

Article

On the Theory of Nonlinear Landau Damping

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Abstract: An exact solution of the collisionless time-dependent Vlasov equation is found. For the first time in a century, an analytical solution to the one-dimensional time-dependent Vlasov–Boltzmann equation has been found. It has been found that instead of the widely discussed damping, waves are subject to instability. By means of this solution, the behavior of the Langmuir waves in the nonlinear stage is considered. A symmetry method is found that allows us to establish the dependence on time of the desired quantity based on the dependence on the previous time. The analysis is restricted by the consideration of the first nonlinear approximation—keeping the second power of the electric strength. It is shown that in general the waves with finite amplitudes are not subjected to the damping. Conditions have been found under which waves can be unstable.

Keywords: plasmawave analysis; nonlinear Landau damping; parametric instability; collisionless Vlasov–Boltzmann kinetic equation

1. Introduction

A large number of papers and textbooks are devoted to the nonlinear theory of the Langmuir waves and the Landau damping [1]. In addition to the standard method for explaining the physics of Landau damping, a large number of diverse approaches and interpretations have been published [2–9] that are more advanced than the standard one. In these papers however, the solution of the linearized Vlasov–Boltzmann kinetic equation is used. After finding the zero-order solution of the main equation, authors construct then approximations of any higher order [10–14]. In the present paper, a new approach is presented that allows the analysis of the problem to be self-consistent in an arbitrary order of nonlinear approximation. A method has been found that allows one to describe the dependence on time based on the dependence at previous moments. In [15], the solution for Vlasov–Boltzmann time-dependent kinetic equation is found. The method used in this article is developed for the case where only the electric field of waves is considered. It is shown that the waves with finite amplitude are not exposed to the damping. Only waves with small amplitude, where the oscillation frequency of captured (in the wave well) electrons is smaller than the damping rate, can damp [16,17]. It is found that at the fulfilment of the resonance condition, when the applied frequency is twice the natural frequency of oscillations, the waves with finite amplitude are unstable. This condition is similar to the condition at the parametric resonance [18].

To solve the problem, i.e., to solve the kinetic equation (in our case representing the partial differential equation), the usual method is used—characteristic equations for



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the kinetic equation are compiled and the solution of the kinetic equation is determined by the constants of these characteristic equations. The article is organized as follows: The formulation of the basic equations and their solutions (which is a pioneering step) are presented in Section 2. Section 3 discusses the possibility of the existence of nonlinear waves under the conditions considered. Section 4 investigates the stability of these waves.

2. Exact Solution of the Vlasov Equation

We start from the one-dimensional Vlasov equation for electrons and Poisson's equation with spatial dependence (z), kinetic electrons, and motionless ions that serve as a neutralizing background.

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial z} - E(z, t) \frac{\partial f}{\partial v} = 0, \quad (1)$$

$$\frac{\partial E}{\partial z} = 1 - n, \quad E = -\frac{\partial \phi}{\partial z}, \quad (2)$$

where the dimensionless values for the time, the coordinate, the velocity and the electric potential are used

$$(\omega_{pe} t) \rightarrow t, \quad (z/\lambda_{De}) \rightarrow z, \quad (v/v_{Te}) \rightarrow v, \quad \text{and} \quad (e\Phi/T_e) \rightarrow \phi, \quad (3)$$

$$\omega_{pe} = \left(\frac{4\pi \cdot e^2 n_0}{m_e} \right)^{1/2}, \quad \lambda_{De} = \frac{v_{Te}}{\omega_{pe}}, \quad v_{Te} = \left(\frac{T_e}{m_e} \right)^{1/2}$$

ω_{pe} is the electron plasma frequency, λ_{De} is the electron Debye length, and v_{Te} is the electron thermal velocity. The dimensionless electric field E and the electron number density n are defined as follows

$$\frac{v_{Te}}{n_0} f \rightarrow f, \quad \frac{E}{\sqrt{4\pi \cdot n_0 T_e}} \rightarrow E, \quad \frac{n}{n_0} \rightarrow n = \int dv \cdot f(z, t, v). \quad (4)$$

