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Bounds for Hermitian Toeplitz and Hankel Determinants for a Certain Subclass of Analytic Functions Related to the Sine Function

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Abstract: This study of Hankel and Hermitian Toeplitz determinants is one of the major areas of interest in Geometric function theory and has wide applications in the areas of signal processing and Applied Mathematics. In our present investigations, we define a new subclass of normalized analytic functions $\mathfrak{H}(\lambda)$ $(\lambda \geq 0)$, defined using a subordination relation with the sine function $\mathcal{K}(z) = 1 + sin\ z$. For the class $\mathfrak{H}(\lambda)$, coefficient estimates, upper and lower bounds for the Hermitian Toeplitz determinants of second and third order are found. In addition, estimates are provided for the second and third-order Hankel determinants for the class $\mathfrak{H}(\lambda)$.

Keywords: analytic functions; sine function; Hermitian Toeplitz determinant; Hankel determinant

MSC: 30C45; 30C50; 30C80

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1. Introduction

Let the class of normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be denoted by A and let S be the subclass of A consisting of univalent functions.

The class of Caratheodory functions denoted by P consists of analytic functions $p(z)$ given by

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \tag{2}$$

where $\mathrm{Re}(p(z)) > 0$ for $z \in \mathbb{D}$ and $p(0) = 1$.

An analytic function f is subordinate to another analytic function g, denoted by $f \prec g$, if there exists an analytic function ω with $|\omega(z)| < |z|$ and $\omega(0) = 0$ such that $f(z) = g(\omega(z))$. If g is univalent then $f(z) \prec g(z) \iff f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

Let $\phi(z)$ be an analytic function with a positive real part on $\mathbb{D}$ with $\phi(0) = 1, \phi'(0) > 0$, which maps the unit disk $\mathbb{D}$ onto a starlike region with respect to 1 and is symmetric with respect to the real axis. In 1992, Ma and Minda [1] defined the classes

$$S^*_\phi = \left\{ f \in A \ : \ \frac{zf'(z)}{f(z)} \prec \phi(z) \ (z \in \mathbb{D}) \right\}$$

and

$$C_\phi = \left\{ f \in A \ : \ 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \ (z \in \mathbb{D}) \right\}.$$

The above functions are called as functions of Ma and Minda type. It is noted that $S^*\left(\frac{1+z}{1-z}\right) = S^*$ and $C\left(\frac{1+z}{1-z}\right) = C$, i.e., the well-known classes of starlike and convex functions.

The class $M_\lambda(\phi)$ of normalized analytic functions $f(z)$ satisfying

$$\frac{zf'(z)}{f(z)} + \lambda \frac{z^2 f''(z)}{f(z)} \prec \phi(z) \ (\lambda \geq 0)$$

was considered by Darus et al. [2], who investigated the bounds of the Fekete–Szegö functional $|a_3 - \mu a_2^2|$, where μ, is real.

For a function $f \in$ A given by (1), Noonan and Thomas [3] defined the qth Hankel determinant as

$$H^n_q(f) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \dots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix} \tag{3}$$

where $a_1 = 1, \ q, \ n \in \mathbb{N} := \{1, 2, 3 \dots\}$.

In particular,

$$H^1_2(f) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2,$$

$$H^2_2(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2$$

and

$$H^1_3(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$
$$= a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2).$$

The study of coefficient problems is indispensable in the area of univalent function theory. In 1960, Pommerenke [4] started researching Hankel determinants for functions that are starlike and univalent. The optimal bound for $|H^n_2(f)|$ of areally mean p-valent functions was computed by Hayman in [5]. Janteng et al. [6] determined the precise upper limit for the second Hankel determinant for classes of starlike and convex functions. Subsequently, for the class of convex functions C, the authors of [7] estimated the bound for the third Hankel determinant as $|H^1_3(f)| \leq \frac{4}{135}$ and established the sharpness of the bound by providing an extremal function $f(z) = arctanz$ for which the equality holds,

and Arif [8] obtained a non-sharp bound for the fourth Hankel determinant for functions with bounded turning. The authors of [9] used the concept of subordination to improve an existing bound of the third Hankel determinant for f, satisfying $\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}$. Further, Vijayalakshmi et al. [10] investigated the bounds of the third-order Hankel determinant for a generalized class of analytic functions, which reduces to a class of functions with bounded turning. Furthermore, Cho and Kumar determined the bound for $|a_5|$ and computed a non-sharp bound on the third and fourth Hankel determinants for starlike functions associated with a lune-shaped region in [11]. Recently, Breaz et al. [12] have obtained the upper bound for the third Hankel determinant for functions f involving the quantities $\frac{zf'(z)}{f(z)}$, $\left|log\frac{zf'(z)}{f(z)}\right|$ and $\frac{(zf'(z))'}{f'(z)}$ associated with the exponential function e^z.

The Hermitian Toeplitz determinant, a special case of the Hankel determinant defined by the coefficients $\{a_k\}_{k\geq 2}$ of the function $f \in A$ given by (1) is described by Hartman and Winter [13] as follows:

$$\det(T_{m,n}(f)) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ \overline{a_{n+1}} & a_n & \cdots & a_{n+m-2} \\ \vdots & \vdots & \cdots & \vdots \\ \overline{a_{n+m-1}} & \overline{a_{n+m-2}} & \cdots & a_n \end{vmatrix} \tag{4}$$

where $a_1 = 1$, m, $n \in \mathbb{N}$.

In particular,

$$\det T_{2,1}(f) = \begin{vmatrix} 1 & a_2 \\ \overline{a_2} & 1 \end{vmatrix} = 1 - |a_2|^2,$$

and

$$\det(T_{3,1}(f)) = \begin{vmatrix} 1 & a_2 & a_3 \\ \overline{a_2} & 1 & a_2 \\ \overline{a_3} & \overline{a_2} & 1 \end{vmatrix} = 2Re(a_2^2\overline{a_3}) - 2|a_2|^2 - |a_3|^2 + 1.$$

Hermitian Toeplitz matrices and their determinants find extensive applications in various fields. In signal processing, Hermitian Toeplitz matrices serve as autocorrelation matrices and their determinants are useful to study statistical properties like power spectra and the presence of periodic components [14], and they are used in linear prediction theory to model signals and estimate their parameters. Furthermore, Hermitian Toeplitz and Hankel matrices are used for efficient data transmission and reception in aircraft communication [15].

