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A Class of Symmetric Harmonic Functions Involving a Specific q -Difference Symmetric Operator

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Abstract: This paper introduces a new class of harmonic functions defined through a generalized symmetric q -differential that acts on both the analytic and co-analytic parts of the function. By combining concepts from symmetric q -calculus and geometric function theory, we develop a framework that extends several well-known operators as special cases. The main contributions of this study include new criteria for harmonic univalence, sharp coefficient bounds, distortion theorems, and covering results. Our operator offers increased flexibility in modeling symmetric structures, with potential applications in complex analysis, fractional calculus, and mathematical physics. To support these theoretical developments, we provide concrete examples and highlight potential directions for future research, including extensions to higher-dimensional settings.

Keywords: analytic functions; univalent functions; harmonic functions; q -symmetric calculus; q -symmetric differential operator

1. Introduction and Preliminaries

The field of symmetric q -calculus has garnered increasing interest due to its applicability in various branches of mathematics and mathematical physics, fractional calculus, geometric function theory (GFT), and quantum mechanics [1]. It is well known that the derivative of a differentiable function f can be approximated by the symmetric q -derivative, which is believed to generally exhibit superior convergence properties compared to those of the q -derivative, although further research is required to confirm this. Originating from the concept of q -calculus developed in the early 20th century, the symmetric variant introduces more refined operator behaviors that are particularly suited to studying function classes with inherent symmetry properties.

Foundational work by Jackson [2] laid the groundwork for the q -analogue of classical differential and integral operators. The concept of q -starlike functions was first introduced by Ismail and his colleagues in 1990 [3]. Their pioneering work laid the foundation for subsequent research in this area, establishing q -starlike functions as a significant topic of study within the broader field of geometric function theory. Since then, the notion of q -starlike functions has been extensively explored and developed, leading to a deeper understanding of their properties and applications in various mathematical contexts. In a related study, Arif et al. [4] employed convolution techniques to define the q -Noor integral operator, subsequently applying it to identifying new subclasses of analytic functions. Furthermore, in [5], the authors applied q -calculus operator theory to establishing the q -analogue of the differential operator and explored a new subclass of analytic functions. Srivastava et al. [6]



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used the q -derivative operator to define a class of k -symmetric harmonic functions. Numerous mathematicians have contributed to this field by defining new subfamilies of analytic functions, often through the use of q -fractional integral and differential operators.

Several notable advancements have stemmed from the application of symmetric q -calculus in geometric function theory. Kanas et al. [7] introduced symmetric q -analogues of differential operators to define novel classes of starlike and convex functions. The foundational work by Kanas et al. set the stage for subsequent studies that have expanded the scope of symmetric q -calculus. Building upon these developments, Khan et al. [8] have revisited and refined the generalized symmetric conic domain by incorporating the principles of symmetric q -calculus. Their research has contributed to the evolution of mathematical techniques and broadened the applicability of symmetric q -calculus in geometric function theory and beyond. In their research, Khan et al. introduced a novel class of q -starlike functions, expanding the scope of geometric function theory by incorporating the principles of symmetric q -calculus. Their study also yielded significant insights into the properties of the symmetric q -operator, various types of q -starlike and q -convex functions, and innovative approaches for extending the conic domain. These findings have contributed to a deeper understanding of the interplay between q -calculus and function theory. In a subsequent study, Sabir et al. [9] developed a systematic approach to analyzing m -fold symmetric functions in relation to symmetric q -calculus. By leveraging this operator, they were able to explore several key results concerning m -fold symmetric bi-univalent functions, further enriching the theoretical framework of q -calculus in geometric function theory. Additionally, Khan et al. [10] advanced the concept of the symmetric q -derivative operator for multivalent functions, uncovering new and significant applications of this operator. Their work provided a foundation for further research into the role of symmetric q -calculus in complex function analyses. In recent work, Khan et al. [11] applied fundamental concepts from symmetric q -calculus and the theory of conic regions to introduce a new form of generalized symmetric conic domains. Building on this framework, they defined a novel subclass of symmetric q -starlike functions within the open unit disk U . Their research not only expanded the theoretical foundation of symmetric q -starlike functions but also led to the establishment of several new results in this area, further contributing to the development of geometric function theory. For more up-to-date work on this topic, consult [12–14].

Despite these advancements, the application of symmetric q -calculus to the theory of harmonic univalent functions has received considerably less attention. Most of the existing results are confined to purely analytic settings, with little exploration of the co-analytic components essential to harmonic mappings. This leaves an important gap in the literature, especially given the broader importance of harmonic functions in complex analyses and their numerous geometric and physical applications.

To address this gap, we introduce a new family of harmonic functions associated with a generalized symmetric q -differential operator, denoted by $\widetilde{\mathcal{R}}_q^{n,s,\lambda}$. This operator extends the classical Ruscheweyh operator and the Al-Oboudi operator through the appropriate selection of the parameters and limits. Our operator is specifically designed to act on both the analytic and co-analytic parts of harmonic mappings, thereby capturing the full complexity of their geometric structure.

This study establishes sufficient and necessary coefficient conditions for harmonic functions to belong to the class $\widetilde{\mathcal{R}}_q^{n,s,\lambda}(\alpha)$. We derive sharp coefficient bounds, distortion results, and covering theorems, offering a comprehensive geometric characterization of this new function class, paving the way for further developments in both theoretical and applied contexts.

To allow for more accurate comprehension of this paper, we will bring up some essential notations and concepts. The assumption we make throughout this paper is that $0 < q < 1$, $\mathbb{N} = \{1, 2, 3, \dots\}$.

The theory of q -analogue or q -extensions of classical formulas and functions originates from the key observation that (see [2])

$$\lim_{q \rightarrow 1^-} \frac{1 - q^n}{1 - q} = n$$

which justifies referring to the quantity $\frac{1 - q^n}{1 - q}$ as the basic number, commonly denoted by $[n]_q$.

The symmetric q -number $\widetilde{[n]}_q$ (see [15]) is defined as

$$\widetilde{[n]}_q = \frac{q^{-n} - q^n}{q^{-1} - q}, \text{ for } n \in \mathbb{N} \text{ and } \widetilde{[0]}_q = 0. \quad (1)$$

It is essential to highlight that as $q \rightarrow 1^-$, the symmetric q -number does not reduce into the standard q -number, which often appears in the context of q -deformed quantum mechanical simple harmonic oscillators [16], but rather approaches n , similar to the standard one, though the definitions differ.

The symmetric q -number shift factorial (see [15]) is defined as

$$\widetilde{[n]}_q! = \begin{cases} \widetilde{[n]}_q \cdot \widetilde{[n-1]}_q \cdot \widetilde{[n-2]}_q \cdots \widetilde{[1]}_q, & \text{for } n = 1, 2, \dots; \\ 1, & \text{for } n = 0. \end{cases} \quad (2)$$

One can see that $\widetilde{[n]}_q! \rightarrow n!$ as $q \rightarrow 1^-$.

Let $\mathcal{A}$ represent the class of functions $h(z)$ which are analytic within the unit disk $U = \{z \in \mathbb{C} \mid |z| < 1\}$, and let $\mathcal{A}^0$ be the subclass of $\mathcal{A}$ consisting of functions h that are normalized such that $h(0) = 0 = h'(0) - 1$ and has the power series representation

$$h(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in U. \quad (3)$$

The symmetric q -derivative operator $\widetilde{D}_q h(z)$ (see [15]) acting on functions $h(z) \in \mathcal{A}^0$ is defined as follows:

$$\widetilde{D}_q h(z) = \begin{cases} \frac{h(qz) - h(q^{-1}z)}{z(q - q^{-1})}, & z \neq 0, q \neq 1, z \in U; \\ h'(0), & \text{for } z = 0. \end{cases} \quad (4)$$

Note that $\widetilde{D}_q h(z) \rightarrow h'(z)$ as $q \rightarrow 1^-$.