Here, T_e is the electron temperature in the energetic units. It is assumed that ions stay in the equilibrium with the density n_0 , which results in the first term ("1") in the right-hand side (rhs) of the Poisson's Equation (2). The characteristic equations for Equation (1) reads

$$\frac{dz}{dt} = v, \quad \frac{dv}{dt} = -E(z, t). \quad (5)$$

For the constants of the integrals R and U ($dR/dt = 0$, $dU/dt = 0$) we find

$$R = z - \int_{t'}^t dt' \cdot H' \left[v + \int_{t'}^t dt'' \cdot E\{z(t''), t''\} \right], \quad (6)$$

$$U = v + \int_{t'}^t dt' \cdot E\{z(t'), t'\}, \quad (7)$$

where the functions $z(t')$ are defined with the following expressions:

$$z(t') = z - \int_{t'}^t dt'' \cdot H' \left[v + \int_{t''}^t dt''' \cdot E\{z(t'''), t'''\} \right], \quad (8)$$

$$z(t'') = z - \int_{t''}^t dt''' \cdot H' \left[v + \int_{t'''}^t dt^{IV} \cdot E\{z(t^{IV}), t^{IV}\} \right], \quad (9)$$

$$\begin{aligned}
 z(t''') &= z - \int_{t'''}^t dt^{IV} \cdot H' \left[v + \int_{t^{IV}}^t dt^V \cdot E \{ z(t^V), t^V \} \right], \\
 &\vdots
 \end{aligned}
 \tag{10}$$

Hence, the chain (8), (9), (10), ..., can be continued by symmetric substitution of corresponding time-moments. Here, $z(t')$, $z(t'')$, $z(t''')$, ..., must be substituted into the arguments of the electric fields' expressions $E\{z(t'), t'\}$, $E\{z(t''), t''\}$, $E\{z(t'''), t'''\}$, ..., and so forth. In Equations (6)–(8), the function $H(x)$ is defined as follows

$$H(x) = \frac{1}{2}x^2, \quad H'(x) = x. \tag{11}$$

In the following, the upper dashes in $H'(x)$ will denote the derivative with the whole argument of the function $H(x)$ and in $E'\{z(t), t\}$ denote derivative only with respect of $z(t)$, $E'\{z(t), t\} = \partial E\{z(t), t\} / \partial z(t)$. The solution of Equation (1) can be represented in the form

$$f = f\{R(v, z, t), U(v, z, t)\}. \tag{12}$$

Substituting Equation (12) into Equation (1), using the definitions (6) and (7) and successively carrying out the derivatives we obtain

$$\begin{aligned}
 \frac{df}{dt} &= \frac{\partial f}{\partial U} \cdot \frac{dU}{dt} + \frac{\partial f}{\partial R} \cdot \frac{dR}{dt} = \\
 &= \frac{\partial f}{\partial U} \left\{ \frac{\partial U}{\partial t} + v \frac{\partial U}{\partial z} - E \frac{\partial U}{\partial v} \right\} + \frac{\partial f}{\partial R} \left\{ \frac{\partial R}{\partial t} + v \frac{\partial R}{\partial z} - E \frac{\partial R}{\partial v} \right\}.
 \end{aligned}
 \tag{13}$$

In the Appendix A the derivatives of the functions U and R are calculated and it is shown that the right-hand (rh) side of Equation (13) is proportional to the expression $H'(v) - v$ (all other terms cancel each other), which according to the second relation from (11) is equal to zero; then, we have,

$$\left\{ \frac{\partial f}{\partial U} \cdot \int_{t'''}^t dt' \cdot E'\{z(t'), t'\} + \frac{\partial f}{\partial R} \right\} \cdot (-H'[v] + v) = 0. \tag{14}$$

