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Initial Value Problem for Mixed Differential Equations of Variable Order with Finite Delay

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Abstract: This study presents a fresh perspective on the existence, uniqueness, and stability of solutions for initial value problems involving variable-order differential equations with finite delay. Departing from conventional techniques that utilize generalized intervals and piecewise constant functions, we introduce a novel fractional operator tailored for this specific problem. Our methodology integrates sophisticated mathematical analysis, including the Schauder fixed-point theorem and Banach's contraction principle, with an examination of the Ulam–Hyers stability of the problem. The strength of our approach is in its simplicity, requiring fewer restrictive assumptions. We conclude with a practical application to illustrate our findings. These results are valuable for understanding complex dynamical systems with time delays, offering applications in diverse fields such as engineering, economics, and medicine, and enhancing numerical methods for solving delay equations.

Keywords: mixed derivatives; finite delay; variable order; ordinary and fractional



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1. Introduction

Fractional calculus extends the traditional concepts of differentiation and integration to include non-integer orders. The theoretical exploration of fractional calculus can branch out into various areas of investigation. While the fundamental ideas trace back to Leibniz's introduction of the concept in 1695, the study of fractional differential calculus has evolved significantly over the last three centuries. Fractional differential equations, in particular, have become a prominent and actively researched area within the broader field of fractional calculus. Functional differential equations find application in diverse scientific and engineering disciplines, including biology, physics, and engineering (see, for example, [1,2]).

The concept and formal definition of variable-order fractional operators have recently gained prominence. In contrast to fixed-order fractional operators, which operate at a constant order, variable-order differentiation and integration enable the order to change continuously as a function of one or more variables. This approach offers enhanced adaptability compared to traditional fractional-order techniques and represents a logical

progression within the mathematical structure [3–5]. These operators have demonstrated their effectiveness in capturing the intricacies of real-world phenomena in diverse fields, including biology, mechanics, control systems, and transport processes. Their capacity to develop evolutionary governing equations has made them a focal point of significant research efforts. Consequently, numerous studies have investigated their applications in modeling engineering and physical systems, as illustrated in several key publications [6–8]. Nevertheless, research concerning the application of nonlinear differential equations with variable fractional order remains relatively scarce (see [9–12]). It is noteworthy that much of the existing research relies significantly on the concept of piecewise constant functions, which is pivotal in addressing these problems. Therefore, the interval of existence $[0, \bar{t}]$ is partitioned into σ intervals as follows: $\{I_1 = [0, j_1], I_2 = (j_1, j_2], I_3 = (j_2, j_3], \dots, I_\sigma = (j_{\sigma-1}, \bar{t}]\}$, where σ is a natural number. A piecewise constant function $\aleph(j): [0, \bar{t}] \rightarrow (0, 1]$ is then defined as follows:

$$\aleph(j) = \begin{cases} \aleph_1, & \text{if } j \in I_1, \\ \aleph_2, & \text{if } j \in I_2, \\ \vdots & \\ \vdots & \\ \aleph_\sigma, & \text{if } j \in I_\sigma, \end{cases}$$

In this context, we have constants $0 < \aleph_k \leq 1$, where k ranges from 1 to σ . A common strategy in existing research involves partitioning the domain of interest into smaller subintervals. Differential and integral operators are then defined based on these subdivisions. This method effectively converts fractional-order problems with variable orders into corresponding problems with constant orders, simplifying the analysis. Boundary value problems involving differential equations with mixed derivatives of variable orders form a vital area of modern mathematical analysis. Their importance stems from their capacity to accurately represent complex dynamical systems that exhibit nonlinear and interconnected behaviors. Symmetry analysis offers a fundamental framework for examining these equations. By employing symmetry techniques, equations can be simplified, potential solutions can be identified, and insights into the evolving dynamics of the systems can be gained. As evidenced by recent publications (e.g., [13–15]), the impact of symmetry on boundary values and solutions for these equations has been thoroughly explored. This research has led to the development of innovative applications in both engineering and the physical sciences, where precise mathematical models are essential. The work in [16] demonstrates the application of the differential transform method (DTM) to differential equations with delayed arguments. Specifically, it focuses on a method designed for solving certain integro-differential equations with a retarded (delayed) argument. The proposed procedure leverages the Taylor differential transformation to convert the given integro-differential equation into a corresponding system of algebraic equations, often nonlinear.

Conventional approaches to variable-order fractional differential equation problems often approximate the variable order using a piecewise constant function. Consequently, these investigations center on equivalent problems involving fractional differential equations with constant orders. In this work, we introduce a new perspective that explores the finite delay of a variable-order initial value problem without resorting to the piecewise constant function approximation. The primary objective is to present an innovative technique that bypasses the need for both a piecewise constant function and the decomposition of existence intervals. Therefore, we directly address the variable order, acknowledging that the established properties of fractional calculus, which are valid only for constant orders, are not applicable. The cornerstone of our strategy is the development of a novel,

more versatile operator that circumvents the requirement for supplementary steps. We apply the developed technique to the following finite delay variable-order initial value problem (FDVOIP):

$$\begin{cases} \mathbb{D}^{\aleph(j)} \iota(j) + \varsigma \iota'(j) = \eta(j, \iota_j), & j \in \tilde{\varphi}_\tau: = [0, \tilde{t}], \\ \iota(j) = \phi(j), & j \in [-\bar{z}, 0], \end{cases} \quad (1)$$

where $\aleph: \tilde{\varphi}_\tau \rightarrow (0, 1)$, $\phi: [-\bar{z}, 0] \rightarrow \mathbb{R}$ are continuous functions with $\phi(0) = 0$ and $\eta: \tilde{\varphi}_\tau \times C([-\bar{z}, 0], \mathbb{R}) \rightarrow \mathbb{R}$ is a specified function. We denote by ι_j the element of $C([-\bar{z}, 0], \mathbb{R})$, defined by

$$\iota_j(\theta) = \iota(j + \theta), \theta \in [-\bar{z}, 0],$$

for any function ι defined on $[-\bar{z}, \tilde{t}]$ and any j in $\tilde{\varphi}_\tau$. Here, $\mathbb{D}^{\aleph(j)}$ represents the variable-order $\aleph(j)$ Riemann–Liouville fractional derivative, and $\iota_j(\cdot)$ describes the state's history, spanning from time $j - \bar{z}$ to the current time j .

