symmetry

MDPI

Article

A Note on Generalized Parabolic Marcinkiewicz Integrals with Grafakos–Stefanov Kernels

Hussain Al-Qassem [1] **and Mohammed Ali** [2,*]

[1] Department of Mathematics and Statistics, Qatar University, Doha P.O. Box 2713, Qatar; husseink@qu.edu.qa
[2] Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan
* Correspondence: myali@just.edu.jo

Abstract: This paper focuses on studying the generalized Marcinkiewicz integral operators with mixed homogeneity over symmetric spaces. By making an appropriate decomposition of the aforementioned operators and tracking certain estimates, the boundedness of these operators is established from the homogeneous Triebel–Lizorkin space $\dot{F}_p^{0,\tau}(\mathbb{R}^d)$ to the $L^p(\mathbb{R}^d)$ space for all $p, \tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$ provided that the kernel functions belong to the Grafakos–Stefanov class. The main results generalize and improve some previously known results on Marcinkiewicz and generalized Marcinkiewicz operators.

Keywords: parabolic Marcinkiewcz integral; mixed homogeneity space; Grafakos–Stefanov class; Triebel–Lizorkin space

MSC: 42B20; 42B25; 42B35

1. Introduction

Let $d \geq 2$ and $\mathbb{S}^{d-1}$ be the unit sphere in the Euclidean space $\mathbb{R}^d$ equipped with the induced Lebesgue surface measure $d\sigma_d(w')$. For $j \in \{1, 2, \cdots, d\}$, let $\kappa_j \geq 1$ be fixed numbers. We define the real mapping Γ on $\mathbb{R}_+ \times \mathbb{R}^d$ as $\Gamma(\gamma, w) = \sum_{j=1}^{d} w_j^2 \gamma^{-2\kappa_j}$, with $w = (w_1, w_2, \ldots, w_d) \in \mathbb{R}^d$. For each fixed $w \in \mathbb{R}^d$, we denote the unique solution to the equation $\Gamma(\gamma, w) = 1$ by $\gamma \equiv \gamma_d(w)$. The metric space $(\mathbb{R}^d, \gamma)$ is known by the mixed homogeneity space associated to $\{\kappa_j\}_{j=1}^d$.

The change in variables concerning the space $(\mathbb{R}^d, \gamma)$ is presented in the following transformation:

$$\begin{aligned} w_1 &= \gamma^{\kappa_1} \cos v_1 \cdots \cos v_{d-2} \cos v_{d-1}, \\ w_2 &= \gamma^{\kappa_2} \cos v_1 \cdots \cos v_{d-2} \sin v_{d-1}, \\ &\vdots \\ w_{d-1} &= \gamma^{\kappa_{d-1}} \cos v_1 \sin v_2, \\ w_d &= \kappa^{\kappa_d} \sin v_1. \end{aligned}$$

So, $dw = \gamma^{\kappa-1} J(w') d\gamma d\sigma_d(w')$, where $\kappa = \sum_{j=1}^{d} \kappa_j$, $J(w') = \sum_{j=1}^{d} \kappa_j (w_j')^2$, $w' = \mathrm{M}_{\gamma^{-1}} w \in \mathbb{S}^{d-1}$, $\gamma^{\kappa-1} J(w')$ is the Jacobian of the transformation, and M_γ is the $d \times d$ diagonal matrix given by

$$\mathrm{M}_\gamma = \begin{bmatrix} \gamma^{\kappa_1} & & 0 \\ & \ddots & \\ 0 & & \gamma^{\kappa_d} \end{bmatrix}.$$

check for updates

Academic Editor: Junesang Choi

Received: 29 January 2025
Revised: 5 March 2025
Accepted: 12 March 2025
Published: 13 March 2025

Citation: Al-Qassem, H.; Ali, M. A Note on Generalized Parabolic Marcinkiewicz Integrals with Grafakos–Stefanov Kernels. *Symmetry* **2025**, *17*, 429. https://doi.org/10.3390/sym17030429

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

In [1], it was proved that $J(w') \in \mathcal{C}^\infty(\mathbb{S}^{d-1})$ and $1 \leq J(w') \leq C$ for some $C > 1$. Let $\mho \in L^1(\mathbb{S}^{d-1})$ be a measurable mapping satisfying

$$\mho(\mathrm{M}_\gamma w) \quad = \quad \mho(w), \ \ \forall \gamma > 0 \tag{1}$$

and

$$\int_{\mathbb{S}^{d-1}} \mho(w') J(w') d\sigma_d(w') \quad = \quad 0. \tag{2}$$

For $f \in \mathcal{S}(\mathbb{R}^d)$ and $\tau \in (1, \infty)$, we define the generalized parabolic Marcinkiewicz integral $\mathcal{G}_\mho^{(\tau)}$ by

$$\mathcal{G}_\mho^{(\tau)}(f)(x) = \left(\int_{\mathbb{R}_+} \left| \frac{1}{r} \int_{\gamma(v) \leq r} f(x-v) \frac{\mho(v)}{(\gamma(v))^{\kappa-1}} dv \right|^\tau \frac{dr}{r} \right)^{1/\tau}.$$

The study of singular integrals, as well as the corresponding Marcinkiewicz integrals, over symmetric spaces has attracted the attention of many authors in the past two decades. These operators fall within the broader category of Littlewood–Paley g-functions. Establishing L^p bounds for them is valuable for analyzing the smoothness characteristics of functions and the behavior of integral transformations, including Poisson integrals, singular integrals, and more broadly, singular Radon transforms.