This study of Hankel and Hermitian Toeplitz in the area of univalent function theory has gained momentum in recent years. For second and third-order Toeplitz determinants whose components are Taylor's coefficients of functions f in S, the sharp upper bounds were estimated by Ali et al. [16] for subclasses of starlike and convex functions. Cudna et al. [17] estimated the sharp lower and upper bounds for second and third-order Hermitian Toeplitz determinants for the classes of starlike and convex functions of order α. The bounds for $|a_n|$, $n = 2, 3, \cdots$ for the classes of starlike and convex functions related to a shell-like curve domain connected with Fibonacci numbers are estimated in [18,19]. Kumar et al. [20] obtained the bounds for $|a_n|$, $n = 2, 3, 4, 5$ for the classes of Janowski-type starlike functions and estimated the sharp lower and upper bounds on second and third-order Hermitian Toeplitz determinants, generalizing the results of [17]. Later, Lecko et. al [21] obtained bounds for second- and third-order Hermitian Toeplitz determinants for certain subclasses of analytic functions. Recently, by using the concept of subordination, Gurusamy et al. [22]

estimated the bounds of second and third-order Hermitian Toeplitz determinants for some subclasses of analytic functions associated with exponential functions.

Consider the function $\mathcal{K}(z) = 1 + sin\ z$, which is analytic and univalent in $\mathbb{D}$. The image of the unit disc under this function is found with the aid of MATLAB R2021a as a sine curve region on the right half plane and is given in Figure 1.

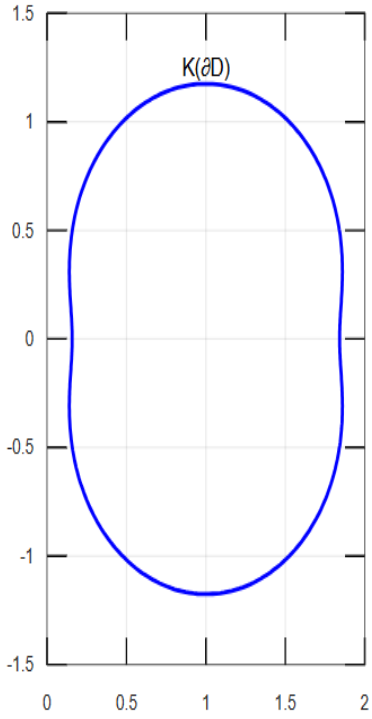


Figure 1. The image of $\mathcal{K}(\partial\mathbb{D})$.

Further , it is noted that the range $\mathcal{K}(\mathbb{D})$ is symmetric with respect to the real axis and $Re(\mathcal{K}(\mathbb{D})) > 0$, for $z \in \mathbb{D}$ with $\mathcal{K}(0) = 1$, $\mathcal{K}'(0) > 0$. Moreover, the domain is starlike with respect to the point $\mathcal{K}(0) = 1$.

In [23], Cho et al. introduced the class S_s^* of univalent functions, which is defined such that the quantity $\frac{zf'(z)}{f(z)}$ is subordinate to the function $\mathcal{K}(z) = 1 + sin\ z$, and they established the radii of starlikeness and convexity for functions in S_s^*. This work propelled the study on the sine curve domain and has gained interest in recent years. Briefly, Arif et al. [24] estimated the bounds of the third Hankel determinant for functions in classes S_s^*, $C^* := \{f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec 1 + sin\ z\ (z \in \mathbb{D})\}$ and $R_{sin} := \{f \in A : f'(z) \prec 1 + sin\ z\ (z \in \mathbb{D})\}$. For f in S_s^*, Zhang and Tang [25] obtained the bounds for the coefficients a_k, $k = 2$ to 7 and established the following: $|H_2^1(f)| \leq \frac{1}{2}$, $|H_2^2(f)| \leq \frac{1}{4}$, $|a_2a_3 - a_4| \leq \frac{1}{3}$, $|a_2a_5 - a_3a_4| \leq \frac{11}{56}$, $|a_5 - a_2a_4| \leq \frac{13}{32}$, $|a_5a_3 - a_4^2| \leq \frac{97}{324}$ and $|H_4^1(f)| \leq 0.81945$. The fourth Hankel determinant for the class C^* was obtained by Zulfiqar et al. [26]. For the class of starlike functions with respect to symmetric points related to the sine function, Ghaffar et al. [27] estimated the upper bounds on the third Hankel determinant. Further, Bilal Khan et al. [28] gave the bounds for the third Hankel determinant for the class S_s^* using logarithmic coefficients. The third Hankel determinant for subclasses of analytic and m-fold symmetric functions involving the sine domain was given by Alahmade et al. [29]. The upper bounds for Hermitian Toeplitz determinants of second and third order for $S^*(\phi)$ and $C(\phi)$ were obtained by Surya Giri and Sivaprasad Kumar [30], who further deduced the results for different choices of ϕ. Recently, Kamali and Alina [31] obtained the bounds for the second and third-order Hankel determinants for functions in S_s^* and C^* that are associated with a differential operator.

Motivated by the class studied by Darus et al. [2] and the aforementioned works, in this paper, we estimate the coefficient bounds, upper and lower bounds for Hermitian Toeplitz determinants, and estimates for second and third-order Hankel determinants for a class of analytic functions related to a sine curve region, using the technique of subordination.

2. Preliminaries

In this section, we provide the definition and Lemmas that are required to establish the main results in the subsequent sections.