The central objective of this paper is to investigate several qualitative properties of the solutions to the FDVOIP (1). The organization of our study is as follows: Section 2 begins by presenting essential preliminary information and characteristics pertaining to the variable-order $\aleph(j)$ Riemann–Liouville fractional derivative. Section 3 provides the existence criteria for solutions to the proposed FDVOIP (1). Section 4 demonstrates the results regarding the uniqueness of these solutions. Section 5 further examines the stability criteria of the derived solutions within the framework of Ulam–Hyers stability. Section 6 illustrates the theoretical conclusions with provided applications. The research concludes with the presentation of a conclusion.

2. Preliminary

This section outlines several notations, definitions, and fundamental terms employed throughout this paper.

It is important to note that $\mathbb{E} = C(\tilde{\varphi}_\tau, \mathbb{R})$ represents a Banach space consisting of continuous functions ι that map from $\tilde{\varphi}_\tau$ into $\mathbb{R}$. The norm for this space is defined as follows:

$$\|\iota\| = \sup\{|\iota(j)|/j \in \tilde{\varphi}_\tau\}.$$

Definition 1 ([7,17,18]). Let us consider a continuous function $\aleph: \tilde{\varphi}_\tau \rightarrow (0, 1)$. In this context, the left R-Liouville FDVO $\aleph(j)$ applied to a function $\iota(j)$ can be expressed as follows:

$$\mathbb{I}_{0+}^{\aleph(j)} \iota(j) = \int_0^j \frac{(j-\alpha)^{\aleph(\alpha)-1}}{\Gamma(\aleph(\alpha))} \iota(\alpha) d\alpha, \quad j > 0, \quad (2)$$

where $\Gamma(\cdot)$ denotes the gamma function.

Definition 2 ([7,17,18]). Consider a continuous function $\aleph: \tilde{\varphi}_\tau \rightarrow (0, 1)$. The left R-Liouville FDVO $\aleph(j)$ applied to a function $\iota(j)$ can be expressed as follows:

$$\mathbb{D}_{0+}^{\aleph(j)} \iota(j) = \left(\frac{d}{dt}\right) \mathbb{I}_{0+}^{1-\aleph(j)} \iota(j) = \left(\frac{d}{dt}\right) \int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} \iota(\alpha) d\alpha, \quad j > 0. \quad (3)$$

Remark 1 ([19–21]). For arbitrary functions $\aleph(j)$ and $v(j)$, it is worth noting that the semigroup property is not generally valid; that is,

$$\mathbb{I}_{a+}^{\aleph(j)} \mathbb{I}_{a+}^{v(j)} \iota(j) \neq \mathbb{I}_{a+}^{\aleph(j)+v(j)} \iota(j).$$

Lemma 1 ([22]). Consider a continuous function $\aleph: \tilde{\varphi}_\tau \rightarrow (0, 1)$. Then, for $y \in C_\delta(\tilde{\varphi}_\tau, \mathbb{R}) = \{y(j) \in C(\tilde{\varphi}_\tau, \mathbb{R}), j^\delta y(j) \in C(\tilde{\varphi}_\tau, \mathbb{R}), (0 \leq \delta < \min \aleph(j))\}$, the variable-order fractional integral $\mathbb{I}_{0+}^{\aleph(j)} y(j)$ exists for all $j \in \tilde{\varphi}_\tau$.

Lemma 2 ([22]). Suppose $\aleph \in C(\tilde{\varphi}_\tau, (0, 1])$ is a continuous function. If $y \in C(\tilde{\varphi}_\tau, \mathbb{R})$, it follows that $\mathbb{I}_{0+}^{\aleph(j)} y(j) \in C(\tilde{\varphi}_\tau, \mathbb{R})$.

Definition 3. Consider $\mathfrak{S}$ as a Banach space and $\wp: \mathfrak{S} \rightarrow \mathfrak{S}$ as an operator. The operator $\wp$ is deemed completely continuous if it satisfies two conditions: first, $\wp$ is continuous; and, second, for every bounded set K in $\mathfrak{S}$ its image, $\wp(K)$, is relatively compact.

Theorem 1 ([23], Ascoli–Arzelà). Let $\mathcal{M}$ a subset of $C(\tilde{\varphi}_\tau, \mathbb{R})$. Then, it is relatively compact provided that

- $\mathcal{M}$ is uniformly bounded.
- $\mathcal{M}$ is equicontinuous.

In order to establish the existence and uniqueness of solutions for a finite delay variable-order initial value problem, denoted as FDVOIVP (1), we will use the following fixed-point theorems.

Theorem 2 ([24,25]). Consider $(\mathfrak{S}, d)$ to be a complete metric space. If the mapping $\wp: \mathfrak{S} \rightarrow \mathfrak{S}$ is a contraction with Lipschitz constant k , then $\wp$ possesses a unique fixed point $\iota \in \mathfrak{S}$.

Theorem 3 ([24,25]). Suppose $\mathfrak{S}$ is a Banach space, and let K be a non-empty, closed, bounded, and convex subset of $\mathfrak{S}$. If $\wp: K \rightarrow K$ is a completely continuous operator, then $\wp$ has at least one fixed point.

3. Existence Criteria

Remark 2. We present the following observations:

1. Since the function $\Gamma(1-\aleph(j))$ is composite by two continuous functions, it is also continuous. Therefore, we can define $M_\Gamma = \max \left| \frac{1}{\Gamma(1-\aleph(j))} \right|$.
2. Considering the continuous nature of the function $\aleph(j, \iota(j))$, the following inequalities hold:
 $\tilde{t}^{-\aleph(j)} \leq 1$ when $1 \leq \tilde{t} < \infty$;
 $\tilde{t}^{-\aleph(j)} \leq \tilde{t}^{-\aleph^*}$ when $0 \leq \tilde{t} \leq 1$.
 From these, we can infer that $\tilde{t}^{-\aleph(j)} \leq \max(1, \tilde{t}^{-\aleph^*}) = \tilde{t}^*$.