Marcinkiewicz integrals have been extensively investigated by numerous authors, tracing back to the foundational work by Zygmund on the circle and that by Stein on $\mathbb{R}^d$. For more information and further applications, readers may refer to [2–4].

We point out that when $\kappa_1 = \kappa_2 = \cdots = \kappa_d = 1$, we have $\kappa = d$, $\gamma(v) = |v|$ and $(\mathbb{R}^d, \gamma) = (\mathbb{R}^d, |\cdot|)$. In this case, we denote $\mathcal{G}_\mho^{(\tau)}$ by $\mathcal{G}_\mho^{(\tau),c}$. In addition, when $\tau = 2$, the operator $\mathcal{G}_\mho^{(\tau),c}$ reduces to the classical Marcinkiewicz integral operator, which is denoted by $\mathcal{M}_\mho$. The operator $\mathcal{M}_\mho$ was introduced by E. Stein in [2], in which he proved that the L^p boundedness of $\mathcal{M}_\mho$ for $p \in (1,2)$ holds under the condition $\mho \in Lip_\nu(\mathbb{S}^{d-1})$, with $\nu \in (0,1]$. Subsequently, the operator $\mathcal{M}_\mho$ has been investigated by many authors. For example, Walsh [5] proved the L^2 boundedness of $\mathcal{M}_\mho$ provided that $\mho \in L(logL)^{1/2}(\mathbb{S}^{d-1}) \supsetneq Lip_\nu(\mathbb{S}^{d-1})$ and proved that the condition $\mho \in L(logL)^{1/2}(\mathbb{S}^{d-1})$ is nearly optimal. Later, this result was extended and improved in [6], where the authors obtained that $\mathcal{M}_\mho$ is of type (p,p) for all $p \in (1,\infty)$ if $\mho \in L(logL)^{1/2}(\mathbb{S}^{d-1})$. The authors in [7] proved that if $\mho$ belongs to the block space $B_q^{(0,-1/2)}(\mathbb{S}^{d-1})$, then $\mathcal{M}_\mho$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1,\infty)$, and this condition on $\mho$ is nearly optimal.

On the other hand, the authors in [8] showed that whenever $\mho$ belongs to the Grafakos–Stefanov class $\mathcal{F}_\alpha(\mathbb{S}^{d-1})$ with $\alpha > 0$, the operator $\mathcal{M}_\mho$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$. For more background information on $\mathcal{M}_\mho$, readers may consult [9–12]; for its extensions and developments, [13–18]; and for recent advances, [19–30].

The generalized Marcinkiewicz operator $\mathcal{G}_\mho^{(\tau),c}$ was introduced by the authors of [31]. They found that if $\mho \in L^q(\mathbb{S}^{d-1})$ with $q > 1$, then the inequality

$$\left\| \mathcal{G}_\mho^{(\tau),c}(f) \right\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{\dot{F}_p^{0,\tau}(\mathbb{R}^d)} \tag{3}$$

holds for all $p, \tau \in (1,\infty)$. Later, Le improved this result in [32] under the weaker condition $\mho \in L(\log L)(\mathbb{S}^{d-1})$. These results were extended and improved by the authors of [33] who showed that if $\mho$ is in either $L(\log L)^{1/\tau}(\mathbb{S}^{d-1})$ or $B_q^{(0,\frac{-1}{\tau})}(\mathbb{S}^{d-1})$, then $\mathcal{G}_\mho^{(\tau),c}$ is bounded on $L^p(\mathbb{R}^d)$ for $p, \tau \in (1,\infty)$. The authors of [34] confirmed inequality (3) for all

$p, \tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$ provided that $\mho \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$ for some $\alpha > 0$. For relevant results, we refer the reader to [35–38].

We point out that the Grafakos–Stefanov class $\mathcal{F}_\alpha(\mathbb{S}^{d-1})$ (for $\alpha > 0$), which is the set of all integrable functions $\mho$ over $\mathbb{S}^{d-1}$ such that

$$\sup_{\zeta \in \mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \log^{\alpha+1}(|\zeta \cdot u|^{-1}) |\mho(u)| d\sigma_d(u) < \infty, \tag{4}$$

was introduced in [5] and then developed in [39].

Now, let us recall the definition of homogeneous Triebel–Lizorkin space $\dot{F}_p^{\epsilon,\tau}(\mathbb{R}^d)$ [40]. For $\epsilon \in \mathbb{R}$ and $\tau, p \in (1, \infty)$, the space $\dot{F}_p^{\epsilon,\tau}(\mathbb{R}^d)$ is the collection of all tempered distribution functions f defined on $\mathbb{R}^d$ such that

$$\|f\|_{\dot{F}_p^{\epsilon,\tau}(\mathbb{R}^d)} = \left\| \left(\sum_{j \in \mathbb{Z}} 2^{j\epsilon\tau} |\Psi_j * f|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)} < \infty,$$

where $\widehat{\Psi_j}(\zeta) = \Gamma(2^{-j}\zeta)$ and $\Gamma \in C_0^\infty(\mathbb{R}^\eta)$ is a radial mapping satisfying the following:

(1) $0 \leq \Gamma(\zeta) \leq 1$;

(2) $supp\,(\Gamma) \subset \left\{ \zeta : |\zeta| \in [\frac{1}{2}, 2] \right\}$;

(3) For $|\zeta| \in [\frac{3}{5}, \frac{5}{3}]$, there is a positive constant C such that $\Gamma(\zeta) \geq C$;

(4) For $\zeta \neq 0$, we have $\sum_{j \in \mathbb{Z}} \Gamma(2^{-j}\zeta) = 1$.