From (4), we have $\widetilde{D}_q z^n = \widetilde{[n]}_q z^{n-1}$, and a power series of $\widetilde{D}_q h$ when $h(z) = z + a_2 z^2 + \dots$, is

$$\widetilde{D}_q h(z) = 1 + \sum_{j=2}^{\infty} \widetilde{[j]}_q a_j z^{j-1}. \quad (5)$$

It is straightforward to verify that the following properties hold [7]:

$$\begin{aligned}
\widetilde{D}_q(h(z) + g(z)) &= \widetilde{D}_qh(z) + \widetilde{D}_qg(z) \\
\widetilde{D}_q(h(z)g(z)) &= g(q^{-1}z)\widetilde{D}_qh(z) + f(qz)\widetilde{D}_qg(z) = g(qz)\widetilde{D}_qh(z) + f(q^{-1}z)\widetilde{D}_qg(z) \\
\widetilde{D}_q\left(\frac{h(z)}{g(z)}\right) &= \frac{h(qz)\widetilde{D}_qg(z) - g(q^{-1}z)\widetilde{D}_qh(z)}{g(q^{-1}z)g(qz)} \\
\widetilde{D}_qh(z) &= \widetilde{D}_{q^2}h(q^{-1}z)
\end{aligned}$$

The applications of the symmetric q -difference operator defined above play a crucial role in deriving the symmetric Ruscheweyh operator, which is specifically formulated for the analytic function h . This operator extends the classical differential techniques by incorporating symmetric q -number properties, ensuring balanced transformations. The definitions of the Al-Oboudi differential operator, the Ruscheweyh differential operator, the symmetric Al-Oboudi operator, and the symmetric Ruscheweyh operator for h are given by

Definition 1 (Al-Oboudi [17]). For $h \in \mathcal{A}$, $h(z) = z + \sum_{j=2}^{\infty} \gamma_j z^j$, $\lambda \geq 0$, and $n \in \mathbb{N}$, the operator D_{λ}^s is defined by $D_{\lambda}^s : \mathcal{A} \rightarrow \mathcal{A}$,

$$\begin{aligned}
D_{\lambda}^0 h(z) &= h(z) \\
D_{\lambda}^1 h(z) &= (1 - \lambda)h(z) + \lambda zh'(z) = D_{\lambda}h(z) \\
&\dots \\
D_{\lambda}^s h(z) &= (1 - \lambda)D_{\lambda}^{s-1}h(z) + \lambda z(D_{\lambda}^{s-1}h(z))' = D_{\lambda}\left(D_{\lambda}^{s-1}h(z)\right), \text{ for } z \in U.
\end{aligned}$$

Remark 1. If $h \in \mathcal{A}$, $h(z) = z + \sum_{j=2}^{\infty} \gamma_j z^j$, then $D_{\lambda}^s h(z) = z + \sum_{j=2}^{\infty} [1 + (j - 1)\lambda]^s \gamma_j z^j$, for $z \in U$.

Note that if $h \in \mathcal{A}$, $\lambda \geq 0$ and $n \in \mathbb{N}$, the symmetric q -operator $\widetilde{D}_{\lambda}^s$ is defined by $\widetilde{D}_{\lambda}^s : \mathcal{A} \rightarrow \mathcal{A}$, $\widetilde{D}_{\lambda}^s h(z) = z + \sum_{j=2}^{\infty} \left(1 - \lambda + [j]_q \lambda\right)^s \gamma_j z^j$, for $z \in U$.

Definition 2 (Ruscheweyh [18]). For $h \in \mathcal{A}$, $h(z) = z + \sum_{j=2}^{\infty} \gamma_j z^j$, $n \in \mathbb{N}$, the Ruscheweyh differential operator R^n is defined by $R^n : \mathcal{A} \rightarrow \mathcal{A}$,

$$\begin{aligned}
R^0 h(z) &= h(z) \\
R^1 h(z) &= zh'(z) \\
&\dots \\
(n + 1)R^{n+1}h(z) &= z(R^n h(z))' + nR^n h(z), \quad z \in U.
\end{aligned}$$

Remark 2. If $h \in \mathcal{A}$, $h(z) = z + \sum_{j=2}^{\infty} \eta_j z^j$, then

$$R^n h(z) = z + \sum_{j=2}^{\infty} \frac{(n + j - 1)!}{n!(j - 1)!} \eta_j z^j, \quad z \in U. \quad (6)$$

Definition 3 ([19]). Let $h \in \mathcal{A}$. Denote by $\widetilde{\mathcal{R}}_q^n$ the symmetric Ruscheweyh q -difference operator defined as

$$\widetilde{\mathcal{R}}_q^n h(z) = z + \sum_{j=2}^{\infty} \frac{[j + n - 1]_q!}{[n]_q! [j - 1]_q!} \eta_j z^j, \quad z \in U, \quad 0 < q < 1, \quad (7)$$

where $[n]_q$ and $[n]_q!$ are defined in (1) and (2).

Remark 3. Since $\widetilde{[n]_q!} \rightarrow n!$ as $q \rightarrow 1^-$, it follows that if $q \rightarrow 1^-$ in the previous definition, we obtain

$$\begin{aligned} \lim_{q \rightarrow 1^-} \widetilde{\mathcal{R}_q^n} h(z) &= z + \lim_{q \rightarrow 1^-} \left[\sum_{j=2}^{\infty} \frac{\widetilde{[j+n-1]_q!}}{\widetilde{[n]_q!} \widetilde{[j-1]_q!}} \eta_j z^j \right] \\ &= z + \sum_{j=2}^{\infty} \frac{[j+n-1]_q!}{[n]_q! [j-1]_q!} \eta_j z^j = \mathcal{R}^n h(z), \end{aligned} \quad (8)$$

where $\mathcal{R}^n h(z)$ is the Ruscheweyh differential operator defined in (6) and has been studied by many authors (see [9,20–24]).

We present a novel symmetric q -operator, denoted as $\widetilde{\mathcal{R}_q^{n,s,\lambda}}$, with the following definition:

Definition 4. Let

$$\begin{aligned} \widetilde{\mathcal{R}_q^{n,0,\lambda}} h(z) &= \mathcal{R}_q^n h(z), \\ \widetilde{\mathcal{R}_q^{n,1,\lambda}} h(z) &= (1-\lambda) \mathcal{R}_q^n h(z) + \lambda z \widetilde{D}_q \left(\mathcal{R}_q^n h(z) \right), \\ \widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) &= \widetilde{\mathcal{R}_q^{n,1,\lambda}} \left(\widetilde{\mathcal{R}_q^{n,s-1,\lambda}} h(z) \right), \end{aligned} \quad (9)$$

for $n, s \in \mathbb{N}$, $0 < q < 1$, $\lambda \geq 0$, $z \in U$.

Assuming $h \in \mathcal{A}$ is represented by (3), we can derive the following from (9):

$$\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) = z + \sum_{j=2}^{\infty} \left(1 - \lambda + \widetilde{[j]_q} \lambda \right)^s \frac{\widetilde{[j+n-1]_q!}}{\widetilde{[n]_q!} \widetilde{[j-1]_q!}} a_j z^j, \quad (10)$$

for $n, s \in \mathbb{N}$, $0 < q < 1$, $\lambda \geq 0$, $z \in U$.

Remark 4. Since $\widetilde{[n]_q!} \rightarrow n!$ as $q \rightarrow 1^-$, it follows that if $q \rightarrow 1^-$ in (10), we obtain

$$\begin{aligned} \lim_{q \rightarrow 1^-} \widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) &= z + \lim_{q \rightarrow 1^-} \left[\sum_{j=2}^{\infty} \left(1 - \lambda + \widetilde{[j]_q} \lambda \right)^s \frac{\widetilde{[j+n-1]_q!}}{\widetilde{[n]_q!} \widetilde{[j-1]_q!}} a_j z^j \right] = \\ &= \mathcal{R}_\lambda^{n,s} h(z), \end{aligned} \quad (11)$$

where the operator $\mathcal{R}_\lambda^{n,s}$ was defined and studied by Andrei L. in [23].

We recall the definition of the convolution (Hadamard) product for two analytic functions in the open unit disk. Given two analytic functions in the open unit disk U , namely $w(z) = z + \sum_{j=2}^{\infty} w_j z^j$ and $v(z) = z + \sum_{j=2}^{\infty} v_j z^j$, their Hadamard product (also known as the convolution) of $w(z)$ and $v(z)$, denoted by $(w * v)(z)$, is defined as

$$w(z) * v(z) = (w * v)(z) = z + \sum_{j=2}^{\infty} w_j v_j z^j.$$

Remark 5. The symmetric q -operator $\widetilde{\mathcal{R}_q^{n,s,\lambda}}$ is the Hadamard product of the symmetric Al-Oboudi differential q -operator $\widetilde{D_{q,\lambda}^s}$ and the symmetric Ruscheweyh q -differential operator $\widetilde{R_q^n}$,

$$\begin{aligned}\widetilde{\mathcal{R}_q^{n,s,\lambda}} f(z) &= \left(\widetilde{D_{q,\lambda}^s} * \widetilde{R_q^n} \right) f(z) = z + \sum_{j=2}^{\infty} \left(1 - \lambda + [j]_q \lambda \right)^s \frac{[j+n-1]_q!}{[n]_q! [j-1]_q!} \lambda_j \gamma_j z^j = \\ &= z + \sum_{j=2}^{\infty} \left(1 - \lambda + [j]_q \lambda \right)^s \frac{[j+n-1]_q!}{[n]_q! [j-1]_q!} a_j z^j,\end{aligned}$$

where we have introduced the notation $a_j = \lambda_j \gamma_j$, for $j \geq 2$.