Note that $\mathbb{E} = C([-z, \tilde{t}], \mathbb{R})$ represents a Banach space. This space comprises continuous functions ι mapping the interval $[-z, \tilde{t}]$ to the real numbers $\mathbb{R}$; the norm defined on this space is given by the following:

$$\|\iota\|_{\mathbb{E}} = \sup\{|\iota(j)|/j \in [-z, \tilde{t}]\}.$$

Definition 4. A function ι belonging to $\mathbb{E}$ is considered a solution to FDVOIP (1) if it fulfills the following criteria: $\mathbb{D}^{\aleph(j)}\iota(j) + \varsigma\iota'(j) = \eta(j, \iota_j)$ on $\tilde{\varphi}_\tau$, and satisfying the condition $\iota(j) = \phi(j)$ on the interval $[-\bar{z}, 0]$.

The subsequent lemma is essential for obtaining a solution to the FDVOIP as defined in (1).

Lemma 3. The function ι represents a solution to the subsequent initial value problem involving a fractional differential equation:

$$\begin{cases} \mathbb{D}^{\aleph(j)}\iota(j) + \varsigma\iota'(j) = v(j), & j \in \tilde{\varphi}_\tau := [0, \bar{t}], \\ \iota(0) = 0, \end{cases} \quad (4)$$

Here, the fractional order $\aleph(j)$ satisfies $0 < \aleph(j) < 1$, and the function v , defined as $v: (0, \bar{t}] \rightarrow \mathbb{R}$, is continuous. The function ι fulfills the initial value problem if and only if it also solves the fractional integral equation given by the following:

$$\int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} \iota(\alpha) \, d\alpha + \varsigma\iota(j) = \int_0^j v(\alpha) \, d\alpha. \quad (5)$$

Proof. Utilizing the definition of the variable-order fractional derivative as presented in (3), the Fractional Differential Variational-Order Initial Value Problem (FDVOIP) (1) can be reformulated as follows:

$$\left(\frac{d}{dt}\right) \int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} \iota(\alpha) \, d\alpha + \varsigma\iota'(j) = v(j).$$

Consequently, we obtain:

$$\int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} \iota(\alpha) \, d\alpha + \varsigma\iota(j) = \int_0^j v(\alpha) \, d\alpha + c_1. \quad (6)$$

Evaluating Equation (7) at $j = 0$ yields $c_1 = 0$. Therefore,

$$\int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} \iota(\alpha) \, d\alpha + \varsigma\iota(j) = \int_0^j v(\alpha) \, d\alpha.$$

Conversely, differentiating both sides of Equation (5) leads to the following:

$$\left(\frac{d}{dt}\right) \int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} \iota(\alpha) \, d\alpha + \varsigma\iota'(j) = v(j),$$

This demonstrates that ι is a solution to FDVOIP (4). $\square$

Theorem 3 provides the groundwork for our initial result. To begin, we establish the subsequent assumptions.

(FY1) The function $\eta: \tilde{\varphi}_\tau \times C([-\bar{z}, 0], \mathbb{R}) \rightarrow \mathbb{R}$ is assumed to be continuous.

(FY2) We postulate the existence of functions $\varrho, \vartheta \in C(\tilde{\varphi}_\tau, \mathbb{R}^+)$ such that $\|\eta(j, \iota) - \varrho(j) + \vartheta(j)\|_C \leq \|\iota\|_C$ holds for all $j \in \tilde{\varphi}_\tau$ and every $\iota \in C([-\bar{z}, 0], \mathbb{R})$.

Theorem 4. Suppose that assumptions (FY1) and (FY2) are valid. Then, there exists at least one solution within the space $\mathbb{E}$ for the FDVOIP (1).

Proof. We transform FDVOIVP (1) into an equivalent fixed-point problem. Let the operator $N: \mathbb{E} \rightarrow \mathbb{E}$.

$$N(\iota)(j) = \begin{cases} \phi(j), & \text{if } j \in [-\bar{z}, 0], \\ \frac{1}{\varsigma} \left[\int_0^j \eta(\alpha, \iota_\alpha) d\alpha - \int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} \iota(\alpha) d\alpha \right], & \text{if } j \in [0, \bar{t}]. \end{cases}$$

Define a constant R such that

$$R \geq \max \left\{ \frac{\frac{\|\varrho\|_\infty \bar{t}}{\varsigma}}{1 - \frac{1}{\varsigma} \left[\frac{M_\Gamma \bar{t}^* \bar{t}}{1-\aleph^*} + \|\vartheta\|_\infty \bar{t} \right]}, \|\phi\|_C \right\}.$$

It is evident that B_R is a nonempty, bounded, convex, and closed set.

We will now demonstrate that N satisfies the conditions stipulated by Theorem 3. The proof will proceed in several stages (four steps).

Step 1: $N(B_R) \subseteq B_R$.