Defining the initial distribution function we assume that at the initial moment $t_0 \rightarrow -\infty$ the electron distribution function depends only on the velocity

$$f\{R(v, z, t_0), U(v, z, t_0)\} = f_0(v)|_{t_0 \rightarrow -\infty}. \tag{15}$$

As $f_0(v)$ we can choose the Maxwell distribution function with a normalizing coefficient, $(1/\sqrt{2\pi}) \cdot \exp(-v^2/2)$. From the relationships at the initial time t_0 ,

$$R(v, z, t_0) = R \quad \text{and} \quad U(v, z, t_0) = U \tag{16}$$

and definitions given by Equations (6) and (7) we can find expressions for the velocity v and the coordinate z :

$$v = U - \int_{t_0}^{t_0} dt' \cdot E\{z(t'), t'\}, \quad (17)$$

$$z = R + \int_{t_0}^{t_0} dt' \cdot H' \left[v + \int_{t'}^{t_0} dt'' \cdot E\{z(t''), t''\} \right]. \quad (18)$$

In Equations (17) and (18), again the expressions (6) and (7) can again be substituted. Hence, using the initial condition (15) for the electron distribution function we find the following:

$$f_0 = f_0 \left\{ v + \int_{t_0 \rightarrow -\infty}^t dt' \cdot E\{z(t'), t'\} \right\}, \quad (19)$$

where the values $z(t')$, $z(t'')$, ..., are defined by Equations (8) and (9), ... Poisson's Equation (2) can then be represented in the following form:

$$\frac{\partial E(z, t)}{\partial z} = 1 - \int_{-\infty}^{\infty} dv \cdot f_0 \left\{ v + \int_{t_0 \rightarrow -\infty}^t dt' \cdot E\{z(t'), t'\} \right\}. \quad (20)$$

It is convenient to introduce a new variable s , defined by the following relation:

$$v + \int_{-\infty}^t dt' \cdot E\{z(t'), t'\} = s, \quad (21)$$

which allows simplification of the argument of f_0 . Then, for the explicit expression for velocity v we have

$$\begin{aligned} v &= s - \int_{t_0 \rightarrow -\infty}^t dt' \cdot E[z(t'), s] \\ &= s - \int_{t_0 \rightarrow -\infty}^t dt' \cdot E[z - s(t - t') + \int_{t'}^t dt'' \int_{t_0}^{t''} dt''' \cdot E\{z - s(t - t'''), s\} \\ &\quad + \int_{t'''}^t dt^{IV} \int_{t_0}^{t^{IV}} dt^V \cdot E\{z - s(t - t^V) + \dots, t'''\} t''], \end{aligned} \quad (22)$$

where by analogy to Equations (8) and (9), ... we have introduced the following symmetric relations:

$$z(t', s) = z - s\{t - t'\} + \int_{t'}^t dt'' \int_{t_0}^{t''} dt''' \cdot E[z(t'''), s, t'''], \quad (23)$$

$$z(t''', s) = z - s\{t - t'''\} + \int_{t'''}^t dt^{IV} \int_{t_0}^{t^{IV}} dt^V \cdot E[z(t^V, s), t^V]. \quad (24)$$

⋮

The Poisson's equation then can be represented in the form

$$\begin{aligned}\frac{\partial E(z, t)}{\partial z} &= 1 - \int_{-\infty}^{\infty} ds \cdot \frac{dv(s)}{ds} f_0(s) \\ &= 1 + \int_{-\infty}^{\infty} ds \cdot \frac{df_0(s)}{ds} \cdot v(s).\end{aligned}\quad (25)$$

Substituting Equation (22) into Equation (23), one can find that for the Maxwell distribution function $f_0(s)$ the first term in (25) is compensated by the positive charge of ions:

$$\frac{\partial E}{\partial z} = - \int_{-\infty}^{+\infty} ds \cdot \frac{\partial f_0(s)}{\partial s} \cdot \int_{t_0 \rightarrow -\infty}^t dt' \cdot E[z(t', s), t']. \quad (26)$$