The authors of [35] proved the following:

(a) The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is dense in $\dot{F}_p^{\epsilon,\tau}(\mathbb{R}^d)$;

(b) For $p \in (1, \infty)$, $\dot{F}_p^{0,2}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$;

(c) $\dot{F}_p^{\epsilon,\tau_1}(\mathbb{R}^d) \subseteq \dot{F}_p^{\epsilon,\tau_2}(\mathbb{R}^d)$ whenever $\tau_1 \leq \tau_2$.

Motivated by the work performed in [34] regarding the L^p boundedness of the generalized Marcinkiewicz integral $\mathcal{G}_\mho^{(\tau),c}$ in the classical form whenever $\mho \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$ and by the work performed in [17] regarding the L^p boundedness of the parabolic Marcinkiewicz integral $\mathcal{G}_\mho^{(2)}$ whenever $\mho \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$, we shall study the L^p boundedness of the generalized parabolic Marcinkiewicz integral $\mathcal{G}_\mho^{(\tau)}$ provided that $\mho \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$. In fact, we shall prove the following.

Theorem 1. *Let $\mho \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$ for some $\alpha > 0$ and satisfy (1)–(2). Then, there exists a positive constant $C_p > 0$ such that*

$$\left\| \mathcal{G}_\mho^{(\tau)}(f) \right\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{\dot{F}_p^{\vec{0},\tau}(\mathbb{R}^d)}$$

for all $p \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$ and $\tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$.

Remark 1. *(1) For the cases $\kappa_j = 1$ $(1 \leq j \leq d)$ and $\tau = 2$, the L^p boundedness of $\mathcal{G}_\mho^{(\tau)}$ was proved in [2] for $p \in (1,2)$ provided that $\mho \in Lip_\nu(\mathbb{S}^{d-1}) \subsetneq \mathcal{F}_\alpha(\mathbb{S}^{d-1})$. Hence, our result generalizes and improves the result in [2].*

(2) Since $\bigcup_{q>1} L^q(\mathbb{S}^{d-1}) \subsetneq \mathcal{F}_\alpha(\mathbb{S}^{d-1})$, Theorem 1 generalizes and improves the result in [15] for the case $\tau = 2$.

(3) If we take $\tau = 2$ in Theorem 1, the main result in [17] is obtained.

(4) *The authors of [18,20] proved the boundedness of* $\mathcal{G}_{\mho}^{(\tau)}$ *whenever* $\mho$ *belongs to the space* $L(\log L)^{1/2}(\mathbb{S}^{d-1})$ *and the space* $B_q^{(0,-\frac{1}{2})}(\mathbb{S}^{d-1})$*, which are totally different from the space* $\mathcal{F}_\alpha(\mathbb{S}^{d-1})$.

(5) *For the special case* $\kappa_j = 1$ $(1 \leq j \leq d)$*, we obtain the main result in [34].*

Henceforth, the letter C stands for a positive constant which is independent of the main parameters and not necessarily the same at each occurrence.

2. Auxiliary Lemmas

We begin this section by presenting several definitions and lemmas. We define the family of measures $\{\lambda_{\mho,r,j} := \lambda_{r,j} : r \in \mathbb{R}_+, j \in \mathbb{Z}\}$ and its corresponding maximal operator λ^* on $\mathbb{R}^d$ as

$$\int_{\mathbb{R}^d} f\, d\lambda_{r,j} \quad = \quad \frac{1}{2^j r} \int_{2^{j-1}r \leq \gamma(u) \leq 2^j r} \mho(u) f(u) du$$

and

$$\lambda^*(f) \quad = \quad \sup_{j\in\mathbb{Z}} \sup_{r\in\mathbb{R}_+} ||\lambda_{r,j}| * f|,$$

where $|\lambda_{r,j}|$ is defined in the same way as $\lambda_{r,j}$ but $\mho$ is replaced by $|\mho|$.

We shall need the following lemma from [41].

Lemma 1. *Let* $P(\gamma) = (v_1\gamma^{\kappa_1}, v_2\gamma^{\kappa_2}, \cdots, v_d\gamma^{\kappa_d})$*, where* $v_j, \kappa_j \in \mathbb{R}$ *and* $j \in \{1, 2, \cdots, d\}$*. Let us suppose that* $\mathcal{M}_P$ *is the maximal function defined on* $\mathbb{R}^d$ *by*

$$\mathcal{M}_P f(w) = \sup_{\rho>0} \frac{1}{\rho} \int_0^\rho |f(w - P(\gamma))| d\gamma.$$

Then, there exists a positive C_p *(independent of* $v_j's$*) such that the estimate*

$$\|\mathcal{M}_P(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$$

holds for all $1 < p \leq \infty$.

We shall need the following lemma by Chen and Ding [16].

Lemma 2. *Let us assume that* ν *denotes the number of distinct elements in* $\{\kappa_j\}_{j=1}^d$ *and that* $\delta \in [0, 1]$*. Then, there exists* $C > 0$ *such that for* $u \in \mathbb{S}^{d-1}$ *and* $\zeta \in \mathbb{R}^d$*,*

$$\left| \int_{1/2}^1 e^{-i\mathrm{M}_\gamma u\cdot\zeta} \frac{d\gamma}{\gamma} \right| \leq C|u \cdot \zeta|^{-\frac{\delta}{\nu}}.$$

By using Lemma 1, we directly deduce the following result.