For $\iota \in B_R$, and for any $j \in [0, \bar{t}]$, we have

$$\begin{aligned} |N(\iota)(j)| &\leq \frac{1}{\varsigma} \left[\left| \int_0^j \eta(\alpha, \iota_\alpha) d\alpha \right| + \left| \int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} \iota(\alpha) d\alpha \right| \right] \\ &\leq \frac{1}{\varsigma} \left[\int_0^j \eta(\alpha, \iota_\alpha) + \int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} |\iota(\alpha)| d\alpha \right] \\ &\leq \frac{1}{\varsigma} \left[\int_0^j \varrho(\alpha) + \vartheta(\alpha) \|\iota_\alpha\|_C d\alpha + M_\Gamma \int_0^j \bar{t}^{-\aleph(\alpha)} \left(\frac{j-\alpha}{\bar{t}} \right)^{-\aleph(\alpha)} |\iota(\alpha)| d\alpha \right] \\ &\leq \frac{1}{\varsigma} \left[\int_0^j \varrho(\alpha) d\alpha + \int_0^j \vartheta(\alpha) \|\iota_\alpha\|_C d\alpha + M_\Gamma \bar{t}^* \int_0^j \left(\frac{j-\alpha}{\bar{t}} \right)^{-\aleph(\alpha)} |\iota(\alpha)| d\alpha \right] \\ &\leq \frac{1}{\varsigma} \left[\|\varrho\|_\infty j + \|\vartheta\|_\infty \|\iota\|_{\mathbb{E}} j + \frac{M_\Gamma \bar{t}^*}{\bar{t}^{-\aleph^*}} \|\iota\|_\infty \int_0^j (j-\alpha)^{-\aleph^*} d\alpha \right] \\ &\leq \frac{1}{\varsigma} \left[\|\varrho\|_\infty \bar{t} + \|\vartheta\|_\infty \|\iota\|_{\mathbb{E}} \bar{t} + \frac{M_\Gamma \bar{t}^*}{\bar{t}^{-\aleph^*}} \frac{j^{1-\aleph^*}}{(1-\aleph^*)} \|\iota\|_{\mathbb{E}} \right] \\ &\leq \frac{1}{\varsigma} \left[\|\varrho\|_\infty \bar{t} + \|\vartheta\|_\infty \|\iota\|_{\mathbb{E}} \bar{t} + \frac{M_\Gamma \bar{t}^* \bar{t}}{(1-\aleph^*)} \|\iota\|_{\mathbb{E}} \right] \\ &\leq \frac{1}{\varsigma} \left[\frac{M_\Gamma \bar{t}^* \bar{t}}{1-\aleph^*} + \|\vartheta\|_\infty \bar{t} \right] \|\iota\|_{\mathbb{E}} + \frac{\|\varrho\|_\infty \bar{t}}{\varsigma} \\ &\leq \frac{1}{\varsigma} \left[\frac{M_\Gamma \bar{t}^* \bar{t}}{1-\aleph^*} + \|\vartheta\|_\infty \bar{t} \right] R + \frac{\|\varrho\|_\infty \bar{t}}{\varsigma} \\ &\leq R. \end{aligned}$$

If $j \in [-\bar{z}, 0]$, we obtain

$$|N(\iota)(j)| \leq \|\phi\|_C \leq R.$$

Consequently, $N(B_R) \subseteq B_R$.

Step 2: Continuity of N .

Consider a sequence (t_n) such that t_n converges to t in $\mathbb{E}$. When $j \in [-z, 0]$, it follows that

$$|N(t_n)(j) - N(t)(j)| = 0.$$

In the case where $j \in \tilde{\varphi}_\tau$, we have

$$\begin{aligned} & |N(t_n)(j) - N(t)(j)| \\ & \leq \frac{1}{\zeta} \left[\int_0^j \sup_{\alpha \in [0, \tilde{t}]} |\eta(\alpha, t_{n\alpha}) - \eta(\alpha, t_\alpha)| d\alpha + \int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} |t(\alpha) - t_n(\alpha)| d\alpha \right] \\ & \leq \frac{1}{\zeta} \left[\|\eta(\cdot, t_n) - \eta(\cdot, t)\|_\infty \int_0^j d\alpha + M_\Gamma \|t_n - t\|_\infty \int_0^j (j-\alpha)^{-\aleph(\alpha)} d\alpha \right] \\ & \leq \frac{1}{\zeta} \left[j \|\eta(\cdot, t_n) - \eta(\cdot, t)\|_\infty + M_\Gamma \tilde{t}^* \|t_n - t\|_\infty \int_0^j \left(\frac{j-\alpha}{\tilde{t}}\right)^{-\aleph^*} d\alpha \right] \\ & \leq \frac{1}{\zeta} \left[\tilde{t} \|\eta(\cdot, t_n) - \eta(\cdot, t)\|_\infty + \frac{M_\Gamma \tilde{t}^*}{\tilde{t}^{-\aleph^*}} \frac{(j)^{1-\aleph^*}}{(1-\aleph^*)} \|t_n - t\|_\infty \right] \\ & \leq \frac{1}{\zeta} \frac{M_\Gamma \tilde{t}^* \tilde{t}}{1-\aleph^*} \|t_n - t\|_\infty + \frac{\tilde{t}}{\zeta} \|\eta(\cdot, t_n) - \eta(\cdot, t)\|_\infty. \end{aligned}$$

Since $\left(\frac{1}{\zeta} \frac{M_\Gamma \tilde{t}^* \tilde{t}}{1-\aleph^*}\right) \|t_n - t\|_\infty$ approaches 0 as n approaches infinity, and $\frac{\tilde{t}}{\zeta} \|\eta(\cdot, t_n) - \eta(\cdot, t)\|_\infty$ also approaches 0 as n approaches infinity, it follows that

$$|N(t_n)(j) - N(t)(j)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore,

$$\|N(t_n)(j) - N(t)(j)\|_{\mathbb{E}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The preceding relation indicates that the operator N exhibits continuity on $\mathbb{E}$.

Step 3: Compactness of N .