Taking the first derivative with time and restricting ourselves by keeping the terms up to second powers of the electric strength (we will maintain such a restriction throughout the calculations) we find

$$\begin{aligned}\frac{\partial}{\partial t} \frac{\partial E}{\partial z} &= - \int_{-\infty}^{+\infty} ds \cdot \frac{\partial f_0(s)}{\partial s} \left\{ -s \int_{t_0 \rightarrow -\infty}^t dt' \cdot E'[z(t', s), t'] \right. \\ &\quad + \int_{t_0 \rightarrow -\infty}^t dt' \cdot E'[z(t', s), t'] \int_{t_0 \rightarrow -\infty}^t dt'' \cdot E[z(t'', s), t''] \\ &\quad \left. - s \int_{t_0 \rightarrow -\infty}^t dt' \cdot \frac{\partial}{\partial z} E[z(t', s), t'] \cdot \int_{t'}^t dt'' \int_{t_0 \rightarrow -\infty}^{t''} dt''' \cdot E'[z(t''', s), t'''] \right\}.\end{aligned}\quad (27)$$

In the last term in the right-hand side of this equation we can change the ordering of integrals in such a manner

$$\int_{t_0 \rightarrow -\infty}^t dt' \int_{t'}^t dt'' \cdot Q(t', t'') = \int_{t_0 \rightarrow -\infty}^t dt'' \int_{t_0 \rightarrow -\infty}^{t''} dt' \cdot Q(t', t''). \quad (28)$$

Then, the second derivative with time of Equation (27) gives

$$\begin{aligned}\frac{\partial^2}{\partial t^2} \frac{\partial E}{\partial z} &= - \frac{\partial E}{\partial z} - \int_{-\infty}^{+\infty} ds \cdot \frac{\partial f_0(s)}{\partial s} \left\{ s^2 \cdot \int_{t_0 \rightarrow -\infty}^t dt' \cdot E''[z(t', s), t'] \right. \\ &\quad - s \cdot \frac{\partial}{\partial z} \cdot \int_{t_0 \rightarrow -\infty}^t dt' \cdot E\{z - s(t - t'), t'\} \int_{t_0 \rightarrow -\infty}^t dt'' \cdot E\{z - s(t - t''), t''\} \\ &\quad \left. + \frac{\partial}{\partial t} \cdot \int_{t_0 \rightarrow -\infty}^t dt' \cdot E\{z - s(t - t'), t'\} \int_{t_0 \rightarrow -\infty}^t dt'' \cdot E\{z - s(t - t''), t''\} \right\}.\end{aligned}\quad (29)$$

In Equation (29), we can transform the last term with the first derivative with time as follows

$$\begin{aligned}
& \frac{\partial}{\partial t} \cdot \int_{t_0 \rightarrow -\infty}^t dt' \cdot \frac{\partial}{\partial z} E[z(t', s), t'] \int_{t_0 \rightarrow -\infty}^t dt''' \cdot E[z(t''', s), t'''] \cong \\
& \cong \frac{\partial}{\partial z} \cdot E(z, t) \int_{t_0 \rightarrow -\infty}^t dt' \cdot E\{z - s(t - t'), t'\} \\
& - s \cdot \frac{\partial}{\partial z} \cdot \int_{t_0 \rightarrow -\infty}^t dt' \cdot E\{z - s(t - t''), t'\} \int_{t_0 \rightarrow -\infty}^t dt''' \cdot E\{z - s(t - t'''), t'''\},
\end{aligned} \quad (30)$$

In the first term in the right-hand side of Equation (30), containing the squared electric field, we can, following Equation (26), use simplified linearized expression

$$\frac{\partial E}{\partial z} \cong - \int_{-\infty}^{+\infty} ds \cdot \frac{\partial f_0(s)}{\partial s} \int_{t_0 \rightarrow -\infty}^t dt' \cdot E\{z - s(t - t'), t'\}. \quad (31)$$