Proposition 1. For natural numbers n and s , where $0 < q < 1$, $\lambda \geq 0$ and $z \in U$, the operator $\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z)$ adheres to the following mathematical equality:

$$z \left(\widetilde{D_q} \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) \right) \right) = z + \sum_{j=2}^{\infty} \frac{\left(1 - \lambda + [j]_q \lambda \right)^s [j+n-1]_q!}{[n]_q! [j-1]_q!} [j]_q a_j z^j \quad (12)$$

Proof. We can infer from (10)

$$\begin{aligned}\widetilde{D_q} \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) \right) &= \frac{\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(qz) - \widetilde{\mathcal{R}_q^{n,s,\lambda}} h(q^{-1}z)}{(q - q^{-1})z} \\ &= \frac{1}{z(q - q^{-1})} \left(qz - q^{-1}z + \sum_{j=2}^{\infty} \frac{\left(1 - \lambda + [j]_q \lambda \right)^s [j+n-1]_q!}{[n]_q! [j-1]_q!} (q^j - q^{-j}) a_j z^j \right) \\ &= 1 + \sum_{j=2}^{\infty} \frac{\left(1 - \lambda + [j]_q \lambda \right)^s [j+n-1]_q!}{[n]_q! [j-1]_q!} \frac{q^j - q^{-j}}{q - q^{-1}} a_j z^{j-1} \\ &= 1 + \sum_{j=2}^{\infty} \frac{\left(1 - \lambda + [j]_q \lambda \right)^s [j+n-1]_q!}{[n]_q! [j-1]_q!} [j]_q a_j z^{j-1}.\end{aligned}$$

Hence, the following identity holds for the operator $\widetilde{\mathcal{R}_q^{n,s,\lambda}}$:

$$z \left(\widetilde{D_q} \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) \right) \right) = z + \sum_{j=2}^{\infty} \frac{\left(1 - \lambda + [j]_q \lambda \right)^s [j+n-1]_q!}{[n]_q! [j-1]_q!} [j]_q a_j z^j. \quad (13)$$

This concludes the proof. $\square$

Proposition 2. For natural numbers n and s , with $0 < q < 1$ and $z \in U$, the operator $\widetilde{\mathcal{R}_q^{n,s,1}}$ fulfills the following equation:

$$z \left(\widetilde{D_q} \left(\widetilde{\mathcal{R}_q^{n,s,1}} h(z) \right) \right) = \widetilde{\mathcal{R}_q^{n,s+1,1}} h(z). \quad (14)$$

Proof. We can derive from (13), with $\lambda = 1$,

$$\begin{aligned} z \left(\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,1} h(z)} \right) \right) &= z + \sum_{j=2}^{\infty} \frac{[\widetilde{j}]_q^s [\widetilde{j+n-1}]_q!}{[\widetilde{n}]_q! [\widetilde{j-1}]_q!} [\widetilde{j}]_q a_j z^j \\ &= z + \sum_{j=2}^{\infty} \frac{[\widetilde{j}]_q^{s+1} [\widetilde{j+n-1}]_q!}{[\widetilde{n}]_q! [\widetilde{j-1}]_q!} a_j z^j \\ &= \widetilde{\mathcal{R}_q^{n,s+1,1} h(z)}. \end{aligned}$$

The proof is completed. $\square$

Remark 6. Based on its definition, it is clear that the operator $\widetilde{\mathcal{R}_q^{n,s,\lambda}}$ can be transformed into several well-known operators by appropriately selecting specific parameter values. Notably, when $q \rightarrow 1^-$, the symmetric q -operator $\widetilde{\mathcal{R}_q^{n,s,\lambda}}$ is converted into the differential operator $\mathcal{R}_\lambda^{n,s}$ introduced and studied in [23]. For $\lambda = 0$, the symmetric q -operator $\widetilde{\mathcal{R}_q^{n,s,\lambda}}$ is reduced into the symmetric Ruscheweyh q -operator [19]. Furthermore, for $q \rightarrow 1^-$, the symmetric q -operator $\widetilde{\mathcal{R}_q^{n,s,\lambda}}$ is converted into the Ruscheweyh operator [18]. Additionally, under the specific conditions where $\lambda = 0$, $n = 0$, and q approaches 1^- , the q -operator $\mathcal{R}_q^{n,s,\lambda}$ assumes a distinct form. In this limiting case, it has been established that the operator coincides with that introduced by Al-Oboudi [17]. These transformations highlight the relevance and novelty of our generalization.

A continuous function $f = u + iv$ is a complex-valued harmonic function defined in U , where u and v are real-valued harmonic functions within U . The function $f(z)$ can be expressed as the sum of two components, $f = h + \bar{g}$, where h and g are both analytic in U . The function h is referred to as the analytic part of f , and g is known as the co-analytic part. The function h belongs to the subclass $\mathcal{A}^0$, defined in (3), and g belongs to the class $\mathcal{A}$, with the following power series expansion:

$$g(z) = \sum_{j=1}^{\infty} b_j z^j, \quad z \in U, |b_1| < 1. \quad (15)$$

A condition that is both necessary and sufficient for the function f to be locally univalent and sense-preserving in the domain U is that $|h'(z)|$ is greater than $|g'(z)|$ at every point in U and the harmonic function $f = h + \bar{g}$ will be normalized if $f(0) = 0 = f'(0) - 1$. This condition ensures that the Jacobian determinant of the function f is positive, which is essential for f to be both locally univalent and orientation-preserving in U .

Let $\mathcal{H}$ the class of sense-preserving univalent harmonic functions $f = h + \bar{g}$ in U , where h and g are defined above. We point out that $\mathcal{H}$ reduces to the well-known class of normalized univalent functions if the co-analytic component of f is identically zero.

Following the pioneering work of Clunie and Sheil-Small [25] on harmonic mappings, a significant body of research emerged focused on various types of complex-valued harmonic univalent functions. Their influential study served as a catalyst for subsequent investigations, leading to numerous publications that delved into the unique characteristics and properties of these functions. Specifically, the distinctive features of several subclasses of harmonic univalent functions were thoroughly examined in a series of works [26–32]. This research has greatly enriched our understanding of harmonic mappings and their applications in complex analyses and geometric function theory.

The primary objective of this article is to introduce and define the symmetric q -derivative operator by employing the principles of symmetric q -calculus in the context of complex functions that are harmonic within the domain U . Furthermore, this study aims to establish precise coefficient bounds, derive distortion theorems, and obtain significant covering results related to these harmonic functions. We introduce and formally define the symmetric q -differential operator specifically for harmonic functions by using the symmetric q -derivative operator defined above (8). The definition is structured to align with the principles of symmetric q -calculus, ensuring its applicability to functions that are harmonic within the given domain. The formulation of this operator is presented as follows.

$$\widetilde{\mathcal{R}_q^{n,s,\lambda}} f(z) = \widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) + \widetilde{\mathcal{R}_q^{n,s,\lambda}} \overline{g(z)}, \quad (16)$$

where

$$\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) = z + \sum_{j=2}^{\infty} \left(1 - \lambda + [j]_q \lambda\right)^s \frac{[j+n-1]_q!}{[n]_q! [j-1]_q!} a_j z^j, \quad (17)$$

$$\widetilde{\mathcal{R}_q^{n,s,\lambda}} g(z) = \sum_{j=1}^{\infty} \left(1 - \lambda + [j]_q \lambda\right)^s \frac{[j+n-1]_q!}{[n]_q! [j-1]_q!} b_j z^j, \quad (18)$$

and

$$z \left(\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} f(z) \right) \right) = z \left(\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) \right) \right) - \overline{z \left(\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} g(z) \right) \right)}. \quad (19)$$

Drawing inspiration from the contributions of Jahangiri [33], Zhang et al. [34], Khan et al. [35], Abubakar et al. [14], and Al-Shbeil et al. [36], this paper introduces novel subclasses of harmonic univalent functions, defined using the newly established symmetric q -differential operator.

