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A Study of Generalized Symmetric Metric Connection on Nearly Kenmotsu Manifolds

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Abstract: The focus of this research is on investigating a new category of generalized symmetric metric connections within nearly Kenmotsu manifolds. The study delves into recognizing the generalized symmetric connections of type (α, β), which represent broader versions of the semi-symmetric metric connection $(\alpha = 1, \beta = 0)$ and the quarter-symmetric metric connection $(\alpha = 0, \beta = 1)$.

Keywords: semi-symmetric metric connection; quarter-symmetric metric connection; M-projective curvature tensor; generalized Einstein manifold; nearly Kenmotsu manifold

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1. Introduction

Nearly Kenmotsu manifolds supply a fascinating class of geometric structures that generalize Kenmotsu manifolds. These manifolds have many applications in areas such as contact geometry in physics, dynamical systems and geometric mechanics. In physics, especially in higher-dimensional models of spacetime, the structure of nearly Kenmotsu manifolds provides a generalization of standard contact structures. These manifolds can be used to model the geometric properties of some physical structures, mainly in the research of dynamical systems with symmetries or in areas like string theory and gauge theory. Nearly Kenmotsu manifolds can be used to examine dynamical systems, especially those with symmetries or conserved quantities. In geometric mechanics, the contact structure has an important role in the analysis of phase spaces and the conduct of mechanical systems. On the other hand, the concept of a semi-symmetric connection on a differentiable manifold was originated by Friedmann and Schouten [1]. Many years later, geometers like Hayden [2] initiated semi-symmetric metric connection on a Riemannian manifold. After that, quarter-symmetric connections on a differentiable manifold were studied by Golab [3]. Based on these connections, many authors including [4–13] have done such work with several interesting results that brought huge contributions to the field of differential geometry. After a millennium, specific types of connections which include the semi-symmetric and quarter-symmetric connections were introduced by Tripathi [14].

A linear connection $\bar{\nabla}$ on a (semi-) Riemannian manifold M is labelled to be a generalized symmetric connection if its torsion tensor T is represented by

$$T(\tau_X, \tau_Y) = \alpha[\eta(\tau_Y)\tau_X - \eta(\tau_X)\tau_Y] + \beta[\eta(\tau_Y)\phi\tau_X - \eta(\tau_X)\phi\tau_Y] \tag{1}$$

for all vector fields τ_X and τ_Y on M, where α and β are smooth functions on M, ϕ is a (1,1) type tensor and η is the 1-form connected with the vector field which has a non-vanishing smooth non-null unit. Moreover, if a Riemannian metric g on the manifold M satisfies $\bar{\nabla}g = 0$, then the corresponding connection is classified as a generalized metric connection; otherwise, it is termed a generalized non-metric connection. Notably, the connection $\bar{\nabla}$ on M serves as a generalization encompassing both the semi-symmetric metric connection and the quarter-symmetric metric connection. These connections hold significant importance in the realm of geometry and find numerous applications in physics. The exploration of nearly Kenmotsu manifolds with semi-symmetric metric connections has been examined by [15], while similar investigations on nearly Kenmotsu manifolds with quarter-symmetric metric connections have been carried out by [16]. Several other studies on nearly Kenmotsu manifolds have been conducted by various geometers.

This research article is organized as follows: Section 1 introduces the topic, while Section 2 covers the preliminaries. In Section 3, a new category of generalized symmetric metric connections on nearly Kenmotsu manifolds is examined. The discussion on curvature tensors concerning generalized symmetric metric connections is presented in Section 4. Section 5 explores nearly Kenmotsu manifolds satisfying $\bar{K}.\bar{Ric} = 0$. Section 6 delves into the M-projective curvature tensor on nearly Kenmotsu manifolds that admit a generalized symmetric metric connection, followed by discussions on quasi M-projectively flat and ϕ-M-projectively flat nearly Kenmotsu manifolds in Section 7. Additionally, Section 8 illustrates nearly Kenmotsu manifolds satisfying $\bar{J}.\bar{Ric} = 0$. Section 9 focuses on ϕ-Ricci symmetric nearly Kenmotsu manifolds in relation to generalized symmetric metric connection. Finally, an example is provided to demonstrate and validate the findings.

2. Preliminaries

Consider $\mathcal{F}$ as an n-dimensional almost contact metric manifold [17], where ϕ represents a (1,1)-type tensor field, υ stands for the structure vector field, η denotes a 1-form, and g symbolizes a Riemannian metric, satisfying the following conditions:

$$\phi^2\tau_X = -\tau_X + \eta(\tau_X)\upsilon, \quad \phi \circ \upsilon = 0, \quad \eta(\phi\tau_X) = 0, \tag{2}$$

$$\eta(\tau_X) = g(\tau_X, \upsilon), \quad \eta(\upsilon) = 0, \tag{3}$$

$$g(\phi\tau_X, \phi\tau_Y) = g(\tau_X, \tau_Y) - \eta(\tau_X)\eta(\tau_Y), \quad g(\phi\tau_X, \tau_Y) = -g(\tau_X, \phi\tau_Y) \tag{4}$$

for all vector fields τ_X, τ_Y on $\mathcal{F}$ and a rank of $\phi = n - 1$. If

$$d\eta(\tau_X, \tau_Y) = g(\tau_X, \phi\tau_Y), \tag{5}$$

then $\mathcal{F}$ is called an almost contact metric manifold.

An almost contact metric manifold is termed a nearly Kenmotsu manifold (abbreviated as NKM) [18] when

$$(\nabla_{\tau_X}\phi)\tau_Y + (\nabla_{\tau_Y}\phi)\tau_X = -\eta(\tau_Y)\phi\tau_X - \eta(\tau_X)\phi\tau_Y, \tag{6}$$

where ∇ denotes the Levi-Civita connection (abbreviated as LCC) of g. Additionally, if the manifold $\mathcal{F}$ fulfills

$$(\nabla_{\tau_X}\phi)\tau_Y = g(\phi\tau_X, \tau_Y)\upsilon - \eta(\tau_Y)\phi\tau_X, \tag{7}$$

then it is called a Kenmotsu manifold [19].

Furthermore, an NKM holds the following relations [20]:

$$\nabla_{\tau_X} \upsilon = \tau_X - \eta(\tau_X)\upsilon, \qquad \nabla_\upsilon \upsilon = 0, \tag{8}$$

$$(\nabla_{\tau_X}\eta)\tau_Y = g(\tau_X, \tau_Y) - \eta(\tau_X)\eta(\tau_Y), \tag{9}$$

$$K(\tau_X, \tau_Y)\upsilon = \eta(\tau_X)\tau_Y - \eta(\tau_Y)\tau_X, \tag{10}$$

$$K(\upsilon, \tau_Y)\tau_Z = \eta(\tau_Z)\tau_Y - g(\tau_Y, \tau_Z)\upsilon, \tag{11}$$

$$K(\upsilon, \tau_Y)\upsilon = \tau_X - \eta(\tau_X)\upsilon = -K(\tau_Y, \upsilon)\upsilon, \tag{12}$$

$$g(K(\tau_X, \tau_Y)\tau_Z, \upsilon) = g(\tau_X, \tau_Z)\eta(\tau_Y) - g(\tau_Y, \tau_Z)\eta(\tau_X), \tag{13}$$

$$Ric(\phi\tau_X, \phi\tau_Y) = Ric(\tau_X, \tau_Y) + (n-1)\eta(\tau_X)\eta(\tau_Y), \tag{14}$$

$$Ric(\phi\tau_X, \upsilon) = -(n-1)\eta(\tau_X), \tag{15}$$

$$Q\tau_X = -(n-1)\tau_X, \tag{16}$$

$$Q\upsilon = -(n-1)\upsilon, \tag{17}$$

where Ric represents the Ricci tensor, K signifies the Riemannian curvature tensor and Q denotes the Ricci operator defined as $g(Q\tau_X, \tau_Y) = Ric(\tau_X, \tau_Y)$ with respect to the Levi-Civita connection ∇.

