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Combinatorial Analysis of k -Oresme and k -Oresme–Lucas Sequences

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Abstract: In this study, firstly the definitions and basic algebraic properties of k -Oresme and k -Oresme–Lucas sequences are given. Then, various summation formulae are derived with the help of the first and second derivatives of two polynomials with k -Oresme and k -Oresme–Lucas number coefficients. The main aim of this study is to establish the relations between the generalized Fibonacci and generalized Lucas sequences and the k -Oresme and k -Oresme–Lucas sequences, respectively. These connections allow us to obtain different combinatorial identities of these sequences using the characteristic equation of the k -Oresme and k -Oresme–Lucas sequences. In this way, the discovered combinatorial identities reveal the arithmetic and structural symmetries in the sequences, through the regularities and recurring patterns observed in the algebraic structures of the considered number sequences. The results obtained in this study enable the development of new symmetric approaches in areas such as numerical analysis, cryptography and optimization algorithms, and the algebraic relations derived in this study can contribute to the solution of different problems in disciplines such as mathematical modelling and theoretical physics.

Keywords: generalized Fibonacci sequence; generalized Lucas sequence; k -Oresme sequence; k -Oresme–Lucas sequence; characteristic equation



Academic Editor: Qing-Wen Wang

Received: 21 March 2025

Revised: 25 April 2025

Accepted: 30 April 2025

Published: 2 May 2025

Citation: Demirtürk, B. Combinatorial Analysis of k -Oresme and k -Oresme–Lucas Sequences. *Symmetry* **2025**, *17*, 697. <https://doi.org/10.3390/sym17050697>

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1. Introduction

Number sequences are among the topics of interest to many researchers. One of the well-known number sequences is the Fibonacci sequence. Fibonacci numbers, which form the mathematical basis of many symmetrical structures observed not only in number theory but also in nature, are used in modelling the symmetrical pattern of many biological and physical structures, especially in the spiral structure of shellfish, in the arrangement of plant leaves, in art and architecture, esthetics, due to its relationship with the golden ratio. The Fibonacci and Lucas sequences, along with their generalizations, are widely used in various areas such as modelling natural systems [1,2], in predictive models for stock market analysis and investment strategies [3], particularly in generating cryptographic keys and designing encryption algorithms [4,5]. Furthermore, they contribute to the design of non-linear optimization problems [6] and heuristic methods. Also, these mathematical sequences inspire rhythmic structures in musical composition [7] and artistic designs.

The k -Fibonacci sequence and its relationship with k -Lucas sequence were first defined by Falcon and Plaza [8,9]. Their work extended beyond the classical Fibonacci sequence, introducing a broader family of number sequences for mathematical exploration. The study of the generalized Fibonacci and the generalized Lucas sequences has since evolved to encompass intriguing aspects such as their connections to Diophantine equations, generating functions, and matrix representations. The combinatorial and numerical analysis

applications of these sequences remain a rich area of research, with numerous studies exploring their theoretical and practical implications [10,11].

Moreover, many variations in this sequence are studied such as generalizations, hybrid numbers, hybrid hyper numbers, bronze numbers, quaternions and spinors. Spreafico et al. introduced the hybrid hyper k-Pell, hybrid hyper k-Pell–Lucas, and hybrid hyper Modified k-Pell numbers and they obtained established new properties, generating functions, and the Binet formula of the hyper k-Pell, hyper k-Pell–Lucas, and hyper Modified k-Pell sequences in [12]. Özkan et al., obtained some properties of Bronze Fibonacci and Bronze Lucas sequences. Moreover, they established the applications of generalized Bronze Fibonacci sequences to hyperbolic quaternions in [13]. Kızılateş et al. defined higher-order generalized Fibonacci hybrid polynomials called higher-order generalized Fibonacci hybrinomials and presented three different matrices whose components are higher-order generalized Fibonacci hybrinomials, higher-order generalized Fibonacci polynomials and Lucas polynomials in [14]. Moreover, Çolak et al. extended the generalized Fibonacci quaternion polynomials to generalized Fibonacci polynomial spinors by relating spinors to quaternions in [15].

After this literature overview, let us recall the definition and fundamental properties of the generalized Fibonacci and generalized Lucas sequences.

The generalized Fibonacci sequence $\{F_{k,n}\}$ is defined by the recurrence relation $F_{k,n} = kF_{k,n-1} - F_{k,n-2}$ for $n \geq 2$ with the initial values $F_{k,0} = 0$ and $F_{k,1} = 1$. $F_{k,n}$ is called the n -th generalized Fibonacci number. The generalized Lucas sequence $\{L_{k,n}\}$ is defined by the recurrence relation $L_{k,n} = kL_{k,n-1} - L_{k,n-2}$ for $n \geq 2$ with the initial values $L_{k,0} = 2$, $L_{k,1} = k$, and $L_{k,n}$ is called the n -th generalized Lucas number. For all $n \geq 1$, it can be shown that $F_{k,(-n)} = -F_{k,n}$ and $L_{k,(-n)} = L_{k,n}$ [16].

The characteristic equation of $\{F_{k,n}\}$ sequence is $x^2 - kx + 1 = 0$, with the roots $\alpha = \frac{k + \sqrt{k^2 - 4}}{2}$ and $\beta = \frac{k - \sqrt{k^2 - 4}}{2}$, for $k > 2$. It can be easily seen that $\alpha + \beta = k$, $\alpha - \beta = \sqrt{k^2 - 4}$, $\alpha \cdot \beta = 1$. Moreover, Binet's formulas for these sequences are defined as $F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_{k,n} = \alpha^n + \beta^n$ [10,11,16–19].

