf{x}\cdot\mathbf{\Omega}U_t d\mathbf{\Omega} \\
&= \int_{\mathbb{R}^3} f\mathbf{x}\cdot\{-\frac{1}{c^2}\int_0^\infty d\nu\int_{\mathbb{S}^2}\mathbf{\Omega}I_t d\mathbf{\Omega} - \nabla p - \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) - \frac{1}{c}\int_0^\infty d\nu\int_{\mathbb{S}^2}|\mathbf{\Omega}|^2\cdot\nabla I d\mathbf{\Omega} \\
&\quad - \frac{\mu}{(1+t)^\lambda}\rho\mathbf{u}\}d\mathbf{x} + \frac{1}{c^2}\int_{\mathbb{R}^3} d\mathbf{x}\int_0^\infty d\nu\int_{\mathbb{S}^2} f\mathbf{x}\cdot\mathbf{\Omega}I_t d\mathbf{\Omega} \qquad \text{(by formula (51))} \\
&= -\int_{\mathbb{R}^3} f\mathbf{x}\cdot\nabla(p-\overline{p})d\mathbf{x} - \int_{\mathbb{R}^3} f\mathbf{x}\cdot\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u})d\mathbf{x} \\
&\quad - \frac{1}{c}\int_{\mathbb{R}^3} d\mathbf{x}\int_0^\infty d\nu\int_{\mathbb{S}^2} f\mathbf{x}\cdot|\mathbf{\Omega}|^2\cdot\nabla(I-\overline{B}(\nu))d\mathbf{\Omega} - \int_{\mathbb{R}^3} f\mathbf{x}\cdot\frac{\mu}{(1+t)^\lambda}\rho\mathbf{u}d\mathbf{x} \\
&= \int_{B(t)}(p-\overline{p})\nabla\cdot(f\mathbf{x})d\mathbf{x} + \int_{B(t)}\left[f\rho|\mathbf{u}|^2 + \frac{f^{'}\rho(\mathbf{x}\cdot\mathbf{u})^2}{r}\right]d\mathbf{x} \\
&\quad + \frac{1}{c}\int_{B(t)} d\mathbf{x}\int_0^\infty d\nu\int_{\mathbb{S}^2}\nabla\cdot(f\mathbf{x})|\mathbf{\Omega}|^2(I-\overline{B}(\nu))d\mathbf{\Omega} - \int_{B(t)} f\mathbf{x}\cdot\frac{\mu}{(1+t)^\lambda}\rho\mathbf{u}d\mathbf{x} \\
&= \int_{B(t)}(p-\overline{p})[rf^{'}(r)+3f]d\mathbf{x} + \int_{B(t)}\left[f\rho|\mathbf{u}|^2 + \frac{f^{'}\rho(\mathbf{x}\cdot\mathbf{u})^2}{r}\right]d\mathbf{x} \\
&\quad + \frac{1}{c}\int_{B(t)} d\mathbf{x}\int_0^\infty d\nu\int_{\mathbb{S}^2}(rf^{'}(r)+3f)|\mathbf{\Omega}|^2(I-\overline{B}(\nu))d\Omega - \int_{B(t)} f\mathbf{x}\cdot\frac{\mu}{(1+t)^\lambda}\rho\mathbf{u}d\mathbf{x} \\
&\geq \int_{B(t)}(p-\overline{p})[rf^{'}(r)+3f]d\mathbf{x} + \int_{B(t)} f\rho|\mathbf{u}|^2 d\mathbf{x} \\
&\quad + \frac{3}{c}\int_{B(t)} d\mathbf{x}\int_0^\infty d\nu\int_{\mathbb{S}^2} f|\mathbf{\Omega}|^2 U\, d\mathbf{\Omega} - \int_{B(t)} f\mathbf{x}\cdot\frac{\mu}{(1+t)^\lambda}\rho\mathbf{u}d\mathbf{x}.
\end{aligned} \tag{52}
$$

Using the spherical coordinate transformation, we estimate the first term of (52) and thereby obtain

$$
\begin{aligned}
&\int_{B(t)}(p-\overline{p})[rf^{'}(r)+3f]d\mathbf{x} \\
&> -\overline{p}\int_{B(t)}[rf^{'}(r)+3f]d\mathbf{x} \\
&= -\overline{p}\int_0^{2\pi} d\theta\int_0^\pi d\varphi\int_0^{R+\sigma t}[rf^{'}(r)+3f]r^2 sin\varphi dr \\
&= -4\pi\overline{p}[(R+\sigma t)^3 f(R+\sigma t) - R^3 f(R)] \\
&> -4\pi\overline{p}(R+\sigma t)^3 f(R+\sigma t).
\end{aligned} \tag{53}
$$

