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Solitons on Submanifolds of Kenmotsu Manifolds with Concurrent Vector Fields

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Abstract: The present research paper investigates submanifolds of Kenmotsu manifolds, focusing on those equipped with concurrent vector fields. It examines the structural and geometric properties of such submanifolds, analyzing the decomposed equations in both vertical and horizontal components. Furthermore, the study generalizes certain results in the context of η -Ricci solitons and η -Yamabe solitons.

Keywords: η -Ricci soliton; η -Yamabe soliton; Kenmotsu manifold; concurrent vector field; Einstein manifold; η -Einstein manifold; umbilical submanifolds

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1. Introduction

During the last few years, owing to the versatile applications of almost-contact geometry in various areas and its pure geometrical perspective, it has captured the attention of various mathematicians. In particular, the study of odd-dimensional manifolds with contact and almost-contact structures plays a crucial role in understanding the structural and geometric features of various manifolds and their submanifolds in differential geometry. Many researchers have studied these manifolds, including Boothby and Wang. In the context of tensor calculus, Sasaki and Hatakeyama reinvestigated such manifolds in 1961. In modern geometry, the concept of contact manifolds has broad applications in mathematical physics, generating a deep understanding of geometric phenomena in a mathematical context.

The Kenmotsu manifold is a prominent and well-studied class in the field of almost-contact manifolds, developed by Kenmotsu in [1]. A Kenmotsu manifold is an odd-dimensional contact manifold equipped with a Riemannian metric, but it does not satisfy the conditions to be a Sasakian manifold. In recent years, the geometry of submanifolds has become a very popular area of interest, which is why many authors have conducted research in this field. Some of the research is defined in [2–5].

In Riemannian geometry, the fundamental concept of a Ricci soliton was introduced by Hamilton in [6] as a special type of solution for the Ricci flow. The Ricci flow is a parabolic partial differential equation that deforms the metric tensor on Riemannian manifolds. A Ricci soliton (a natural generalization of the Einstein metric) on Riemannian manifold $\tilde{M}$ is denoted by $(\tilde{M}, g, U, \lambda)$ with Riemannian metric g , a vector field U , and a scalar λ , and it is represented by

$$\mathcal{L}_U g + 2Ric + 2\lambda g = 0, \quad (1)$$

where $\mathcal{L}_U$ and Ric represent the Lie derivative along U and the Ricci tensor, respectively. Thereafter, various authors have worked in this field and published their research work in the context of Ricci solitons in different kinds of classes of almost-contact manifolds.

Cho and Kimura [7] introduced the extension of the Ricci soliton, namely, the η -Ricci soliton, which provides a more generalized and flexible form of solitons. In [8], Calin and Crasmareanu explored the study of such solitons on Hopf hypersurfaces embedded in complex space forms. An η -Ricci soliton (g, U, λ, μ) on a Riemannian manifold $(\tilde{M}, g)$ is given by the following equation:

$$\mathcal{L}_U g + 2Ric + 2\lambda g + 2\mu\eta \otimes \eta = 0, \quad (2)$$

where $\mathcal{L}_U$, Ric , and λ, μ represent a Lie derivative along U , a Ricci tensor, and real constants, respectively. The values of the constant λ as either negative, zero, or positive determine whether the η -Ricci soliton is shrinking, steady, or expanding.

Furthermore, in conjunction with the Ricci flow, Hamilton also inaugurated the concept of Yamabe flow in [9]. This corresponds to a self-similar solution of the Yamabe flow, known as the Yamabe soliton. Again, the η -Yamabe soliton, as a generalization of the Yamabe soliton on Riemannian manifolds $(\tilde{M}, g)$, is defined below

$$\mathcal{L}_U g = 2(r - \lambda)g - 2\mu\eta \otimes \eta, \quad (3)$$

where λ and μ are constants and η is 1-form. It is to be noted that for $\mu = 0$, the η -Yamabe soliton becomes a Yamabe soliton. Research on geometry flow involving different solitons in contact and complex manifolds has been developed by several researchers [10–17], providing new perspectives for understanding the geometry of diverse Riemannian manifolds. A number-theoretic perspective, utilizing Pontryagin numbers, was applied to the study of (ε) -Kenmotsu manifolds endowed with a semi-symmetric metric connection in [18].

Applications of Solitons in Physics

1. General Relativity and Cosmology

- (a) η -Ricci solitons contribute to the understanding of self-similar solutions in spacetime evolution and black hole physics.
- (b) Ricci soliton flows help model gravitational collapse and cosmic evolution.