2. Definitions and Properties of k-Oresme and k-Oresme–Lucas Sequences

In this section, we will analyze the basic algebraic properties of the k-Oresme and k-Oresme–Lucas sequences and show their relationship with the generalized Fibonacci and generalized Lucas sequences. Among the main contributions of the paper is the derivation of new sum formulae and combinatorial identities for these special number sequences. Furthermore, the characteristic equations of these sequences are analyzed, the connections between their roots are determined and various mathematical relations are derived from these relations.

In terms of practical advantages, the results obtained in this study may enable the development of new approaches in areas such as numerical analysis, cryptography, and optimization algorithms. In particular, the combinatorial structure of k-Oresme sequences has the potential to be applied in areas such as cryptographic security protocols, network theory and data compression techniques. In addition, the derived algebraic relations can contribute to the solution of different problems in disciplines such as mathematical modelling and theoretical physics.

Oresme number sequences are one of the number sequences that play an important role in the analysis of mathematical structures and various applications due to their recursive structure and have similar properties to the generalized Fibonacci sequence. These sequences are widely used in disciplines such as algebraic number theory, cryptography, differential equations, graph theory, dynamical systems, and optimization.

The Oresme number sequence was first studied in the fourteenth century by Nicole Oresme, a French philosopher, mathematician, physicist, economist, and astronomer. Oresme worked on the sum of the sequence of rational numbers. Later, Horadam utilized Oresme's work to define the Oresme sequence, a sequence with rational coefficients [20].

In the literature, there are many papers that are interested in Oresme numbers and their variations. In [21], Liana and Wloch defined the Oresme hybrid numbers based on the known Oresme sequence and gave some properties of Oresme hybrid numbers. Besides these, Halıcı and Sayın analyzed some algebraic properties and applications of these numbers and gave some summation formulas for k -Oresme hybrid numbers that were defined by Liana and Wloch in [22].

In [23], Goy and Zatorsky considered determinants of Toeplitz–Hessenberg matrices whose entries are the Oresme numbers. Also, Aktaş and Soykan were interested in Toeplitz matrices with Oresme number components and found the Frobenius row and column norms of these matrices in [24].

Moreover, in [25], Ertaş and Yılmaz considered Gaussian Oresme numbers and quaternions with Gaussian Oresme coefficients and obtained some of their characteristic properties.

In 2019, Morales gave the well-known identities related to Oresme polynomials by matrix methods and the derivatives of these polynomials [26].

Halıcı and Gür obtained some summation formulae using k -Oresme polynomial and its derivative in [27]. In another study, a geometric approach to Oresme numbers was made by Sayın and Halıcı. Moreover, they defined k -Oresme polynomials with negative indices as a generalization of k -Oresme numbers with negative indices in [28,29].

Besides these, Oresme number sequences are not only limited to theoretical mathematics studies but also have a wide range of applications in modern engineering and informatics fields.

The specific patterns and modular arithmetic structures produced by number sequences are very useful for random number generation in encryption algorithms [30]. Like Fibonacci sequence-based encryption methods, the Oresme and k -Oresme sequences can be integrated into similar algorithms. Some variations in the Oresme sequence are used specifically in encryption and secure communication systems.

In the field of differential equations and mathematical physics, Oresme sequences can be used to solve non-linear systems. Especially in systems such as propagation and wave equations, analytical and numerical solutions are obtained through such sequences [31,32]. In addition, these sequences can also play a role in dynamical systems and in the analysis of chaotic motions.

The Oresme sequences also have an important place in graph theory and network analyses. Such sequences are integrated into certain group theoretical structures and used as analysis tools for hierarchical network systems, tree structures, optimization algorithms and error correction codes [33]. Furthermore, some generalized versions of Oresme sequences can be used instead of the Fibonacci sequences in metaheuristic algorithms for solving engineering and optimization problems. In combination with methods such as genetic algorithms, particle swarm optimization and artificial bee colony algorithms, parameters derived from these sequences contribute to the optimization process [34].

The Oresme sequence is defined by $\{O_n\}_{n \geq 1} = \left\{\frac{n}{2^n}\right\} = \left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \dots, \frac{n}{2^n}, \dots\right\}$, with the initial values $O_0 = 0, O_1 = \frac{1}{2}$, and the recurrence relation $O_{n+2} = O_{n+1} - \frac{1}{4}O_n$ for all $n \geq 0$. The Oresme–Lucas sequence is defined by the recurrence relation $OL_{n+2} = OL_{n+1} - \frac{1}{4}OL_n$ with the initial conditions $OL_0 = 2, OL_1 = 1$, for all $n \geq 0$ [20,35,36].

Özkan and Akkuş defined the k -Oresme and k -Oresme–Lucas sequences in [37]. Also, some applications of these sequence are studied by Halıcı and Gür [27]. Throughout this

paper, for convenience of notation, the k -Oresme sequence and k -Oresme–Lucas sequences will be denoted by $\{A_n\}$ and $\{B_n\}$, respectively.

Now we can give the definitions of these sequences.

Definition 1. The k -Oresme sequence $\{A_n\}$ is given by the recurrence relation $A_{n+2} = A_{n+1} - \frac{1}{k^2}A_n$ for all $n \geq 0$, with the initial conditions $A_0 = 0, A_1 = \frac{1}{k}$.