Using the relations (30) and (31), we can represent the fourth derivative with time of Equation (29) in the form

$$\begin{aligned}
& \frac{\partial^4 E}{\partial t^4} + \frac{\partial^2 E}{\partial t^2} + 3 \frac{\partial^2 E}{\partial z^2} + \frac{\partial^3}{\partial z^3} \cdot \int_{-\infty}^{+\infty} ds \cdot s^4 \cdot \frac{\partial f_0(s)}{\partial s} \int_{t_0 \rightarrow -\infty}^t dt' \cdot E\{z - s(t - t'), t'\} \\
& + \frac{\partial}{\partial z} \frac{\partial^2}{\partial t^2} \left[\int_{t_0 \rightarrow -\infty}^t dt' \cdot E(z, t') \cdot \int_{t_0 \rightarrow -\infty}^t dt' \cdot E(z, t') - \frac{1}{2} \cdot E(z, t) \cdot E(z, t) \right] = 0.
\end{aligned} \quad (32)$$

At obtaining Equation (32), the derivatives $(\partial/\partial z)$ in front of every term were canceled down and the explicit expressions for the integrals from the Maxwell distribution function $f_0(z)$ were used.

3. Waves in the Weak Nonlinear Case

The first three terms of Equation (32) should obviously describe the Landau damping in the linear approximation; one can be easily convinced of this as follows. By means of the Fourier expansion of these three terms for the corresponding amplitudes, we find

$$E(\omega, k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dz \cdot E(z, t) \cdot \exp\{i(\omega t - kz)\}, \quad (33)$$

which results in

$$\omega^4 - \omega^2 - 3k^2 + \int_{-\infty}^{+\infty} ds \cdot \frac{k^3 s^3}{\omega - ks} \cdot s \frac{\partial f_0(s)}{\partial s} = 0. \quad (34)$$

Simple transformation of the last equation gives

$$\omega^4 + \omega^3 \int_{-\infty}^{+\infty} ds \frac{s}{\omega - ks} \frac{\partial f_0(s)}{\partial s} = 0. \quad (35)$$

After expanding the denominator of (35) in powers of (ks/ω) for the real and the imaginary parts of the frequency, $\omega = \omega_0 + i\gamma$, we obtain

$$\omega_0^2 = 1 + 3k^2 \quad \text{and} \quad \gamma = -\sqrt{\frac{\pi}{8}} \frac{1}{k^3} \exp\left\{-\frac{1}{2k^2} - \frac{3}{2}\right\}, \quad (36)$$

which determines the Landau damping of the high-frequency plasma Langmuir waves [19].

In fact, the Landau damping describes the initial stage of the electrons' capturing by the wave cavity (by the minimum region of the wave), herewith the amplitude must be very small, smaller than the value proportional to γ , namely $\sqrt{eE_0/T_e} \cdot k \ll \gamma/k$ [16] (here E_0 is the amplitude of the wave). This inequality, in other words, means that the oscillation frequency of captured electrons' in the wave well, must be much smaller than the damping rate of the wave γ —the electrons are pushed by the back-side wall of the wave cavity and during the time-interval of passing the cavity width by electrons the wave should be damped [16,17].

Below, we will consider the case when the inverse inequality is fulfilled, that means the predominance of the electrons' oscillation frequency in the well over the damping rate— γ . Then, the captured electrons are reflected many times from the cavity walls (getting and losing the energy) and on the average the wave keeps its energy—hence at $\sqrt{eE_0/T_e} \cdot k \cdot k \gg \gamma$ the damping of the wave does not take place [16,17].