Based on the above inequality, inequality (52) can be rewritten as

$$
\begin{aligned}
F^{'}(t) \geq &\int_{B(t)} f\rho|\mathbf{u}|^2 d\mathbf{x} + \frac{3}{c}\int_{B(t)} d\mathbf{x}\int_0^\infty d\nu\int_{\mathbb{S}^2} f|\mathbf{\Omega}|^2 U\, d\mathbf{\Omega} \\
&- 4\pi\overline{p}(R+\sigma t)^3 f(R+\sigma t) - \frac{\mu}{(1+t)^\lambda}\int_{B(t)} f\mathbf{x}\cdot\rho\mathbf{u}d\mathbf{x}.
\end{aligned} \tag{54}
$$

By squaring the functional $F(t)$ and using the compact support property, we can restrict the domain of integration from $\mathbb{R}^3$ to $B(t)$, thus obtaining the following

$$
\begin{aligned}
F^2(t) =& \left[\int_{\mathbb{R}^3} f\mathbf{x}\cdot\rho\mathbf{u}d\mathbf{x} + \frac{1}{c^2}\int_{\mathbb{R}^3} d\mathbf{x}\int_0^\infty d\nu \int_{\mathbb{S}^2} f\mathbf{x}\cdot\mathbf{\Omega}U d\mathbf{\Omega}\right]^2 \\
=& \left[\int_{B(t)} f\mathbf{x}\cdot\rho\mathbf{u}d\mathbf{x} + \frac{1}{c^2}\int_{B(t)} d\mathbf{x}\int_0^\infty d\nu \int_{\mathbb{S}^2} f\mathbf{x}\cdot\mathbf{\Omega}U\, d\mathbf{\Omega}\right]^2 \\
\le& 2\left[\int_{B(t)} f\mathbf{x}\cdot\rho\mathbf{u}d\mathbf{x}\right]^2 + \frac{2}{c^4}\left[\int_{B(t)} d\mathbf{x}\int_0^\infty d\nu \int_{\mathbb{S}^2} f\mathbf{x}\cdot\mathbf{\Omega}U\, d\mathbf{\Omega}\right]^2 \\
\le& \frac{2}{c^4}\left[\int_{B(t)} d\mathbf{x}\int_0^\infty d\nu \int_{\mathbb{S}^2} f|\mathbf{x}|^2 U\, d\mathbf{\Omega}\right]\left[\int_{B(t)} d\mathbf{x}\int_0^\infty d\nu \int_{\mathbb{S}^2} f|\mathbf{\Omega}|^2 U\, d\mathbf{\Omega}\right] \\
&+ 2\left(\int_{B(t)} f\rho|\mathbf{x}|^2 d\mathbf{x}\right)\left(\int_{B(t)} f\rho|\mathbf{u}|^2 d\mathbf{x}\right) \quad \text{(by Hölder's inequality)} \\
\le& \left[2\int_{B(t)} f\rho|\mathbf{x}|^2 d\mathbf{x} + \frac{2}{3c^3}\int_{B(t)} d\mathbf{x}\int_0^\infty d\nu \int_{\mathbb{S}^2} f|\mathbf{x}|^2 U\, d\mathbf{\Omega}\right] \\
&\times \left[\int_{B(t)} f\rho|\mathbf{u}|^2 d\mathbf{x} + \frac{3}{c}\int_{B(t)} d\mathbf{x}\int_0^\infty d\nu \int_{\mathbb{S}^2} f|\mathbf{\Omega}|^2 U\, d\mathbf{\Omega}\right].
\end{aligned} \tag{55}
$$

The second-to-last step in Equation (55) involves two product terms derived from four positive components. By appropriately reorganizing these components into a product of four individual terms, we establish the transformation between the preceding and current steps. For the first term of (55), by the total mass conservation result, we have

$$
\begin{aligned}
& 2\int_{B(t)} f\rho|\mathbf{x}|^2 d\mathbf{x} \\
& \le 2(R+\sigma t)^2 f(R+\sigma t)\int_{B(t)} \rho d\mathbf{x} \\
& = 2(R+\sigma t)^2 f(R+\sigma t)\left[\int_{B(t)} (\rho-\overline{\rho}) d\mathbf{x} + \int_{B(t)} \overline{\rho} d\mathbf{x}\right] \\
& = 2(R+\sigma t)^2 f(R+\sigma t)[m(t) + \overline{\rho}|B(t)|] \\
& = 2(R+\sigma t)^2 f(R+\sigma t)[m(0) + \overline{\rho}|B(t)|].
\end{aligned} \tag{56}
$$