Definition 2. The k -Oresme–Lucas sequence $\{B_n\}$ is defined by the recurrence relation $B_{n+2} = B_{n+1} - \frac{1}{k^2}B_n$ for all $n \geq 0$, with the initial conditions $B_0 = 2, B_1 = 1$.

The characteristic equation of these sequences is given by $x^2 - x + \frac{1}{k^2} = 0$ with the roots $\alpha^* = \frac{k + \sqrt{k^2 - 4}}{2k}$ and $\beta^* = \frac{k - \sqrt{k^2 - 4}}{2k}$, where $k > 2$. It is clear from here that $\alpha = k\alpha^*$ and $\beta = k\beta^*$. The following proposition can be given by this relation:

Proposition 1. For all $n \in \mathbb{Z}$, $A_n = \left(\frac{1}{k}\right)^n F_{k,n}$ and $B_n = \left(\frac{1}{k}\right)^n L_{k,n}$.

Binet formulas for these sequences are defined as $A_n = \frac{(\alpha^*)^n - (\beta^*)^n}{(\alpha^* - \beta^*)k}$ and $B_n = (\alpha^*)^n + (\beta^*)^n$ for $n \in \mathbb{N}$, and $k > 2$ [36,37].

Proposition 2. For all $n \in \mathbb{Z}$, $A_{-n} = -k^{2n}A_n$ and $B_{-n} = k^{2n}B_n$.

Proposition 3. For all $n \in \mathbb{Z}$, $(\alpha^*)^n = k\alpha^* A_n - \frac{1}{k}A_{n-1}$ and $(\beta^*)^n = k\beta^* A_n - \frac{1}{k}A_{n-1}$.

Now we will list some identities of k -Oresme and k -Oresme–Lucas sequences, since we will use them in the next section. These identities can be easily proved using Binet's formulas and are given in [37]. For all $n \in \mathbb{Z}$,

$$B_n = kA_{n+1} - \frac{1}{k}A_{n-1} \quad (1)$$

$$B_n^2 = (k^2 - 4)A_n^2 + 4k^{-2n} \quad (2)$$

$$B_{2n} = B_n^2 - 2k^{-2n} \quad (3)$$

$$A_n B_n = A_{2n} \quad (4)$$

3. Combinatorial Identities of the k -Oresme and k -Oresme–Lucas Sequences

The k -Oresme sequence, k -Oresme–Lucas sequence, and variations in these sequences have recently become a subject of interest. In this section of the paper, several sum formulas and various equations related to k -Oresme and k -Oresme–Lucas numbers will be derived. The approach presented here offers a distinct perspective. This exploration not only enhances the understanding of k -Oresme and k -Oresme–Lucas numbers but also opens new avenues for further research in this area. In the first part of this section, we will consider two new polynomials related to characteristic equations of these sequences and in the second part binomial properties are used to obtain some new binomial sums of these sequences.

3.1. Sum Formulas from Characteristic Equation

In [38], Amini gave the polynomial identity as follows:

$$x^n = (x^2 - x - 1) \left(\sum_{j=0}^{n-1} F_j x^{n-1-i} \right) + F_n x + F_{n-1},$$

for $n \geq 1$, where F_n is the n -th Fibonacci number. Then, in [39], Cerin, Demirtürk and Keskin revealed the generalized form of Amini's equation and showed that the following polynomial equality:

$$2x^{n+1} - px^n = (x^2 - px + q) \left(\sum_{j=0}^{n-1} V_j x^{n-1-i} \right) + V_n x - qV_{n-1}$$

holds for $n \geq 1$, where U_n and V_n are the n -th (p, q) -Fibonacci and (p, q) -Lucas numbers, respectively.

In this study, we revealed two polynomial equations as follows:

$$x^n = \left(x^2 - x + \frac{1}{k^2} \right) \left(\sum_{i=0}^{n-1} kA_i x^{n-1-i} \right) + kA_n x - \frac{1}{k} A_{n-1} \quad (5)$$

and

$$2x^{n+1} - x^n = \left(x^2 - x + \frac{1}{k^2} \right) \left(\sum_{i=0}^{n-1} B_i x^{n-1-i} \right) + B_n x - \frac{1}{k^2} B_{n-1} \quad (6)$$

that hold for $n \geq 1$, where A_n and B_n are n -th k -Oresme and k -Oresme–Lucas numbers, respectively. By using (5), (6) and derivatives of them, we stated some sum formulas of the k -Oresme and k -Oresme–Lucas sequences.

Theorem 1. For all $n \in \mathbb{N}$,

$$nB_n - kA_n = (k^2 - 4) \sum_{i=0}^n A_i A_{n-i}, \quad (7)$$

$$(n+1)A_n = \sum_{i=0}^n A_i B_{n-i}. \quad (8)$$

Proof of Theorem 1. The first derivative of Equation (5) is given as follows:

$$nx^{n-1} = (2x-1) \left(\sum_{i=0}^{n-1} kA_i x^{n-1-i} \right) + \left(x^2 - x + \frac{1}{k^2} \right) \left(\sum_{i=0}^{n-2} (n-1-i)kA_i x^{n-2-i} \right) + kA_n. \quad (9)$$