2. Quantum Field Theory and String Theory

- (a) Solitons describe stable particle-like excitations in nonlinear field equations.
- (b) Kenmotsu manifold structures may offer insights into extra dimensions and compactifications in string theory.

3. Optical and Plasma Physics

- (a) Solitons in optical fibers enable long-distance, distortion-free communication.
- (b) Plasma physics uses solitonic models to study ion-acoustic and Alfvén waves in astrophysical environments.

4. Fluid Dynamics and Nonlinear Waves

- (a) Stable wave propagation in shallow water and atmospheric dynamics is modeled using solitonic solutions.
- (b) η -Yamabe solitons may describe energy transport in nonlinear wave systems.

5. Condensed Matter Physics

- (a) Solitons explain defects in superconductors, liquid crystals, and magnetic materials.

- (b) Submanifold structures of Kenmotsu manifolds may offer new approaches to studying geometric defects in materials.

The study of η -Ricci and η -Yamabe solitons in Kenmotsu manifolds with concurrent vector fields is motivated by their role in geometric flows and curvature evolution. These solitons generalize the classical Ricci and Yamabe solitons by incorporating the Reeb vector field, making them well suited for contact metric geometries. Their interaction with concurrent vector fields leads to structured geometric flows, often resulting in Einstein-like conditions or self-similar evolution. Additionally, they have significant applications in physics, particularly in general relativity and wave mechanics. Their study enhances the understanding of curvature-induced variations and conformal structures in Kenmotsu geometry.

In [19], sufficient conditions for almost Yamabe solitons corresponding to Yamabe metrics—metrics of constant scalar curvature—are established, with minimal topological constraints. By utilizing the properties of conformal vector fields, the study derives the necessary conditions for the soliton vector fields of gradient almost-Yamabe solitons, ensuring the existence of Yamabe metrics.

In [20], the behavior of concurrent vector fields on immersed manifolds was examined, and the characterization of concurrent vector fields with a constant length was established.

A concurrent vector field on a differentiable manifold is a vector field whose integral curves converge (or appear to meet at a single point in a well-defined sense). Formally, a vector field U on $\tilde{M}$ is said to be concurrent if its covariant derivative satisfies

$$\tilde{\nabla}_W U = W \quad (4)$$

for any $W \in T_p \tilde{M}$, $p \in \tilde{M}$.

This equation implies that the flow generated by U behaves in a controlled manner, resembling a homothetic transformation centered at some fixed point or infinity.

A concurrent vector field on a manifold can be visualized as a field whose integral curves intersect at a common point. In $\mathbb{R}^n$, a simple example of a concurrent vector field is the radial vector field given by

$$X = x^i \frac{\partial}{\partial x^i}.$$

This field describes vectors pointing outward from the origin. Every integral curve is a straight line passing through the origin, illustrating the concurrent nature. In $\mathbb{R}^2$, this would look like arrows radiating outward from the origin, resembling a starburst pattern.

On a Riemannian manifold $(\tilde{M}, g)$, a concurrent vector field is one where the integral curves all converge to or emanate from a single point. This behavior generalizes radial vector fields but is influenced by the curvature of $\tilde{M}$. In Kenmotsu geometry, concurrent vector fields provide insight into the relationship between the metric, connection, and curvature. Their existence, if possible, can impose geometric constraints on both the ambient manifold and its submanifolds, influencing their overall structure. This makes them relevant in the study of contact geometry, Riemannian submersions, and geometric flows. However, in a Kenmotsu manifold, the Reeb vector field does not satisfy the concurrent vector field condition.

From the above discussion, the outline of this article is as follows: In Section 2, we provide some information about the notations and definitions of the Kenmotsu manifolds and their submanifolds. In Section 3, we determine the decomposition of the vector field to submanifolds of Kenmotsu manifolds into its normal and tangent components, and we also establish some conditions for umbilical submanifolds with concurrent vector fields.

Additionally, we prove the applications of such submanifolds in the context of the η -Ricci soliton and η -Yamabe soliton with respect to concurrent vector fields. This section deals with the results of the tangent vector field to submanifolds endowed with concurrent vector fields. We also explore the conditions under which such submanifolds admit an η -Ricci and η -Yamabe soliton with concurrent vector fields. Section 4 concludes with some examples and conclusions.

2. Preliminaries

This section presents some fundamental definitions, notations, and formulas related to Kenmotsu manifolds and their submanifolds.