Our aim here is to establish the relative compactness of $N(B_R)$, which will demonstrate the compactness of N . As a consequence of Step 1, $N(B_R)$ is uniformly bounded, as we have the following:

$$N(B_R) = \{N(t) : t \in B_R\} \subset B_R.$$

Thus, for every $t \in B_R$, we obtain $\|N(t)\| \leq R$ implying the uniform boundedness of $N(B_R)$. Initially, consider the function $w(j) = a^j - b^j$, $j \in (-1, 0)$, where $0 < a < b < 1$. This function is decreasing. Indeed, given that $\ln a < \ln b < 0$ and $a^j > b^j > 0$, we have

$$w'(j) = a^j \ln a - b^j \ln b < b^j \ln a - b^j \ln b = b^j (\ln a - \ln b) < 0,$$

which confirms that $w(j)$ is a decreasing function. Consequently, for $\varkappa(\alpha) = \left(\frac{j_1-\alpha}{\tilde{t}}\right)^{-\aleph(\alpha)} - \left(\frac{j_2-\alpha}{\tilde{t}}\right)^{-\aleph(\alpha)}$ (where $0 < \frac{j_1-\alpha}{\tilde{t}} < \frac{j_2-\alpha}{\tilde{t}} < 1$), we can consider $\varkappa(\alpha)$ to be of the same form as $w(\alpha)$. Therefore, $\varkappa(\alpha)$ decreases with respect to its exponent $-\aleph(\alpha)$. Hence, for $j_1, j_2 \in \tilde{\varphi}_\tau$, with $j_1 < j_2$ and for $t \in B_R$, we have

$$|Nt(j_2) - Nt(j_1)|$$

$$\begin{aligned}
&\leq \frac{1}{\varsigma} \left[\left| \int_0^{j_2} \eta(\alpha, \iota_\alpha) d\alpha - \int_0^{j_1} \eta(\alpha, \iota_\alpha) d\alpha \right| \right. \\
&\quad \left. + \left| \int_0^{j_1} \frac{(j_1 - \alpha)^{-\aleph(\alpha)}}{\Gamma(1 - \aleph(\alpha))} \iota(\alpha) - \int_0^{j_2} \frac{(j_2 - \alpha)^{-\aleph(\alpha)}}{\Gamma(1 - \aleph(\alpha))} \iota(\alpha) d\alpha \right| \right] \\
&\leq \frac{1}{\varsigma} \left[\left| \int_{j_1}^{j_2} \eta(\alpha, \iota_\alpha) d\alpha \right| \right. \\
&\quad \left. + \left| \int_0^{j_1} \frac{(j_2 - \alpha)^{-\aleph(\alpha)}}{\Gamma(1 - \aleph(\alpha))} \iota(\alpha) - \frac{(j_1 - \alpha)^{-\aleph(\alpha)}}{\Gamma(1 - \aleph(\alpha))} \iota(\alpha) d\alpha \right| \right. \\
&\quad \left. + \left| \int_{j_1}^{j_2} \frac{(j_2 - \alpha)^{-\aleph(\alpha)}}{\Gamma(1 - \aleph(\alpha))} \iota(\alpha) d\alpha \right| \right] \\
&\leq \frac{1}{\varsigma} \left[\int_{j_1}^{j_2} |\eta(\alpha, \iota_\alpha)| d\alpha \right. \\
&\quad + \int_0^{j_1} \left| \frac{1}{\Gamma(1 - \aleph(\alpha))} \right| |(j_2 - \alpha)^{-\aleph(\alpha)} - (j_1 - \alpha)^{-\aleph(\alpha)}| |\iota(\alpha)| d\alpha \\
&\quad + \int_{j_1}^{j_2} \left| \frac{(j_2 - \alpha)^{-\aleph(\alpha)}}{\Gamma(1 - \aleph(\alpha))} \right| |\iota(\alpha)| d\alpha \Big] \\
&\leq \frac{1}{\varsigma} \left[\int_{j_1}^{j_2} \varrho(\alpha) + \vartheta(\alpha) \|\iota\|_C + M_\Gamma \|\iota\|_\mathbb{E} \int_0^{j_1} (j_1 - \alpha)^{-\aleph(\alpha)} - (j_2 - \alpha)^{-\aleph(\alpha)} d\alpha \right. \\
&\quad + \int_{j_1}^{j_2} \left| \frac{(j_2 - \alpha)^{-\aleph(\alpha)}}{\Gamma(1 - \aleph(\alpha))} \right| |\iota(\alpha)| d\alpha \Big] \\
&\leq \frac{1}{\varsigma} \left[\|\varrho\|_\infty (j_2 - j_1) + \|\vartheta\|_\infty \|\iota\|_\mathbb{E} (j_2 - j_1) \right. \\
&\quad + M_\Gamma \|\iota\|_\mathbb{E} \int_0^{j_1} \tilde{t}^{-\aleph(\alpha)} \left[\left(\frac{j_1 - \alpha}{\tilde{t}} \right)^{-\aleph(\alpha)} - \left(\frac{j_2 - \alpha}{\tilde{t}} \right)^{-\aleph(\alpha)} \right] d\alpha \\
&\quad + M_\Gamma \|\iota\|_\mathbb{E} \int_{j_1}^{j_2} \tilde{t}^{-\aleph(\alpha)} \left(\frac{j_2 - \alpha}{\tilde{t}} \right)^{-\aleph(\alpha)} d\alpha \Big] \\
&\leq \frac{1}{\varsigma} \left[\left(\|\varrho\|_\infty + \|\vartheta\|_\infty \|\iota\|_\mathbb{E} \right) (j_2 - j_1) \right. \\
&\quad + M_\Gamma \|\iota\|_\mathbb{E} \tilde{t}^* \int_0^{j_1} \left[\left(\frac{j_1 - \alpha}{\tilde{t}} \right)^{-\aleph^*} - \left(\frac{j_2 - \alpha}{\tilde{t}} \right)^{-\aleph^*} \right] d\alpha \\
&\quad + M_\Gamma \|\iota\|_\mathbb{E} \tilde{t}^* \int_{j_1}^{j_2} \left(\frac{j_2 - \alpha}{\tilde{t}} \right)^{-\aleph^*} d\alpha \Big] \\
&\leq \frac{1}{\varsigma} \left[\left(\|\varrho\|_\infty + \|\vartheta\|_\infty \|\iota\|_\mathbb{E} \right) (j_2 - j_1) \right. \\
&\quad + \left. \left[(j_1)^{1-\aleph^*} - (j_2)^{1-\aleph^*} + 2(j_2 - j_1)^{1-\aleph^*} \right] \frac{M_\Gamma \|\iota\|_\mathbb{E} \tilde{t}^*}{\tilde{t}^{-\aleph^*} (1 - \aleph^*)} \right].
\end{aligned}$$

Therefore, $\|N\iota(j_2) - N\iota(j_1)\|$ tends to 0 as j_2 approaches j_1 , indicating that $N(B_R)$ is equicontinuous.