Taking $x = \alpha^*$ and $x = \beta^*$ in Equation (9), and considering the following: $2\alpha^* - 1 = \frac{\sqrt{k^2-4}}{k}$ and $2\beta^* - 1 = \frac{-\sqrt{k^2-4}}{k}$, we have

$$n(\alpha^*)^{n-1} = \frac{\sqrt{k^2-4}}{k} \left(\sum_{i=0}^{n-1} kA_i (\alpha^*)^{n-1-i} \right) + kA_n, \quad (10)$$

$$n(\beta^*)^{n-1} = \frac{-\sqrt{k^2-4}}{k} \left(\sum_{i=0}^{n-1} kA_i (\beta^*)^{n-1-i} \right) + kA_n. \quad (11)$$

The sum of Equations (10) and (11) gives the following:

$$nB_{n-1} - 2kA_n = (k^2 - 4) \left(\sum_{i=0}^{n-1} A_i A_{n-1-i} \right). \quad (12)$$

Taking $n + 1$ instead of n in Equation (12), and using (1), Equation (7) follows.

On the other hand, by taking the difference of Equations (10) and (11), we have the following:

$$n\sqrt{k^2 - 4}A_{n-1} = \frac{\sqrt{k^2 - 4}}{k} \left(\sum_{i=0}^{n-1} kA_i B_{n-1-i} \right). \quad (13)$$

Thus, it follows that $(n + 1)A_n = \sum_{i=0}^n A_i B_{n-i}$. $\square$

Corollary 1. For all $n \in \mathbb{N}$,

$$\sum_{i=0}^n A_i (\alpha^*)^{-i} = \frac{(n + 1)(\alpha^*)^n - kA_{n+1}}{(\alpha^*)^n \cdot \sqrt{k^2 - 4}}, \quad (14)$$

$$\sum_{i=0}^n A_i (\beta^*)^{-i} = \frac{-(n + 1)(\beta^*)^n + kA_{n+1}}{(\beta^*)^n \cdot \sqrt{k^2 - 4}}. \quad (15)$$

Proof of Corollary 1. The proof is obvious by taking $x = \alpha^*$ and $x = \beta^*$ in (9), and using Binet formulas of the k -Oresme and k -Oresme–Lucas sequences. $\square$

Theorem 2. For all $n \in \mathbb{N}$,

$$(n + 1)A_n = \sum_{i=0}^n B_i A_{n-i}, \quad (16)$$

$$(n + 1)B_n + 2kA_{n+1} = \sum_{i=0}^n B_i B_{n-i} \quad (17)$$

Proof of Theorem 2. By the first derivative of (6), we obtain the following:

$$2(n + 1)x^n - nx^{n-1} = (2x - 1) \left(\sum_{i=0}^{n-1} B_i x^{n-1-i} \right) + \left(x^2 - x + \frac{1}{k^2} \right) \left(\sum_{i=0}^{n-2} (n - 1 - i) B_i x^{n-2-i} \right) + B_n. \quad (18)$$

Hence, for $x = \alpha^*$ and $x = \beta^*$ in Equation (18), and taking the sum of new equations using Binet formulas and replacing n with $n + 1$, we have the expected Equation (16). On the other hand, we obtain (17), starting from the difference in the new equations obtained from taking $x = \alpha^*$ and $x = \beta^*$ in Equation (18). $\square$

Theorem 3. For all $n \in \mathbb{N}$,

$$\sum_{i=0}^n A_i B_{-i} = k^{2n+1} A_n A_{n+1}, \quad (19)$$

$$(k^2 - 4) \sum_{i=0}^n A_i A_{-i} = 2(n + 1) - k^{2n+1} A_{n+1} B_n, \quad (20)$$

$$(k^2 - 4) \sum_{i=0}^n k^{2i} A_i^2 = k^{2n+1} A_{n+1} B_n - 2(n + 1), \quad (21)$$

$$\sum_{i=0}^n k^{2i} B_i^2 = k^{2n+1} A_{n+1} B_n + 2(n + 1), \quad (22)$$

$$\sum_{i=0}^n k^{2i} B_{2i} = k^{2n+1} B_n A_{n+1}, \quad (23)$$

$$\sum_{i=0}^n k^{2i} A_{2i} = k^{2n+1} A_n A_{n+1}. \quad (24)$$

Proof of Theorem 3. Firstly, if we take the sum of β^* multiple of Equation (14) and α^* multiple of Equation (15) with the fact that $\alpha^* \cdot \beta^* = k^{-2}$, then we achieve (19). Secondly, Equation (20) follows from taking the difference of β^* multiple of Equation (14) and α^* multiple of Equation (15) with the fact that $\alpha^* \cdot \beta^* = k^{-2}$. Then, by using Proposition 2 in Equation (20), we obtain (21). Moreover, using (2) with Equation (21), we achieve (22). Also considering (3) with Equation (22) gives (23). Furthermore, using Proposition 2 and Equation (4) in (19) we achieve (24). $\square$

The following results will concern sums that are obtained from the second derivatives of (5) and (6).