If an odd-dimensional $n (= 2a + 1)$ manifold follows a (ψ, ζ, η, g) structure, also known as an almost-contact metric structure, the following relations hold:

$$\psi^2 U = U - \eta(U)\zeta, \quad (5)$$

$$\eta(\zeta) = 1, \quad \psi(\zeta) = 0, \quad \eta(\psi\zeta) = 0, \quad (6)$$

$$g(U, \psi V) = -g(\psi U, V), \quad (7)$$

$$g(U, \zeta) = \eta(U), \quad g(\psi U, \psi V) = g(U, V) - \eta(U)\eta(V), \quad (8)$$

for all $U, V \in \chi(\tilde{M})$, where ψ denotes a $(1, 1)$ -tensor field, ζ is a vector field, η is 1-form, and g is the compatible Riemannian metric on $\tilde{M}$. The structure is called an almost-contact metric manifold.

Let $\tilde{\nabla}$ be the Levi-Civita connection of an almost-contact metric manifold $(\tilde{M}, \psi, \zeta, \eta, g)$, which satisfies the following condition:

$$(\tilde{\nabla}_U \psi)V = -g(U, \psi V)\zeta - \eta(V)\psi U, \quad (9)$$

for all $U, V \in T\tilde{M}$. Then, $\tilde{M}$ is said to be a Kenmotsu manifold [1].

In addition, Kenmotsu manifolds also satisfy the relation defined as

$$\tilde{\nabla}_U \zeta = U - \eta(U)\zeta. \quad (10)$$

Furthermore, consider M as a submanifold of Kenmotsu manifold $\tilde{M}$, endowed with the induced metric g . We denote ∇ and $\nabla^\perp$ as the induced connections on the tangent and normal bundles of M , respectively. Then, the Gauss and Weingarten formulas hold as follows [21]:

$$\tilde{\nabla}_U V = \nabla_U V + h(U, V), \quad (11)$$

and

$$\tilde{\nabla}_U X = -A_X U + \nabla_U^\perp X, \quad (12)$$

for all U, V belonging to a tangent bundle and X belonging to a normal bundle. Here, h and A_X characterize the second fundamental form and the shape operator, respectively, corresponding to the immersion of M into $\tilde{M}$.

The following equation expresses the relation between the second fundamental form h and the shape operator A_X , as follows:

$$g(h(U, V), X) = g(A_X U, V), \quad (13)$$

for all $U, V \in T_p M$ and $X \in T_p^\perp M$.

The notion of a submanifold being totally umbilical in a Kenmotsu manifold is characterized as follows:

$$h(U, V) = g(U, V)H, \quad (14)$$

where H denotes the mean curvature on M . Furthermore, a submanifold M is said to be umbilical with respect to $X \in T^\perp M$ if

$$g(h(U, V), X) = \gamma g(U, V) \quad (15)$$

for some function γ .

More specifically, M is said to be pseudo-umbilical if the normal vector field X is replaced with mean curvature H in (15), and we have

$$g(h(U, V), H) = \gamma g(U, V) \quad (16)$$

A vector field U of $\tilde{M}$ is said to be conformal if it satisfies

$$\mathcal{L}_U g = 2\gamma g, \quad (17)$$

where γ is a function on $\tilde{M}$. More specifically, when $\gamma = 0$, the conformal vector field reduces to a Killing vector field. If γ is constant, the conformal vector field is homothetic.

Now, consider a mapping $\psi : M \rightarrow \tilde{M}$ defined by

$$\psi U = PU + QU. \quad (18)$$

Here, PU and QU denote the tangential and normal components of ψU , respectively. In the context of differential geometry, a submanifold M is said to be a generalized self-similar submanifold of $\tilde{M}$ if

$$QU = fH, \quad (19)$$

where f is a scalar function on M that determines the proportionality between the normal component QU and the mean curvature H .

3. Solitonic View of Submanifolds in Kenmotsu Manifolds

Theorem 1. *Let M be a submanifold in a Kenmotsu manifold $\tilde{M}$ associated with concurrent vector field U such that ζ is normal to M ; then, U is a homothetic vector field.*

Proof. Since U is a concurrent vector field, it follows from (11) and (4) that

$$\nabla_W U + h(U, W) = W. \quad (20)$$

Comparing the tangent and normal components of (20), we obtain

$$\nabla_W U = W, \quad h(U, W) = 0. \quad (21)$$

Now, using the Lie derivative and (21), we obtain

$$(\mathcal{L}_U g)(V, W) = 2g(V, W). \quad (22)$$

This shows that U is a conformal vector field with $\gamma = 1$. Since a conformal vector field is homothetic when the conformal factor is constant, it follows from (22) that U is a homothetic vector field. $\square$

Theorem 2. Let M be a submanifold in a Kenmotsu manifold $\tilde{M}$ equipped with a concurrent vector such that ζ is normal to M . Then,