Based on the preceding three steps and applying the Arzelà–Ascoli theorem, we conclude that N is a completely continuous operator. Consequently, by virtue of Theorem 3, N possesses a fixed point ι within B_R , that represents a solution to FDVOIP (1). $\square$

4. Uniqueness Results

The subsequent result relies on Theorem (2). We initiate this section by establishing the following assumption.

(FY3) Assume that $\eta: \tilde{\varphi}_\tau \times C([-z, 0], \mathbb{R}) \rightarrow \mathbb{R}$ and that there exist constants $0 \leq \sigma < \min_{j \in \tilde{\varphi}_\tau} |\aleph(j)|$, and $\tilde{p} > 0$, such that $j^\sigma |\eta(j, \iota) - \eta(j, y)| \leq \tilde{p} \|\iota - y\|_C$ holds for all $\iota, y \in C([-z, 0], \mathbb{R})$, and $j \in \tilde{\varphi}_\tau$.

Theorem 5. Suppose that assumption (FY3) holds. If the following inequality is satisfied,

$$\frac{1}{\varsigma} \left(\frac{M_{\Gamma} \tilde{t}^* \tilde{t}}{1 - \aleph^*} + \tilde{p} \frac{\tilde{t}^{-\sigma+1}}{-\sigma+1} \right) < 1, \quad (7)$$

then FDVOIVP (1) possesses a unique solution within the space $\mathbb{E}$.

Proof. Let the operator

$$N: \mathbb{E} \rightarrow \mathbb{E},$$

defined as follows,

$$N(\iota)(j) = \begin{cases} \phi(j), & \text{if } j \in [-\bar{z}, 0], \\ \frac{1}{\varsigma} \left[\int_0^j \eta(\alpha, \iota_{\alpha}) d\alpha - \int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} \iota(\alpha) d\alpha \right], & \text{if } j \in [0, \tilde{t}]. \end{cases}$$

Let ι and ι^* be elements of $\mathbb{E}$. If j belongs to the interval $[-\bar{z}, 0]$, then

$$|N(\iota)(j) - N(\iota^*)(j)| = 0.$$

In the case where $j \in \tilde{\varphi}_{\tau}$, we have

$$\begin{aligned} & |N(\iota)(j) - N(\iota^*)(j)| \\ & \leq \frac{1}{\varsigma} \left[\int_0^j \alpha^{-\sigma} \alpha^{\sigma} |\eta(\alpha, \iota_{\alpha}) - \eta(\alpha, \iota_{\alpha}^*)| d\alpha \right. \\ & \quad \left. + \int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} |\iota^*(\alpha) - \iota(\alpha)| d\alpha \right] \\ & \leq \frac{1}{\varsigma} \left[\tilde{p} \int_0^j \alpha^{-\sigma} \|\iota_{\alpha} - \iota_{\alpha}^*\|_C d\alpha + M_{\Gamma} \|\iota - \iota^*\|_{\mathbb{E}} \int_0^j (j-\alpha)^{-\aleph(\alpha)} d\alpha \right] \\ & \leq \frac{1}{\varsigma} \left[\tilde{p} \|\iota - \iota^*\|_{\mathbb{E}} \frac{j^{-\sigma+1}}{-\sigma+1} + M_{\Gamma} \tilde{t}^* \|\iota - \iota^*\|_{\mathbb{E}} \int_0^j \left(\frac{j-\alpha}{\tilde{t}} \right)^{-\aleph^*} d\alpha \right] \\ & \leq \frac{1}{\varsigma} \left[\tilde{p} \frac{\tilde{t}^{-\sigma+1}}{-\sigma+1} \|\iota - \iota^*\|_{\mathbb{E}} + \frac{M_{\Gamma} \tilde{t}^*}{\tilde{t}^{-\aleph^*} (1-\aleph^*)} \|\iota - \iota^*\|_{\mathbb{E}} \right] \\ & \leq \frac{1}{\varsigma} \left[\tilde{p} \frac{\tilde{t}^{-\sigma+1}}{-\sigma+1} \|\iota - \iota^*\|_{\mathbb{E}} + \frac{M_{\Gamma} \tilde{t}^* \tilde{t}}{1-\aleph^*} \|\iota - \iota^*\|_{\mathbb{E}} \right] \\ & \leq \frac{1}{\varsigma} \left(\frac{M_{\Gamma} \tilde{t}^* \tilde{t}}{1-\aleph^*} + \tilde{p} \frac{\tilde{t}^{-\sigma+1}}{-\sigma+1} \right) \|\iota - \iota^*\|_{\mathbb{E}}. \end{aligned}$$

Therefore, as a consequence of Inequality (7), the operator N is a contraction mapping. Applying the Banach contraction principle, we conclude that the operator N possesses a unique fixed point, which corresponds to the unique solution of the FDVOIVP (1). $\square$

5. Ulam–Hyers Stability

Definition 5 ([26]). With regard to FDVOIVP (1), consider the following inequality:

$$|D_{0+}^{\aleph(j)} \kappa(j) + \varsigma \kappa'(j) - \eta(j, \kappa_j)| \leq \epsilon, \quad j \in \tilde{\varphi}_{\tau}. \quad (8)$$

We define FDVOIVP (1) to be UHS if there exists a constant $c_{\eta} > 0$ with the following property: for every $\epsilon > 0$ and for any solution $\kappa \in C(\tilde{\varphi}_{\tau}, \mathbb{R})$ satisfying the inequality (8), one can find a solution $\iota \in C(\tilde{\varphi}_{\tau}, \mathbb{R})$ of FDVOIVP (1), satisfying the following condition:

$$|\kappa(j) - \iota(j)| \leq c_{\eta} \epsilon, \quad j \in \tilde{\varphi}_{\tau}.$$

Theorem 6. Suppose that assumption (FY3) holds and that inequality (7) is valid. Then, FDVOIVP (1) exhibits Ulam–Hyers stability.