A nearly Kenmotsu manifold $\mathcal{F}$ is classified as a generalized η-Einstein manifold if the Ricci tensor Ric takes the form of

$$Ric(\tau_X, \tau_Y) = ag(\tau_X, \tau_Y) + b\eta(\tau_X)\eta(\tau_Y) + cg(\phi\tau_X, \tau_Y), \tag{18}$$

where a, b, and c are scalar functions on $\mathcal{F}$. When $c = 0$, (18) simplifies to an η-Einstein manifold. Furthermore, if $b = c = 0$, (18) transforms into an Einstein manifold.

3. Generalized Symmetric Metric Connection on NKM

Bahadir and Chaubey [21] introduced the concept of a generalized symmetric metric connection on Lorentzian para-Sasakian manifolds, which was later expanded upon by [22–24]. Building on the ideas presented in [21], we establish a new category of generalized symmetric metric connections denoted by $\bar{\nabla}$ (abbreviated as GSMC $\bar{\nabla}$) on an n-dimensional NKM $\mathcal{F}$.

Let us consider $\bar{\nabla}$ as a linear connection and ∇ as the Levi-Civita connection on an NKM $\mathcal{F}$, satisfying

$$\bar{\nabla}_{\tau_X}\tau_Y = \nabla_{\tau_X}\tau_Y + N(\tau_X, \tau_Y) \tag{19}$$

for all $\tau_X, \tau_Y \in \chi(M)$ and N is a tensor of type (1, 2). For a linear connection $\bar{\nabla}$ to be a GSMC on $\mathcal{F}$, we have

$$2N(\tau_X, \tau_Y) = T(\tau_X, \tau_Y) + F(\tau_X, \tau_Y) + F(\tau_Y, \tau_X), \tag{20}$$

where T is a torsion tensor of $\bar{\nabla}$ defined by

$$g(F(\tau_X, \tau_Y), \tau_Z) = g(T(\tau_Z, \tau_X)\tau_Y). \tag{21}$$

From Equations (1) and (21), we have

$$F(\tau_X, \tau_Y) = \alpha[\eta(\tau_X)\tau_Y - g(\tau_X, \tau_Y)\upsilon] - \beta[\eta(\tau_X)\phi\tau_Y + g(\phi\tau_X, \tau_Y)\upsilon]. \tag{22}$$

If we interchange τ_X and τ_Y in Equation (22), then $F(\tau_Y, \tau_X)$ is obtained. From Equations (1), (20) and (22), we obtain

$$N(\tau_X, \tau_Y) = \alpha[\eta(\tau_Y)\tau_X - g(\tau_X, \tau_Y)v] - \beta\eta(\tau_X)\phi\tau_Y. \tag{23}$$

From Equations (19) and (23), we obtain

$$\bar{\nabla}_{\tau_X}\tau_Y = \nabla_{\tau_X}\tau_Y + \alpha[\eta(\tau_Y)\tau_X - g(\tau_X, \tau_Y)v] - \beta\eta(\tau_X)\phi\tau_Y, \tag{24}$$

which shows that Equation (24) is a generalized symmetric metric connection $\bar{\nabla}$ on $\mathcal{F}$ such that $(\bar{\nabla}_{\tau_X} g)(\tau_Y, \tau_Z) = 0$. Therefore, the following theorem can be stated as follows:

Theorem 1. *For an n-dimensional NKM $\mathcal{F}$, the GSMC $\bar{\nabla}$ of type (α, β) satisfying Equation (1) is given by Equation (24).*

Remark 1. *Equation (24) reduces to an α-semi-symmetric connection if $\alpha \neq 0, \beta = 0$.*

Remark 2. *Equation (24) reduces to a β-quarter-symmetric connection if $\alpha = 0, \beta \neq 0$.*

Remark 3. *If $\alpha = 1, \beta = 0$, then Equation (24) becomes semi-symmetric metric connection [3].*

Remark 4. *If $\alpha = 0, \beta = 1$, then Equation (24) turns into a quarter-symmetric metric connection [3].*

Proposition 1. *If $\mathcal{F}$ is an NKM with respect to GSMC $\bar{\nabla}$, then the following conditions satisfy*

$$(\bar{\nabla}_{\tau_X}\phi)\tau_Y = (\alpha + 1)[g(\phi\tau_X, \tau_Y)v - \eta(\tau_Y)\phi\tau_X], \tag{25}$$

$$\bar{\nabla}_{\tau_X} v = (\alpha + 1)[\tau_X - \eta(\tau_X)v], \tag{26}$$

$$(\bar{\nabla}_{\tau_X}\eta)\tau_Y = (\alpha + 1)[g(\tau_X, \tau_Y) - \eta(\tau_X)\eta(\tau_Y)] \tag{27}$$

for all $\tau_X, \tau_Y \in \chi(\mathcal{F})$.

4. Curvature Tensors on NKM with Respect to GSMC

If $\mathcal{F}$ is an n-dimensional NKM, then the curvature tensor $\bar{K}$ with respect to GSMC $\bar{\nabla}$ is defined by

$$\bar{K}(\tau_X, \tau_Y)\tau_Z = \bar{\nabla}_{\tau_X}\bar{\nabla}_{\tau_Y}\tau_Z - \bar{\nabla}_{\tau_Y}\bar{\nabla}_{\tau_X}\tau_Z - \bar{\nabla}_{[\tau_X, \tau_Y]}\tau_Z. \tag{28}$$

From Equations (24)–(28), we obtain

$$\begin{aligned}\bar{K}(\tau_X, \tau_Y)\tau_Z =& K(\tau_X, \tau_Y)\tau_Z \\ &+ \alpha(\alpha + 2)[g(\tau_X, \tau_Z)\tau_Y - g(\tau_Y, \tau_Z)\tau_X \\ &+ \eta(\tau_Y)\eta(\tau_Z)\tau_X - \eta(\tau_X)\eta(\tau_Z)\tau_Y \\ &+ \eta(\tau_X)g(\tau_Y, \tau_Z)v - \eta(\tau_Y)g(\tau_X, \tau_Z)v] \\ &+ \beta(\alpha + 1)[\eta(\tau_Y)\eta(\tau_Z)\phi\tau_X - \eta(\tau_X)\eta(\tau_Z)\phi\tau_Y \\ &+ \eta(\tau_X)g(\phi\tau_Y, \tau_Z)v - \eta(\tau_Y)g(\phi\tau_X, \tau_Z)v].\end{aligned} \tag{29}$$