1. The tangent and normal components are given as

$$PW = \nabla_W PU - A_{QU}W, \quad (23)$$

and

$$QW = h(W, PU) + \nabla_W^\perp QU - g(PW, U)\zeta. \quad (24)$$

2. PU is conformal if and only if M is umbilical with respect to QU .

Proof. Since U is a concurrent vector field, we obtain

$$\psi W = \psi \tilde{\nabla}_W U = \tilde{\nabla}_W \psi U - (\tilde{\nabla}_W \psi)U. \quad (25)$$

Using the Expressions (9), (11), (12), and (18) in (25), we obtain

$$PW + QW = \nabla_W PU + h(W, PU) - A_{QU}W + \nabla_W^\perp QU - g(PW, U)\zeta. \quad (26)$$

After equating the tangent and normal components of (26), we obtain the Relations (23) and (24). Furthermore, the Lie derivative is defined as

$$(\mathcal{L}_{PU}g)(V, W) = g(\nabla_V PU, W) + g(V, \nabla_W PU) \quad (27)$$

From Relations (23), (13), and (27), we have

$$(\mathcal{L}_{PU}g)(V, W) = 2g(h(V, W), QU), \quad (28)$$

Now, if we assume that PU is conformal, then by (17) and (28), we arrive at

$$g(h(V, W), QU) = \gamma g(V, W). \quad (29)$$

Thus, the above condition implies that M is umbilical with respect to the normal component QU .

Conversely, suppose M is umbilical with respect to QU . Then, from (15) and (28), we obtain

$$(\mathcal{L}_{PU}g)(V, W) = 2\gamma g(V, W), \quad (30)$$

which implies that PU is conformal. $\square$

Theorem 3. Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ admitting an η -Ricci soliton equipped with a concurrent vector field U such that ζ is normal to M and symmetric. Then,

1. M is an Einstein manifold.
2. M is always shrinking.

Proof. Let (g, U, λ, μ) be an η -Ricci soliton on M . Then, with the help of (2) and (21), we arrive at

$$\text{Ric}(V, W) = -(\lambda + 1)g(V, W) - \mu\eta(V)\eta(W), \quad (31)$$

for all $V, W \in T_p M$. Now, if we take ζ as a normal vector field, then, from (31), we conclude that M is an Einstein manifold.

By the definition of the Riemannian curvature tensor R and using (21), we compute

$$\begin{aligned} R(V, W)U &= \nabla_V \nabla_W U - \nabla_W \nabla_V U - \nabla_{[V, W]} U \\ &= \nabla_V W - \nabla_W V - [V, W] \\ &= V - W - [V, W] \end{aligned}$$

Again, using the condition of symmetry and (21), we obtain $R(V, W)U = 0$. Consequently, we have $\text{Ric}(V, W) = 0$, which implies that $\lambda = -1 < 0$. Hence, the η -Ricci soliton is always shrinking. $\square$

Theorem 4. Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ admitting an η -Ricci soliton equipped with a concurrent vector field U such that ζ is normal to M . Then, the Ricci tensor is given by

$$\text{Ric}(V, W) = -\lambda g(V, W) - g(h(V, W), QU), \quad (32)$$

for all V, W tangent to M .

Proof. Let (g, PU, λ, μ) be an η -Ricci soliton on M . Then, combining (2) and (28), we obtain the Relation (32). $\square$

Theorem 5. Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ admitting an η -Ricci soliton equipped with a concurrent vector field U such that ζ is normal to M . Then, PU is conformal if and only if M is umbilical with respect to QU .

Proof. Let (g, PU, λ, μ) be an η -Ricci soliton on M , and suppose that PU is conformal. Then, (2) can be written as

$$\text{Ric}(V, W) = -\lambda g(V, W) - \gamma g(V, W). \quad (33)$$

From (32) and (33), we obtain

$$\gamma g(V, W) = g(h(V, W), QU),$$

which implies that M is umbilical.

Conversely, assume that M is umbilical. Then, using the definition, along with (32) and (33), we derive

$$(\mathcal{L}_{PU}g)(V, W) = 2\gamma g(V, W), \quad (34)$$

which shows that PU is conformal. $\square$

Theorem 6. Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ admitting an η -Yamabe soliton equipped with concurrent vector field U such that ζ is normal to M . Then, such a soliton is shrinking, steady, and expanding in accordance with $r < 1$, $r = 1$, and $r > 1$, respectively.