Proof. Let $\epsilon > 0$ be an arbitrary positive number, and let $\kappa(j)$ be a function in $C(\tilde{\varphi}_\tau, \mathbb{R})$ that satisfies the following inequality:

$$|D_{0+}^{\aleph(j)} \kappa(j) + \varsigma \kappa'(j) - \eta(j, \kappa_j)| \leq \epsilon, \quad j \in \tilde{\varphi}_\tau. \quad (9)$$

Integrating both sides of the aforementioned inequality (9), we arrive at

$$\left| \kappa(j) + \frac{1}{\varsigma} \left[\int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} \kappa(\alpha) \, d\alpha - \int_0^j \eta(\alpha, \kappa_\alpha) \, d\alpha \right] \right| \leq \epsilon \tilde{\iota}.$$

Now, consider $j \in \tilde{\varphi}_\tau$; then,

$$\begin{aligned} |\kappa(j) - \iota(j)| &= \left| \kappa(j) + \frac{1}{\varsigma} \left[\int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} \iota(\alpha) \, d\alpha - \int_0^j \eta(\alpha, \iota_\alpha) \, d\alpha \right] \right| \\ &\leq \left| \kappa(j) + \frac{1}{\varsigma} \left[\int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} \kappa(\alpha) \, d\alpha - \int_0^j \eta(\alpha, \kappa_\alpha) \, d\alpha \right] \right| \\ &\quad + \frac{1}{\varsigma} \left| \int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} \iota(\alpha) \, d\alpha - \int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} \kappa(\alpha) \, d\alpha \right| \\ &\quad + \frac{1}{\varsigma} \int_0^j |\eta(\alpha, \kappa_\alpha) - \eta(\alpha, \iota_\alpha)| \, d\alpha \\ &\leq \epsilon \tilde{\iota} + \frac{1}{\varsigma} \left[\int_0^j \frac{(j-\alpha)^{-\aleph(\alpha)}}{\Gamma(1-\aleph(\alpha))} |\iota(\alpha) - \kappa(\alpha)| \, d\alpha \right. \\ &\quad \left. + \int_0^j \alpha^{-\sigma} \alpha^\sigma |\eta(\alpha, \kappa_\alpha) - \eta(\alpha, \iota_\alpha)| \, d\alpha \right] \\ &\leq \epsilon \tilde{\iota} + \frac{1}{\varsigma} \left[M_\Gamma \tilde{\iota}^* \|\kappa - \iota\|_{\mathbb{E}} \int_0^j \left(\frac{j-\alpha}{\tilde{\iota}} \right)^{-\aleph^*} d\alpha + \tilde{p} \|\kappa - \iota\|_{\mathbb{E}} \frac{j^{-\sigma+1}}{-\sigma+1} \right] \\ &\leq \epsilon \tilde{\iota} + \frac{1}{\varsigma} \left[\frac{M_\Gamma \tilde{\iota}^*}{\tilde{\iota}^{-\aleph^*}} \frac{(j)^{1-\aleph^*}}{(1-\aleph^*)} \|\kappa - \iota\|_{\mathbb{E}} + \tilde{p} \frac{\tilde{\iota}^{-\sigma+1}}{-\sigma+1} \|\kappa - \iota\|_{\mathbb{E}} \right] \\ &\leq \epsilon \tilde{\iota} + \frac{1}{\varsigma} \left[\frac{M_\Gamma \tilde{\iota}^* \tilde{\iota}}{1-\aleph^*} \|\kappa - \iota\|_{\mathbb{E}} + \tilde{p} \frac{\tilde{\iota}^{-\sigma+1}}{-\sigma+1} \|\kappa - \iota\|_{\mathbb{E}} \right] \\ &\leq \epsilon \tilde{\iota} + \frac{1}{\varsigma} \left(\frac{M_\Gamma \tilde{\iota}^* \tilde{\iota}}{1-\aleph^*} + \tilde{p} \frac{\tilde{\iota}^{-\sigma+1}}{-\sigma+1} \right) \|\kappa - \iota\|_{\mathbb{E}}. \end{aligned}$$

Consequently,

$$\|\kappa - \iota\|_{\mathbb{E}} \left(1 - \frac{1}{\varsigma} \left[\frac{M_\Gamma \tilde{\iota}^* \tilde{\iota}}{1-\aleph^*} + \tilde{p} \frac{\tilde{\iota}^{-\sigma+1}}{-\sigma+1} \right] \right) \leq \epsilon \tilde{\iota}.$$

From this, we deduce that, for every $j \in \tilde{\varphi}_\tau$,

$$|\kappa(j) - \iota(j)| \leq \|\kappa - \iota\|_{\mathbb{E}} \leq \frac{\tilde{\iota}}{1 - \frac{1}{\varsigma} \left[\frac{M_\Gamma \tilde{\iota}^* \tilde{\iota}}{1-\aleph^*} + \tilde{p} \frac{\tilde{\iota}^{-\sigma+1}}{-\sigma+1} \right]} \epsilon = c_\eta \epsilon.$$

Therefore, FDVOIVP (1) exhibits Ulam–Hyers stability. $\square$

6. Numerical Examples

Example 1. Let the following FDVOIVP

$$\begin{cases} D^{\frac{1}{3}+\frac{1}{2}} \iota(j) + 5\iota'(j) = \log(j+1) + \pi \frac{\exp(-j+4)}{\sqrt{2\pi j+\frac{5}{6}}} (j^2) + 4 \cos(j) + \frac{\sqrt{\pi}}{3} \iota, & j \in [0, \tilde{\iota}] = [0, 1], \\ \iota(j) = \phi(j), & j \in [-\tilde{z}, 0]. \end{cases} \quad (10)$$