Theorem 4. For all $n \in \mathbb{N}$,

$$2(k^2 - 4) \sum_{i=0}^n i A_i A_{n-i} = n(n B_n - k A_n), \quad (25)$$

$$2(k^2 - 4) \sum_{i=0}^n i A_i B_{n-i} = 2kn B_{n+1} + \left((k^2 - 4)n(n+1) - 4 \right) A_n, \quad (26)$$

$$2(k^2 - 4) \sum_{i=0}^n i B_i A_{n-i} = \left(n^2(k^2 - 4) + 4 \right) A_n - kn B_n, \quad (27)$$

$$2 \sum_{i=0}^n i B_i B_{n-i} = n[(n+1) B_n + 2k A_n] - \frac{2n}{k^2}. \quad (28)$$

Proof of Theorem 4. Let us denote the polynomial $\left(\sum_{i=0}^{n-1} k A_i x^{n-1-i} \right)$ with $G(x)$. Taking the second derivative of (5), we have the following:

$$n(n-1)x^{n-2} = 2G(x) + 2(2x-1)G'(x) + \left(x^2 - x + \frac{1}{k^2} \right) G''(x). \quad (29)$$

For $x = \alpha^*$ and $x = \beta^*$ in Equation (29), and taking the sum of them, we achieve the following:

$$n(n-1)B_{n-2} = 2(G(\alpha^*) + G(\beta^*)) + 2 \frac{\sqrt{k^2 - 4}}{k} (G'(\alpha^*) - G'(\beta^*)). \quad (30)$$

Thus, it follows that

$$n(n-1)B_{n-2} = 2kn A_{n-1} + 2 \left(k^2 - 4 \right) \sum_{i=0}^{n-2} (n-1-i) A_i A_{n-2-i}. \quad (31)$$

Replacing n with $n+2$ in (31), we obtain (25). $\square$

Moreover, for $x = \alpha^*$ and $x = \beta^*$ in Equation (29) and taking the difference in them we achieve the expected formula (26) by taking $n+2$ instead of n .

Let us denote $\left(\sum_{i=0}^{n-1} B_i x^{n-1-i}\right)$ with the polynomial $H(x)$ in Equation (18), and take the derivative of (18) to achieve the following:

$$2(n+1)nx^{n-1} - n(n-1)x^{n-2} = 2H(x) + 2(2x-1)H'(x) + \left(x^2 - x + \frac{1}{k^2}\right)H''(x). \quad (32)$$

Taking $x = \alpha^*$ and $x = \beta^*$ in Equation (32), by the sum of those equations and using (16) gives (27). Moreover, taking $x = \alpha^*$ and $x = \beta^*$ in Equation (32), by the difference in those equations and using (17) gives (28).

Corollary 2. For all $n \in \mathbb{N}$,

$$\sum_{i=0}^n ik^{2i} A_{2i} = k^{2n+1} \left[(k^2 - 4)(n+1)A_n A_{n+1} - k B_n B_{n+2} + (n+2)k^{-2n} \right], \quad (33)$$

$$(k^2 - 4) \sum_{i=0}^n ik^{2i} A_i^2 = k^{2n+1} [(n+1)B_n A_{n+1} - k A_n A_{n+2}] - (n^2 + 2n + 2), \quad (34)$$

$$\sum_{i=0}^n ik^{2i} B_i^2 = k^{2n+1} [(n+1)B_n A_{n+1} - k A_n A_{n+2}] + (n^2 - 2), \quad (35)$$

$$\sum_{i=0}^n ik^{2i} B_{2i} = k^{2n+1} [(n+1)B_n A_{n+1} - k A_n A_{n+2}] - (n+2). \quad (36)$$

Proof of Corollary 2. Taking $x = \alpha^*$ in (29) and multiplying with $(\beta^*)^{n-1}$, we achieve the following:

$$\sum_{i=0}^{n-2} (n-1-i) A_i (\alpha^*)^i = \frac{(k^2 - 4)n(n+1) - 2\alpha^* (nk - A_n (\alpha^*)^{1-n})}{2(k^2 - 4)}. \quad (37)$$

Also, taking $x = \beta^*$ in (29) and multiplying with $(\alpha^*)^{n-1}$, we achieve the following:

$$\sum_{i=0}^{n-2} (n-1-i) A_i (\beta^*)^i = \frac{-(k^2 - 4)n(n-1) - 2\beta^* (nk - A_n (\beta^*)^{1-n})}{2(k^2 - 4)}. \quad (38)$$

Taking the sum of (37) and (38), by replacing n with $n+2$ and using (24), we obtain (33). Similarly, we obtain (34) by taking the difference in (37) and (38), and then replacing n with $n+2$ and using (21). Also, if we use (2) with (34), then we obtain Equation (35). Moreover, Equation (36) follows from the equations together with (3) and (35). $\square$

3.2. Sum Formulas from Binomial Properties

In this subsection, we will consider the following equations:

$$(\alpha^*)^n = k\alpha^* A_n - \frac{1}{k} A_{n-1}, \quad (39)$$

$$B_n - 2(\alpha^*)^n = -\sqrt{k^2 - 4} A_n, \quad (40)$$

$$B_n - 2(\beta^*)^n = \sqrt{k^2 - 4} A_n \quad (41)$$

with the binomial properties to obtain binomial sum formulas of the k -Oresme and k -Oresme–Lucas sequences.

Theorem 5. For all $n \in \mathbb{N}$ and $m, r \in \mathbb{Z}$,

$$k^n B_{mn+r} = \sum_{i=0}^n \binom{n}{i} k^{2i} A_m^i (-A_{m-1})^{n-i} B_{i+r}, \quad (42)$$

$$k^n A_{mn+r} = \sum_{i=0}^n \binom{n}{i} k^{2i} A_m^i (-A_{m-1})^{n-i} A_{i+r}. \quad (43)$$

Proof of Theorem 5. Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, we can write the equation

$$\left(kA_mx - \frac{1}{k}A_{m-1}\right)^n = \sum_{i=0}^n \binom{n}{i} (kA_mx)^i \left(-\frac{1}{k}A_{m-1}\right)^{n-i} \quad (44)$$

by considering the binomial sum formula.