For the second term of (55), noting inequality (50), we have

$$
\begin{aligned}
& \frac{2}{3c^2}\int_{B(t)} d\mathbf{x}\int_0^\infty d\nu \int_{\mathbb{S}^2} f|\mathbf{x}|^2 U\, d\mathbf{\Omega} \\
& \le \frac{2}{3c^2}(R+\sigma t)^2 f(R+\sigma t)\int_{B(t)} d\mathbf{x}\int_0^\infty d\nu \int_{\mathbb{S}^2} U\, d\mathbf{\Omega} \\
& \le \frac{2}{3c^2}(R+\sigma t)^2 f(R+\sigma t)\int_{B(t)} d\mathbf{x}\int_0^\infty d\nu \int_{\mathbb{S}^2} U_0\, d\mathbf{\Omega} \\
& \le \frac{2}{3c^2}(R+\sigma t)^2 f(R+\sigma t)|B(t)|\cdot \max\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} U_0\, d\mathbf{\Omega}\right).
\end{aligned} \tag{57}
$$

Here, we denote

$$
\begin{aligned}
G(t) &= 2(R+\sigma t)^2 f(R+\sigma t)[m(0) + \overline{\rho}|B(t)|] > 0, \\
Q(t) &= \frac{2}{3c^2}(R+\sigma t)^2 f(R+\sigma t)|B(t)|\cdot \max\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} U_0\, d\mathbf{\Omega}\right) > 0.
\end{aligned} \tag{58}
$$

Based on the above analysis, inequality (55) can be rewritten as

$$
F^2(t) \le (G(t)+Q(t))\left(\int_{B(t)} f\rho|\mathbf{u}|^2 d\mathbf{x} + \frac{3}{c}\int_{B(t)} d\mathbf{x}\int_0^\infty d\nu \int_{\mathbb{S}^2} f|\mathbf{\Omega}|^2 U\, d\mathbf{\Omega}\right). \tag{59}
$$

Then, $\int_{B(t)} f\rho|\mathbf{u}|^2 d\mathbf{x} + \frac{3}{c}\int_{B(t)} d\mathbf{x}\int_0^\infty dv\int_{\mathbb{S}^2} f|\mathbf{\Omega}|^2 U\, d\mathbf{\Omega} \geq \frac{F^2(t)}{G(t)+Q(t)}$.

Noting the first two terms of the inequality of (54) and Lemma 4, we have

$$\begin{aligned}
F'(t) \geq & \frac{F^2(t)}{G(t)+Q(t)} - 4\pi\overline{p}(R+\sigma t)^3 f(R+\sigma t) - \int_{B(t)} f\mathbf{x}\cdot\frac{\mu}{(1+t)^\lambda}\rho\mathbf{u}d\mathbf{x} \\
= & \frac{F^2(t)}{G(t)+Q(t)} - 4\pi\overline{p}(R+\sigma t)^3 f(R+\sigma t) - \frac{\mu}{(1+t)^\lambda}\int_{B(t)} f\mathbf{x}\cdot\rho\mathbf{u}d\mathbf{x} \\
\geq & \frac{F^2(t)}{G(t)+Q(t)} - 4\pi\overline{p}(R+\sigma t)^3 f(R+\sigma t) - \frac{\mu}{(1+t)^\lambda}F(t) \\
& + \frac{\mu}{c^2(1+t)^\lambda}\int_{\mathbb{R}^3} d\mathbf{x}\int_0^\infty dv\int_{\mathbb{S}^2} f\mathbf{x}\cdot\mathbf{\Omega}\, U\, d\mathbf{\Omega} \\
\geq & \frac{F^2(t)}{G(t)+Q(t)} - 4\pi\overline{p}(R+\sigma t)^3 f(R+\sigma t) - \frac{\mu}{(1+t)^\lambda}F(t) \\
\geq & \frac{(1-a)F^2(t)}{G(t)+Q(t)} + \left(\frac{aF^2(t)}{G(t)+Q(t)} - 4\pi\overline{p}(R+\sigma T^*)^3 f(R+\sigma T^*) - \frac{\mu}{(1+t)^\lambda}F(t)\right) \\
=: & \frac{(1-a)F^2(t)}{G(t)+Q(t)} + J_1(t),
\end{aligned} \tag{60}$$

where the constant $a \in (0,1)$. By Lemma 4, the fourth term in the third step of the above expression is always non-negative. More specifically, U is always non-negative. A special case occurs when $\mathbf{x}\cdot\mathbf{\Omega} < 0$, in this case, as long as the initial data $U_0 = 0$, we have $U = 0$.

Actually, if $F(0) \geq 0$ and $J_1(t) \geq 0$ for $t \in [0, T^*]$, we can obtain

$$F'(t) \geq \frac{(1-a)F^2(t)}{G(t)+Q(t)}, \quad t \in [0, T^*]. \tag{61}$$

Therefore, $F(t)$ is an increasing function and $F(t) > F(0) \geq 0$.