Setting $\varsigma = 5$ and defining $\aleph(j, \iota(j)) = \frac{1}{3} + \frac{1}{2}$, we observe that $\aleph$ is a continuous function with $0 < \aleph(j, \iota(j)) < \frac{1}{3} + \frac{1}{2} = \frac{5}{6} = \aleph^* < 1$. Furthermore, $\min_{j \in \tilde{\varphi}_\tau} |\aleph(j, \iota(j))| = \frac{1}{2}$. Now, consider the following:

$$\begin{aligned} j^{\frac{1}{3}} |\eta(j, \iota) - \eta(j, y)| &= j^{\frac{1}{3}} \left| \log(j+1) + \pi \frac{\exp(-j+4)}{\sqrt{2\pi j + \frac{5}{6}}} (j^2) + 4 \cos(j) + \frac{\sqrt{\pi}}{3} \iota \right. \\ &\quad \left. - \log(j+1) - \pi \frac{\exp(-j+4)}{\sqrt{2\pi j + \frac{5}{6}}} (j^2) - 4 \cos(j) - \frac{\sqrt{\pi}}{3} y \right| \\ &= j^{\frac{1}{3}} \left| \frac{\sqrt{\pi}}{3} \iota - \frac{\sqrt{\pi}}{3} y \right| \\ &\leq j^{\frac{1}{3}} \frac{\sqrt{\pi}}{3} |\iota - y| \\ &\leq \frac{\sqrt{\pi}}{3} |\iota - y|. \end{aligned}$$

Therefore, condition (FY 3) is satisfied with $\sigma = \frac{1}{3}$ and $\tilde{p} = \frac{\sqrt{\pi}}{3}$. In addition,

$$\frac{1}{\varsigma} \left[\tilde{p} \frac{\tilde{\tau}^{-\sigma+1}}{-\sigma+1} + 4M_\Gamma \tilde{\tau}^* \tilde{\iota} \right] = \frac{1}{5} \left[\frac{\sqrt{\pi}}{3} \frac{1}{\frac{2}{3}} + 4 \frac{1}{\sqrt{\pi}} \right] = \frac{1}{5} \left[\frac{\sqrt{\pi}}{2} + \frac{4}{\sqrt{\pi}} \right] = \frac{1}{5} \left[\frac{\pi+8}{2\sqrt{\pi}} \right] = \frac{11.14}{17.72} \approx 0.63 < 1.$$

According to Theorem (5), FDVOIVP (10) admits a unique solution. Further, by Theorem (6), FDVOIVP (10) is UHS.

Example 2. Let the FDVOIVP

$$\begin{cases} D^{\frac{1}{2} + \frac{1}{4}} \iota(j) + 8\iota'(j) = \frac{\log(j+1)}{j+1} \frac{8(j+1)}{4\sqrt{2\pi}} + (\exp(\sqrt{j+1})) + 3 \sin(j) + \frac{\pi}{2} \iota, & j \in [0, \tilde{\iota}] = [0, 1], \\ \iota(j) = \phi(j), & j \in [-\tilde{z}, 0]. \end{cases} \quad (11)$$

Setting $\varsigma = 8$, $\aleph(j, \iota(j)) = \frac{1}{2} + \frac{1}{4}$, it follows that $\aleph$ is a continuous function satisfying $0 < \aleph(j, \iota(j)) < \frac{1}{2} + \frac{1}{4} = \frac{6}{8} = \aleph^* < 1$. Moreover, we have $\min_{j \in \tilde{\varphi}_\tau} |\aleph(j, \iota(j))| = \frac{1}{4}$. Now, we examine the following:

$$\begin{aligned} j^{\frac{1}{8}} |\eta(j, \iota) - \eta(j, y)| &= j^{\frac{1}{8}} \left| \frac{\log(j+1)}{j+1} \frac{8(j+1)}{4\sqrt{2\pi}} + (\exp(\sqrt{j+1})) + 3 \sin(j) + \frac{\pi}{2} \iota \right. \\ &\quad \left. - \frac{\log(j+1)}{j+1} \frac{8(j+1)}{4\sqrt{2\pi}} - (\exp(\sqrt{j+1})) - 3 \sin(j) - \frac{\pi}{2} y \right| \\ &= j^{\frac{1}{8}} \left| \frac{\pi}{2} \iota - \frac{\pi}{2} y \right| \\ &\leq j^{\frac{1}{8}} \frac{\pi}{2} |\iota - y| \\ &\leq \frac{\pi}{2} |\iota - y|. \end{aligned}$$

Consequently, condition (FY 3) is fulfilled with $\sigma = \frac{1}{8}$ and $\tilde{p} = \frac{\pi}{2}$. Furthermore,

$$\frac{1}{8} \left[\tilde{p} \frac{\tilde{\tau}^{-\sigma+1}}{-\sigma+1} + 4M_\Gamma \tilde{\tau}^* \tilde{\iota} \right] = \frac{1}{8} \left[\frac{\pi}{2} \frac{1}{\frac{7}{8}} + 4 \times \left(\frac{1}{1.2254} \right) \right] = \frac{1}{8} \left[\frac{4\pi}{7} + 3.26 \right] = \frac{5.05}{8} \approx 0.63 < 1.$$

In light of Theorem (5), FDVOIVP (11) possesses a unique solution. Additionally, according to Theorem (6), FDVOIVP (11) is Ulam–Hyers stable.

7. Conclusions

This study explored the existence, uniqueness, and stability properties of solutions for a finite delay variable-order initial value problem, specifically FDVOIVP (1). Our

analysis relied on the Ulam–Hyers stability theorem (Theorem 6) and fixed-point theory (Theorems 4 and 5) to establish these results. We presented numerical examples to illustrate the applicability of the derived theoretical framework. The methodology employed in this work offers a notable departure from existing techniques, providing a more accessible and streamlined approach. The presented results furnish a practical means of analysis, minimizing the need for intricate computational procedures. Furthermore, this investigation opens avenues for future research, including the extension of these findings through the application of alternative fractional-order operators such as the (k, φ) -Hilfer operator and the modified Atangana–Baleanu–Caputo (ABC) operator. These results carry significant implications across diverse scientific disciplines and offer valuable tools for a range of applications.

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