Definition 1. *For $\lambda \geq 0$, let $\mathfrak{H}(\lambda)$ denote the class of functions $f \in A$, which satisfies the subordination relation*

$$\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)} \prec 1 + sin\, z,\ z \in \mathbb{D}.$$

In order to prove that class $\mathfrak{H}(\lambda)$ is non-empty, we provide the following example:

Example 1. *Let $f : \mathbb{D} \to \mathbb{C}$ be defined by*

$$f(z) = z + az^2,\ 0 \leq a \leq \frac{1}{10^{\lambda+1}},\ \lambda \geq 0,\ f \in \mathfrak{H}(\lambda).$$

Therefore, for $z \in \mathbb{D}$,

$$\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)} = \frac{1 + 2a(\lambda+1)z}{1 + az} := g_\lambda(z).$$

For $0 \leq a \leq \frac{1}{10^{\lambda+1}}$, we see that g_λ maps the unit disc onto the open disc that is symmetric about the real axis. The image is obtained using MATLAB R2021a SOFTWARE. The region inside the red colour curve denotes the image of unit disk $\mathbb{D}$ under the mapping $\mathcal{K}(z) = 1 + sin\, z$ and the region inside the green colour curve denotes the image of unit disk $\mathbb{D}$ under the mapping $g_\lambda(z)$. The images of unit disk under $\mathcal{K}(z)$ and $g_\lambda(z)$ for different values of λ and a in the range specified are given in Figures 2–4.

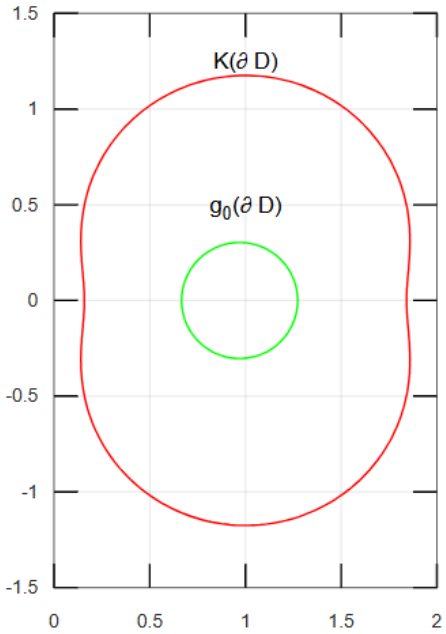


Figure 2. $\lambda = 0$, a = $\frac{1}{10}$.

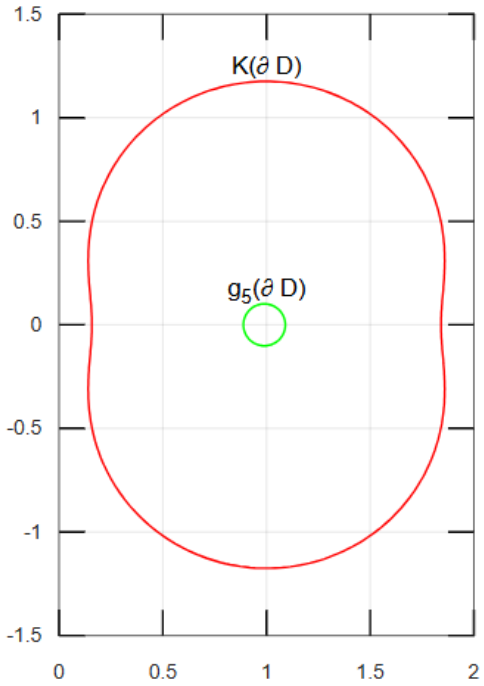


Figure 3. $\lambda = 5$, a = $\frac{1}{10^6}$.

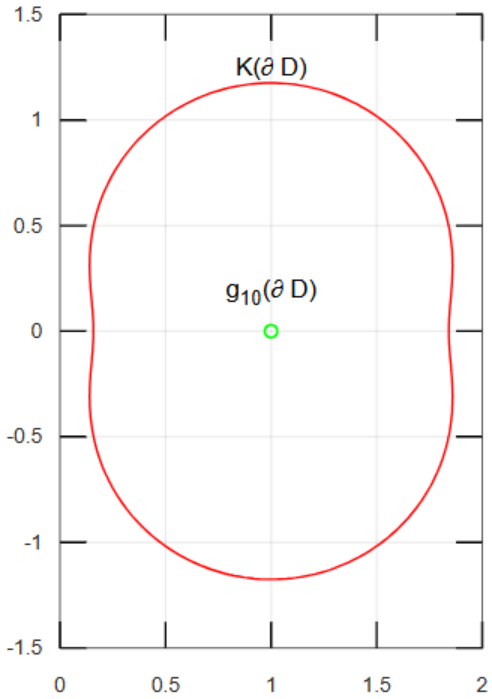


Figure 4. $\lambda = 10$, $a = \frac{1}{10^{11}}$.

Remark 1. *For $\lambda = 0$, the class $\mathfrak{H}(\lambda)$ reduces to the class $\mathbb{S}_s^*$ studied by Zhang and Tang [25].*

Lemma 1 ([32]). *If $p \in P$ is of the form (2) with $Re(p(z)) > 0$ $(z \in \mathbb{D})$, then*

$$(i)\ 2p_2 = p_1^2 + (4 - p_1^2)x, \tag{5}$$

$$(ii)\ 4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)z, \tag{6}$$

for some complex number x, z satisfying $|x| \leq 1$ and $|z| \leq 1$.

Lemma 2 ([33]). *If $p \in P$, then for all $m, n \in \mathbb{N}$,*

$$|\mu p_n p_m - p_{m+n}| \leq \begin{cases} 2, & 0 \leq \mu \leq 1 \\ 2|2\mu - 1|, & \text{elsewhere} \end{cases}. \tag{7}$$

Lemma 3 ([34]). *If $p \in P$, then*

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1^2|}{2}. \tag{8}$$

Finally, we recall that if $f \in A$ and $g(z) = e^{-i\theta} f(e^{i\theta} z)$, $\theta \in \mathbb{R}$ then, $g \in A$. It can be easily seen that $\mathfrak{H}(\lambda)$ and the functionals $|a_4 - a_2 a_3|$, $\det(T_{(2,1)} f)$, $\det(T_{(3,1)}(f))$, $|H_2^1(f)|$, $|H_2^2(f)|$ and $|H_3^1(f)|$ are rotationally invariant. Due to this condition, without loss of generality, we may assume that the second coefficient, namely p_1, is positive and $p_1 \in [0, 2]$ is used in all considerations.