Taking $x = \alpha^*$ and $x = \beta^*$ in Equation (44), and summation of these new equations give us the following:

$$((\alpha^*)^m)^n + ((\beta^*)^m)^n = \sum_{i=0}^n \binom{n}{i} (kA_m)^i \left((\alpha^*)^i + (\beta^*)^i\right) \left(-\frac{1}{k}A_{m-1}\right)^{n-i}. \quad (45)$$

Since the left-hand side of (45) is B_{mn} by Binet formulas, we obtain the following:

$$k^n B_{mn} = \sum_{i=0}^n \binom{n}{i} k^{2i} A_m^i (-A_{m-1})^{n-i} B_i. \quad (46)$$

Similarly taking $x = \alpha^*$ and $x = \beta^*$ in Equation (44), and difference in these new equations give us the following:

$$k^n A_{mn} = \sum_{i=0}^n \binom{n}{i} k^{2i} A_m^i (-A_{m-1})^{n-i} A_i. \quad (47)$$

For a more general formula, the following pattern can be followed. If we take $x = \alpha^*$ in (44) and multiply the new equation by $(\alpha^*)^r$, then we have the following:

$$((\alpha^*)^m)^n (\alpha^*)^r = \sum_{i=0}^n \binom{n}{i} (kA_m)^i \left(-\frac{1}{k}A_{m-1}\right)^{n-i} (\alpha^*)^{i+r}. \quad (48)$$

If we take $x = \beta^*$ in (44) and multiply the new equation by $(\beta^*)^r$, then we achieve the following:

$$((\beta^*)^m)^n (\beta^*)^r = \sum_{i=0}^n \binom{n}{i} (kA_m)^i \left(-\frac{1}{k}A_{m-1}\right)^{n-i} (\beta^*)^{i+r}. \quad (49)$$

Taking the sum and difference of Equations (48) and (49), we obtain the following:

$$(\alpha^*)^{mn+r} + (\beta^*)^{mn+r} = \sum_{i=0}^n \binom{n}{i} (kA_mx)^i \left((\alpha^*)^{i+r} + (\beta^*)^{i+r}\right) \left(-\frac{1}{k}A_{m-1}\right)^{n-i} \quad (50)$$

and

$$(\alpha^*)^{mn+r} - (\beta^*)^{mn+r} = \sum_{i=0}^n \binom{n}{i} (kA_mx)^i \left((\alpha^*)^{i+r} - (\beta^*)^{i+r}\right) \left(-\frac{1}{k}A_{m-1}\right)^{n-i}. \quad (51)$$

which gives Equations (50) and (51), respectively. $\square$

Theorem 6. For all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$,

$$nk^{n+1}A_mB_{mn} = \sum_{i=0}^n \binom{n}{i} ik^{2i}A_m^i(-A_{m-n})^{n-i}B_{m+i-1}, \quad (52)$$

$$nk^{n+1}A_mA_{mn} = \sum_{i=0}^n \binom{n}{i} ik^{2i}A_m^i(-A_{m-n})^{n-i}A_{m+i-1}. \quad (53)$$

Proof of Theorem 6. The first derivative of Equation (44) is given by the following:

$$nkA_m \left(kA_mx - \frac{1}{k}A_{m-1} \right)^{n-1} = \sum_{i=0}^n \binom{n}{i} i(kA_m)^i \left(-\frac{1}{k}A_{m-1} \right)^{n-i} x^{i-1}. \quad (54)$$

Then, taking $x = \alpha^*$ in (54) and multiplying the new equation by $(\alpha^*)^m$ and taking $x = \beta^*$ in (54) and multiplying the new equation by $(\beta^*)^m$, and the sum and the difference in the newest equations give us the following:

$$nkA_m \left[((\alpha^*)^m)^{n-1}(\alpha^*)^m + ((\beta^*)^m)^{n-1}(\beta^*)^m \right] = \sum_{i=0}^n \binom{n}{i} i(kA_m)^i \left[(\alpha^*)^{i-1}(\alpha^*)^m + (\beta^*)^{i-1}(\beta^*)^m \right] \left(-\frac{1}{k}A_{m-1} \right)^{n-i} \quad (55)$$

and

$$nkA_m \left[((\alpha^*)^m)^{n-1}(\alpha^*)^m - ((\beta^*)^m)^{n-1}(\beta^*)^m \right] = \sum_{i=0}^n \binom{n}{i} i(kA_m)^i \left[(\alpha^*)^{i-1}(\alpha^*)^m - (\beta^*)^{i-1}(\beta^*)^m \right] \left(-\frac{1}{k}A_{m-1} \right)^{n-i}, \quad (56)$$

respectively. By using Binet's formulas in (55) and (56), we can conclude the proof. $\square$

Theorem 7. For all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$,

$$\left(\sqrt{k^2 - 4} \right)^n [(-1)^n + 1] \left(\frac{A_m}{B_m} \right)^n = \sum_{i=0}^n \binom{n}{i} i \left(\frac{-2}{B_m} \right)^i B_{mi}, \quad (57)$$

$$\left(\sqrt{k^2 - 4} \right)^n [(-1)^n - 1] \left(\frac{A_m}{B_m} \right)^n = \sum_{i=0}^n \binom{n}{i} i \left(\frac{-2}{B_m} \right)^i A_{mi}. \quad (58)$$