Lemma 3. *Let* $\mho \in L^1(\mathbb{S}^{d-1})$ *and* $f \in L^p(\mathbb{R}^d)$*. Then, for each* $p \in (1, \infty)$*, there exists a positive constant* C_p *such that*

$$\|\lambda^*(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|\mho\|_{L^1(\mathbb{S}^{d-1})} \|f\|_{L^p(\mathbb{R}^d)}. \tag{5}$$

Lemma 4. *Let* $\mho \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$ *for some* $\alpha > 0$ *and satisfy* (1)–(2)*. Then, there is a positive constant* C *such that*

$$\|\lambda_{r,j}\| \quad \leq \quad C, \tag{6}$$

$$|\hat{\lambda}_{r,j}(\zeta)| \leq C \min\Big\{|\mathrm{M}_{2^j r}\zeta|, (\log|\mathrm{M}_{2^j r}\zeta|)^{-(\alpha+1)}\Big\}, \tag{7}$$

where $\|\lambda_r\|$ *is the total variation of* λ_r.

Proof. By the definition of $\lambda_{r,j}$, it is easy to verify that (6) holds. By Fubini's theorem and Hölder's inequality, we obtain

$$\begin{aligned} |\hat{\lambda}_{r,j}(\zeta)| &\leq C\int_{1/2}^{1}\left|\int_{\mathbb{S}^{d-1}} \Omega(u)J(u)\, e^{-i\mathrm{M}_{2^j r\gamma}u\cdot\zeta}\, d\sigma_d(u)\right|\frac{d\gamma}{\gamma} \\ &\leq C\int_{\mathbb{S}^{d-1}}|\Omega(u)|\int_{1/2}^{1} I(\zeta,u,r)\frac{d\gamma}{\gamma}d\sigma_d(u), \end{aligned} \tag{8}$$

where

$$I(\zeta,u,r) = \int_{1/2}^{1} e^{-i\mathrm{M}_{2^j r\gamma}u\cdot\zeta}\,\frac{d\gamma}{\gamma}.$$

Thanks to Lemma 2, we deduce that

$$I(\zeta,u,r) \leq \left|\mathrm{M}_{r2^j}u\cdot\zeta\right|^{-\frac{\delta}{\nu}}$$

When the above is combined with the trivial estimate $|I(\zeta,u,r)| \leq 1$ and by using the fact $(t/\log^{\alpha} t)$ is increasing over the interval $(2^{\alpha},\infty)$, we have that

$$|I(\zeta,u,r)| \leq C\frac{\log^{\alpha+1}(|\eta\cdot u|^{-1})}{\log^{\alpha+1}(|\mathrm{M}_{2^j r}\zeta|)} \quad if\ |\mathrm{M}_{2^j r}\zeta| > 1, \tag{9}$$

where $\eta = \frac{\mathrm{M}_{2^j r}\zeta}{|\mathrm{M}_{2^j r}\zeta|}$. Thus, by (8)–(9), we obtain

$$\begin{aligned} |\hat{\lambda}_{r,j}(\zeta)| &\leq C\big(\log|\mathrm{M}_{2^j r}\zeta|\big)^{-\alpha-1}\int_{\mathbb{S}^{d-1}}|\Omega(u)|\log^{\alpha}(|\eta\cdot u|^{-1})d\sigma_d(u) \\ &\leq C\big(\log|\mathrm{M}_{2^j r}\zeta|\big)^{-\alpha-1} \quad if\ |\mathrm{M}_{2^j r}\zeta| > 1. \end{aligned} \tag{10}$$

Now, by condition (2), we obtain

$$\begin{aligned} |\hat{\lambda}_{r,j}(\zeta)| &\leq C\int_{\mathbb{S}^{d-1}}|\Omega(u)|\int_{1/2}^{1}|I(\zeta,u,r)-1|\frac{d\gamma}{\gamma}d\sigma_d(u) \\ &\leq C|\mathrm{M}_{2^j r}\zeta|. \end{aligned} \tag{11}$$

Consequently, by (10)–(11), we obtain (7). The proof of the lemma is complete. □

Now, we need to prove the following result.

Lemma 5. *Let* $\Omega \in \mathcal{F}_{\alpha}(\mathbb{S}^{d-1})$ *for some* $\alpha > 0$ *satisfy* (1)–(2). *Then, for any class of functions* $\{\mathcal{H}_j(\cdot),\ j\in\mathbb{Z}\}$ *defined on* $\mathbb{R}^d$, *we have*

$$\left\|\left(\sum_{j\in\mathbb{Z}}\int_{1}^{2}|\lambda_{r,j}*\mathcal{H}_j|^{\tau}\frac{dr}{r}\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^d)} \leq C\|\Omega\|_{L^1(\mathbb{S}^{d-1})}\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{H}_j|^{\tau}\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^d)} \tag{12}$$

for all $p\in(1,\infty)$ *and* $\tau\in(1,\infty)$.