3. Coefficient Estimates

In the following result, we find the coefficient bounds for $f \in \mathfrak{H}(\lambda)$.

Theorem 1. *If f of the form (1) is in $\mathfrak{H}(\lambda)$, then, for $\lambda \geq 0$,*

$$|a_2| \leq \frac{1}{1 + 2\lambda},$$

$$|a_3| \leq \frac{1}{2(1 + 3\lambda)},$$

$$|a_4| \leq \begin{cases} \frac{1}{3(1+4\lambda)} \left[1 + \frac{102\lambda^2 + 37\lambda + 2}{18\sqrt{3}(1+2\lambda)(1+3\lambda)} \right] & \text{if } \lambda \in [0, \frac{5}{8}] \\ \frac{1}{3(1+4\lambda)} \left[1 + \frac{54\lambda^2 + 13\lambda}{6(1+2\lambda)(1+3\lambda)} \right] & \text{if } \lambda > \frac{5}{8} \end{cases},$$

$$|a_5| \leq \frac{3(1 + 4\lambda)}{8(1 + 3\lambda)(1 + 5\lambda)}.$$

The first two bounds are the best possible.

Proof. Let the function $f \in \mathfrak{H}(\lambda)$. Then, for some $p \in P$, we can write

$$\frac{z f'(z) + \lambda z^2 f''(z)}{f(z)} = 1 + sin\left(\frac{p(z) - 1}{p(z) + 1} \right).$$

From the above equality, we have

$$z + \sum_{n=2}^{\infty} (1 + (n-1)\lambda) n a_n z^n$$
$$= z + \left(\frac{p_1}{2} + a_2 \right) z^2 + \left(\left(\frac{p_2}{2} - \frac{p_1^2}{4} \right) + \frac{p_1}{2} a_2 + a_3 \right) z^3 + \left(\left(\frac{p_3}{2} + \frac{5p_1^3}{48} - \frac{p_1 p_2}{2} \right) + \right.$$

$$\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)a_2+\frac{p_1}{2}a_3+a_4\Big)z^4+\left(\left(\frac{p_4}{2}-\frac{p_1p_3}{2}+\frac{5p_1^2p_2}{16}-\frac{p_2^2}{4}-\frac{p_1^4}{32}\right)\right.$$
$$\left.+\left(\frac{p_3}{2}+\frac{5p_1^3}{48}-\frac{p_1p_2}{2}\right)a_2+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)a_3+\frac{p_1}{2}a_4+a_5\right)z^5+\cdots.$$

Equating the coefficients of z^n on both sides, we obtain

$$a_2=\frac{p_1}{2(1+2\lambda)}, \tag{9}$$

$$a_3=\frac{1}{2(1+3\lambda)}\left[\frac{p_2}{2}-\frac{\lambda p_1^2}{2(1+2\lambda)}\right], \tag{10}$$

$$a_4=\frac{1}{3(1+4\lambda)}\left[\frac{p_3}{2}-\frac{p_1p_2(24\lambda^2+12\lambda+1)}{8(1+2\lambda)(1+3\lambda)}+\frac{p_1^3(30\lambda^2+\lambda-1)}{48(1+2\lambda)(1+3\lambda)}\right], \tag{11}$$

$$a_5=\frac{1}{4(1+5\lambda)}\left[\frac{p_4}{2}-\frac{p_1p_3(24\lambda^2+11\lambda+1)}{8(1+4\lambda)(1+2\lambda)}-\frac{p_2^2(1+6\lambda)}{8(1+3\lambda)}\right.$$
$$\left.-\frac{p_1^2p_2(360\lambda^3+180\lambda^2+18\lambda+1)}{48(1+4\lambda)(1+2\lambda)(1+3\lambda)}-\frac{p_1^4(216\lambda^3-48\lambda^2-43\lambda-5)}{288(1+4\lambda)(1+2\lambda)(1+3\lambda)}\right]. \tag{12}$$

Since $|p_n|\leq 2$ (see [35]) from (9), we have

$$|a_2|\leq\frac{1}{1+2\lambda}. \tag{13}$$

Applying Lemma 2 to (10), we obtain

$$|a_3|\leq\frac{1}{2(1+3\lambda)}. \tag{14}$$

Now, by rearranging the terms and making use of triangle inequality, (11) becomes

$$|a_4|\leq\frac{1}{3(1+4\lambda)}\left[\frac{1}{2}\left|\frac{(24\lambda^2+12\lambda+1)p_1p_2}{3(1+2\lambda)(1+3\lambda)}-p_3\right|+\right.$$
$$\left.\left|\frac{p_1p_2(24\lambda^2+12\lambda+1)}{24(1+3\lambda)(1+2\lambda)}+\frac{p_1^3(30\lambda^2+\lambda-1)}{48(1+2\lambda)(1+3\lambda)}\right|\right].$$

In virtue of Lemma 2, the above reduces to

$$|a_4|\leq\frac{1}{3(1+4\lambda)}\left[1+\left|\frac{p_1p_2(24\lambda^2+12\lambda+1)}{24(1+3\lambda)(1+2\lambda)}+\frac{p_1^3(30\lambda^2+\lambda-1)}{48(1+2\lambda)(1+3\lambda)}\right|\right].$$

Applying Lemma 1 and Lemma 3, and letting $p_1=p,\ p\in[0,2]$, we have

$$|a_4|\leq\frac{1}{3(1+4\lambda)}\left[1+\frac{p(24\lambda^2+12\lambda+1)}{12(1+3\lambda)(1+2\lambda)}+\frac{p^3(30\lambda^2+\lambda-1)}{48(1+2\lambda)(1+3\lambda)}\right]:=F_1(p,\lambda).$$

Now,

$$\frac{\partial F_1}{\partial p}=\frac{1}{3(1+4\lambda)}\left[\frac{(24\lambda^2+12\lambda+1)}{12(1+3\lambda)(1+2\lambda)}+\frac{p^2(30\lambda^2+\lambda-1)}{16(1+2\lambda)(1+3\lambda)}\right].$$

For $\lambda \in [0, \frac{5}{8}]$, we see that $\frac{\partial F_1}{\partial p} > 0$ for $p \in [0, \frac{2}{\sqrt{3}})$ and $\frac{\partial F_1}{\partial p} < 0$ for $p \in (\frac{2}{\sqrt{3}}, 2]$. This implies that $F_1(p, \lambda)$ increases in $[0, \frac{2}{\sqrt{3}})$, decreases in $(\frac{2}{\sqrt{3}}, 2]$ and the maximum value of $F_1(p, \lambda)$ occurs at $p = \frac{2}{\sqrt{3}}$.