For simplifying the further calculations, it is convenient to introduce the following auxiliary value,

$$I(z, t) = \int_{t_0 \rightarrow -\infty}^t dt' \cdot E(z, t'), \quad (37)$$

for which from Equation (32) we can construct the following equation neglecting the term corresponding to the wave damping:

$$\frac{\partial^4 I}{\partial t^4} + \frac{\partial^2 I}{\partial t^2} + 3 \frac{\partial^2 I}{\partial z^2} + \frac{\partial}{\partial z} \frac{\partial}{\partial t} \left\{ I^2(z, t) - \frac{1}{2} \left(\frac{\partial I(z, t)}{\partial t} \right)^2 \right\} = 0. \quad (38)$$

By means of Equation (38), we can construct the expression for the value $\frac{1}{2} \left(\frac{\partial I(z, t)}{\partial t} \right)^2$. The straightforward calculations give

$$\begin{aligned} & \frac{1}{2} \left(\frac{\partial I(z, t)}{\partial t} \right)^2 + \frac{1}{2} I(z, t)^2 = \\ & \int_{-\infty}^t dt' \frac{\partial I(z, t')}{\partial t'} \frac{\partial}{\partial z} \int_{-\infty}^{t'} dt'' \left\{ \frac{1}{2} \left(\frac{\partial I(z, t'')}{\partial t''} \right)^2 - I^2(z, t'') \right\} \\ & - 3 \int_{-\infty}^t dt' \frac{\partial I(z, t')}{\partial t'} \frac{\partial^2}{\partial z^2} \int_{-\infty}^{t'} dt'' \int_{-\infty}^{t''} dt''' \cdot I(z, t'''). \end{aligned} \quad (39)$$

Furthermore, we will (a) hold on to the approximation usually used in the theory of the Landau damping—the assumption of the smooth dependence of the electric field on the spatial coordinate. Therefore, in Equation (38), the terms only up to second derivative with the spatial coordinate z are kept, and (b) uses the relation $(\partial I/\partial t) = E(z, t) = -\partial\varphi/\partial z$, which follows from Equation (37) and we restrict ourselves with the quadratic term I^2 in the last term in the rhs of Equation (39). Assuming the dependence of all unknown values on the argument $\xi = z - V \cdot t$, we find that $I(\xi) = (1/V) \cdot \varphi(\xi)$, where V is the dimensionless velocity of the waves, which is assumed to be large, $V \gg 1$. Under restrictions (a) and (b) the last two terms in the rhs of Equation (39) give negligibly small contributions. Substituting the remaining first term in the rhs into Equation (38) after integration two times we obtain

$$V^4 \frac{\partial^2 \varphi}{\partial \xi^2} + (V^2 + 3) \cdot \varphi - \frac{3}{2} \varphi^2 = \frac{1}{2} C_1. \quad (40)$$

Here, the constant C_1 does not depend on time. In the rhs of Equation (40), we have neglected the term proportional to ξ , leading to the nonphysical result at $\xi \rightarrow \infty$. Multiplying Equation (40) by $(\partial \varphi / \partial \xi)$ after integration we find the following:

$$V^4 \left(\frac{\partial \varphi}{\partial \xi} \right)^2 = \varphi^3 - (V^2 + 3) \cdot \varphi^2 + C_1 \cdot \varphi + C_2, \quad (41)$$

where the constants of the integrations C_1 and C_2 can be expressed in terms of the minimum φ_n and the maximum φ_m values of the potential, $(\partial \varphi / \partial \xi)|_{\varphi_m, \varphi_n} = 0$. As a result, we obtain

$$V^4 \left(\frac{\partial \varphi}{\partial \xi} \right)^2 = (\varphi_m - \varphi) \cdot (\varphi_s - \varphi) \cdot (\varphi - \varphi_n). \quad (42)$$