Proof. As $p > 1$, by duality, a non-negative mapping $\varphi \in L^{p'}(\mathbb{R}^d)$ exists such that $\|\varphi\|_{L^{p'}(\mathbb{R}^d)} \leq 1$ and

$$\begin{aligned}\left\| \left(\sum_{j\in\mathbb{Z}} \int_1^2 |\lambda_{r,j} * \mathcal{H}_j| \frac{dr}{r} \right) \right\|_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \sum_{j\in\mathbb{Z}} \int_1^2 |\lambda_{r,j} * \mathcal{H}_j(w)| \frac{dr}{r} \varphi(w) dw \\ &\leq \int_{\mathbb{R}^d} \sum_{j\in\mathbb{Z}} |\mathcal{H}_j(w)| \lambda^*(\tilde{\varphi})(-w) dw, \end{aligned} \tag{13}$$

where $\tilde{\varphi}(w) = \varphi(-w)$. Thus, by Lemma 3, Hölder's inequality, and (13), we obtain that

$$\left\| \left(\sum_{j\in\mathbb{Z}} \int_1^2 |\lambda_{r,j} * \mathcal{H}_j| \frac{dr}{r} \right) \right\|_{L^p(\mathbb{R}^d)} \leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \left\| \sum_{j\in\mathbb{Z}} |\mathcal{H}_j| \right\|_{L^p(\mathbb{R}^d)} \tag{14}$$

for $1 < p < \infty$. On the other hand, by inequality (5), we obtain

$$\begin{aligned}\left\| \sup_{j\in\mathbb{Z}} \sup_{r\in[1,2]} |\lambda_{r,j} * \mathcal{H}_j| \right\|_{L^p(\mathbb{R}^d)} &\leq \left\| \lambda^* \left(\sup_{j\in\mathbb{Z}} |\mathcal{H}_j| \right) \right\|_{L^p(\mathbb{R}^d)} \\ &\leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \left\| \sup_{j\in\mathbb{Z}} |\mathcal{H}_j| \right\|_{L^p(\mathbb{R}^d)} \end{aligned} \tag{15}$$

for all $p \in (1, \infty)$. Therefore, by interpolating between (13) and (15), we have

$$\left\| \left(\sum_{j\in\mathbb{Z}} \int_1^2 |\lambda_{r,j} * \mathcal{H}_j|^\tau \frac{dr}{r} \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)} \leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{H}_j|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)}$$

for all $p, \tau \in (1, \infty)$. □

3. Proof of Theorem 1

Let us assume that $\mathcal{U} \in \mathcal{F}_\alpha(\mathbb{S}^{d-1})$ for some $\alpha > 0$. By Minkowski's inequality, we obtain

$$\begin{aligned}\mathcal{G}_{\mathcal{U}}^{(\tau)}(f)(x) &= \left(\int_{\mathbb{R}_+} \left| \sum_{j=0}^{\infty} \frac{1}{r} \int_{2^{-j-1}r < \gamma(v) \leq 2^{-j}r} f(x-v) \frac{\mathcal{U}(v)}{(\gamma(v))^{\kappa-1}} dv \right|^\tau \frac{d\gamma}{\gamma} \right)^{1/\tau} \\ &\leq \sum_{j=0}^{\infty} \left(\int_{\mathbb{R}_+} \left| \frac{1}{r} \int_{2^{-j-1}r < \gamma(v) \leq 2^{-j}r} f(x-v) \frac{\mathcal{U}(v)}{(\gamma(v))^{\kappa-1}} dv \right|^\tau \frac{d\gamma}{\gamma} \right)^{1/\tau} \\ &\leq C \left(\sum_{j=0}^{\infty} \int_1^2 |\lambda_{r,j} * f(x)|^\tau \frac{d\gamma}{\gamma} \right)^{1/\tau}. \end{aligned} \tag{16}$$

Let us choose a set of smooth mappings $\{\Lambda_j\}_{j\in\mathbb{Z}}$ on $\mathbb{R}_+$ satisfying the following:

$$\begin{aligned}\Lambda_j(w) = \Lambda_j(\gamma(w)), \quad \Lambda_j &\subset [0,1], \quad \sum_{j\in\mathbb{Z}} \Lambda_j(v)^2 = 1, \\ \text{supp}\,(\Lambda_j) &\subseteq [2^{-1-j}, 2^{1-j}], \text{ and } \left| \frac{d^m \Lambda_j(v)}{dv^m} \right| \leq \frac{C_m}{v^m}. \end{aligned}$$

For $\zeta \in \mathbb{R}^d$, we define the operators $\widehat{(\Psi_j}(\zeta)) = \Lambda_j(\gamma(\zeta))$ and $S_j(\zeta) = (\Psi_j * f)(\zeta)$. Hence, for any $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\left(\sum_{j=0}^{\infty}\int_1^2 |\lambda_{r,j} * f(x)|^\tau \frac{d\gamma}{\gamma}\right)^{1/\tau} \leq \left(\sum_{j\in\mathbb{Z}}\int_1^2 \left|\sum_{n\in\mathbb{Z}} S_{j+n}(\lambda_{r,j} * (S_{j+n}f))(x)\right|^\tau \frac{d\gamma}{\gamma}\right)^{1/\tau} := D(f)(x).$$

Thus, to prove our main result, it suffices to prove that

$$\|D(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{\dot{F}_p^{\vec{0},\tau}(\mathbb{R}^d)} \tag{17}$$

for all $p \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$ and $\tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$. We define the mapping T as

$$T : \{\mathcal{H}_{j,n}(x)\}_{n,j\in\mathbb{Z}} \longrightarrow \left\{\sum_{n=1}^{\infty} S_{j+n}(\mathcal{H}_{j,n})(x)\right\}_{j\in\mathbb{Z}}.$$