Setting $\tau_Z = v$ in Equation (29), we obtain

$$\bar{K}(\tau_X, \tau_Y)v = (\alpha + 1)[\eta(\tau_X)\tau_Y - \eta(\tau_Y)\tau_X] + \beta(\alpha + 1)[\eta(\tau_Y)\phi\tau_X - \eta(\tau_X)\phi\tau_Y]. \tag{30}$$

Similarly, by setting $\tau_X = v$ in Equation (29), it becomes

$$\begin{aligned}\bar{K}(v,\tau_Y)\tau_Z =&(\alpha+1)[\eta(\tau_Z)\tau_Y - g(\tau_Y,\tau_Z)v]\\ &-\beta(\alpha+1)[\eta(\tau_Z)\phi\tau_Y - g(\phi\tau_Y,\tau_Z)v].\end{aligned} \tag{31}$$

If we replace τ_Z with v in Equation (31), we obtain

$$\bar{K}(v,\tau_Y)v = (\alpha+1)[\tau_Y - \eta(\tau_Y)v] - \beta(\alpha+1)\phi\tau_Y \tag{32}$$

for all $\tau_X, \tau_Y, \tau_Z \in \chi(\mathcal{F})$.
By taking the inner product of Equation (29) with the vector field τ_U, we obtain

$$\begin{aligned}{}'\bar{K}(\tau_X,\tau_Y,\tau_Z,\tau_U) =&K(\tau_X,\tau_Y,\tau_Z,\tau_U)\\ &+\alpha(\alpha+2)[g(\tau_X,\tau_Z)g(\tau_Y,\tau_U) - g(\tau_Y,\tau_Z)g(\tau_X,\tau_U)\\ &+\eta(\tau_Y)\eta(\tau_Z)g(\tau_X,\tau_U) - \eta(\tau_X)\eta(\tau_Z)g(\tau_Y,\tau_U)\\ &+\eta(\tau_X)g(\tau_Y,\tau_Z)g(v,\tau_U) - \eta(\tau_Y)g(\tau_X,\tau_Z)g(v,\tau_U)]\\ &+\beta(\alpha+1)[\eta(\tau_Y)\eta(\tau_Z)g(\phi\tau_X,\tau_U)\\ &-\eta(\tau_X)\eta(\tau_Z)g(\phi\tau_Y,\tau_U) + \eta(\tau_X)g(\phi\tau_Y,\tau_Z)g(v,\tau_U)\\ &-\eta(\tau_Y)g(\phi\tau_X,\tau_Z)g(v,\tau_U)],\end{aligned} \tag{33}$$

where ${}'\bar{K}(\tau_X,\tau_Y,\tau_Z,\tau_U) = g(\bar{K}(\tau_X,\tau_Y)\tau_Z,\tau_U)$.
By contracting Equation (33) over τ_X and τ_U, we obtain

$$\begin{aligned}\bar{Ric}(\tau_Y,\tau_Z) =&Ric(\tau_Y,\tau_Z) + \alpha(2\alpha+3-2n-\alpha n)g(\tau_Y,\tau_Z)\\ &+\alpha(\alpha+1)(n-2)\eta(\tau_Y)\eta(\tau_Z) + \beta(\alpha+1)g(\phi\tau_Y,\tau_Z),\end{aligned} \tag{34}$$

where $\bar{Ric}$ and Ric are the Ricci tensors with respect to GSMC $\bar{\nabla}$ and LCC ∇, respectively. So, in an NKM $\mathcal{F}$, the Ricci tensor with respect to GSMC $\bar{\nabla}$ is not symmetric.

On the contraction of Equation (34) over τ_Y and τ_Z, we have

$$\bar{r} = r + \alpha n(2\alpha+3-2n-\alpha n) + \alpha(\alpha+1)(n-2), \tag{35}$$

where $\bar{r}$ and r are the scalar curvatures of the connections $\bar{\nabla}$ and ∇, respectively. Thus, we can state the following:

Theorem 2. *For an NKM $\mathcal{F}$ with respect to GSMC $\bar{\nabla}$,*

(i) *the curvature tensor $\bar{K}$ is given by Equation (29),*
(ii) *the Ricci tensor $\bar{Ric}$ is given by Equation (34),*
(iii) $\bar{K}(\tau_X,\tau_Y)\tau_Z = -\bar{K}(\tau_Y,\tau_X)\tau_Z$,
(iv) *the scalar curvature $\bar{r}$ is given by Equation (35),*
(v) *the Ricci tensor $\bar{Ric}$ is not symmetric.*

If we take $\tau_Z = v$ in Equation (34), we have

$$\bar{Ric}(\tau_Y,v) = (\alpha+1)(1-n)\eta(\tau_Y). \tag{36}$$

By replacing τ_Y with $\phi\tau_Y$ and τ_Z with $\phi\tau_Z$ in Equation (34), we obtain

$$\begin{aligned}\bar{Ric}(\phi\tau_Y,\phi\tau_Z) =&Ric(\tau_Y,\tau_Z) + \alpha(2\alpha+3-2n-\alpha n)g(\tau_Y,\tau_Z)\\ &+(n-1-2\alpha^2-3\alpha+2\alpha n+\alpha^2 n)\eta(\tau_Y)\eta(\tau_Z)\\ &+\beta(\alpha+1)g(\phi\tau_Y,\tau_Z).\end{aligned} \tag{37}$$

From Equation (37), it follows that

$$\bar{Q}\tau_Y = (\alpha+1)(1-n)\tau_Y, \qquad \bar{Q}v = (\alpha+1)(1-n)v. \tag{38}$$

Let us consider that in an n-dimensional NKM $\mathcal{F}$ with respect to GSMC $\bar{\nabla}$, $\bar{Ric} = 0$. Then, from Equation (34), we obtain

$$\begin{aligned} Ric(\tau_Y,\tau_Z) =& \alpha(2n+\alpha n-2\alpha-3)g(\tau_Y,\tau_Z) \\ &+\alpha(\alpha+1)(2-n)\eta(\tau_Y)\eta(\tau_Z)-\beta(\alpha+1)g(\phi\tau_Y,\tau_Z). \end{aligned} \tag{39}$$

Therefore, from Equation (39), the statement can be written as follows:

Theorem 3. *If an n-dimensional NKM $\mathcal{F}$ with respect to GSMC $\bar{\nabla}$ is Ricci flat, then the manifold $\mathcal{F}$ of the (α,β) type is a generalized η-Einstein manifold.*

5. NKM Satisfying $\bar{K}.\bar{Ric} = 0$

Theorem 4. *If an n-dimensional NKM $\mathcal{F}$ with respect to GSMC $\bar{\nabla}$ satisfies $\bar{K}.\bar{Ric} = 0$, then the manifold $\mathcal{F}$ of the (α,β) type is an Einstein manifold.*

Proof. Let $\mathcal{F}$ be an n-dimensional NKM satisfying $\bar{K}.\bar{Ric} = 0$, then we have

$$\bar{Ric}(\bar{K}(\tau_X,\tau_Y)\tau_Z,\tau_U)+\bar{Ric}(\tau_Z,\bar{K}(\tau_X,\tau_Y)\tau_U) = 0. \tag{40}$$