Proof of Theorem 7. From Equations (40) and (41) we can obtain the following:

$$\left(-\sqrt{k^2 - 4}A_m \right)^n = [B_m - 2(\alpha^*)^m]^n = \sum_{i=0}^n \binom{n}{i} (-2)^i B_m^{n-i} (\alpha^*)^{mi}, \quad (59)$$

$$\left(\sqrt{k^2 - 4}A_m \right)^n = [B_m - 2(\beta^*)^m]^n = \sum_{i=0}^n \binom{n}{i} (-2)^i B_m^{n-i} (\beta^*)^{mi} \quad (60)$$

Then, taking the sum of (59) and (60) and using Binet formulas we obtain (57). Similarly, Equation (58) follows from the difference in (59) and (60). $\square$

4. Conclusions

This paper investigates the basic algebraic structures of the k-Oresme and k-Oresme–Lucas sequences and establishes their connections with the generalized Fibonacci and

generalized Lucas sequences. The main contributions of the paper focus on the derivation of new sum formulae and combinatorial identities for the k -Oresme and k -Oresme–Lucas sequences by using the polynomials with the k -Oresme and k -Oresme–Lucas number coefficients. Furthermore, the characteristic equations of the sequences are analyzed and used to obtain new sum formulas, and various mathematical results are obtained from the binomial properties applied to these sequences.

The results presented here could motivate other researchers for further research on sum formulas, identities, and recurrence relations about different number sequences. In addition to this study, additional relations can be considered to prove new sum formulas and identities in the future work. Also, we can give some suggestions to support theoretical studies with practical applications.

Numerical simulations and computational models can be obtained by using programming languages such as MATLAB, Python, Mathematica, etc. to test the accuracy of the theoretical results obtained in this study and to demonstrate their applicability. The asymptotic behavior, convergence rates, and growth rates of k -Oresme sequences can be analyzed and performance comparisons with other optimization algorithms can be made. Error analysis and sensitivity tests can be performed to understand how the arrays behave in different systems with different initial conditions and parameter values.

Moreover, new methods based on the k -Oresme sequences can be developed instead of existing encryption methods based on Fibonacci and Lucas sequences in random number and key generation algorithms. Besides these, new parameter update methods can be designed with the k -Oresme sequences in metaheuristic algorithms. The effectiveness of the k -Oresme-based algorithms can be tested in problems such as function minimization, route planning and big data analysis. By developing the Oresme- and k -Oresme-based algorithms instead of traditional Fibonacci optimization, new approaches can be developed in stock market prediction algorithms, exchange rate modelling, and risk management systems, and performance comparisons can be made.

In addition, the relationship of the k -Oresme sequences with signal processing, data compression, and error correction codes can be examined.

Thus, integrating the theoretically developed k -Oresme sequences with applied sciences can lead to innovative solutions in areas such as number theory, computer science, engineering, finance, and cryptography. Numerical simulations, optimization algorithms, cryptographic systems, and economic models can contribute to the academic literature and technology by testing the applicability of theoretical studies to the real-world problems.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Boman, B.M.; Dinh, T.N.; Decker, K.; Emerick, B.; Raymond, C.; Schleiniger, G. Why do Fibonacci Numbers Appear in Patterns of Growth in Nature? *Fibonacci Q.* **2017**, *55*, 30–41. [[CrossRef](#)]
2. Zhu, Y. Visualization techniques for the design and analysis of dynamic programming algorithms. In Proceedings of the 28th International Conference on Information Visualization (IV), Coimbra, Portugal, 22–26 July 2024; pp. 1–6.
3. Allahyari Soeini, R.; Niroomand, A.; Kheyrmand Parizi, A. Using Fibonacci numbers to forecast the stock market. *Int. J. Manag. Sci. Eng. Manag.* **2012**, *7*, 268–279. [[CrossRef](#)]
4. Aydinuz, S.; Aşci, M. Error detection and correction for coding theory on k -order Gaussian Fibonacci matrices. *Math. Biosci. Eng.* **2022**, *20*, 1993–2010. [[CrossRef](#)]
5. Bellini, E.; Marcolla, C.; Murru, N. An application of p -Fibonacci error-correcting codes to cryptography. *Mathematics* **2021**, *9*, 789. [[CrossRef](#)]