For the case that decay rate $\mu > 0$ and the damping coefficient $\lambda \in [0, +\infty)$, it is apparent that $-\frac{\mu}{(1+t)^\lambda}F(t) \geq -\mu F(t)$. It follows that

$$\begin{aligned}
J_1(t) \geq & \frac{aF^2(t)}{G(T^*)+Q(T^*)} - 4\pi\overline{p}(R+\sigma T^*)^3 f(R+\sigma T^*) - \mu F(t) \\
=: & I_1F^2(t) - \mu F(t) - I_2 \\
=: & J_2(F(t)).
\end{aligned} \tag{62}$$

Apparently, $J_2(F(t))$ is a quadratic equation about $F(t)$ and $F(t) \in [F(0), F(T^*)]$ for $t \in [0, T^*]$. It follows that $J_2(F(t)) \geq 0$ if $F(0) \geq \frac{\mu}{2I_1}$ and $J_2(F(0)) \geq 0$. We can obtain from initial condition (26) that $J_1(t) \geq J_2(F(t)) \geq 0$ for $t \in [0, T^*]$.

For the case that decay rate $\mu > 0$ and the damping coefficient $\lambda \in (-\infty, 0)$, it is apparent that $-\frac{\mu}{(1+t)^\lambda}F(t) \geq -\frac{\mu}{(1+T^*)^\lambda}F(t)$. Therefore, we have

$$\begin{aligned}
J_1(t) \geq & \frac{aF^2(t)}{G(T^*)+Q(T^*)} - 4\pi\overline{p}(R+\sigma T^*)^3 f(R+\sigma T^*) - \frac{\mu}{(1+T^*)^\lambda}F(t) \\
=: & I_1F^2(t) - I_3F(t) - I_2 \\
=: & J_3(F(t)).
\end{aligned} \tag{63}$$

If $F(0) \geq \frac{I_3}{2I_1}$ and $J_3(F(0)) \geq 0$, we can obtain $J_3(F(t)) \geq 0$. From the initial condition (37), we can obtain $J_1(t) \geq J_3(F(t)) \geq 0$ for $t \in [0, T^*]$.

From inequality (61), we have

$$0 < \frac{1}{F(t)} < \frac{1}{F(0)} - (1-a)\int_0^{T^*} \frac{dt}{G(t)+Q(t)}. \tag{64}$$

By the initial condition (27), we have

$$\frac{1}{F(0)} \leq (1-a)\int_0^{T^*} \frac{dt}{G(t)+Q(t)}. \tag{65}$$

Equation (65) shows that the right-hand side of Equation (64) is nonpositive (i.e., less than or equal to zero), which leads to a contradiction. Therefore, as long as the initial dataset is sufficiently large (satisfying condition (27)), the classical solution will blowup on or before the finite time T^*. □

4. Discussion

In this paper, we explore the influence of time-dependent damping on the singularity formation of solutions. Jiang [10] investigated the blowup phenomenon for solutions of the three-dimensional compressible Euler–Boltzmann equations using a specific weighted functional (6). However, this weighted functional does not apply to similar equations with time-dependent damping. By taking into account the functional (14) proposed by Yuen et al. [25] and the effects of radiation, we construct a new weighted functional (17). Our results demonstrate that the classical solutions of the three-dimensional compressible Euler–Boltzmann equations truly blowup in finite time, despite the presence of time-dependent damping.

5. Conclusions

This paper focuses on investigating the singularity formation of classical solutions for the compressible Euler–Boltzmann equations with time-dependent damping in $\mathbb{R}^3$. We first establish the local existence of classical solutions. Subsequently, by introducing a functional $F(t)$ weighted by general test function f, we demonstrate that, under the assumption of sufficiently large initial data, the classical solutions inevitably blowup in finite time. This study includes two innovations. First, the introduction of a time-dependent damping term enhances the physical accuracy of the model compared to traditional Euler–Boltzmann systems, making it more suitable for practical scenarios, such as the flow of radiative fluids in porous media. Second, we derived the sufficient condition for the finite-time blowup of classical solutions, with the type of blowup related to functionals of the solution, providing new insights into the formation of singularities in the radiative fluid model with damping, and extending the existing blowup theory under a physically relevant damping framework.

Author Contributions: Conceptualization, M.Y.; methodology, J.L., M.L. and M.Y.; validation, J.L., M.L. and M.Y.; investigation, M.L.; writing—original draft preparation, M.L.; writing—review and editing, J.L. and M.Y.; supervision, J.L.; project administration, M.Y.; funding acquisition, M.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Natural Science Foundation of Shanghai 20ZR1419400. This research paper is partially funded by the Departmental Research Grant 2024-25 (MIT/DRG10/24-25) from The Education University of Hong Kong.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: We express our gratitude to the esteemed professors whose insightful reviews and feedback have significantly contributed to the refinement of our paper.

Conflicts of Interest: The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

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