Replacing τ_X and τ_Z with v in Equation (40), we obtain

$$\bar{Ric}(\bar{K}(v,\tau_Y)v,\tau_U)+\bar{Ric}(v,\bar{K}(v,\tau_Y)\tau_U) = 0. \tag{41}$$

From Equation (41), it can be easily seen that

$$\begin{aligned} \bar{Ric}(\bar{K}(v,\tau_Y)v,\tau_U) =& (\alpha+1)[\bar{Ric}(\tau_Y,\tau_U)-(\alpha+1-n-\alpha n)\eta(\tau_Y)\eta(\tau_U) \\ &-\beta\bar{Ric}(\phi\tau_Y,\tau_U)], \end{aligned}$$

$$\begin{aligned} \bar{Ric}(v,\bar{K}(v,\tau_Y)\tau_U) =& (\alpha+1)(\alpha+1-n-\alpha n)[\eta(\tau_Y)\eta(\tau_U)-g(\tau_Y,\tau_U) \\ &+\beta g(\phi\tau_Y,\tau_U)]. \end{aligned}$$

Using the above two equations in Equation (41) and with the help of Equations (31), (32) and (36) in Equation (41), we obtain

$$\begin{aligned} (\alpha+1)\bar{Ric}(\tau_Y,\tau_U) =& \beta(\alpha+1)\bar{Ric}(\phi\tau_Y,\tau_U) \\ &+(\alpha+1)(\alpha+1-n-\alpha n)g(\tau_Y,\tau_U) \\ &-\beta(\alpha+1)(\alpha+1-n-\alpha n)g(\phi\tau_Y,\tau_U). \end{aligned} \tag{42}$$

Again, by replacing τ_Y with $\phi\tau_Y$ in Equation (42) and applying Equation (36), we obtain

$$\begin{aligned} \beta(\alpha+1)\bar{Ric}(\phi\tau_Y,\tau_U) =& -\beta(\alpha+1)\bar{Ric}(\tau_Y,\tau_U) \\ &+\beta(\alpha+1)(\alpha+1-n-\alpha n)g(\tau_Y,\tau_U) \\ &+(\alpha+1)(\alpha+1-n-\alpha n)g(\phi\tau_Y,\tau_U). \end{aligned} \tag{43}$$

From Equations (42) and (43), we can easily find that

$$(1+\beta^2)\bar{Ric}(\tau_Y,\tau_U) = (\alpha+1-n-\alpha n)(1+\beta^2)g(\tau_Y,\tau_U). \tag{44}$$

Therefore, Equation (44) shows that $\mathcal{F}$ is an Einstein manifold. □

Theorem 5. *If a 3-dimensional Ricci semi-symmetric NKM $\mathcal{F}$ admitting GSMC $\bar{\nabla}$ satisfies $\bar{K}.\bar{Ric} = 0$, then the scalar curvature is either $\bar{r} = -12$ of the $(\alpha, \beta = 1, 0)$ type or $\bar{r} = -6$ of the $(\alpha, \beta = 0, 1)$ type, respectively.*

Proof. Let us consider a three-dimensional Ricci semi-symmetric NKM $\mathcal{F}$ equipped with GSMC $\bar{\nabla}$ satisfying $\bar{K}.\bar{Ric} = 0$.

Let $\{e_i\}, i = 1, 2, 3$ be an orthonormal basis of the tangent space at each point of the manifold $\mathcal{F}$.

Setting $\tau_Y = \tau_U = e_i$ in Equation (44) and performing summation over $i, 1 \leq i \leq 3$, we obtain

$$(1 + \beta^2)\bar{r} = 3(\alpha - 2 - 3\alpha)(1 + \beta^2). \tag{45}$$

From Equation (45), we get either $\bar{r} = -12$ for $\alpha, \beta = 1, 0$ or $\bar{r} = -6$ for $\alpha, \beta = 0, 1$, respectively.

Thus, the proof is completed. □

6. *M*-Projective Curvature Tensor on NKM Admitting GSMC

Pokhariyal and Mishra [25] introduced M-projective curvature tensor J on a Riemannian manifold given by

$$\begin{aligned} J(\tau_X, \tau_Y)\tau_Z =& K(\tau_X, \tau_Y)\tau_Z - \frac{1}{2(n-1)}\big[Ric(\tau_Y, \tau_Z)\tau_X - Ric(\tau_X, \tau_Z)\tau_Y \\ &+ g(\tau_Y, \tau_Z)Q\tau_X - g(\tau_X, \tau_Z)Q\tau_Y\big], \end{aligned} \tag{46}$$

where K denotes the Riemannian curvature tensor, Ric denotes the Ricci tensor and Q denotes the Ricci operator defined by $g(Q\tau_X, \tau_Y) = Ric(\tau_X, \tau_Y)$ with respect to the LCC, respectively.

Let us consider that in an n-dimensional NKM $\mathcal{F}$, the M-projective curvature tensor $\bar{J}$ with respect to GSMC $\bar{\nabla}$ is given by

$$\begin{aligned} \bar{J}(\tau_X, \tau_Y)\tau_Z =& \bar{K}(\tau_X, \tau_Y)\tau_Z - \frac{1}{2(n-1)}\big[\bar{Ric}(\tau_Y, \tau_Z)\tau_X - \bar{Ric}(\tau_X, \tau_Z)\tau_Y \\ &+ g(\tau_Y, \tau_Z)\bar{Q}\tau_X - g(\tau_X, \tau_Z)\bar{Q}\tau_Y\big], \end{aligned} \tag{47}$$

where $\bar{K}$, $\bar{Ric}$ and $\bar{Q}$ are the Riemannian curvature tensor, the Ricci tensor and the Ricci operator with respect to GSMC $\bar{\nabla}$, respectively.

Definition 1. *An n-dimensional NKM $\mathcal{F}$ with respect to GSMC $\bar{\nabla}$ is said to be M-projectively flat if $\bar{J}(\tau_X, \tau_Y)\tau_Z = 0$ for all $\tau_X, \tau_Y, \tau_Z \in \chi(\mathcal{F})$.*

Theorem 6. *An M-projectively flat NKM $\mathcal{F}$ of the n-dimension admitting GSMC $\bar{\nabla}$ is a generalized η-Einstein manifold.*

Proof. Let $\mathcal{F}$ be an n-dimensional M-projectively flat NKM with respect to GSMC $\bar{\nabla}$, that is, $\bar{J} = 0$. Then, from (47), we obtain

$$\begin{aligned} \bar{K}(\tau_X, \tau_Y)\tau_Z =& \frac{1}{2(n-1)}\big[\bar{Ric}(\tau_Y, \tau_Z)\tau_X - \bar{Ric}(\tau_X, \tau_Z)\tau_Y \\ &+ g(\tau_Y, \tau_Z)\bar{Q}\tau_X - g(\tau_X, \tau_Z)\bar{Q}\tau_Y\big]. \end{aligned} \tag{48}$$