For $0 \leq \alpha < 1$, let $\widetilde{\mathcal{RH}_q^{n,s,\lambda}}(\alpha)$ represent the family of harmonic functions $f = h + \overline{g}$ that fulfills the requirements

$$\operatorname{Re} \left\{ \frac{z \left(\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) \right) \right)}{\widetilde{\mathcal{R}_q^{n,s,\lambda}} f(z)} \right\} \geq \alpha, \quad z \in U, \quad (20)$$

where $\widetilde{\mathcal{R}_q^{n,s,\lambda}} f(z)$ is given by (16). Additionally, let the subclass of $\widetilde{\mathcal{RH}_q^{n,s,\lambda}}(\alpha)$ be denoted by $\mathcal{RH}_q^{n,s,\lambda}(\alpha)$, which comprises the harmonic functions $f = h + \overline{g}$, where h and g take the following form:

$$h(z) = z - \sum_{j=2}^{\infty} a_j z^j \quad \text{and} \quad g(z) = \sum_{j=1}^{\infty} b_j z^j, \quad z \in U, |b_1| < 1, \text{ where } a_j, b_j \geq 0. \quad (21)$$

By means of (13), $\widetilde{\mathcal{RH}_q^{n,s,1}}(\alpha)$ represents the family of harmonic functions $f = h + \overline{g}$ that complies with the conditions

$$\operatorname{Re} \left\{ \frac{\widetilde{\mathcal{R}_q^{n,s+1,1} f(z)}}{\widetilde{\mathcal{R}_q^{n,s,1} f(z)}} \right\} \geq \alpha, \quad z \in U. \quad (22)$$

In the next section, we derive necessary and sufficient conditions that characterize the harmonic functions within the class $\widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$, providing a comprehensive framework for their classification. Furthermore, we explore the geometric properties of the functions in $\widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$ by proving distortion theorems, which offer crucial insights into the behavior and bounds of these functions. Additionally, we identify and analyze the extreme points of the functions in $\widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$, further enriching our understanding of their structural properties.

2. The Main Results

In the theorems below, we will establish coefficient bounds for harmonic functions that belong to the classes $\widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$ and $\mathcal{RH}_q^{n,s,\lambda}(\alpha)$.

Theorem 1. Let $f = h + \overline{g}$, $0 \leq \alpha < 1$, $0 < q < 1$, $\lambda \geq 0$, $z \in U$ and

$$\frac{1}{1-\alpha} \sum_{j=1}^{\infty} \left((\widetilde{[j]_q} - \alpha) |a_j| + (\widetilde{[j]_q} + \alpha) |b_j| \right) (1 - \lambda + \widetilde{[j]_q} \lambda)^s \frac{[j+n-1]_q!}{[\widetilde{n}]_q! [\widetilde{j-1}]_q!} \leq 2, \quad (23)$$

where the functions h and g are defined in (3) and (15). In this case, the function f is harmonic and univalent in U and belongs to the class $\widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$ if it satisfies the inequality (23). The equality in (23) applies for the harmonic function

$$\begin{aligned} f(z) = z + \sum_{j=2}^{\infty} \frac{(1-\alpha) [\widetilde{n}]_q! [\widetilde{j-1}]_q!}{(1-\lambda + \widetilde{[j]_q} \lambda)^s [j+n-1]_q! (\widetilde{[j]_q} - \alpha)} A_j z^j \\ + \sum_{j=1}^{\infty} \frac{(1-\alpha) [\widetilde{n}]_q! [\widetilde{j-1}]_q!}{(1-\lambda + \widetilde{[j]_q} \lambda)^s [j+n-1]_q! (\widetilde{[j]_q} + \alpha)} \overline{B_j} \overline{z}^j, \end{aligned} \quad (24)$$

$$\text{where } \sum_{j=2}^{\infty} |A_j| + \sum_{j=1}^{\infty} |B_j| = 1.$$

Proof. Initially, it is necessary to demonstrate that the function $f = h + \overline{g}$ is locally univalent and preserves the orientation within U . To this end, it suffices to prove that the

second complex dilatation, w , of f satisfies the condition $|w| = \left| \frac{\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda} g(z)} \right)}{\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda} h(z)} \right)} \right| < 1$ in U .

This holds true for $z = re^{i\theta} \in U$. We obtain

$$\begin{aligned}
\left| \widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) \right) \right| &\geq 1 - \sum_{j=2}^{\infty} \frac{(1-\lambda + [\widetilde{j}]_q \lambda)^s [\widetilde{j+n-1}]_q!}{[\widetilde{n}]_q! [\widetilde{j-1}]_q!} [\widetilde{j}]_q |a_j| r^{j-1} \\
&\geq 1 - \sum_{j=2}^{\infty} \frac{(1-\lambda + [\widetilde{j}]_q \lambda)^s [\widetilde{j+n-1}]_q!}{[\widetilde{n}]_q! [\widetilde{j-1}]_q!} [\widetilde{j}]_q |a_j| \\
&\geq 1 - \sum_{j=2}^{\infty} \frac{1}{1-\alpha} \frac{[\widetilde{j+n-1}]_q!}{[\widetilde{n}]_q! [\widetilde{j-1}]_q!} (1-\lambda + [\widetilde{j}]_q \lambda)^s ([\widetilde{j}]_q - \alpha) |a_j| \\
&\geq \sum_{j=1}^{\infty} \frac{1}{1-\alpha} \frac{[\widetilde{j+n-1}]_q!}{[\widetilde{n}]_q! [\widetilde{j-1}]_q!} (1-\lambda + [\widetilde{j}]_q \lambda)^s ([\widetilde{j}]_q + \alpha) |b_j| \\
&\geq \sum_{j=1}^{\infty} \frac{(1-\lambda + [\widetilde{j}]_q \lambda)^s [\widetilde{j+n-1}]_q!}{[\widetilde{n}]_q! [\widetilde{j-1}]_q!} [\widetilde{j}]_q |b_j| \geq \\
&\geq \sum_{j=1}^{\infty} \frac{(1-\lambda + [\widetilde{j}]_q \lambda)^s [\widetilde{j+n-1}]_q!}{[\widetilde{n}]_q! [\widetilde{j-1}]_q!} [\widetilde{j}]_q |b_j| r^{j-1} \\
&\geq \left| \widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} g(z) \right) \right|, \quad z \in U.
\end{aligned}$$

So, if $q \rightarrow 1^-$, then $\left| \widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) \right) \right| \geq \left| \widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} g(z) \right) \right|$ in U , and it follows that the function $f = h + \overline{g}$ is locally univalent and sense-preserving in U . To establish that f is univalent in U , we employ an argument originally presented by the author in reference [31]. Consider two distinct points z_1 and z_2 in U , meaning $z_1 \neq z_2$. Since U is simply connected and convex, we can construct a parameterized path between these points given by $z(t) = (1-t)z_1 + tz_2$, for $0 \leq t \leq 1$. For the case where $z_1 \neq z_2$, we can express

$$\begin{aligned}
&\operatorname{Re} \left(\frac{\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} f(z_2) \right) - \widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} f(z_1) \right)}{z_2 - z_1} \right) > \\
&\int_0^1 \operatorname{Re} \left(\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) \right) \right) - \left| \widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} g(z) \right) \right| dt.
\end{aligned}$$

Conversely, we note that

$$\begin{aligned}
&\operatorname{Re} \left(\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) \right) \right) - \left| \widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} g(z) \right) \right| \\
&\geq \operatorname{Re} \left(\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) \right) \right) - \sum_{j=1}^{\infty} \frac{[\widetilde{j+n-1}]_q!}{[\widetilde{n}]_q! [\widetilde{j-1}]_q!} (1-\lambda + [\widetilde{j}]_q \lambda)^s [\widetilde{j}]_q |b_j| \\
&\geq 1 - \sum_{j=2}^{\infty} \frac{[\widetilde{j+n-1}]_q!}{[\widetilde{n}]_q! [\widetilde{j-1}]_q!} (1-\lambda + [\widetilde{j}]_q \lambda)^s [\widetilde{j}]_q |a_j| - \sum_{j=1}^{\infty} \frac{[\widetilde{j+n-1}]_q!}{[\widetilde{n}]_q! [\widetilde{j-1}]_q!} (1-\lambda + [\widetilde{j}]_q \lambda)^s [\widetilde{j}]_q |b_j| \\
&\geq 1 - \sum_{j=2}^{\infty} \frac{1}{1-\alpha} (1-\lambda + [\widetilde{j}]_q \lambda)^s \frac{[\widetilde{j+n-1}]_q!}{[\widetilde{n}]_q! [\widetilde{j-1}]_q!} ([\widetilde{j}]_q - \alpha) |a_j| - \\
&- \sum_{j=1}^{\infty} \frac{1}{1-\alpha} (1-\lambda + [\widetilde{j}]_q \lambda)^s \frac{[\widetilde{j+n-1}]_q!}{[\widetilde{n}]_q! [\widetilde{j-1}]_q!} ([\widetilde{j}]_q + \alpha) |b_j| \geq 0.
\end{aligned}$$