Proof. Let (g, U, λ, μ) be an η -Yamabe soliton, and suppose that ζ is normal to M . Then, (3) becomes

$$\mathcal{L}_U g(V, W) = 2(r - \lambda)g(V, W). \quad (35)$$

By using (22) and (35), we obtain

$$(r - \lambda - 1)g(V, W) = 0. \quad (36)$$

From the above equation, we determine $\lambda = r - 1$, which provides the required result. $\square$

Theorem 7. Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ admitting an η -Yamabe soliton (g, PU, λ, μ) equipped with concurrent vector field U such that ζ is normal to M . Then, PU is conformal.

Proof. Let (g, PU, λ, μ) be an η -Yamabe soliton, and suppose that ζ is normal to M . Then, (3) becomes

$$(\mathcal{L}_{PU} g)(V, W) = 2(r - \lambda)g(V, W). \quad (37)$$

Now, for $V, W \in T_p M$, using (28) and (37), we obtain

$$g(h(V, W), QU) = 2(r - \lambda)g(V, W), \quad (38)$$

which implies that M is umbilical with respect to QU . Furthermore, in the context of Theorem 2, we deduce that PU is conformal by using Equations (28) and (38). $\square$

Theorem 8. Let M be a generalized self-similar submanifold of Kenmotsu manifold $\tilde{M}$, associated with a concurrent vector field U such that ζ is normal to M . Then, M is pseudo-umbilical if and only if PU is a conformal vector field.

Proof. Suppose that M is a generalized self-similar submanifold of $\tilde{M}$. If M is a pseudo-umbilical submanifold, then, from (19), we deduce that M is umbilical with respect to QU , which implies that PU is conformal in the context of Theorem 2.

Conversely, suppose that PU is a conformal vector field. Then, combining (19) and (29), we obtain the following condition:

$$g(h(V, W), fH) = \gamma g(V, W). \quad (39)$$

From this, we conclude that M is pseudo-umbilical. $\square$

Theorem 9. Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ admitting an η -Ricci soliton equipped with concurrent vector field U such that ζ is tangent to M . Then,

1. M is an η -Einstein manifold.
2. λ and μ are related as $\lambda + \mu = -1$.

Proof. Let (g, U, λ, μ) be an η -Ricci soliton on M . Taking ζ as a tangent vector field, we conclude from (31) that M is an η -Einstein manifold. Furthermore, by virtue of Theorem 3, we obtain

$$Ric(V, W) = 0, \quad (40)$$

$$-(\lambda + 1)g(V, W) - \mu\eta(V)\eta(W) = 0. \quad (41)$$

Setting $W = \zeta$ in (41), we obtain $\lambda + \mu = -1$, which proves our second result. $\square$

Theorem 10. Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ admitting an η -Ricci soliton with concurrent vector field U such that ζ is tangent to M . Then,

1. The tangent and normal components are given by

$$PW = \nabla_W PU - A_{QU}W - g(PW, U)\zeta + \eta(U)PW, \quad (42)$$

and

$$QW = h(W, PU) + \nabla_W^\perp QU + \eta(U)QW. \quad (43)$$

2. The Ricci tensor is given by

$$\begin{aligned} Ric(V, W) = & -\lambda g(V, W) - \mu \eta(V)\eta(W) - g(h(V, W), QU) \\ & - \frac{1}{2}[g(PV, U)\eta(W) + g(PW, U)\eta(V)], \end{aligned} \quad (44)$$

for any V, W tangent to M .

Proof. Let (g, PU, λ, μ) be an η -Ricci soliton on M . From relations (9), (11), (12), (18), and (25), we derive

$$\begin{aligned} PW + QW = & \nabla_W PU + h(W, PU) - A_{QU}W + \nabla_W^\perp QU \\ & - g(PW, U)\zeta + \eta(U)PW + \eta(U)QW. \end{aligned} \quad (45)$$

By equating the tangent and normal components of (45), we obtain relations (42) and (43). Moreover, using (13), (27), and (42), we have

$$(\mathcal{L}_{PU}g)(V, W) = 2g(h(V, W), QU) + g(PV, U)\eta(W) + g(PW, U)\eta(V). \quad (46)$$

Again, by the definition of an η -Ricci soliton and using (46), we obtain (44). $\square$

Theorem 11. Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ admitting an η -Yamabe soliton equipped with concurrent vector field U such that ζ is tangent to M . Then, the scalar curvature is constant.

Proof. Let (g, U, λ, μ) be an η -Yamabe soliton on M . Then, combining (3) and (22), we obtain

$$(r - \lambda - 1)g(V, W) - \mu \eta(V)\eta(W) = 0. \quad (47)$$

Setting $W = \zeta$ in (47), we obtain

$$r = \lambda + \mu + 1.$$

This shows that r is constant as λ and μ are constant. $\square$

Corollary 1. Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ equipped with concurrent vector field U such that ζ is tangent to M and (g, PU, λ, μ) is an η -Yamabe soliton. Then, the scalar curvature is constant.