By taking the inner product with a vector field of τ_U in Equation (48), we obtain

$$\begin{aligned} g(\bar{K}(\tau_X,\tau_Y)\tau_Z,\tau_U) =& \frac{1}{2(n-1)}\big[\bar{Ric}(\tau_Y,\tau_Z)g(\tau_X,\tau_U) \\ &- \bar{Ric}(\tau_X,\tau_Z)g(\tau_Y,\tau_U) + g(\tau_Y,\tau_Z)\bar{Ric}(\tau_X,\tau_U) \\ &- g(\tau_X,\tau_Z)\bar{Ric}(\tau_Y,\tau_U)\big]. \end{aligned} \tag{49}$$

by taking a frame field of $\mathcal{F}$ and contracting over τ_X and τ_U in Equation (49) and taking summation over i, $1 \le i \le n$, we obtain

$$\begin{aligned} \sum_{i=1}^{n} g(\bar{K}(e_i,\tau_Y)\tau_Z,e_i) =& \sum_{i=1}^{n} \frac{1}{2(n-1)}\big[\bar{Ric}(\tau_Y,\tau_Z)g(e_i,e_i) \\ &- \bar{Ric}(e_i,\tau_Z)g(\tau_Y,e_i) + g(\tau_Y,\tau_Z)\bar{Ric}(e_i,e_i) \\ &- g(e_i,\tau_Z)\bar{Ric}(\tau_Y,e_i)\big], \end{aligned}$$

which yields

$$\bar{Ric}(\tau_Y,\tau_Z) = \frac{\bar{r}}{n}g(\tau_Y,\tau_Z). \tag{50}$$

By inserting Equations (34) and (35) into Equation (50), we obtain

$$Ric(\tau_Y,\tau_Z) = \lambda_1 g(\tau_Y,\tau_Z) + \lambda_2\eta(\tau_Y)\eta(\tau_Z) + \lambda_3 g(\phi\tau_Y,\tau_Z), \tag{51}$$

where

$$\lambda_1 = \frac{\alpha n(2+3\alpha) - 2\alpha(\alpha+1) - 3n + r}{n},$$
$$\lambda_2 = \alpha(\alpha+1)(2-n) \quad \text{and} \quad \lambda_3 = -\beta(\alpha+1).$$

Therefore, Equation (51) indicates that $\mathcal{F}$ is a generalized η-Einstein manifold. □

Definition 2. *Let $\mathcal{F}$ be an n-dimensional NKM, then $\mathcal{F}$ is said to be v-M-projectively flat with respect to GSMC $\bar{\nabla}$ if $\bar{J}(\tau_X,\tau_Y)v = 0$ for all $\tau_X,\tau_Y,\tau_Z \in \chi(\mathcal{F})$, where $\bar{J}$ is the M-projective curvature tensor with respect to GSMC $\bar{\nabla}$.*

Theorem 7. *A v-M-projectively flat NKM $\mathcal{F}$ of the n-dimension admitting GSMC $\bar{\nabla}$ is a generalized η-Einstein manifold.*

Proof. Let $\mathcal{F}$ be an n-dimensional v-M-projectively flat NKM with respect to GSMC $\bar{\nabla}$, that is, $\bar{J}(\tau_X,\tau_Y)v = 0$. Then, from Equation (47), we obtain

$$\begin{aligned} \bar{K}(\tau_X,\tau_Y)v =& \frac{1}{2(n-1)}\big[\bar{Ric}(\tau_Y,v)\tau_X - \bar{Ric}(\tau_X,v)\tau_Y \\ &+ \eta(\tau_Y)\bar{Q}\tau_X - \eta(\tau_X)\bar{Q}\tau_Y\big]. \end{aligned} \tag{52}$$

by applying Equations (30) and (36) into Equation (52), we have

$$\begin{aligned} &(\alpha+1)[\eta(\tau_X)\tau_Y - \eta(\tau_Y)\tau_X] + \beta(\alpha+1)[\eta(\tau_Y)\phi\tau_X - \eta(\tau_X)\phi\tau_Y] \\ =& \frac{1}{2(n-1)}[(\alpha+1)(1-n)\eta(\tau_Y)\tau_X - (\alpha+1)(1-n)\eta(\tau_X)\tau_Y \\ &+ \eta(\tau_Y)\bar{Q}\tau_X - \eta(\tau_X)\bar{Q}\tau_Y]. \end{aligned} \tag{53}$$

By taking the inner product with a vector field τ_U in (53), we have

$$\begin{aligned}&(\alpha+1)[\eta(\tau_X)g(\tau_Y,\tau_U)-\eta(\tau_Y)g(\tau_X,\tau_U)]\\&+\beta(\alpha+1)[\eta(\tau_Y)g(\phi\tau_X,\tau_U)-\eta(\tau_X)g(\phi\tau_Y,\tau_U)]\\=&\frac{1}{2(n-1)}[(\alpha+1)(1-n)\eta(\tau_Y)g(\tau_X,\tau_U)\\&-(\alpha+1)(1-n)\eta(\tau_X)g(\tau_Y,\tau_U)+\eta(\tau_Y)\bar{Ric}(\tau_X,\tau_U)\\&-\eta(\tau_X)\bar{Ric}(\tau_Y,\tau_U)].\end{aligned}\tag{54}$$

By setting $\tau_Y = v$ and using Equations (2), (3), (34) and (35) in Equation (54), we obtain

$$Ric(\tau_X,\tau_U)=\gamma_1 g(\tau_X,\tau_U)+\gamma_2\eta(\tau_X)\eta(\tau_U)+\gamma_3 g(\phi\tau_X,\tau_U),\tag{55}$$

where

$$\gamma_1 = 2(n-1)[\alpha(\alpha+1)(n-2)+1-n],$$
$$\gamma_2 = \alpha(\alpha+1)(2-n)=\lambda_2 \quad \text{and} \quad \gamma_3=\beta(\alpha+1)(2n-3).$$

Therefore, Equation (55) verifies that the manifold $\mathcal{F}$ is a generalized η-Einstein manifold. □

Let $\{e_i\}(1\le i\le n)$ be an orthonormal basis of the tangent space at any point of the manifold $\mathcal{F}$.

By setting $\tau_X=\tau_U=e_i$ in (55) and taking summation over $i, 1\le i\le n$, we obtain

$$r=2n(n-1)[\alpha(\alpha+1)(n-2)+1-n]-\alpha(\alpha+1)(n-2),\tag{56}$$

which shows that the scalar curvature r with respect to the LCC ∇ is constant.
This leads to the following:

Theorem 8. *If an n-dimensional NKM $\mathcal{F}$ with respect to GSMC $\bar{\nabla}$ is v-M-projectively flat, then the scalar curvature with respect to the LCC ∇ is constant.*

7. Quasi *M*-Projectively Flat and ϕ-*M*-Projectively Flat NKM with Respect to GSMC

Definition 3. *An n-dimensional NKM $\mathcal{F}$ is said to be quasi M-projectively flat with respect to GSMC $\bar{\nabla}$ if*

$$g(\bar{J}(\phi\tau_X,\tau_Y)\tau_Z,\phi\tau_U)=0\tag{57}$$

for all $\tau_X,\tau_Y,\tau_Z,\tau_U\in\chi(\mathcal{F})$, where $\bar{J}$ is the M-projective curvature tensor with respect to GSMC $\bar{\nabla}$.