Thus, the function f is univalent in U . However, we still need to establish that inequality (20) holds under the assumption that the coefficients of the univalent harmonic function $f = h + \bar{g}$ satisfy the condition given in (23). We are required to prove that

$$\operatorname{Re} \left\{ \frac{z \left(\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} f(z) \right) \right)}{\widetilde{\mathcal{R}_q^{n,s,\lambda}} f(z)} \right\} = \operatorname{Re} \left\{ \frac{z \left(\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) \right) \right) - \overline{z \left(\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} g(z) \right) \right)}}{\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) + \overline{\widetilde{\mathcal{R}_q^{n,s,\lambda}} g(z)}} \right\} \geq \alpha. \quad (25)$$

Let

$$\begin{aligned} X(z) &= z \left(\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) \right) \right) - \overline{z \left(\widetilde{D}_q \left(\widetilde{\mathcal{R}_q^{n,s,\lambda}} g(z) \right) \right)} \\ &= z + \sum_{j=2}^{\infty} [\widetilde{j}]_q \left(1 - \lambda + [\widetilde{j}]_q \lambda \right)^s \frac{[j+n-1]_q!}{[\widetilde{n}]_q! [\widetilde{j}-1]_q!} a_j z^j - \\ &\quad \sum_{j=1}^{\infty} [\widetilde{j}]_q \left(1 - \lambda + [\widetilde{j}]_q \lambda \right)^s \frac{[j+n-1]_q!}{[\widetilde{n}]_q! [\widetilde{j}-1]_q!} \overline{b_j z^j}, \end{aligned}$$

and

$$\begin{aligned} Y(z) &= \widetilde{\mathcal{R}_q^{n,s,1}} f(z) = \widetilde{\mathcal{R}_q^{n,s,\lambda}} h(z) + \overline{\widetilde{\mathcal{R}_q^{n,s,\lambda}} g(z)} \\ &= z + \sum_{j=2}^{\infty} \left(1 - \lambda + [\widetilde{j}]_q \lambda \right)^s \frac{[j+n-1]_q!}{[\widetilde{n}]_q! [\widetilde{j}-1]_q!} a_j z^j + \sum_{j=1}^{\infty} \left(1 - \lambda + [\widetilde{j}]_q \lambda \right)^s \frac{[j+n-1]_q!}{[\widetilde{n}]_q! [\widetilde{j}-1]_q!} \overline{b_j z^j}, \end{aligned}$$

Taking into account that $\operatorname{Re}(w) \geq \alpha$ necessarily and sufficiently if $|1+w-\alpha| \geq |1-w+\alpha|$, it is adequate to establish that

$$|X(z) + (1-\alpha)Y(z)| - |X(z) - (1+\alpha)Y(z)| \geq 0 \quad (26)$$

The left-hand side of the inequality (26) becomes

$$\begin{aligned} &|X(z) + (1-\alpha)Y(z)| - |X(z) - (1+\alpha)Y(z)| \\ &= \left| (2-\alpha)z + \sum_{j=2}^{\infty} \left([\widetilde{j}]_q + 1 - \alpha \right) \left(1 - \lambda + [\widetilde{j}]_q \lambda \right)^s \frac{[j+n-1]_q!}{[\widetilde{n}]_q! [\widetilde{j}-1]_q!} a_j z^j \right. \\ &\quad \left. - \sum_{j=1}^{\infty} \left([\widetilde{j}]_q - 1 + \alpha \right) \left(1 - \lambda + [\widetilde{j}]_q \lambda \right)^s \frac{[j+n-1]_q!}{[\widetilde{n}]_q! [\widetilde{j}-1]_q!} \overline{b_j z^j} \right| \\ &\quad - \left| -\alpha z + \sum_{j=2}^{\infty} \left([\widetilde{j}]_q - 1 - \alpha \right) \left(1 - \lambda + [\widetilde{j}]_q \lambda \right)^s \frac{[j+n-1]_q!}{[\widetilde{n}]_q! [\widetilde{j}-1]_q!} a_j z^j \right. \\ &\quad \left. - \sum_{j=1}^{\infty} \left([\widetilde{j}]_q + 1 + \alpha \right) \left(1 - \lambda + [\widetilde{j}]_q \lambda \right)^s \frac{[j+n-1]_q!}{[\widetilde{n}]_q! [\widetilde{j}-1]_q!} \overline{b_j z^j} \right| \\ &\geq (2-\alpha)|z| - \sum_{j=2}^{\infty} \left([\widetilde{j}]_q + 1 - \alpha \right) \left(1 - \lambda + [\widetilde{j}]_q \lambda \right)^s \frac{[j+n-1]_q!}{[\widetilde{n}]_q! [\widetilde{j}-1]_q!} |a_j| |z|^j \\ &\quad - \sum_{j=1}^{\infty} \left([\widetilde{j}]_q - 1 + \alpha \right) \left(1 - \lambda + [\widetilde{j}]_q \lambda \right)^s \frac{[j+n-1]_q!}{[\widetilde{n}]_q! [\widetilde{j}-1]_q!} |b_j| |z|^j \\ &\quad - \alpha |z| \sum_{j=2}^{\infty} \left([\widetilde{j}]_q - 1 - \alpha \right) \left(1 - \lambda + [\widetilde{j}]_q \lambda \right)^s \frac{[j+n-1]_q!}{[\widetilde{n}]_q! [\widetilde{j}-1]_q!} |a_j| |z|^j \\ &\quad - \sum_{j=1}^{\infty} \left([\widetilde{j}]_q + 1 + \alpha \right) \left(1 - \lambda + [\widetilde{j}]_q \lambda \right)^s \frac{[j+n-1]_q!}{[\widetilde{n}]_q! [\widetilde{j}-1]_q!} |b_j| |z|^j \\ &\geq 2(1-\alpha)|z| \left[2 - \sum_{j=1}^{\infty} \left(\frac{[\widetilde{j}]_q - \alpha}{1-\alpha} |a_j| + \frac{[\widetilde{j}]_q + \alpha}{1-\alpha} |b_j| \right) \left(1 - \lambda + [\widetilde{j}]_q \lambda \right)^s \frac{[j+n-1]_q!}{[\widetilde{n}]_q! [\widetilde{j}-1]_q!} |z|^{j-1} \right]. \end{aligned}$$

$$\geq 2(1-\alpha) \left[2 - \frac{1}{1-\alpha} \sum_{j=1}^{\infty} \left((\widetilde{[j]_q} - \alpha) |a_j| + (\widetilde{[j]_q} + \alpha) |b_j| \right) (1-\lambda + \widetilde{[j]_q} \lambda)^s \frac{\widetilde{[j+n-1]_q!}}{[\widetilde{n}]_q! \widetilde{[j-1]_q!}} \right] \quad (27)$$

Applying condition (23), we see that this expression is non-negative, which finalizes the proof that $f \in \widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$.

The functions expressed by (24) are in $\widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$ because

$$\begin{aligned} \sum_{j=1}^{\infty} \left(\frac{\widetilde{[j+n-1]_q!} (\widetilde{[j]_q} - \alpha)}{(1-\alpha) [\widetilde{n}]_q! \widetilde{[j-1]_q!}} (1-\lambda + \widetilde{[j]_q} \lambda)^s |a_j| + \frac{\widetilde{[j+n-1]_q!} (\widetilde{[j]_q} + \alpha)}{(1-\alpha) [\widetilde{n}]_q! \widetilde{[j-1]_q!}} (1-\lambda + \widetilde{[j]_q} \lambda)^s |b_j| \right) \\ = 1 + \sum_{j=2}^{\infty} |A_j| + \sum_{j=1}^{\infty} |B_j| = 2. \quad (28) \end{aligned}$$

This completes the proof of our theorem. $\square$

Several examples of functions satisfying the conditions of this class are provided below.