Proof. By (28) and by virtue of Theorem 11, we establish the desired result. $\square$

4. Concluding Remarks and Examples

In mathematical physics and general relativity, Kenmotsu manifolds are recognized as a vital class of manifolds. This study emphasizes Kenmotsu manifolds and their submanifolds in relation to concurrent vector fields. Note that concurrent vector fields naturally arise in the study of conformal geometry and soliton theory, as they often relate to structures like η -Ricci solitons or η -Yamabe solitons, where the geometry evolves under certain flows. Furthermore, it explores the characterization of Kenmotsu manifolds and their submanifolds that admit specific solitons, such as η -Ricci solitons and η -Yamabe solitons. This study aims to enhance the understanding of these submanifolds through the analysis of decomposed equations, both vertically and horizontally, particularly in the context of η -Ricci and η -Yamabe solitons. Beyond Kenmotsu manifolds, the study highlights the broader scope of this framework, uncovering novel insights into various types of manifolds and their submanifolds.

Example 1. Let $\tilde{M} = \mathbb{R}^{2a+1}$ with coordinates $(x^1, y^1, x^2, y^2, \dots, x^a, y^a, z)$. Define the following tensor fields on $\tilde{M}$:

1. Structure tensor : ψ , a $(1, 1)$ -tensor, acting on the coordinate frame $\{\partial x^i, \partial y^i, \partial z\}$ as

$$\psi(\partial x^i) = \partial y^i, \quad \psi(\partial y^i) = -\partial x^i, \quad \psi(\partial z) = 0.$$

2. Reeb vector field : $\zeta = \partial z$.
 3. Contact 1-form : $\eta = dz - \sum_{i=1}^a x^i dy^i$. This satisfies $\eta(\zeta) = 1$.
 4. Metric tensor : $g = \sum_{i=1}^a ((dx^i)^2 + (dy^i)^2) + \eta^2$. Furthermore, it satisfies $g(\zeta, \zeta) = 1$.
- Let us compute (9) for the basis vector fields.

Case 1. $U = \partial x^i, V = \partial x^j$. Using the definition of ψ , we have $\psi(\partial x^j) = \partial y^j$. So, we compute

$$g(U, \psi V) = g(\partial x^i, \partial y^j) = 0.$$

Furthermore,

$$\eta(V) = \eta(\partial x^j) = 0.$$

Thus,

$$-g(U, \psi V)\zeta - \eta(V)\psi U = 0.$$

We now compute $(\tilde{\nabla}_U \psi)V$, which is indeed zero in this case.

Case 2. $U = \partial x^i, V = \partial y^j$.

$$\psi V = \psi(\partial y^j) = -\partial x^j.$$

Then,

$$g(U, \psi V) = -g(\partial x^i, \partial x^j) = -\delta_{ij}.$$

Similarly,

$$\eta(\partial y^j) = -x^j.$$

Thus,

$$-g(U, \psi V)\zeta - \eta(V)\psi U = \delta_{ij}\zeta + x^j\partial x^i.$$

Now computing

$$\begin{aligned}(\tilde{\nabla}_U\psi)V &= \tilde{\nabla}_U(\psi V) - \psi(\tilde{\nabla}_U V) \\ &= \tilde{\nabla}_{\partial x^i}(\partial x^j) - \psi(x^j\partial y^i) \\ &= \delta_{ij}\zeta + x^j\partial x^i.\end{aligned}$$

Next, we compute (10) for the basic vector fields.

Case 3. $U = \partial x^i$. We compute $\eta(\partial x^i) = 0$. Thus, $U - \eta(U)\zeta = \partial x^i$. From the standard connection properties, we have $\tilde{\nabla}_U\zeta = \partial x^i$.

Case 4. $U = \partial y^i$. We compute $\eta(\partial y^i) = -x^i$. Thus, $U - \eta(U)\zeta = \partial y^i + x^i\zeta$. From the standard results, we obtain $\tilde{\nabla}_U\zeta = \partial y^i + x^i\zeta$.

Similarly

Case 5. $U = \partial z$. We compute $\eta(\partial z) = 1$. Thus, $U - \eta(U)\zeta = 0$. It is easy to find that $\tilde{\nabla}_U\zeta = 0$.

Thus, the manifold $(\tilde{M}, \psi, \zeta, \eta, g)$ satisfies the defining properties of a Kenmotsu manifold, that is, Relations (9) and (10).