The constant φ_s (together with φ_m and φ_n) defines the wave velocity V :

$$V^2 = \varphi_m + \varphi_s + \varphi_n - 3. \quad (43)$$

At $\varphi_m > \varphi_s > \varphi_n$, the solution to Equation (42) represents a periodic wave, described by the expression (see Figure 1)

$$\varphi = \varphi_m - (\varphi_m - \varphi_n) \cdot dn^2\{x, s\}, \quad (44)$$

where $dn\{x, s\}$ is a Jacobi elliptic function with the module s [20],

$$dn\{x, s\} = \frac{\pi}{2 \cdot K(s)} + \frac{2\pi}{K(s)} \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}} \cos \left[\frac{\pi \cdot n \cdot x}{K(s)} \right], \quad (45)$$

$$\begin{aligned} q &= \exp \left[-\pi \frac{K(s')}{K(s)} \right], \quad s' = \sqrt{1 - s^2}, \\ x &= \frac{\sqrt{\varphi_m - \varphi_n}}{2 \cdot V^2} \cdot \xi, \quad s^2 = \frac{\varphi_s - \varphi_n}{\varphi_m - \varphi_n}. \end{aligned} \quad (46)$$

The function $dn^2\{x, s\}$ is a periodic function with the period $2K(s)$, where $K(s)$ is a complete elliptic integral of the first kind,

$$K(s) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - s^2 \cdot \sin^2 \alpha}}, \quad (47)$$

therefore, the wavelength of the periodic solution (44) can be defined according to the relation

$$\lambda = \frac{4}{\sqrt{\varphi_m - \varphi_n}} \cdot V^2 K(s) \quad (48)$$

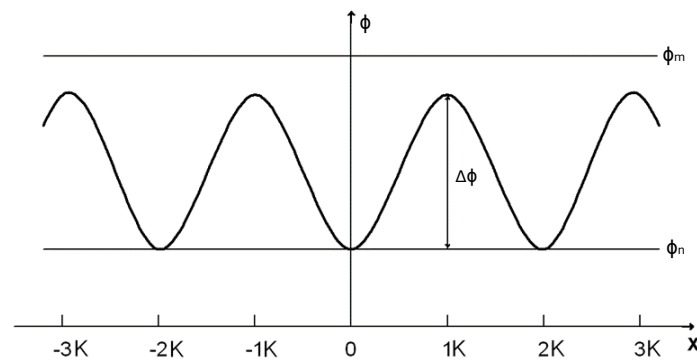


Figure 1. Sketch of the dependence of $dn^2\{x, s\}$ on x at $s = \text{const}$; $\Delta\phi = (1 - a^2) \cdot (\phi_m - \phi_n)$, a is the minimum value of the curve $dn\{x, s\}$

4. Instability of the Waves

We can now analyze the stability of the waves (44). Starting from Equation (40) we introduce the potential perturbation $\delta\phi$ according to the relation $\phi \rightarrow \phi + \delta\phi$. For the unperturbed part of the potential ϕ remains the expression (44) and for $\delta\phi$ we obtain

$$\frac{\partial^2 \delta\phi}{\partial \xi^2} + \frac{1}{V^2} \left\{ 1 + \frac{3}{V^2} (1 - \phi_m) + \frac{3}{V^2} (\phi_m - \phi_n) \cdot dn^2(z, s^2) \right\} \cdot \delta\phi = 0. \quad (49)$$

An explicit solution of this equation with the periodic coefficient can be found applying Hill's method [21]. This method is rather cumbersome with the long and difficult calculations. Here, we simplify the equation recalling the condition used above $V \gg 1$, and assuming that the parameter s is small, $s \ll 1$. Then, Equation (49) can be reduced to the Mathieu's equation [18,21], which describes the phenomena known as a parametric resonance,

$$\frac{\partial^2 \delta\phi}{\partial \xi^2} + \frac{1}{V^2} \{1 + h \cdot \cos k\xi\} \cdot \delta\phi = 0, \quad (50)$$

where $h \cong \frac{12}{V^2} \frac{\phi_m - \phi_n}{Ch\pi}$ and $k \cong \frac{\sqrt{\phi_m - \phi_n}}{V^2}$. Applying a standard method at the fulfillment of the resonance condition $k = 2/V$, Equation (50) gives the following expression for the rate of the instability; $\bar{\gamma} = \frac{1}{4} \frac{h}{V}$ [18].