Example 1. The function $f = h + \overline{g}$ expressed as

$$f(z) = z + \sum_{j=2}^{\infty} X_j z^j + \sum_{j=1}^{\infty} Y_j \overline{z}^j$$

with

$$X_j = \frac{(1+v)(1-\alpha) [\widetilde{n}]_q! \widetilde{[j-1]_q!} c_j}{2(j+v)(j+v-1) (1-\lambda + \widetilde{[j]_q} \lambda)^s \widetilde{[j+n-1]_q!} (\widetilde{[j]_q} - \alpha)}$$

and

$$Y_j = \frac{v(1-\alpha) [\widetilde{n}]_q! \widetilde{[j-1]_q!} c_j}{2(j+v)(j+v-1) (1-\lambda + \widetilde{[j]_q} \lambda)^s \widetilde{[j+n-1]_q!} (\widetilde{[j]_q} + \alpha)}$$

belongs to the class $\widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$, for $v > -1, 0 \leq \alpha < 1, q \in (0, 1), c_j \in \mathbb{C}$ with $|c_j| = 1$. Indeed, we are aware that

$$\begin{aligned} & \sum_{j=2}^{\infty} (1-\lambda + \widetilde{[j]_q} \lambda)^s \frac{\widetilde{[j+n-1]_q!}}{[\widetilde{n}]_q! \widetilde{[j-1]_q!}} (\widetilde{[j]_q} - \alpha) |X_j| \\ & + \sum_{j=1}^{\infty} (1-\lambda + \widetilde{[j]_q} \lambda)^s \frac{\widetilde{[j+n-1]_q!}}{[\widetilde{n}]_q! \widetilde{[j-1]_q!}} (\widetilde{[j]_q} + \alpha) |Y_j| \\ & \leq \sum_{j=2}^{\infty} \frac{(1+v)(1-\alpha)}{2(j+v)(j+v-1)} + \sum_{j=1}^{\infty} \frac{v(1-\alpha)}{2(j+v)(j+v-1)} \\ & = \frac{(1+v)(1-\alpha)}{2} \sum_{j=2}^{\infty} \frac{1}{(j+v)(j+v-1)} + \frac{v(1-\alpha)}{2} \sum_{j=1}^{\infty} \frac{1}{(j+v)(j+v-1)} \\ & = \frac{(1+v)(1-\alpha)}{2} \sum_{j=2}^{\infty} \left(\frac{1}{j+v-1} - \frac{1}{j+v} \right) + \frac{v(1-\alpha)}{2} \sum_{j=1}^{\infty} \left(\frac{1}{j+v-1} - \frac{1}{j+v} \right) = 1 - \alpha. \end{aligned}$$

So,

$$\frac{1}{1-\alpha} \sum_{j=1}^{\infty} \left(\left(\widetilde{[j]_q} - \alpha \right) |a_j| + \left(\widetilde{[j]_q} + \alpha \right) |b_j| \right) \left(1 - \lambda + \widetilde{[j]_q} \lambda \right)^s \frac{\widetilde{[j+n-1]_q}!}{\widetilde{[n]_q}! \widetilde{[j-1]_q}!} \leq 2$$

Example 2. To illustrate the sufficient coefficient condition stated in Theorem 1, consider the parameter values $q = 0.5, n = 1, s = 1, \lambda = 0.5, \alpha = 0.2, j = 2$, and $v = 1$ in the previous example. Let $c_2 = 1$. So,

$$X_2 = \frac{2(1-0.2)}{12 \left(0.5 + \widetilde{[2]_{0.5}} \right) \widetilde{[2]_{0.5}}! \left(\widetilde{[2]_{0.5}} - 0.2 \right)} \approx 0.01325$$

and

$$Y_2 = \frac{(1-0.2)}{12 \left(1 - 0.5 + \widetilde{[2]_{0.5}} 0.5 \right) \widetilde{[2]_{0.5}}! \left(\widetilde{[2]_{0.5}} + 0.2 \right)} \approx 0.00564$$

belong to the class $\widetilde{\mathcal{RH}}_{0.5}^{1,1,0.5}(0.2)$. For $j = 2$, the corresponding term in the inequality from Theorem 1 evaluates to approximately 0.25, which is well within the admissible upper bound of 2. This confirms that the condition is satisfied for this choice of parameters and supports the validity of the theorem in practice.

Example 3. The function $f = z + \beta \bar{z}$, with $|\beta| < 1$, belongs to the class $\widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$, for $0 \leq \alpha < 1, q \in (0, 1)$. Indeed, as the analytic part of f is $h(z) = z$ and the co-analytic part is $g(z) = \beta \bar{z}$, it is straightforward to show that the relation (24) is satisfied.

The following theorem provides the necessary and sufficient conditions for the harmonic functions $f = h + \bar{g}$ to belong to $\widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$

Theorem 2. Let $f = h + \bar{g}$ and condition (23) be fulfilled for $0 \leq \alpha < 1$, where the functions h and g are defined in (3) and (15). The function f is harmonic and univalent in U and is a member of the class $\widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$ if and only if it satisfies inequality (23).

Proof. Let $f \in \widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$. Due to the fact that $\widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$ is a subclass of $\widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$, the proof is complete once we demonstrate the “only if” part of the theorem. Put differently, for functions $f = h + \bar{g}$, we will demonstrate that if condition (25) is satisfied, then the coefficients of the function f fulfill the inequality.

Let us first denote

$$M = \left(1 - \lambda + \widetilde{[j]_q} \lambda \right)^s \frac{\widetilde{[j+n-1]_q}!}{\widetilde{[n]_q}! \widetilde{[j-1]_q}!}$$

Additionally, we observe that condition is equivalent to

$$\operatorname{Re} \left(\frac{(1-\alpha)z - \sum_{j=2}^{\infty} M \left(\widetilde{[j]_q} - \alpha \right) a_j z^j - \overline{\sum_{j=1}^{\infty} M \left(\widetilde{[j]_q} + \alpha \right) b_j z^j}}{z - \sum_{j=2}^{\infty} M a_j z^j + \overline{\sum_{j=1}^{\infty} M b_j z^j}} \right) \geq 0, \quad z \in U.$$

Accordingly, by taking z on the positive real axis, where $0 \leq z = r < 1$, we arrive at

$$\frac{1 - \alpha - \sum_{j=2}^{\infty} M\left(\widetilde{[j]}_q - \alpha\right) a_j r^{j-1} - \sum_{j=1}^{\infty} M\left(\widetilde{[j]}_q + \alpha\right) b_j r^{j-1}}{1 - \sum_{j=2}^{\infty} M a_j r^{j-1} + \sum_{j=1}^{\infty} M b_j r^{j-1}} \geq 0, z \in U. \quad (29)$$

If condition (23) is not satisfied, then the numerator in Equation (29) becomes negative for values of r that are sufficiently close to 1. As a result, some $z = r$ in the interval $(0, 1)$ exists for which the left-hand side of inequality (23) is negative. However, this does not align with the imposed condition that $f \in \widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$. Since this contradiction arises from assuming that condition (23) does not hold, we conclude that condition (23) must indeed be valid. Thus, the proof is complete. $\square$

Remark 7. When $q \rightarrow 1^-$, Theorems 1 and 2 reduce to the classical setting previously established in the literature. In this context, the “classical case” specifically refers to the study of harmonic starlike functions defined in the open unit disk. In contrast, the theorems established in this work offer greater flexibility for modeling harmonic mappings and allow for a more general coefficient structure. As an illustrative example, Theorems 1 and 2, under the parameter specialization $q \rightarrow 1^-$, $\lambda = 0$, and $s = 1$, coincide with well-known criteria for harmonic starlike functions, as found in the work of [31] and others.

Corollary 1. Let $f = h + \overline{g}$, $0 \leq \alpha < 1$, $0 < q < 1$, $z \in U$. Then, the function f belongs to the class $\widetilde{\mathcal{RH}_q^{n,s,1}(\alpha)}$ if and only if

$$\frac{1}{1 - \alpha} \sum_{j=1}^{\infty} \left(\left(\widetilde{[j]}_q - \alpha \right) |a_j| + \left(\widetilde{[j]}_q + \alpha \right) |b_j| \right) [j]_q^s \frac{[j + n - 1]_q!}{[n]_q! [j - 1]_q!} \leq 2.$$

Proof. For $\lambda = 1$ in Theorem 1, we obtain the corollary $\square$

The closed convex hull of $\widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$, denoted as $clco \widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$, represents the smallest closed set that fully encompasses $\widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$. More formally, it is defined as the intersection of all closed convex sets that contain $\widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$. This construction ensures that $clco \widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$ is the minimal closed convex superset of $\widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$ with respect to set inclusion. In the following theorem, we establish a characterization of the extreme points of the closed convex hull $clco \widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$, which play a fundamental role in understanding the geometric and structural properties of this set.