Then, it is easy to show the following:
(i) For $1 < p < \tau$ and $1 < s < p$,

$$\left\|\left(\sum_{j\in\mathbb{Z}}\int_1^2 \left|\sum_{n\in\mathbb{Z}} S_{j+n}(\mathcal{H}_{j,n})\right|^\tau \frac{d\gamma}{\gamma}\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^d)}$$

$$\leq C\left(\sum_{n\in\mathbb{Z}}\left\|\left(\sum_{j\in\mathbb{Z}}\int_1^2 |\mathcal{H}_{j,n}|^\tau \frac{d\gamma}{\gamma}\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^d)}^s\right)^{1/s}. \tag{18}$$

(ii) For $\tau < p < \infty$ and $1 < s < p'$,

$$\left\|\left(\sum_{j\in\mathbb{Z}}\int_1^2 \left|\sum_{n\in\mathbb{Z}} S_{j+n}(\mathcal{H}_{j,n})\right|^\tau \frac{d\gamma}{\gamma}\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^d)}$$

$$\leq C\left(\sum_{n\in\mathbb{Z}}\left(\int_1^2\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{H}_{j,n}|^\tau\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^d)}^\tau \frac{d\gamma}{\gamma}\right)^{s/\tau}\right)^{1/s}. \tag{19}$$

To prove (17), we need to consider three cases.

Case 1. $p \in (\frac{2+2\alpha}{1+2\alpha}, \tau)$, and $\tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$. By (18), we deduce that

$$\begin{aligned}\|D(f)\|_{L^p(\mathbb{R}^d)} \leq\ & C\left(\sum_{n\in\mathbb{Z}}\left\|\left(\sum_{j\in\mathbb{Z}}\int_1^2 |\lambda_{r,j} * (S_{j+n}f)|^\tau \frac{d\gamma}{\gamma}\right)^{1/\tau}\right\|_{L^p(\mathbb{R}^d)}^s\right)^{1/s} \\ \leq\ & C\left(\sum_{n\in\mathbb{Z}} \|\mathrm{A}_{n,\tau}f\|_{L^p(\mathbb{R}^d)}^s\right)^{1/s}\end{aligned} \tag{20}$$

for all $1 < s < p$, where

$$\mathrm{A}_{n,\tau}f = \left(\sum_{j\in\mathbb{Z}}\int_1^2 |\lambda_{r,j} * (S_{j+n}f)|^\tau \frac{d\gamma}{\gamma}\right)^{1/\tau}. \tag{21}$$

Let us estimate the L^p norm of $\mathrm{A}_{n,\tau}f$ whenever $p = \tau = 2$. For this case, we have that $\dot{F}_2^{\vec{0},\vec{2}}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. Thus, by Plancherel's theorem along with Fubini's theorem and Lemma 4, we obtain

$$\begin{aligned}
\|\mathrm{A}_{n,\tau}f\|^2_{L^2(\mathbb{R}^d)} &\leq \sum_{j\in\mathbb{Z}} \int_{\Omega_{n+j}} \left(\int_1^2 \left|\widehat{\lambda_{r,j}}(\zeta)\right|^2 \frac{d\gamma}{\gamma} \right) \left|\hat{f}(\zeta)\right|^2 d\zeta \\
&\leq C \sum_{j\in\mathbb{Z}} \int_{\Omega_{n+j}} \left|\hat{f}(\zeta)\right|^2 \min\left\{ |\mathrm{M}_{2^j r}\zeta|^2, \left(\log|\mathrm{M}_{2^j r}\zeta|\right)^{-2(\alpha+1)} \right\} d\zeta \\
&\leq C(1+|n|)^{-2(1+\alpha)} \|f\|^2_{L^2(\mathbb{R}^d)} \\
&= C(1+|n|)^{-2(1+\alpha)} \|f\|_{\dot{F}_2^{\vec{0},2}(\mathbb{R}^d)},
\end{aligned} \tag{22}$$

where $\Omega_n = \left\{ \zeta \in \mathbb{R}^d : |\zeta| \in [2^{-1-n}, 2^{1-n}] \right\}$.

It is clear that by Lemma 5,

$$\|\mathrm{A}_{n,\tau}f\|_{L^p(\mathbb{R}^d)} \leq C\|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \left\| \left(\sum_{j\in\mathbb{Z}} |\Psi_j * f|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)} \sim C\,\|f\|_{\dot{F}_p^{\vec{0},\tau}(\mathbb{R}^d)} \tag{23}$$

for all $1 < p, \tau < \infty$. Therefore, by interpolation of (22) with (23), there exists a constant $\theta \in (1/(1+\alpha), 1)$ such that for all $p \in (\frac{2+2\alpha}{1+2\alpha}, \tau)$,

$$\|\mathrm{A}_{n,\tau}f\|_{L^p(\mathbb{R}^d)} \leq C(1+|n|)^\theta \|f\|_{\dot{F}_p^{\vec{0},\tau}(\mathbb{R}^d)}. \tag{24}$$

For a fixed $p \in (\frac{2+2\alpha}{1+2\alpha}, \tau)$, let us choose $1 < s < p$ so that $s\theta > 1/(\alpha+1)$ and employ (20) along with (24); we confirm (17), which, by (16), proves our main result for the case $p \in (\frac{2+2\alpha}{1+2\alpha}, \tau)$ and $\tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$.