Theorem 9. *A quasi M-projectively flat NKM $\mathcal{F}$ of n-dimension with respect to GSMC $\bar{\nabla}$ is a generalized η-Einstein manifold.*

Proof. The (0,4)-type tensor field of Equation (47) is given by

$$\begin{aligned}g(\bar{J}(\tau_X,\tau_Y)\tau_Z,\tau_U)=&g(\bar{K}(\tau_X,\tau_Y)\tau_Z,\tau_U)\\&-\frac{1}{2(n-1)}\big[\bar{Ric}(\tau_Y,\tau_Z)g(\tau_X,\tau_U)\\&-\bar{Ric}(\tau_X,\tau_Z)g(\tau_Y,\tau_U)+g(\tau_Y,\tau_Z)\bar{Ric}(\tau_X,\tau_U)\\&-g(\tau_X,\tau_Z)\bar{Ric}(\tau_Y,\tau_U)\big].\end{aligned}\tag{58}$$

By replacing τ_X with $\phi\tau_X$ and τ_U with $\phi\tau_U$ in Equation (58), we obtain

$$\begin{aligned} g(\bar{J}(\phi\tau_X,\tau_Y)\tau_Z,\phi\tau_U) =& g(\bar{K}(\phi\tau_X,\tau_Y)\tau_Z,\phi\tau_U) \\ &- \frac{1}{2(n-1)}\big[\bar{Ric}(\tau_Y,\tau_Z)g(\phi\tau_X,\phi\tau_U) \\ &- \bar{Ric}(\phi\tau_X,\tau_Z)g(\tau_Y,\phi\tau_U) \\ &+ g(\tau_Y,\tau_Z)\bar{Ric}(\phi\tau_X,\phi\tau_U) \\ &- g(\phi\tau_X,\tau_Z)\bar{Ric}(\tau_Y,\phi\tau_U)\big]. \end{aligned} \tag{59}$$

Now, let us assume that $\mathcal{F}$ is a quasi M-projectively flat NKM with respect to GSMC $\bar{\nabla}$. Then, from Equations (57) and (59), we have

$$\begin{aligned} g(\bar{K}(\phi\tau_X,\tau_Y)\tau_Z,\phi\tau_U) =& \frac{1}{2(n-1)}\big[\bar{Ric}(\tau_Y,\tau_Z)g(\phi\tau_X,\phi\tau_U) \\ &- \bar{Ric}(\phi\tau_X,\tau_Z)g(\tau_Y,\phi\tau_U) \\ &+ g(\tau_Y,\tau_Z)\bar{Ric}(\phi\tau_X,\phi\tau_U) \\ &- g(\phi\tau_X,\tau_Z)\bar{Ric}(\tau_Y,\phi\tau_U)\big]. \end{aligned} \tag{60}$$

Let $\{e_1, e_2, \ldots, e_{n-1}, v\}$ be a local orthonormal moving frame of n-dimensional manifold $\mathcal{F}$, then $\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, v\}$ is also a local orthonormal moving frame of n-dimensional manifold $\mathcal{F}$.

By setting $\tau_X = \tau_U = e_i$ in Equation (60) and taking summation over $i, 1 \leq i \leq n-1$, we obtain

$$\begin{aligned} \sum_{i=1}^{n-1} g(\bar{K}(\phi e_i,\tau_Y)\tau_Z,\phi e_i) =& \frac{1}{2(n-1)}\sum_{i=1}^{n-1}\big[\bar{Ric}(\tau_Y,\tau_Z)g(\phi e_i,\phi e_i) \\ &- \bar{Ric}(\phi e_i,\tau_Z)g(\tau_Y,\phi e_i) \\ &+ g(\tau_Y,\tau_Z)\bar{Ric}(\phi e_i,\phi e_i) \\ &- g(\phi e_i,\tau_Z)\bar{Ric}(\tau_Y,\phi e_i)\big]. \end{aligned} \tag{61}$$

Also,

$$\sum_{i=1}^{n-1} g(\bar{K}(\phi e_i,\tau_Y)\tau_Z,\phi e_i) = \bar{Ric}(\tau_Y,\tau_Z) + g(\tau_Y,\tau_Z), \tag{62}$$

$$\sum_{i=1}^{n-1} g(\phi e_i,\phi e_i) = n-1, \tag{63}$$

$$\sum_{i=1}^{n-1} \bar{Ric}(\phi e_i,\phi e_i) = \sum_{i=1}^{n-1} \bar{Ric}(e_i,e_i) = \bar{r}, \tag{64}$$

$$\sum_{i=1}^{n-1} \bar{Ric}(\phi e_i,\tau_Z)g(\tau_Y,\phi e_i) = \bar{Ric}(\tau_Y,\tau_Z). \tag{65}$$

Applying Equation (62) to Equation (65) in Equation (61) gives

$$(n+1)\bar{Ric}(\tau_Y,\tau_Z) = (\bar{r}-2n+2)g(\tau_Y,\tau_Z). \tag{66}$$

Again, with the help of Equations (34) and (35) in Equation (66), we obtain

$$Ric(\tau_Y,\tau_Z) = \delta_1 g(\tau_Y,\tau_Z) + \delta_2\eta(\tau_Y)\eta(\tau_Z) + \delta_3 g(\phi\tau_Y,\tau_Z), \tag{67}$$

where

$$\delta_1 = \frac{r + \alpha^2 n(1-n) + 2\alpha(n-2\alpha) - 5\alpha - 2n + 2 + \alpha n(n+2)}{n+1},$$
$$\delta_2 = \alpha(\alpha+1)(2-n) = \lambda_2 = \gamma_2,$$
$$\delta_3 = -\beta(\alpha+1) = \lambda_3.$$

Thus, Equation (67) shows that $\mathcal{F}$ is a generalized η-Einstein manifold. □

Next, we will consider whether an NKM $\mathcal{F}$ admitting GSMC $\bar{\nabla}$ is ϕ-M-projectively flat.

Definition 4. *An n-dimensional NKM $\mathcal{F}$ admitting GSMC $\bar{\nabla}$ is said to be ϕ-M-projectively flat if $g(\bar{J}(\phi\tau_X, \phi\tau_Y, \phi\tau_Z, \phi\tau_U) = 0$ for all $\tau_X, \tau_Y, \tau_Z, \tau_U \in \chi(\mathcal{F})$.*

Theorem 10. *A ϕ-M-projectively flat NKM $\mathcal{F}$ of the n-dimension admitting GSMC $\bar{\nabla}$ is a generalized η-Einstein manifold.*

Proof. Suppose that $g(\bar{J}(\phi\tau_X, \phi\tau_Y, \phi\tau_Z, \phi\tau_U) = 0$, then the (0,4)-type tensor of Equation (47) is given by

$$\begin{aligned} g(\bar{K}(\phi\tau_X, \phi\tau_Y)\phi\tau_Z, \phi\tau_U) =& \frac{1}{2(n-1)}\big[\bar{Ric}(\phi\tau_Y, \phi\tau_Z)g(\phi\tau_X, \phi\tau_U) \\ &- \bar{Ric}(\phi\tau_X, \phi\tau_Z)g(\phi\tau_Y, \phi\tau_U) \\ &+ g(\phi\tau_Y, \phi\tau_Z)\bar{Ric}(\phi\tau_X, \phi\tau_U) \\ &- g(\phi\tau_X, \phi\tau_Z)\bar{Ric}(\phi\tau_Y, \phi\tau_U)\big]. \end{aligned} \tag{68}$$