Therefore,

$$|a_4| \leq F_1(\frac{2}{\sqrt{3}}, \lambda) = \frac{1}{3(1+4\lambda)}\left[1 + \frac{102\lambda^2 + 37\lambda + 2}{18\sqrt{3}(1+2\lambda)(1+3\lambda)}\right]. \tag{15}$$

For $\lambda > \frac{5}{8}$, we see that $\frac{\partial F_1}{\partial p} > 0$ for $p \in [0, 2]$ and the maximum value of $F_1(p, \lambda)$ occurs at $p = 2$.

Therefore,

$$|a_4| \leq F_1(2, \lambda) = \frac{1}{3(1+4\lambda)}\left[1 + \frac{54\lambda^2 + 13\lambda}{6(1+2\lambda)(1+3\lambda)}\right]. \tag{16}$$

After a suitable rearrangement of terms in (12), applying the triangle inequality followed by Lemma 2, we have

$$|a_5| \leq \frac{1}{4(1+5\lambda)}\left[1 + \left|\frac{p_1^4(216\lambda^3 - 48\lambda^2 - 43\lambda - 5)}{288(1+4\lambda)(1+2\lambda)(1+3\lambda)} - \frac{p_2^2(1+6\lambda)}{8(1+3\lambda)} - \frac{p_1^2 p_2(360\lambda^3 + 180\lambda^2 + 18\lambda + 1)}{48(1+4\lambda)(1+2\lambda)(1+3\lambda)}\right|\right].$$

By making use of Lemma 1 and letting $p_1 = p$, $p \in [0, 2]$, we obtain

$$|a_5| \leq \frac{1}{4(1+5\lambda)}\left[1 + \left|\frac{p^4(216\lambda^3 - 48\lambda^2 - 43\lambda - 5)}{288(1+4\lambda)(1+2\lambda)(1+3\lambda)} - \frac{p^2(360\lambda^3 + 180\lambda^2 + 18\lambda + 1)}{24(1+4\lambda)(1+2\lambda)(1+3\lambda)} - \frac{(1+6\lambda)}{2(1+3\lambda)}\right|\right] := \frac{1}{4(1+5\lambda)}(1 + |F_2(p, \lambda)|).$$

By the second derivative test, we see that $F_2(p, \lambda)$ attains the maximum at $p = 0$.

Thus,

$$|a_5| \leq \frac{3(1+4\lambda)}{8(1+3\lambda)(1+5\lambda)}. \tag{17}$$

The equality for the bounds given in (13) and (14) is obtained by taking

$$\frac{zf_n'(z) + \lambda z^2 f_n''(z)}{f_n(z)} = 1 + sin\Big((\omega(z))^{n-1}\Big).$$

For the given functions

$$f_2(z) = z + \frac{z^2}{1+2\lambda},\ n = 2,$$

$$f_3(z) = z + \frac{z^3}{2(1+3\lambda)},\ n = 3.$$

The proof of our theorem is complete. □

For the choice of $\lambda = 0$, the above theorem reduces to the results of Zhang and Tang [25], as given in the following corollary.

Corollary 1 ([25]). *If f of the form (1) is in $\mathbb{S}_s^*$, then $|a_2| \leq 1$, $|a_3| \leq \frac{1}{2}$, $|a_4| \leq 0.344$, $|a_5| \leq \frac{3}{8}$.*

4. Hermitian Toeplitz Determinant

In this section, the upper and lower bounds for $\det(T_{(2,1)}(f))$ and $\det(T_{(3,1)}(f))$ for $f \in \mathfrak{H}(\lambda)$ are estimated.

Theorem 2. *If f of the form (1) is in $\mathfrak{H}(\lambda)$, then, for $\lambda \geq 0$,*

$$1 - \frac{1}{(1+2\lambda)^2} \leq \det(T_{(2,1)}(f)) \leq 1.$$

The bounds are sharp.

Proof. Consider $\det T_{2,1}(f) = 1 - |a_2|^2$.

From (9), we have

$$\det T_{2,1}(f) = 1 - \left| \frac{p_1^2}{4(1+2\lambda)^2} \right|.$$

Since $|p_1| \leq 2$ (see in [35]), it is observed that

$$1 - \left| \frac{p_1^2}{4(1+2\lambda)^2} \right| \leq 1 \text{ and } 1 - \left| \frac{p_1^2}{4(1+2\lambda)^2} \right| \geq 1 - \frac{1}{(1+2\lambda)^2}.$$

Therefore,

$$1 - \frac{1}{(1+2\lambda)^2} \leq \det T_{2,1}(f) \leq 1, \tag{18}$$

which gives the desired result.

The upper bound is sharp for $f_0(z) = z$, and the lower bound is sharp for $f_1(z) = z + \frac{1}{1+2\lambda} z^2$. □

For $\lambda = 0$ we have the following corollary due to Surya Giri and Sivaprasad Kumar [30].