Example 2. Let $M \subset \tilde{M}$ be the submanifold defined by fixing $z = \text{constant}$, e.g., $z = 0$. Then, M is locally parameterized by $(x^1, y^1, x^2, y^2, \dots, x^a, y^a)$. The induced Riemannian metric on M is

$$g = \sum_{i=1}^a ((dx^i)^2 + (dy^i)^2).$$

The tangent space $T_p M$ at any point p is spanned by

$$\{\partial x^1, \partial y^1, \dots, \partial x^a, \partial y^a\}.$$

Since the metric satisfies

$$g(\zeta, \partial x^i) = 0, \quad g(\zeta, \partial y^i) = 0,$$

it follows that ζ is orthogonal to all tangent vectors of M . Moreover, the structure η vanishes identically on M , implying that ζ is not tangent to M . Hence, M is not an almost-contact submanifold but can be viewed as a $2a$ -dimensional Riemannian submanifold of $\tilde{M}$, where ζ is normal to M everywhere.

Example 3. We define $M \subset \tilde{M}$ as the submanifold, as follows:

$$M = \{(x^1, y^1, \dots, x^k, y^k, z) | x^{k+1} = \dots = x^a = 0, y^{k+1} = \dots = y^a = 0\},$$

where $k \leq a$ determines the dimension of M . The Reeb vector field $\zeta = \partial z$ is tangent to M because the coordinate z is unrestricted in M . The induced Riemannian metric on M is as follows:

$$g = \sum_{i=1}^k (dx^i)^2 + (dy^i)^2 + \eta^2,$$

where $\eta = dz - \sum_{i=1}^k x^i dy^i$. The restriction of ψ to M satisfies

$$\psi(\partial x^i) = \partial y^i, \quad \psi(\partial y^i) = -\partial x^i, \quad \psi(\partial z) = 0.$$

Verifying $g(U, \zeta) = \eta(U)$ of (8): For any vector field U , write that following

$$U = \sum_{i=1}^a (A_i \partial x^i + B_i \partial y^i) + C \partial z, \quad (48)$$

where A, B , and C are components of U . Then, using the metric

$$g(U, \zeta) = g\left(\sum (A_i \partial x^i + B_i \partial y^i) + C \partial z, \partial z\right). \quad (49)$$

Since $g(\partial z, \partial z) = 1$ and $g(\partial z, \partial x^i) = g(\partial z, \partial y^i) = 0$, we obtain

$$g(U, \zeta) = C. \quad (50)$$

On the other hand, from the definition of η ,

$$\eta(U) = dz(U) - \sum x^i dy^i(U). \quad (51)$$

Since $dz(U) = C$ and the second term vanishes when U does not have ∂y^i components, we obtain

$$\eta(U) = C = g(U, \zeta). \quad (52)$$

Thus, the first equation holds.

Verifying $g(\psi U, \psi V) = g(U, V) - \eta(U)\eta(V)$ of (8). Let U, V be two arbitrary vector fields

$$U = \sum_{i=1}^a (A_i \partial x^i + B_i \partial y^i) + C \partial z, \quad (53)$$

$$V = \sum_{i=1}^a (P_i \partial x^i + Q_i \partial y^i) + R \partial z, \quad (54)$$

where P, Q , and R are components of V . Now, compute $g(\psi U, \psi V)$

$$g(\psi U, \psi V) = \sum_{i=1}^a (A_i P_i + B_i Q_i). \quad (55)$$

On the other hand,

$$g(U, V) = \sum_{i=1}^a (A_i P_i + B_i Q_i) + CR. \quad (56)$$

Furthermore,

$$\eta(U) = C, \quad \eta(V) = R. \quad (57)$$

Thus,

$$g(U, V) - \eta(U)\eta(V) = \sum_{i=1}^a (A_i P_i + B_i Q_i). \quad (58)$$

Since both sides are equal, we conclude that the second equation holds. Thus, M inherits an almost-contact metric structure.

Example 4. Let $\tilde{M} = \mathbb{R}^{2a+1}$ with coordinates $(x^1, y^1, x^2, y^2, \dots, x^a, y^a, z)$. The Kenmotsu structure is defined as follows:

$$\begin{aligned}\zeta &= \partial z, \quad \eta = dz - \sum_{i=1}^a x^i dy^i, \\ \psi(\partial x^i) &= \partial y^i, \quad \psi(\partial y^i) = -\partial x^i, \quad \psi(\partial z) = 0, \\ g &= \sum_{i=1}^a ((dx^i)^2 + (dy^i)^2) + (dz)^2.\end{aligned}$$

Let M be the 3-dimensional submanifold of $\tilde{M} = \mathbb{R}^5$ (for $a = 2$) defined by

$$M = \{(x^1, y^1, x^2, y^2, z) \in \mathbb{R}^5 \mid (x^1)^2 + (y^1)^2 = 1, z = 0\}.$$