Let $\{e_1, e_2, \ldots, e_{n-1}, v\}$ be a local orthonormal moving frame of n-dimensional manifold $\mathcal{F}$, then $\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, v\}$ is also a local orthonormal moving frame of n-dimensional manifold $\mathcal{F}$.
By setting $\tau_X = \tau_U = e_i$ in Equation (68) and taking summation over $i, 1 \le i \le n-1$, we obtain

$$\begin{aligned} \sum_{i=1}^{n-1} g(\bar{K}(\phi e_i, \phi\tau_Y)\phi\tau_Z, \phi e_i) =& \frac{1}{2(n-1)}\sum_{i=1}^{n-1}\big[\bar{Ric}(\phi\tau_Y, \phi\tau_Z)g(\phi e_i, \phi e_i) \\ &- \bar{Ric}(\phi e_i, \phi\tau_Z)g(\phi\tau_Y, \phi e_i) \\ &+ g(\phi\tau_Y, \phi\tau_Z)\bar{Ric}(\phi e_i, \phi e_i) \\ &- g(\phi e_i, \phi\tau_Z)\bar{Ric}(\phi\tau_Y, \phi e_i)\big]. \end{aligned} \tag{69}$$

Also,

$$\sum_{i=1}^{n-1} g(\bar{K}(\phi e_i, \phi\tau_Y)\phi\tau_Z, \phi e_i) = \bar{Ric}(\phi\tau_Y, \phi\tau_Z) + g(\phi\tau_Y, \phi\tau_Z), \tag{70}$$

$$\sum_{i=1}^{n-1} \bar{Ric}(\phi e_i, \phi\tau_Z)g(\phi\tau_Y, \phi e_i) = \bar{Ric}(\phi\tau_Y, \phi\tau_Z). \tag{71}$$

With the help of Equations (63), (64), (70) and (71) in Equation (69), we obtain

$$(n+1)\bar{Ric}(\phi\tau_Y, \tau_Z) = (\bar{r} - 2n + 2)g(\phi\tau_Y, \phi\tau_Z). \tag{72}$$

From Equations (4), (35), (37) and (72), we have

$$Ric(\tau_Y, \tau_Z) = \omega_1 g(\tau_Y, \tau_Z) + \omega_2\eta(\tau_Y)\eta(\tau_Z) + \omega_3 g(\phi\tau_Y, \tau_Z), \tag{73}$$

where

$$\omega_1 = \frac{r + 2\alpha^2 n - 4\alpha^2 + 3\alpha n - 5\alpha - 2n + 2}{n+1},$$
$$\omega_2 = -\frac{r + 2\alpha^2 n - 4\alpha + 3\alpha n - 5\alpha - 2n + n^2 + 1}{n+1},$$
$$\omega_3 = -\beta(\alpha+1) = \lambda_3 = \delta_3.$$

Therefore, Equation (73) reveals that $\mathcal{F}$ is a generalized η-Einstein manifold. □

8. NKM Satisfying $\bar{J}.\bar{Ric} = 0$

Definition 5. *An n-dimensional NKM $\mathcal{F}$ with respect to GSMC $\bar{\nabla}$ satisfies*

$$(\bar{J}(\tau_X, \tau_Y).\bar{Ric})(\tau_Z, \tau_U) = 0 \tag{74}$$

for all $\tau_X, \tau_Y, \tau_Z, \tau_U \in \chi(\mathcal{F})$.

From Equation (74), we are able to find that

$$\bar{Ric}(\bar{J}(\tau_X, \tau_Y)\tau_Z, \tau_U) + \bar{Ric}(\tau_Z, \bar{J}(\tau_X, \tau_Y)\tau_U) = 0. \tag{75}$$

Arranging $\tau_X = \tau_Z = v$ in Equation (75), we obtain

$$\bar{Ric}(\bar{J}(v, \tau_Y)v, \tau_U) + \bar{Ric}(v, \bar{J}(v, \tau_Y)\tau_U) = 0. \tag{76}$$

By inserting Equation (30) into Equations (32), (36) and (38) in Equation (76), we obtain

$$\begin{aligned}\beta(\alpha+1)\bar{Ric}(\phi\tau_Y, \tau_U) =& \frac{(\alpha+1)}{2}\bar{Ric}(\tau_Y, \tau_U) + \frac{(\alpha+1)^2(n-1)}{2}g(\tau_Y, \tau_U) \\ &- \beta(\alpha+1)^2(n-1)g(\phi\tau_Y, \tau_U).\end{aligned} \tag{77}$$

By changing τ_Y with $\phi\tau_Y$ in Equation (77) and using Equation (36), we obtain

$$\begin{aligned}\bar{Ric}(\phi\tau_Y, \tau_U) =& -2\beta\bar{Ric}(\tau_Y, \tau_U) - 2\beta(\alpha+1)(n-1)g(\tau_Y, \tau_U) \\ &- 2(\alpha+1)(n-1)g(\phi\tau_Y, \tau_U).\end{aligned} \tag{78}$$

From Equations (77) and (78), we have

$$\bar{Ric}(\tau_Y, \tau_U) = \frac{t_2}{t_1}g(\tau_Y, \tau_U) + \frac{t_3}{t_1}g(\phi\tau_Y, \tau_U), \tag{79}$$

where

$$t_1 = \frac{(\alpha+1)(4\beta^2+1)}{2},$$
$$t_2 = -\frac{(\alpha+1)^2(4\beta^2+1)(n-1)}{2},$$
$$t_3 = 3\beta(\alpha+1)^2(n-1).$$

Finally, from Equations (34) and (79) we obtain

$$Ric(\tau_Y, \tau_U) = \mu_1 g(\tau_Y, \tau_U) + \mu_2\eta(\tau_Y)\eta(\tau_U) + \mu_3 g(\phi\tau_Y, \tau_U), \tag{80}$$

where

$$\mu_1 = \frac{t_2}{t_1} - \alpha(2\alpha + 3 - 2n - \alpha n),$$

$$\mu_2 = \alpha(\alpha+1)(2-n) = \lambda_2 = \gamma_2 = \delta_2,$$
$$\mu_3 = \frac{t_3}{t_1} - \beta(\alpha+1).$$

Therefore, we can state the following:

Theorem 11. *An n-dimensional NKM $\mathcal{F}$ with respect to GSMC $\bar{\nabla}$ satisfying $\bar{J}.\bar{Ric} = 0$ is a generalized η-Einstein manifold.*

9. ϕ-Ricci Symmetric NKM

Definition 6 ([26]). *An n-dimensional NKM $\mathcal{F}$ with respect to GSMC $\bar{\nabla}$ is said to be ϕ-Ricci symmetric if the Ricci operator satisfies*

$$\phi^2((\bar{\nabla}_{\tau_X}\bar{Q})(\tau_Y)) = 0 \tag{81}$$

for any vector field τ_X, τ_Y on $\mathcal{F}$ and $\bar{S}(\tau_X, \tau_Y) = g(\bar{Q}\tau_X, \tau_Y)$.