Corollary 2 ([30]). *If f of the form (1) is in $\mathbb{S}_s^*$, then $0 \leq \det(T_{(2,1)}(f)) \leq 1$.*

Theorem 3. *If f of the form (1) is in $\mathfrak{H}(\lambda)$, then, for $\lambda \geq 0$,*

$$\sigma_1(\lambda) \leq \det(T_{3,1}(f)) \leq 1,$$

where

$$\sigma_1(\lambda) = 1 - \left(\frac{48\lambda^2 + 30\lambda + 5\lambda + 5}{4(1+2\lambda)^3(1+3\lambda)^2} \right).$$

Proof. Consider

$$\det(T_{3,1}(f)) = 2Re(a_2^2 \overline{a_3}) - 2|a_2|^2 - |a_3|^2 + 1. \tag{19}$$

Using (9), (10), and Lemma 1,

$$2Re(a_2^2 \overline{a_3}) = \frac{1}{16(1+2\lambda)^2(1+3\lambda)} \left[\frac{p_1^4}{1+2\lambda} + p_1^2(4 - p_1^2) Re(\overline{x}) \right]. \tag{20}$$

Now, from (9),

$$2|a_2|^2 = \frac{p_1^2}{2(1+2\lambda)^2}. \tag{21}$$

Further, using (10) and Lemma 1,

$$|a_3|^2 = \frac{1}{16(1+3\lambda)^2}\left[\frac{p_1^4}{4(1+2\lambda)^2} + \frac{(4-p_1)^2|x|^2}{4} + \frac{2(Re(\overline{x}))(4-p_1^2)p_1^2}{4} - \frac{\lambda p_1^2(4-p_1^2)|x|}{(1+2\lambda)}\right]. \tag{22}$$

Substituting (20)–(22) in (19), we obtain

$$\begin{aligned}\det(T_{3,1}(f)) =& 1 + \frac{1}{16(1+2\lambda^2)(1+3\lambda)}\left[\frac{p_1^4}{1+2\lambda} + p_1^2(4-p_1)^2 Re(\overline{x})\right] - \frac{p_1^2}{2(1+2\lambda)^2} \\ &- \frac{1}{16(1+3\lambda)^2}\left[\frac{p_1^4}{4(1+2\lambda)^2} + \frac{(4-p_1)^2|x|^2}{4} + \frac{2(Re(\overline{x}))(4-p_1^2)p_1^2}{4} \right. \\ &\left. - \frac{\lambda p_1^2(4-p_1^2)|x|}{(1+2\lambda)}\right] := F(p_1^2, |x|, Re(\overline{x}), \lambda).\end{aligned} \tag{23}$$

Since $|Re(\overline{x})| \leq |x|$, from (22) we obtain

$$\begin{aligned}&\det(T_{3,1}(f)) \leq F(p_1^2, |x|, |x|, \lambda) \text{ and} \\ &\det(T_{3,1}(f)) \geq F(p_1^2, |x|, -|x|, \lambda).\end{aligned}$$

Letting $p_1^2 := t \in [0,4]$ and $|x| := y \in [0,1]$, we have

$$\begin{aligned}\det(T_{3,1}(f)) \leq& 1 + \frac{1}{16(1+2\lambda)^2(1+3\lambda)}\left[\frac{t^2}{1+2\lambda} + t(4-t)y\right] - \frac{t}{2(1+2\lambda)^2} \\ &- \frac{1}{16(1+3\lambda)^2}\left[\frac{t^2}{4(1+2\lambda)^2} + \frac{(4-t)^2y^2}{4} + \frac{2yt(4-t)}{4} \right. \\ &\left. - \frac{\lambda t(4-t)y}{(1+2\lambda)}\right] := F(t, y, \lambda)\end{aligned}$$

and

$$\begin{aligned}\det(T_{3,1}(f)) \geq& 1 + \frac{1}{16(1+2\lambda)^2(1+3\lambda)}\left[\frac{t^2}{1+2\lambda} - t(4-t)y\right] - \frac{t}{2(1+2\lambda)^2} \\ &- \frac{1}{16(1+3\lambda)^2}\left[\frac{t^2}{4(1+2\lambda)^2} + \frac{(4-t)^2y^2}{4} - \frac{2yt(4-t)}{4} \right. \\ &\left. + \frac{\lambda t(4-t)y}{(1+2\lambda)}\right] := G(t, y, \lambda).\end{aligned}$$

We observe that

$$F(0, y, \lambda) = 1 - \frac{y^2}{16(1+3\lambda)^2} \leq 1,$$

$$F(4, y, \lambda) = 1 - \frac{48\lambda^2 + 30\lambda + 5}{4(1+2\lambda)^3(1+3\lambda)^2} \leq 1,$$

$$F(t,0,\lambda) = 1 + \frac{t^2}{16(1+2\lambda)^3(1+3\lambda)} - \frac{t}{4(1+2\lambda)^2} - \frac{t^2}{64(1+2\lambda)^2(1+3\lambda)^2} \leq 1$$

and

$$\begin{aligned} F(t,1,\lambda) =& 1 + \frac{1}{16(1+2\lambda)^2(1+3\lambda)}\left[\frac{t^2}{1+2\lambda} + t(4-t)\right] - \frac{t}{2(1+2\lambda)^2} \\ & - \frac{1}{16(1+3\lambda)^2}\left[\frac{t^2}{4(1+2\lambda)^2} + \frac{(4-t)^2}{4} + \frac{2t(4-t)}{4} - \frac{\lambda t(4-t)}{(1+2\lambda)}\right] \leq 1. \end{aligned}$$

By the second derivative test, we see that there are no critical points for $F(t,y,\lambda)$ in the domain $(0,4)\times(0,1)$.

Therefore,

$$\max_{(0\leq t\leq 4, 0\leq y\leq 1)} F(t,y,\lambda) = 1.$$

By a similarly proceeding, we obtain the minimum of $G(t,y,\lambda)$ as

$$\min_{(0\leq t\leq 4, 0\leq y\leq 1)} G(t,y,\lambda) = 1 - \left(\frac{48\lambda^2+30\lambda+5}{4(1+2\lambda)^3(1+3\lambda)^2}\right) := \sigma_1(\lambda).$$

Hence,

$$\sigma_1(\lambda) \leq \det(T_{3,1}(f)) \leq 1, \tag{24}$$

which completes the proof. □

For $\lambda = 0$, we have the following corollary obtained by Surya Giri and Sivaprasad Kumar [30].

Corollary 3 ([30]). *If f of the form (1) is in $\mathbb{S}_s^*$, then $\frac{-1}{4} \leq \det(T_{3,1}(f)) \leq 1$.*

5. Hankel Determinant

In this section, the upper bounds for the second- and third-order Hankel determinants for $f \in \mathfrak{H}(\lambda)$ are determined.