5. Summary

In the present manuscript, an attempt is made to analytically solve the one-dimensional collisionless Vlasov–Boltzmann kinetic equation and analyze the presence and the stability of waves in a plasma using the found solution. It is shown that this solution has no poles and the Landau method of bypassing the pole is not applicable.

By means of the found kinetic solution, the nonlinear stage of the Langmuir waves is analyzed. The Langmuir waves with finite amplitude and an oscillation frequency (of electrons in the wave well) larger than the damping rate (found in the linear approximation) do not damp and generally keep the periodic structure. Only at the fulfillment of the definite resonance conditions are the waves unstable. The discovery of the corresponding rate is quite similar to the procedure for investigation of parametric instability.

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Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Let us present the relations (6) and (7) for R and U more in expanded forms:

$$R = z - \int dt' H' \left\{ v + \int_{t'}^t dt'' E \left[z - \int_{t''}^t dt''' H' \left\langle v + \int_{t'''}^t dt^{IV} E \left(z - \int_{t^{IV}}^t dt^V H' \{v + \dots\} \right) \right\rangle \right] \right\}, \quad (A1)$$

$$U = v + \int dt' \cdot E \left\{ z - \int_{t'}^t dt'' H' \left[v + \int_{t''}^t dt''' E \left\langle z - \int_{t'''}^t dt^{IV} H' \left(v + \int_{t^{IV}}^t dt^V E \{z - \dots\} \right) \right\rangle \right] \right\}. \quad (A2)$$

For the derivatives for these functions, we find the following:

$$\begin{aligned} \frac{\partial R}{\partial t} = & -H'[v] - \int dt' H''[\dots] \cdot \left[E(z, t) + \int_{t'}^t dt'' \frac{\partial E\{\dots\}}{\partial z} \left\{ -H'(v) \right. \right. \\ & - \int_{t''}^t dt''' H''(\dots) \cdot \left\langle E(z, t) + \int_{t'''}^t dt^{IV} \frac{\partial E(\dots)}{\partial z} \left(-H'(v) \right. \right. \\ & \left. \left. \left. - \int_{t^{IV}}^t dt^V H''[\dots] \cdot \{E(z, t) + \dots\} \right) \right\rangle \right] \Bigg], \end{aligned} \quad (A3)$$

$$\begin{aligned} \frac{\partial R}{\partial z} = & 1 - \int dt' \cdot H''[\dots] \cdot \left[\int_{t'}^t dt'' \frac{\partial E\{\dots\}}{\partial z} \left\{ 1 - \int_{t''}^t dt''' H''[\dots] \right. \right. \\ & \cdot \left\langle \int_{t'''}^t dt^{IV} \frac{\partial E(\dots)}{\partial z} \left(1 - \int_{t^{IV}}^t dt^V H''[\dots] \cdot \left\{ \frac{\partial E\{\dots\}}{\partial z} \{1 - \dots\} \right\} \right) \right\rangle \Bigg], \end{aligned} \quad (A4)$$

$$\begin{aligned} \frac{\partial R}{\partial v} = & - \int dt' \cdot H''[\dots] \cdot \left[1 + \int_{t'}^t dt'' \frac{\partial E\{\dots\}}{\partial z} \left\{ - \int_{t''}^t dt''' H''[\dots] \right. \right. \\ & \cdot \left\langle 1 - \int_{t'''}^t dt^{IV} \frac{\partial E(\dots)}{\partial z} \left(- \int_{t^{IV}}^t dt^V H''[\dots] \{1 - \dots\} \right) \right\rangle \Bigg]. \end{aligned} \quad (A5)$$

Using Equation (A2), one can also find expressions for the same derivatives of the function U . Substituting all these derivatives into the Equation (13) (separately for functions R and U) we will make sure that (dR/dt) and (dU/dt) are proportional to $H'[v] - v$. All other terms in expressions (dR/dt) and (dU/dt) cancel each other and the equality $H'[v] - v = 0$ shows that the functions R and U are indeed solutions of the Equation (1).

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