At any point $p \in M$, the tangent space is spanned by

$$T_p M = \text{span}\{\partial x^1, \partial y^1, \partial x^2\}.$$

Since $z = 0$ on M , ζ is normal to M . The induced metric g on M is

$$g = \sum_{i=1}^2 (dx^i)^2 + (dy^1)^2.$$

Let the vector field U on $\tilde{M}$ be

$$U = \sum_{i=1}^a (x^i \partial x^i + y^i \partial y^i) + z \partial z.$$

On M , the component of U tangent to M is

$$U_M = x^1 \partial x^1 + y^1 \partial y^1.$$

Since the Levi-Civita connection is compatible with the metric, it follows that the coordinate basis $\{\partial x^i, \partial y^i\}$ is orthonormal and satisfies

$$\nabla_{\partial x^i} \partial x^j = 0, \quad \nabla_{\partial y^i} \partial y^j = 0, \quad \nabla \partial x^i \partial y^j = 0.$$

Applying the connection to U_M , we have

$$\nabla_{\partial x^j} U_M = \nabla_{\partial x^j} (x^1 \partial x^1 + y^1 \partial y^1).$$

Since $\partial x^j(x^1) = \delta_j^1$ and $\partial x^j(y^1) = 0$, we obtain

$$\nabla_{\partial x^j} U_M = \delta_j^1 \partial x^1.$$

Similarly, for ∂y^j , the following applies:

$$\nabla_{\partial y^j} U_M = \partial y^j(x^1) \partial x^1 + \partial y^j(y^1) \partial y^1.$$

Since $\partial y^j(y^1) = \delta_j^1$ and $\partial y^j(x^1) = 0$, we obtain

$$\nabla_{\partial y^j} U_M = \delta_j^1 \partial y^1 = \partial y^j$$

when $j = 1$. Similarly,

$$\nabla_{\partial x^j} U_M = \delta_j^1 \partial x^1 = \partial x^j$$

when $j = 1$. Thus, for any any $X \in T_p M$, $\nabla_X U_M = X$, verifying that U_M is a homothetic vector field on M .

Lastly, we demonstrate a concrete case where Theorem 3 of the article holds, verifying the Einstein nature and shrinking property of η -Ricci solitons in submanifolds of Kenmotsu manifolds.

Example 5. Consider the Kenmotsu manifold $\tilde{M} = \mathbb{R}^5$ with the coordinates (x, y, z, u, v) and the following structure:

1. Contact 1-form: $\eta = dz - xdy$.
2. Reeb vector field: $\zeta = \partial_z$.
3. Metric tensor: $g = dx^2 + dy^2 + du^2 + dv^2 + \eta^2$.
4. Almost contact structure tensor: Defined by $\psi(\partial_x) = \partial_y$, $\psi(\partial_y) = -\partial_x$, $\psi(\partial_z) = 0$.

Let M be the 3-dimensional submanifold of $\tilde{M}$ given by

$$M = \{(x, y, z, u, v) \in \mathbb{R}^5 | u = v = 0\}.$$

The induced metric on M is

$$g_M = dx^2 + dy^2 + \eta^2.$$

Since $\zeta = \partial_z$ is normal to M , this satisfies the assumption of Theorem 3. Consider the concurrent vector field on $\tilde{M}$ as

$$U = x\partial_x + y\partial_y + z\partial_z.$$

For any tangent vector W on M , we compute $\nabla_W U = W$. This confirms that U is a concurrent vector field. By computing the Lie derivative $\mathcal{L}_U g$ and the Ricci tensor on M , and ensuring the Ricci tensor is symmetric, we find

$$\text{Ric}(V, W) = -(\lambda + 1)g(V, W),$$

which confirms that M is an Einstein manifold. Moreover, using the curvature properties of M , we derive $\lambda = -1 < 0$, which proves that the η -Ricci soliton is always shrinking, supporting Theorem 3.

5. Future Directions

This study opens several avenues for further research. Extending the η -Ricci and η -Yamabe solitons to Kenmotsu statistical manifolds could provide insights into dual connections, statistical curvature, and divergence geometry. Investigating geometric flows, such as Ricci and Yamabe flows, in this setting, may reveal new stability conditions for α -connections. The role of concurrent vector fields in statistical structures and their effect on Fisher information-like tensors remains an open problem. Additionally, studying submanifolds of Kenmotsu statistical manifolds could uncover interactions between induced statistical connections and solitonic structures. Potential applications in machine learning, Bayesian inference, and optimal transport theory could further connect these geometric concepts to applied mathematics.

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