Theorem 4. *If f of the form (1) is in $\mathfrak{H}(\lambda)$, then, for $\lambda \geq 0$,*

$$|H_2^1(f)| = |a_3 - a_2^2| \leq \frac{1}{2(1+3\lambda)}.$$

Proof. From (9) and (10), we have

$$\begin{aligned} |H_2^1(f)| &= |a_3 - a_2^2| \\ &= \left|\frac{(2\lambda^2+4\lambda+1)}{4(1+2\lambda)^2(1+3\lambda)}p_1^2 - \frac{p_2}{4(1+3\lambda)}\right| \\ &\leq \frac{1}{4(1+3\lambda)}\left|\frac{(2\lambda^2+4\lambda+1)}{(1+2\lambda)^2}p_1^2 - p_2\right|. \end{aligned}$$

In view of Lemma 2, the above inequality becomes

$$|H_2^1(f)| \leq \frac{1}{2(1+3\lambda)}, \tag{25}$$

which gives the desired result. □

For $\lambda = 0$, the above Theorem 4 reduces to the following corollary, obtained earlier by Zhang and Tang [25].

Corollary 4 ([25]). *If f of the form (1) is in $\mathbb{S}_s^*$, then $|H_2^1(f)| \leq \frac{1}{2}$.*

Theorem 5. *If f of the form (1) is in $\mathfrak{H}(\lambda)$, then for, $\lambda \geq 0$,*

$$|H_2^2(f)| \leq \frac{1}{4(1+3\lambda)^2}.$$

Proof. Consider

$$|H_2^2(f)| = |a_2a_4 - a_3^2|.$$

From (9)–(11), we have

$$|a_2a_4 - a_3^2| = \left| \frac{p_1p_3}{12(1+2\lambda)(1+3\lambda)} - \frac{p_1^2p_2(6\lambda^2+6\lambda+1)}{48(1+2\lambda)^2(1+3\lambda)^2} + \frac{p_1^4(30\lambda^2-17\lambda-1)}{48(1+2\lambda)(1+3\lambda)} \right| := |\mathcal{F}(p_1,p_2,p_3,\lambda)|.$$

In virtue of Lemma 1, the above reduces to

$$\begin{aligned}\mathcal{F}(p_1,p_2,p_3,\lambda) = {} & \frac{p_1^4 + 2p_1^2x(4-p_1^2) - (4-p_1^2)p_1^2x^2}{48(1+2\lambda)(1+3\lambda)} + \frac{p_1(4-p_1^2)(1-|x|^2)z}{24(1+2\lambda)(1+3\lambda)} \\ & - \frac{(p_1^2x(4-p_1^2)(6\lambda^2+6\lambda+1) - 3p_1^4(30\lambda^2-17\lambda-1))}{288(1+2\lambda)^2(1+3\lambda)^2} \\ & - \frac{p_1^4(6\lambda^2+6\lambda+1)}{96(1+2\lambda)^2(1+3\lambda)^2} - \frac{(p_1^4+x^2(4-p_1^2)^2}{64(1+3\lambda)^2} - \frac{2p_1^2x(4-p_1^2)}{48(1+3\lambda)^2} \\ & := \mathcal{F}(p_1,x,\lambda).\end{aligned}$$

Letting $p_1 = p, p \in [0,2], |x| = t, t \in [0,1]$, we have

$$\begin{aligned}\mathcal{F}(p_1,x,\lambda) = {} & \frac{p^4 + 2p^2t(4-p^2) - (4-p^2)p^2t^2}{48(1+2\lambda)(1+3\lambda)} + \frac{p(4-p^2)(1-t^2)z}{24(1+2\lambda)(1+3\lambda)} \\ & - \frac{(p^2t(4-p^2)(6\lambda^2+6\lambda+1) - 3p^4(30\lambda^2-17\lambda-1))}{288(1+2\lambda)^2(1+3\lambda)^2} \\ & - \frac{p^4(6\lambda^2+6\lambda+1)}{96(1+2\lambda)^2(1+3\lambda)^2} - \frac{(p^4+x^2(4-p^2)^2}{64(1+3\lambda)^2} - \frac{2p^2x(4-p^2)}{48(1+3\lambda)^2} \\ & := \mathcal{F}(p,t,\lambda).\end{aligned}$$

For any $t \in (0,1)$, $p \in (0,2)$ and $\lambda \geq 0$, we have

$$\frac{\partial \mathcal{F}}{\partial t} = \frac{4t(4-p^2)}{48(1+2\lambda)(1+3\lambda)} + \frac{t(4-p^2)^2}{32(1+3\lambda)^2} + \frac{2p^2\lambda(4-p^2)}{48(1+3\lambda)^2(1+2\lambda)} > 0.$$

This implies that $\mathcal{F}(p,t,\lambda)$ is an increasing function with respect to t, for $t \in [0,1]$. By the second derivative test, we see that there are no critical points for $\mathcal{F}(p,t,\lambda)$ in (0,1) and $\mathcal{F}(p,t,\lambda)$ attains a maximum at $t = 1$ and

$$\mathcal{F}(p,1,\lambda) = \frac{(12\lambda^2-22\lambda-5)p^4}{576(1+2\lambda)^2(1+3\lambda^2)} + \frac{(4-p^2)^2}{64(1+3\lambda)^2} + \frac{2p^2\lambda(4-p^2)}{48(1+3\lambda)^2(1+2\lambda)} := \mathcal{G}(p,\lambda).$$

Now, by applying the second derivative test, we see that $\mathcal{G}$ attains the maximum at $p = 0$ and, therefore,

$$|a_2a_4 - a_3^2| \leq \mathcal{G}(0,\lambda) = \frac{1}{4(1+3\lambda)^2}, \tag{26}$$

which completes the proof. □

For $\lambda = 0$, the above Theorem 5 reduces to the result of Zhang and Tang [25] given by the following corollary.