Theorem 12. *An n-dimensional ϕ-Ricci symmetric NKM $\mathcal{F}$ equipped with GSMC $\bar{\nabla}$ is a generalized η-Einstein manifold.*

Proof. Let us consider that the manifold M is ϕ-Ricci symmetric. Then, we have

$$\phi^2((\bar{\nabla}_{\tau_X}\bar{Q})(\tau_Y)) = 0.$$

Using Equation (2) and taking the inner product with τ_Z in the above equation, we obtain

$$-g((\bar{\nabla}_{\tau_X}\bar{Q})(\tau_Y), \tau_Z) + \eta((\bar{\nabla}_{\tau_X}\bar{Q})(\tau_Y))\eta(\tau_Z) = 0. \tag{82}$$

By inserting Equation (38) and setting $\tau_Y = v$ in (82), we have

$$-g(\bar{\nabla}_{\tau_X}\bar{Q}v, \tau_Z) + \bar{Ric}(\bar{\nabla}_{\tau_X}v, \tau_Z) + \eta((\bar{\nabla}_{\tau_X}\bar{Q})(v))\eta(\tau_Z) = 0. \tag{83}$$

Again, by using Equations (26), (36) and (38) in Equation (83), we have

$$(\alpha+1)\bar{Ric}(\tau_X, \tau_Z) + (\alpha+1)^2(n-1)g(\tau_X, \tau_Z) + \eta((\bar{\nabla}_{\tau_X}\bar{Q})(v))\eta(\tau_Z) = 0. \tag{84}$$

By replacing τ_X with $\phi\tau_X$ and τ_Z with $\phi\tau_Z$ in Equation (84), we obtain

$$(\alpha+1)\bar{Ric}(\phi\tau_X, \phi\tau_Z) + (\alpha+1)^2(n-1)g(\phi\tau_X, \phi\tau_Z) = 0. \tag{85}$$

From Equations (4), (37) and (85), we have

$$\bar{Ric}(\tau_X, \tau_Z) = \psi_1 g(\tau_X, \tau_Z) + \psi_2\eta(\tau_X)\eta(\tau_Z) + \psi_3 g(\phi\tau_X, \tau_Z), \tag{86}$$

where

$$\psi_1 = 3\alpha n - 4\alpha - 2\alpha^2 + \alpha^2 n + n - 1,$$
$$\psi_2 = -(3\alpha n - 4\alpha - 2\alpha^2 + \alpha^2 n) - 2(n-1),$$
$$\psi_3 = -\beta(\alpha+1) = \lambda_3 = \gamma_3 = \delta_3.$$

Thus, the proof is completed. □

10. Example

A three-dimensional manifold $\mathcal{F} = \{(x_1, x_2, x_3) \in \mathbb{R}^3, (x_1, x_2, x_3) \neq (0,0,0)\}$ is considered, where (x_1, x_2, x_3) are regarded as the standard coordinates in $\mathbb{R}^3$.

Suppose that e_1, e_2, e_3 are linearly independent global frames on $\mathcal{F}$ as given by

$$e_1 = x_3 \frac{\partial}{\partial x_1}, \quad e_2 = x_3 \frac{\partial}{\partial x_2}, \quad e_2 = -x_3 \frac{\partial}{\partial x_3}.$$

Let us assume that g is a Lorentzian metric defined as

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0, \quad g(e_i, e_i) = 1 \ \forall \ i = 1,2,3.$$

When we can consider that η is a 1-form represented as $\eta(\tau_Z) = g(\tau_Z, e_3)$ for every $\tau_Z \in \chi(\mathcal{F})$ and ϕ is the (1,1)-type tensor field determined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Using the linearity property of ϕ and g, we have

$$\eta(e_3) = 1, \quad \phi^2 \tau_Z = -\tau_Z + \eta(\tau_Z) e_3, \quad g(\phi\tau_Z, \phi\tau_U) = g(\tau_Z, \tau_U) - \eta(\tau_Z)\eta(\tau_U)$$

for all $\tau_Z, \tau_U \in \chi(\mathcal{F})$. Thus, $e_3 = \upsilon$, the structure $(\phi, \upsilon, \eta, g)$ defines an almost contact metric structure on $\mathcal{F}^3$.

Let ∇ be the LCC with respect to the metric g, then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

Koszul's formula is given by

$$\begin{aligned} 2g(\nabla_{\tau_X} \tau_Y, \tau_Z) =& \tau_X g(\tau_Y, \tau_Z) + \tau_Y g(\tau_Z, \tau_X) - \tau_Z g(\tau_X, \tau_Y) - g(\tau_X, [\tau_Y, \tau_Z]) \\ & - g(\tau_Y, [\tau_X, \tau_Z]) + g(\tau_Z, [\tau_X, \tau_Y]). \end{aligned}$$

From the above formula, the results are obtained as

$$\begin{aligned} &\nabla_{e_1} e_1 = e_1, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \\ &\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = e_2, \\ &\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \end{aligned}$$

Thus, $\mathcal{F}^3$ satisfies $\nabla_{\tau_X} \upsilon = \tau_X - \eta(\tau_X)\upsilon$, where $\upsilon = e_3$.
With the help of (24), the calculations become

$$\begin{aligned} &\bar{\nabla}_{e_1} e_1 = -\alpha e_3, \quad \bar{\nabla}_{e_1} e_2 = 0, \quad \bar{\nabla}_{e_1} e_3 = (\alpha + 1) e_1, \\ &\bar{\nabla}_{e_2} e_1 = 0, \quad \bar{\nabla}_{e_2} e_2 = -\alpha e_3, \quad \bar{\nabla}_{e_2} e_3 = (\alpha + 1) e_2, \\ &\bar{\nabla}_{e_3} e_1 = \beta e_2, \quad \bar{\nabla}_{e_3} e_2 = -\beta e_1, \quad \bar{\nabla}_{e_3} e_3 = 0. \end{aligned}$$

From the above relations, it can be seen that

$$T(e_1, e_3) = \bar{\nabla}_{e_1} e_3 - \bar{\nabla}_{e_3} e_1 - \bar{\nabla}_{[e_1, e_3]} = \alpha e_1 - \beta e_2.$$

Therefore, the above equation shows that Equation (1) is satisfied. Also, it can be easily verified that $(\bar{\nabla}_{e_i} g)(e_j, e_k) = 0$ for $i, j, k = 1,2,3$, meaning that the connection referred to in Equation (24) is a metric connection.

Thus, a linear connection $\bar{\nabla}$ is a GSMC of the (α, β) type on $\mathcal{F}$.

11. Conclusions

The findings of this investigation shed light on the intricate interplay between geometry and symmetry within nearly Kenmotsu manifolds, offering a nuanced perspective on the underlying mathematical structures at play. The identification and characterization of generalized symmetric connections of type (α, β) present a versatile framework that enriches the existing literature on metric connections and their applications.

Moreover, the delineation of these generalized connections paves the way for a deeper understanding of the geometric properties and invariants associated with nearly Kenmotsu manifolds, thereby contributing to the broader field of Riemannian geometry. By elucidating the relationships between different classes of symmetric connections, this research not only extends the theoretical foundations of differential geometry but also provides a basis for practical applications in various scientific disciplines, such as theoretical physics, cosmology, and mathematical biology.

In essence, this study represents a significant step forward in the exploration of symmetric metric connections in the context of nearly Kenmotsu manifolds, offering a comprehensive framework that has the potential to inspire further research endeavors and foster new insights into the geometric and algebraic structures that underpin these complex mathematical spaces.

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