Theorem 3. The function $f = h + \overline{g} \in clco \widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$ if and only if

$$f(z) = \sum_{j=1}^{\infty} \left(x_j h_j(z) + \overline{y_j g_j(z)} \right),$$

where

$$h_1(z) = z \text{ and } h_n(z) = z - \frac{(1 - \alpha) [\widetilde{n}]_q! [\widetilde{j - 1}]_q!}{\left((1 - \lambda + \widetilde{[j]}_q \lambda)^s [\widetilde{j + n - 1}]_q! (\widetilde{[j]}_q - \alpha) \right)} z^j, \text{ for } j = 2, 3, \dots \quad (30)$$

$$g_n(z) = z + \frac{(1-\alpha)[n]_q![j-1]_q!}{\left(1-\lambda + [j]_q\lambda\right)^s [j+n-1]_q! ([j]_q + \alpha)} z^j, \text{ for } j = 1, 2, 3, \dots \quad (31)$$

and

$$\sum_{j=1}^{\infty} (x_j + y_j) = 1, x_j \geq 0, y_j \geq 0. \quad (32)$$

Notably, the extreme points of $\widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$ are the functions designated in (30) and (31).

Proof. For x_j, y_j described (32), we acquire

$$\begin{aligned} f(z) &= \sum_{j=1}^{\infty} (x_j + y_j) z - \sum_{j=2}^{\infty} \frac{(1-\alpha)[n]_q![j-1]_q!}{\left(1-\lambda + [j]_q\lambda\right)^s [j+n-1]_q! ([j]_q - \alpha)} x_j z^n \\ &\quad + \sum_{j=1}^{\infty} \frac{(1-\alpha)[n]_q![j-1]_q!}{\left(1-\lambda + [j]_q\lambda\right)^s [j+n-1]_q! ([j]_q + \alpha)} y_j z^n. \end{aligned}$$

This leads to

$$\begin{aligned} &\sum_{j=2}^{\infty} \frac{\left(1-\lambda + [j]_q\lambda\right)^s [j+n-1]_q! ([j]_q - \alpha)}{(1-\alpha)[n]_q![j-1]_q!} a_j + \sum_{j=1}^{\infty} \frac{\left(1-\lambda + [j]_q\lambda\right)^s [j+n-1]_q! ([j]_q + \alpha)}{(1-\alpha)[n]_q![j-1]_q!} b_j \\ &= \sum_{j=2}^{\infty} x_j + \sum_{j=1}^{\infty} y_j = 1 - x_1 \leq 1. \end{aligned}$$

Hence, $f = h + \bar{g} \in \text{clco } \widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$.

Reversely, allow $f = h + \bar{g} \in \text{clco } \widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$. Consequently, by assigning

$$\begin{aligned} x_j &= \frac{\left(1-\lambda + [j]_q\lambda\right)^s [j+n-1]_q! ([j]_q - \alpha)}{(1-\alpha)[n]_q![j-1]_q!} a_j, \quad j = 2, 3, \dots \\ y_j &= \frac{\left(1-\lambda + [j]_q\lambda\right)^s [j+n-1]_q! ([j]_q + \alpha)}{(1-\alpha)[n]_q![j-1]_q!} b_j, \quad j = 2, 3, \dots \end{aligned}$$

where $\sum_{j=1}^{\infty} (x_j + y_j) = 1, x_j \geq 0, y_j \geq 0$, we arrive at functions of the form (30) and (31), fulfilling the requirement. $\square$

Remark 8. When $q \rightarrow 1^-$, Theorem 3 naturally transitions to a classical result, aligning with earlier findings in the literature. See, for instance, the exposition provided by Jahangiri [31].

Moving forward, we present the subsequent distortion bounds, which play a crucial role in establishing a covering result for $\widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$. These bounds provide fundamental constraints on the geometric properties of the set and serve as key tools in analyzing its structure. By leveraging these results, we derive a covering theorem that ensures an effective description of $\widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$ in terms of well-defined bounding regions.

Theorem 4. If the function $f = h + \overline{g} \in \widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$, then for $|z| = r < 1$, we obtain the distortion bounds

$$|f(z)| \geq (1 - |b_1|)r - \frac{[n]_q!}{\left(1 - \lambda + [\widetilde{2}]_q\lambda\right)^s [\widetilde{1+n}]_q!} \left(\frac{1 - \alpha}{[\widetilde{2}]_q - \alpha} - \frac{1 + \alpha}{[\widetilde{2}]_q - \alpha} |b_1| \right) r^2,$$

$$|f(z)| \leq (1 + |b_1|)r + \frac{[n]_q!}{\left(1 - \lambda + [\widetilde{2}]_q\lambda\right)^s [\widetilde{1+n}]_q!} \left(\frac{1 - \alpha}{[\widetilde{2}]_q - \alpha} - \frac{1 + \alpha}{[\widetilde{2}]_q - \alpha} |b_1| \right) r^2. \quad (33)$$

Proof. We will prove only the right-hand inequality, as the proof for the left-hand inequality follows similarly from the argument used for the right-hand side. By taking the absolute value of $f(z)$, we obtain

$$\begin{aligned} |f(z)| &= \left| z + \sum_{j=2}^{\infty} a_j z^j + \sum_{j=1}^{\infty} b_j \overline{z}^j \right| \leq (1 + |b_1|)r + \sum_{j=2}^{\infty} (a_j + b_j) |z|^j \leq (1 + |b_1|)r + \sum_{j=2}^{\infty} (a_j + b_j) r^2 \\ &\leq (1 + |b_1|)r + \frac{1 - \alpha}{[\widetilde{2}]_q - \alpha} \frac{[n]_q!}{\left(1 - \lambda + [\widetilde{2}]_q\lambda\right)^s [\widetilde{1+n}]_q!} \\ &\quad \cdot \sum_{j=2}^{\infty} \left(\frac{[\widetilde{2}]_q - \alpha}{1 - \alpha} \frac{\left(1 - \lambda + [\widetilde{2}]_q\lambda\right)^s [\widetilde{1+n}]_q!}{[n]_q!} a_j + \frac{[\widetilde{2}]_q - \alpha}{1 - \alpha} \frac{\left(1 - \lambda + [\widetilde{2}]_q\lambda\right)^s [\widetilde{1+n}]_q!}{[n]_q!} b_j \right) r^2 \\ &\leq (1 + |b_1|)r + \frac{1 - \alpha}{[\widetilde{2}]_q - \alpha} \frac{[n]_q!}{\left(1 - \lambda + [\widetilde{2}]_q\lambda\right)^s [\widetilde{1+n}]_q!} \left(1 - \frac{1 + \alpha}{1 - \alpha} |b_1| \right) r^2 \\ &\leq (1 + |b_1|)r + \frac{[n]_q!}{\left(1 - \lambda + [\widetilde{2}]_q\lambda\right)^s [\widetilde{1+n}]_q!} \left(\frac{1 - \alpha}{[\widetilde{2}]_q - \alpha} - \frac{1 + \alpha}{[\widetilde{2}]_q - \alpha} |b_1| \right) r^2. \end{aligned}$$

The proof is now complete. $\square$

Example 4. The function $f = z + \beta \overline{z}$, with $|\beta| < 1$, belongs to $\widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$, so for $|z| = r < 1$, we obtain the distortion bounds $(1 - |\beta|)r \leq |f(z)| \leq (1 + |\beta|)r$.

Accordingly, we obtain the following covering property.