6. Vivek; Kumar, M. Solution of linear and nonlinear singular value problems using operational matrix of integration of Fibonacci wavelets. *J. Eng. Math.* **2024**, *145*, 19. [\[CrossRef\]](#)
7. Ulaşan, A. The Relationship Between the Golden Ratio and Fibonacci Numbers with Music. Master's Thesis, Bingöl University, Bingöl, Turkey, 2023.
8. Falcon, S.; Plaza, A. On the Fibonacci k-numbers. *Chaos Solitons Fractals* **2007**, *32*, 1615–1624. [\[CrossRef\]](#)
9. Horadam, A.F. A generalized Fibonacci sequence. *Am. Math. Mon.* **1961**, *68*, 455–459. [\[CrossRef\]](#)
10. Koshy, T. *Fibonacci and Lucas Numbers with Applications*; John Wiley & Sons: New York, NY, USA, 2001.
11. Vajda, S. *Fibonacci and LUCAS NUMBERS and the Golden Section*; Ellis Horwood Limited: Chichester, UK, 1989.
12. Spreafico, E.V.; Catarino, P.M.M.C. On generalized Oresme numbers and the Fibonacci fundamental system. *Rev. Sergip. Mat. Educ. Mat.* **2024**, *9*, 68–80. [\[CrossRef\]](#)
13. Özkan, E.; Akkuş, H.; Özkan, A. Properties of Generalized Bronze Fibonacci Sequences and Their Hyperbolic Quaternions. *Axioms* **2025**, *14*, 14. [\[CrossRef\]](#)
14. Kızılateş, C.; Du, W.-S.; Terzioğlu, N. On Higher-Order Generalized Fibonacci Hybrinomials: New Properties, Recurrence Relations and Matrix Representations. *Mathematics* **2024**, *12*, 1156. [\[CrossRef\]](#)
15. Çolak, E.G.; Gönül Bilgin, N.; Soykan, Y. On Generalized Fibospinomials: Generalized Fibonacci Polynomial Spinors. *Symmetry* **2024**, *16*, 694. [\[CrossRef\]](#)
16. Kalman, D.; Mena, R. The Fibonacci numbers-exposed. *Math. Mag.* **2003**, *76*, 167–181.
17. Rabinowitz, S. Algorithmic manipulation of Fibonacci identities. In *Applications of Fibonacci Numbers*; Kluwer Academic Publishers: Totnes, UK, 1996; Volume 6, pp. 389–408.
18. Ribenboim, P. *The Little Book of Big Primes*; Springer: Berlin, Germany, 1991.
19. Ribenboim, P. *My Numbers, My Friends*; Springer: Berlin, Germany, 2000.
20. Horadam, A.F. Oresme numbers. *Fibonacci Q.* **1974**, *12*, 267–271. [\[CrossRef\]](#)
21. Liana, S.A.; Wloch, I. Oresme hybrid numbers and hybrationals. *Kragujevac J. Math.* **2024**, *48*, 747–753. [\[CrossRef\]](#)
22. Halıcı, S.; Sayın, E. On some k-Oresme hybrid numbers. *Util. Math.* **2023**, *120*, 1–11.
23. Goy, T.; Zatorsky, R. On Oresme numbers and their connection with Fibonacci and Pell numbers. *Fibonacci Q.* **2019**, *57*, 238–245. [\[CrossRef\]](#)
24. Aktaş, S.; Soykan, Y. On the Norms of Toeplitz Matrices with the Generalized Oresme Numbers. *Asian J. Adv. Res. Rep.* **2023**, *17*, 41–57. [\[CrossRef\]](#)
25. Ertaş, A.; Yılmaz, F. On Quaternions with Gaussian Oresme Coefficients. *Turk. J. Math. Comput. Sci.* **2023**, *15*, 192–202.
26. Morales, C.G. Oresme Polynomials and Their Derivatives. *arXiv* **2019**, arXiv:1904.01165.
27. Halıcı, S.; Gür, Z.B. On some derivatives of k-Oresme polynomials. *Bull. Int. Math. Virtual Inst.* **2023**, *13*, 41–50.
28. Sayın, E.; Halıcı, S. On Oresme numbers and their geometric interpretations. *J. Inst. Sci. Technol.* **2024**, *14*, 1291–1300.
29. Halıcı, S.; Sayın, E.; Gür, Z.B. k-Oresme numbers and k-Oresme numbers with negative indices. In *International Conference on Mathematics and Its Applications in Science and Engineering*; Springer International Publishing: Cham, Switzerland, 2022; pp. 211–223.
30. Knuth, D.E. *The Art of Computer Programming, Volume 2: Seminumerical Algorithms*, 3rd ed.; Addison-Wesley Longman Publishing Co., Inc.: Boston, MA, USA, 1997.
31. Sastry, S.S. *Introductory Methods of Numerical Analysis*, 5th ed.; PHI Learning Pvt. Ltd.: New Delhi, India, 2012.
32. Alves, F.R.V. Sequência de Oresme e algumas propriedades (matriciais) generalizadas. *CQD-Rev. Electron. Paul. Math.* **2019**, *16*, 28–52.
33. Gross, J.L.; Yellen, J.; Zhang, P. (Eds.) *Handbook of Graph Theory*, 2nd ed.; Chapman and Hall/CRC: Boca Raton, FL, USA, 2013.
34. Dorigo, M.; Stützle, T. *Ant Colony Optimization*; The MIT Press: Cambridge, MA, USA, 2004; p. 305.
35. Busard, H.L.L. *Nicole Oresme: Quaestiones Super Geometrian Euclidis*; Franz Steiner Verlag: Stuttgart, Germany, 1961.
36. Cook, C.K. Some sums related to sums of Oresme numbers. In *Proceedings of the 10th International Research Conference on Fibonacci Numbers and Their Applications*; Springer: Dordrecht, The Netherlands, 2004; Volume 9, pp. 87–99.
37. Özkan, E.; Akkuş, H. A New Approach to k-Oresme and k-Oresme-Lucas Sequences. *Symmetry* **2024**, *16*, 1407. [\[CrossRef\]](#)
38. Amini, A.R. Fibonacci numbers from a long division formula. *Math. Spectrum.* **2008**, *40*, 59–61.
39. Demirtürk, B. A Matrix-Based Examination of k-Fibonacci and k-Oresme Sequences: Identities and Some Equation Solutions. *Commun Algebra*. **2025**; *in submitted*.

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