Case 2. $p \in (\tau, 2\alpha+2)$, and $\tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$. We construct the proof of Theorem 1 by following a similar argument as above, except we need to invoke (19) instead of (18). The details are omitted.

Case 3. $p = \tau$, and $\tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$. By the definition of $D(f)$ and Lemma 1, we have

$$\begin{aligned}
\|Df\|_{L^p(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} \sum_{j\in\mathbb{Z}}^{\infty} \int_1^2 \left| \sum_{n\in\mathbb{Z}} S_{j+n}\left(\lambda_{r,j} * (S_{j+n}f)\right)(x) \right|^\tau \frac{dr}{r} dx \right)^{1/\tau} \\
&\leq C\|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \left(\sum_{j\in\mathbb{Z}} \int_1^2 \left\| \mathcal{M}_P(|S_{j+n}f|) \right\|^\tau_{L^p(\mathbb{R}^d)} \frac{dr}{r} \right)^{1/\tau} \\
&\leq C\|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \left\| \left(\sum_{j\in\mathbb{Z}} |\Psi_j * f|^\tau \right)^{1/\tau} \right\|_{L^p(\mathbb{R}^d)} \\
&\leq C\|\mathcal{U}\|_{L^1(\mathbb{S}^{d-1})} \|f\|_{\dot{F}_p^{\vec{0},\tau}(\mathbb{R}^d)}.
\end{aligned}$$

Consequently, the proof of Theorem 1 is complete.

4. Conclusions

In this work, we introduced the generalized parabolic Marcinkiewicz operator $\mathcal{G}^{(\tau)}_{\mathcal{U}}$ under a very weak condition on the rough kernel $\mathcal{U}$. Whenever $\mathcal{U}$ belongs to the Grafakos–Stefanov class $\mathcal{F}_\alpha(\mathbb{S}^{d-1})$ for some $\alpha > 0$, we proved that the operator $\mathcal{G}^{(\tau)}_{\mathcal{U}}$

is bounded from homogeneous Triebel–Lizorkin space $\dot{F}_p^{0,\tau}(\mathbb{R}^d)$ to the $L^p(\mathbb{R}^d)$ space for all $p, \tau \in (\frac{2+2\alpha}{1+2\alpha}, 2\alpha+2)$. The results obtained in this paper improve and generalize a number of previously known results (see [2,8,15,17,31,34]).

Author Contributions: Formal analysis and writing—original draft preparation: M.A. and H.A.-Q. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data were used to support this study.

Acknowledgments: This work was conducted during the second author's sabbatical leave from Jordan University of Science and Technology at Abdullah Al-Salem University in Kuwait. We would like to expresses our gratitude to both universities for offering a conducive research environment.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Fabes, E.; Riviére, N. Singular integrals with mixed homogeneity. *Studia Math.* **1966**, *27*, 19–38. [CrossRef]
2. Stein, E. On the functions of Littlewood-Paley, Lusin and Marcinkiewicz. *Trans. Amer. Math. Soc.* **1958**, *88*, 430–466. [CrossRef]
3. Stein, E. Problems in harmonic analysis related to curvature and oscillatory integrals. In Proceedings of the International Congress of Mathematicians, Berkeley, CA, USA, 3–11 August 1986; Volume 1, pp. 196–221.
4. Stein, E. Some geometrical concepts arising in harmonic analysis. *Geom. Funct. Anal.* **2011**, *1*, 434–453.
5. Walsh, T. On the function of Marcinkiewicz. *Studia Math.* **1972**, *44*, 203–217. [CrossRef]
6. Al-Salman, A.; Al-Qassem, H.; Cheng, L.; Pan, Y. L^p bounds for the function of Marcinkiewicz. *Math. Res. Lett.* **2002**, *9*, 697–700.
7. Al-Qassem, H.; Al-Salman, A. A note on Marcinkiewicz integral operators. *J. Math. Anal. Appl.* **2003**, *282*, 698–710. [CrossRef]
8. Chen, J.; Fan, D.; Pan, Y. A note on a Marcinkiewicz integral operator. *Math. Nachr.* **2001**, *227*, 33–42. [CrossRef]
9. Hörmander, L. Estimates for translation invariant operators in L^p space. *Acta Math.* **1960**, *104*, 93–139. [CrossRef]
10. Ding, Y.; Lu, S.; Yabuta, K. A problem on rough parametric Marcinkiewicz functions. *J. Aust. Math. Soc.* **2002**, *72*, 13–21. [CrossRef]
11. Sakamota, M.; Yabuta, K. Boundedness of Marcinkiewicz functions. *Studia Math.* **1999**, *135*, 103–142.
12. Torchinsky, A.; Wang, S. A note on the Marcinkiewicz integral. *Colloq. Math.* **1990**, *61*, 235–243. [CrossRef]
13. Ali, M. L^p Estimates for Marcinkiewicz Integral Operators and Extrapolation. *J. Ineq. Appl.* **2014**, *1*, 269. [CrossRef]
14. Liu, F.; Fu, Z.; Jhang, S. Boundedness and continuity of Marcinkiewicz integrals associated to homogeneous mappings on Triebel-Lizorkin spaces. *Front. Math. China* **2019**, *14*, 95–122. [CrossRef]
15. Xue, Q.; Ding, Y.; Yabuta, K. Parabolic Littlewood-Paley g-function with rough kernels. *Acta Math. Sin. (Engl. Ser.)* **2008**, *24*, 2049–2060. [CrossRef]
16. Chen, Y.; Ding, Y. L^p Bounds for the parabolic Marcinkiewicz integral with rough kernels. *J. Korean Math. Soc.* **2007**, *44*, 733–745. [CrossRef]
17. Al-Salman, A. A note on parabolic Marcinkiewicz integrals along surfaces. *Trans. A Razmadze Math. Inst.* **2010**, *154*, 21–36.
18. Wang, F.; Chen, Y.; Yu, W. L^p bounds for the parabolic Littlewood-Paley operator associated to surfaces of revolution. *Bull. Korean Math. Soc.* **2012**, *29*, 787–797. [CrossRef]
19. Liu, F. A note on Marcinkiewicz integral associated to surfaces of revolution. *J. Aust. Math. Soc.* **2018**, *104*, 380–402. [CrossRef]
20. Ali, M.; Katatbeh, Q. Generalized parabolic Marcinkiewicz integrals associated with polynomial compound curves with rough kernels. *Demon. Math.* **2020**, *53*, 44–57. [CrossRef]
21. Liu, F.; Mao, S. L^p bounds for nonisotropic Marcinkiewicz integrals associated to surfaces. *J. Aust. Math. Soc.* **2015**, *99*, 380–398. [CrossRef]
22. Zhang, J.; He, Q.; Xue, Q. On weighted boundedness and compactness of commutators of Marcinkiewicz integral associated with Schrödinger operators. *Ann. Func. Anal.* **2023**, *14*, 59. [CrossRef]
23. Gao, W.; Tang, L. BLO estimates for Marcinkiewicz integrals associated with Schrödinger operators. *Proc. Indian Acad. Sci. Math. Sci.* **2019**, *129*, 1–10. [CrossRef]
24. Gürbüz, F. A note concerning Marcinkiewicz integral with rough kernel. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **2021**, *24*, 53. [CrossRef]
25. Khalil, O.; Tao, S.; Bechi, A. Boundedness of Marcinkiewicz integral with rough kernel and their commutator on weighted Herz space with variable exponent. *Acta Math. Univ. Comen.* **2023**, *2*, 145–163.
26. Jia, H.; Yang, D.; Yuan, W.; Zhang, Y. Estimates for Littlewood–Paley operators on ball Campanato-type function spaces. *Results Math.* **2023**, *78*, 1–37. [CrossRef]