Corollary 2. Let $f \in \widetilde{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$. We acquire

$$\left\{ w : |w| < \frac{([\widetilde{2}]_q - \alpha) [\widetilde{n+1}]_q (1 - \lambda + [\widetilde{2}]_q\lambda)^s - (1 - \alpha)}{([\widetilde{2}]_q - \alpha) [\widetilde{n+1}]_q (1 - \lambda + [\widetilde{2}]_q\lambda)^s} - \frac{([\widetilde{2}]_q - \alpha) [\widetilde{n+1}]_q (1 - \lambda + [\widetilde{2}]_q\lambda)^s - (1 + \alpha)}{([\widetilde{2}]_q - \alpha) [\widetilde{n+1}]_q (1 - \lambda + [\widetilde{2}]_q\lambda)^s} b_1 \right\} \subset f(U). \quad (34)$$

Proof. By applying the left-hand inequality from the above theorem and taking the limit as $r \rightarrow 1$, we obtain that

$$\begin{aligned}
 & (1 - b_1) - \frac{1}{[n+1]_q (1 - \lambda + [2]_q \lambda)^s} \left(\frac{1 - \alpha}{[2]_q - \alpha} - \frac{1 + \alpha}{[2]_q - \alpha} b_1 \right) \\
 = & (1 - b_1) - \frac{1}{[n+1]_q (1 - \lambda + [2]_q \lambda)^s ([2]_q - \alpha)} [(1 - \alpha) - (1 + \alpha) b_1] \\
 = & \frac{(1 - b_1) ([2]_q - \alpha) [n+1]_q (1 - \lambda + [2]_q \lambda)^s - (1 - \alpha) + (1 + \alpha) b_1}{[n+1]_q (1 - \lambda + [2]_q \lambda)^s ([2]_q - \alpha)} \\
 = & \left(\frac{([2]_q - \alpha) [n+1]_q (1 - \lambda + [2]_q \lambda)^s - (1 - \alpha)}{([2]_q - \alpha) [n+1]_q (1 - \lambda + [2]_q \lambda)^s} \right. \\
 & \left. - \frac{([2]_q - \alpha) [n+1]_q (1 - \lambda + [2]_q \lambda)^s - (1 + \alpha)}{([2]_q - \alpha) [n+1]_q (1 - \lambda + [2]_q \lambda)^s} |b_1| \right) \subset f(U).
 \end{aligned}$$

Thus, we obtain the required result. $\square$

In what follows, we demonstrate that $\widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$ remains closed under convex combinations of its elements, meaning that if two or more elements belong to $\widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$, then any convex combination of these elements also remains within the set. This property ensures the preservation of the structural characteristics under linear blending, reinforcing the convex nature of $\widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$ within the given mathematical framework.

Theorem 5. The family $\widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$ is closed under convex combination.

Proof. Considering $i = 1, 2, \dots$, assume that $f_i \in \widetilde{\mathcal{RH}}_q^{n,s,\lambda}(\alpha)$, with

$$f_i(z) = z - \sum_{j=2}^{\infty} a_{i,j} z^j + \sum_{j=1}^{\infty} \overline{b_{i,j}} \bar{z}^j. \quad (35)$$

Consequently, according to Theorem 1, we establish that

$$\sum_{j=2}^{\infty} \frac{([n]_q - \alpha) (1 - \lambda + [j]_q \lambda)^s [j + n - 1]_q!}{[n]_q! [j - 1]_q!} a_{i,j} + \sum_{j=1}^{\infty} \frac{([n]_q + \alpha) (1 - \lambda + [j]_q \lambda)^s [j + n - 1]_q!}{[n]_q! [j - 1]_q!} b_{i,j} \leq 1. \quad (36)$$

Considering $\sum_{i=1}^{\infty} p_i = 1$, with $0 \leq p_i \leq 1$, the convex combinations of f_i can be expressed in the form of

$$\sum_{i=1}^{\infty} p_i f_i(z) = z - \sum_{j=2}^{\infty} \left(\sum_{i=1}^{\infty} p_i a_{i,j} \right) z^j + \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} p_i b_{i,j} \right) \bar{z}^j. \quad (37)$$

Employing inequality (23), we determine

$$\sum_{j=2}^{\infty} \frac{(\widetilde{[n]_q} - \alpha)(1 - \lambda + \widetilde{[j]_q}\lambda)^s [j + \widetilde{n} - 1]_q!}{\widetilde{[n]_q}! \widetilde{[j - 1]_q}!} \left(\sum_{i=1}^{\infty} p_i a_{i,j} \right) \quad (38)$$

$$+ \sum_{j=1}^{\infty} \frac{(\widetilde{[n]_q} + \alpha)(1 - \lambda + \widetilde{[j]_q}\lambda)^s [j + \widetilde{n} - 1]_q!}{\widetilde{[n]_q}! \widetilde{[j - 1]_q}!} \left(\sum_{i=1}^{\infty} p_i b_{i,j} \right) \quad (39)$$

$$= \sum_{i=1}^{\infty} p_i \left(\sum_{j=2}^{\infty} \frac{(\widetilde{[n]_q} - \alpha)(1 - \lambda + \widetilde{[j]_q}\lambda)^s [j + \widetilde{n} - 1]_q!}{\widetilde{[n]_q}! \widetilde{[j - 1]_q}!} a_{i,j} \right. \quad (40)$$

$$\left. + \sum_{j=1}^{\infty} \frac{(\widetilde{[n]_q} + \alpha)(1 - \lambda + \widetilde{[j]_q}\lambda)^s [j + \widetilde{n} - 1]_q!}{\widetilde{[n]_q}! \widetilde{[j - 1]_q}!} b_{i,j} \right) \leq \sum_{i=1}^{\infty} p_i \leq 1. \quad (41)$$

Hence, $\sum_{i=1}^{\infty} p_i f_i \in \overline{\mathcal{RH}_q^{n,s,\lambda}(\alpha)}$. $\square$

Closing Comments and Reflections

The study of symmetric q -calculus in connection with geometric function theory (GFT), particularly in the context of harmonic univalent functions, is a relatively recent and developing area of mathematical research. Harmonic functions are central to modern complex analyses, with applications in minimal surface theory, fluid dynamics, and elasticity. Despite its potential significance, this field remains largely unexplored, with only a limited body of published work available.

Some recent articles have contributed to the expanding field of harmonic function theory by incorporating symmetric q -calculus operators, offering new perspectives and generalizations of classical results. In [35], Khan et al. introduced a new subclass of harmonic functions using symmetric q -calculus theory. Their study establishes novel results that generalize the existing findings, particularly by defining harmonic q -starlike functions associated with symmetrical points and Janowski functions. The recent study [14] develops a new generalized symmetric q -difference operator and applies it to defining a subclass of meromorphic harmonic functions. This paper investigates the properties of this subclass and its relation to existing function classes. The article [36] develops a new generalized symmetric q -difference operator and applies it to defining a subclass of meromorphic harmonic functions. This paper investigates the properties of this subclass and its relation to existing function classes.

In the present paper, we employ symmetric q -calculus to introduce and analyze new classes of harmonic univalent functions, expanding the existing framework of geometric function theory. Our methodology is centered around a newly defined symmetric q -differential operator, which is specifically designed to handle the complexities of harmonic functions in the complex plane. The theoretical implications of this operator are manifold: it allows for a unifying framework encompassing several known operators as special cases; it facilitates the derivation of sharp bounds and structural theorems; and it is particularly suitable for modeling physical systems exhibiting time-reversal or bilateral symmetry, as encountered in quantum mechanics and fractional calculus. Thus, the present work represents a novel contribution to both symmetric q -calculus and harmonic function theory. This operator serves as a powerful tool for characterizing and studying these functions.

Using this approach, we establish a series of fundamental results that contribute to the understanding of harmonic univalent functions. In particular, we derive sharp coefficient bounds, which offer precise constraints on the function parameters; distortion

theorems, which describe how functions behave under conformal mappings; and covering results, which provide insights into the geometric structure and range of these functions. These findings not only enhance the theory of harmonic univalent functions but also open new possibilities for further developments in symmetric q -calculus and its applications in geometric function theory.

The framework developed in this paper opens several promising directions for future research. One natural extension involves the generalization of the proposed symmetric q -differential operator to harmonic functions in higher-dimensional complex spaces, such as $\mathbb{C}^n$, where the theory of several complex variables and pluriharmonic mappings may offer new challenges and applications. Additionally, the interplay between the operator $\widetilde{\mathcal{R}}_q^{n,s,\lambda}$ and fractional symmetric q -calculus suggests the potential for further investigations into fractional harmonic mappings and their geometric behavior. Another line of inquiry could focus on applying the current results to problems in mathematical physics, particularly in models exhibiting bilateral symmetry or nonlocal interactions. Finally, exploring other subclasses of harmonic functions through convolution techniques or subordination principles may reveal deeper structural insights and unify different areas within geometric function theory.

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