Corollary 5 ([25]). *If f of the form (1) is in $\mathbb{S}_s^*$, then $|a_2a_4 - a_3^2| \leq \frac{1}{4}$.*

Theorem 6. *If f of the form (1) is in $\mathfrak{H}(\lambda)$, then, for $\lambda \geq 0$,*

$$|a_4 - a_2a_3| \leq \frac{1}{3(1+4\lambda)}.$$

Proof. From (9)–(11), we have

$$|a_4 - a_2a_3| = \left| \frac{(24\lambda^2+24\lambda+4)p_1p_2}{24(1+2\lambda)(1+3\lambda)(1+4\lambda)} - \frac{(60\lambda^3+104\lambda^2+17\lambda-1)p_1^3}{144(1+2\lambda)^2(1+3\lambda)(1+4\lambda)} - \frac{p_3}{6(1+4\lambda)} \right|.$$

Using Lemma 1, we have

$$|a_4 - a_2a_3| = \left| \frac{(24\lambda^2+24\lambda+4)p_1(p_1^2+x(4-p_1^2)))}{24(1+2\lambda)(1+3\lambda)(1+4\lambda)} - \frac{(60\lambda^3+104\lambda^2+17\lambda-1)p_1^3}{144(1+2\lambda)^2(1+3\lambda)(1+4\lambda)} - \frac{p_1^3+2p_1x(4-p_1^2)-(4-p_1^2)p_1x^2+2(4-p_1^2)(1-|x|^2)z}{24(1+4\lambda)} \right|.$$

Letting $p_1 = p, p \in [0,2]$ and $|x| = t,\ x \in [0,1]$, we have

$$|a_4 - a_2a_3| \leq \left| \frac{(24\lambda^2+24\lambda+4)p(p^2+t(4-p^2))}{24(1+2\lambda)(1+3\lambda)(1+4\lambda)} - \frac{(60\lambda^3+104\lambda^2+17\lambda-1)p^3}{144(1+2\lambda)^2(1+3\lambda)(1+4\lambda)} - \frac{p^3+2pt(4-p^2)-(4-p^2)pt^2+2(4-p^2)(1-t^2)z}{24(1+4\lambda)} \right| := |H(p,t,\lambda)|.$$

For any $t \in (0,1)$, $p \in (0,2)$ and $\lambda \geq 0$, we have

$$\frac{\partial H}{\partial t} = \frac{(4-p^2)(p-2)t}{24(1+4\lambda)} < 0,\ t \in (0,1).$$

This implies $H(p,t,\lambda)$ is a decreasing function with respect to t, and by the second derivative test, we see that there are no critical points in (0,1), and $H(p,t,\lambda)$ attains the maximum at $t = 0$ and

$$H(p,0,\lambda) = \frac{(12\lambda^3+6\lambda^2+37\lambda+7)P^3}{144(1+2\lambda)^2(1+3\lambda)(1+4\lambda)} + \frac{4-p^2}{12(1+4\lambda)} := K(p,\lambda).$$

By applying the second derivative test, we see that $K(p,\lambda)$ attains the maximum at $p = 0$ and

$$|a_4 - a_2a_3| \leq K(0,\lambda) = \frac{1}{3(1+4\lambda)}, \tag{27}$$

which completes the proof. □

For $\lambda = 0$, the above Theorem 6 reduces to the following corollary, obtained earlier by Zhang and Tang [25].

Corollary 6 ([25]). *If f of the form (1) is in $\mathbb{S}_s^*$, then*

$$|a_4 - a_2a_3| \leq \frac{1}{3}.$$

Remark 2. *(i) From (14) and (26), we have*

$$|a_3||H_2^2(f)| \leq \frac{1}{8(1+3\lambda)^3} := K_1(\lambda).$$

(ii) From (15), (16), and (27), we have

$$|a_4||a_4 - a_2a_3| \leq \begin{cases} \frac{1}{9(1+4\lambda)^2}\left[1 + \frac{102\lambda^2+37\lambda+2}{18\sqrt{3}(1+2\lambda)(1+3\lambda)}\right] & \text{if } \lambda \in [0, \frac{5}{8}] \\ \frac{1}{9(1+4\lambda)^2}\left[1 + \frac{54\lambda^2+13\lambda}{6(1+2\lambda)(1+3\lambda)}\right] & \text{if } \lambda > \frac{5}{8} \end{cases}$$
$$:= K_2(\lambda).$$

(iii) From (17) and (24), we have

$$|a_5||H_2^1(f)| \leq \frac{3}{8(1+3\lambda)(1+5\lambda)} := K_3(\lambda).$$

Theorem 7. *If f of the form (1) is in $\mathfrak{H}(\lambda)$, then, for $\lambda \geq 0$,* $|H_3^1(f)| \leq K_1(\lambda) + K_2(\lambda) + K_3(\lambda)$.

Proof. Consider

$$\begin{aligned} |H_3^1(f)| &= a_3|H_2^2(f)| - a_4(a_4 - a_2a_3) + a_5|H_2^1(f)| \\ &\leq |a_3||H_2^2(f)| + |a_4||a_4 - a_2a_3| + |a_5||H_2^1(f)|. \end{aligned}$$

From Remark 2, we obtain the required result. □

For $\lambda = 0$ in the above theorem, we obtain the following corollary.

Corollary 7. *If f of the form (1) is in $\mathbb{S}_s^*$, then*

$$|H_3^1(f)| \leq \frac{11}{18} + \frac{\sqrt{3}}{24}.$$

Remark 3. *The sharpness of the results in Theorems 3–7 are to be examined.*

6. Conclusions

In the present investigation, we determined the coefficient estimates, bounds for the second and third-order Hankel determinants for a new subclass $\mathfrak{H}(\lambda)$ of analytic functions defined by subordination to a domain involving the sine function. Further, bounds for the second and third-order Hermtian Toeplitz determinants for the class $\mathfrak{H}(\lambda)$ are estimated. However, the sharpness of some of the results obtained are yet to be explored. It would be intriguing to determine the estimates of $|H_4^1(f)|$ and $det(T_{4,1}(f))$ for the class $\mathfrak{H}(\lambda)$.

Additionally, an attempt to apply these observations in areas under the broad spectra of Physics and Engineering will serve as a future scope of this study.

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