27. Ayman-Mursaleen, M.; Nasiruzzaman, M.; Sharma, S.; Cai, Q. Invariant means and lacunary sequencespaces of order (α, β). *Demonstr. Math.* **2024**, *57*, 1–12.
28. Rao, N.; Ayman-Mursaleen, M.; Aslan, R. A note on a general sequence of λ-Szász Kantorovich type operators. *Comput. Appl. Math.* **2024**, *43*, 428. [CrossRef]
29. Ayman-Mursaleen, M.; Nasiruzzaman, M.; Rao, N.; Dilshad, M.; Nisar, K. Approximation by the modified λ-Bernstein-polynomial in terms of basis function. *AIMS Math.* **2024**, *9*, 4409–4426. [CrossRef]
30. Ayman-Mursaleen, M.; Lamichhane, B.; Kilicman, A.; Senu, N. On q-statistical approximation of wavelets aided Kantorovich q-Baskakov operators. *FILOMAT* **2024**, *38*, 3261–3274.
31. Chen, J.; Fan, D.; Ying, Y. Singular integral operators on function spaces. *J. Math. Anal. Appl.* **2002**, *276*, 691–708. [CrossRef]
32. Le, H. Singular integrals with mixed homogeneity in Triebel-Lizorkin spaces. *J. Math. Anal. Appl.* **2008**, *345*, 903–916. [CrossRef]
33. Al-Qassem, H.; Cheng, L.; Pan, Y. On rough generalized parametric Marcinkiewicz integrals. *J. Math. Ineq.* **2017**, *11*, 763–780. [CrossRef]
34. Al-Qassem, H.; Cheng, L.; Pan, Y. Boundedness of rough integral operators on Triebel-Lizorkin space. *Publ. Mat.* **2012**, *56*, 261–277. [CrossRef]
35. Fan, D.; Wu, H. On the generalized Marcinkiewicz integral operators with rough kernels. *Can. Math. Bull.* **2011**, *54*, 100–112. [CrossRef]
36. Ali, M.; Al-Mohammed, O. Boundedness of a class of rough maximal functions. *J. Ineq. Appl.* **2018**, *305*, 1900. [CrossRef]
37. Liu, F. A note on generalized parametric Marcinkiewicz integrals. *Bull. Korean Math. Soc.* **2019**, *56*, 1099–1115.
38. Ali, M.; Al-Qassem, H. Estimates for certain class of rough generalized Marcinkiewicz functions along submanifolds. *Open Math.* **2023**, *21*, 603. [CrossRef]
39. Grafakos, L.; Stefanov, A. L^p bounds for singular integrals and maximal singular integrals with rough kernel. *Indiana Univ. Math. J.* **1998**, *47*, 455–469. [CrossRef]
40. Triebel, H. *Theory of Function Spaces*, 1st ed.; Birkhäuser: Basel, Switzerland, 1983; pp. 1–285.
41. Ricci, R.; Stein, E. Harmonic analysis on nilpotent groups and singular integrals. I. Oscillatory integrals. *J. Funct. Anal.* **1987**, *73*, 179–194. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.